ELEMENTARY AMENABLE GROUPS OF COHOMOLOGICAL DIMENSION 3

JONATHAN A. HILLMAN

Abstract. We show that torsion-free elementary amenable groups of Hirsch length $\leqslant 3$ are solvable, of derived length $\leqslant 3$. This class includes all solvable groups of cohomological dimension 3. We show also that groups in the latter subclass are either polycyclic, semidirect products $BS(1,n) \rtimes \mathbb{Z}$, or properly ascending HNN extensions with base $\mathbb{Z}^2$ or $\pi_1(Kb)$.

We show that finitely generated, torsion-free elementary amenable groups of Hirsch length 3 are in fact solvable minimax groups, of derived length $\leqslant 3$. We show also that such a group is finitely presentable if and only if it is constructible, and such groups are either polycyclic, semidirect products with base a solvable Baumslag-Solitar group, or properly ascending HNN extensions with base $\mathbb{Z}^2$ or $\pi_1(Kb)$. Our interest in this class of groups arose from recent work on aspherical 4-manifolds with non-empty boundary and elementary amenable fundamental group [4]. Such groups have cohomological dimension $\leqslant 3$ and are of type FP, and thus are in the class considered here. (One of the results of [4] is that the groups arising there are all either polycyclic or solvable Baumslag-Solitar groups, and so may be considered well understood.)

1. background

Let G be a torsion-free elementary amenable group of finite Hirsch length $h = h(G)$. Then G is virtually solvable [6], and so has a subgroup of finite index which is an extension of a finitely generated free abelian group $\mathbb{Z}^v$ by a nilpotent group [3]. Since $v \leqslant h < \infty$ we may assume that v is the virtual first Betti number of G, i.e., the maximum of the ranks of abelian quotients of subgroups of finite index in G. If $G \neq 1$ then $0 < v \leqslant h = h(G) \leqslant c.d.G \leqslant h + 1$.

We recall that the *Hirsch-Plotkin radical* $\sqrt{G}$ of a group G is the (unique) maximal locally nilpotent normal subgroup of the group. (For the groups G considered below, either $\sqrt{G}$ is abelian or G is virtually

Key words and phrases. cohomological dimension, elementary amenable, finitely presentable, Hirsch length, solvable, torsion-free.

nilpotent.) If G is solvable then $\sqrt{G}$ is its own centralizer in G (by the maximality assumption), and so the homomorphism from $G/\sqrt{G}$ to $Aut(\sqrt{G})$ induced by conjugation in G is a monomorphism.

A solvable group is *minimax* if it has a composition series whose sections are either finite or isomorphic to $\mathbb{Z}[\frac{1}{m}]$, for some $m > 0$. A group is *constructible* if it is in the smallest class containing the trivial group which is closed under finite extensions and HNN extensions [1]. If G is a torsion-free virtually solvable group group then $c.d.G = h \Leftrightarrow G$ is of type $FP \Leftrightarrow G$ is constructible [7].

Let $BS(m,n)$ be the Baumslag-Solitar group with presentation

$$\langle a, t \mid ta^m t^{-1} = a^n \rangle,$$

and let $\overline{BS}(m,n)$ be the metabelian quotient $BS(m,n)/\langle\langle a \rangle\rangle'$, where $\langle\langle a \rangle\rangle'$ is the commutator subgroup of the normal closure of the image of a in $BS(m,n)$. We may assume that $m > 0$ and $|n| \geqslant m$. (When $m = 1$ and $n = \pm 1$ we get $\mathbb{Z}^2$ and $\pi_1(Kb)$.) Since we are only interested in torsion-free groups we shall assume also that $(m,n) = 1$.

2. HIRSCH LENGTH 2

In this section we shall consider groups of Hirsch length 2, which arise naturally in the analysis of groups of Hirsch length 3. (Note also that some groups of Hirsch length 2 have cohomological dimension 3.)

Theorem 1. *Let G be a torsion-free elementary amenable group of Hirsch length 2. Then $\sqrt{G}$ is abelian, and either $\sqrt{G}$ has rank 1 and $G \cong \sqrt{G} \rtimes \mathbb{Z}$ or $\sqrt{G}$ has rank 2 and $[G : \sqrt{G}] \leqslant 2$.*

Proof. Since G is virtually solvable [6] and the lowest non-trivial term of the derived series of a solvable group is a non-trivial abelian normal subgroup, $\sqrt{G} \neq 1$. Since any two members of $\sqrt{G}$ generate a torsion-free nilpotent group of Hirsch length $\leqslant 2$ they commute. Hence $\sqrt{G}$ is abelian, of rank $r = 1$ or 2, say, and $h(G/\sqrt{G}) = 2 - r$.

Let $C = C_G(\sqrt{G})$ be the centralizer of $\sqrt{G}$ in G. If $N \leqslant C$ is a normal subgroup of G with locally finite image in $G/\sqrt{G}$ then N' is locally finite, by an easy extension of Schur's Theorem [8, 10.1.4]. Hence $N' = 1$, so N is abelian, and then $N \leqslant \sqrt{G}$, by the maximality of $\sqrt{G}$. Therefore any locally finite normal subgroup of $G/\sqrt{G}$ must act effectively on $\sqrt{G}$.

If $\sqrt{G}$ has rank 1 then $G/\sqrt{G}$ can have no non-trivial torsion normal subgroup. If $C \neq \sqrt{G}$ is infinite then it has an infinite abelian normal subgroup (since it is non-trivial, virtually solvable, and has no non-trivial torsion normal subgroup). But the preimage of any such subgroup in G is nilpotent (since it is a central extension of an abelian group). This contradicts the maximality of $\sqrt{G}$. Hence $= \sqrt{G}$ and so $G/\sqrt{G}$ acts effectively on $\sqrt{G}$. Since $h(G/\sqrt{G}) = 1$ and $Aut(\sqrt{G}) \leqslant \mathbb{Q}^\times$, and $G/\sqrt{G}$ has no normal torsion subgroup, we see that $G/\sqrt{G} \cong \mathbb{Z}$.

If $\sqrt{G}$ has rank 2 then $G/\sqrt{G}$ is a torsion group, and $Aut(\sqrt{G})$ is isomorphic to a subgroup of $GL(2, \mathbb{Q})$. If $G/\sqrt{G}$ is infinite then it must have an infinite locally finite normal subgroup (since it is a virtually solvable torsion group). But finite subgroups of $GL(2, \mathbb{Q})$ have order dividing 24, and so $G/\sqrt{G}$ is finite. If g in G has image of finite order $p > 1$ in $G/\sqrt{G}$ then conjugation by g fixes $g^p \in \sqrt{G}$. It follows that g must have order 2 and its image in $GL(2, \mathbb{Q})$ must have determinant -1. Hence $[G : \sqrt{G}] \leqslant 2$. □

If G is finitely generated then $\sqrt{G}$ is finitely generated as a module over $\mathbb{Z}[G/\sqrt{G}]$, with respect to the action induced by conjugation in G. If $h(\sqrt{G}) = 1$ then $\sqrt{G}$ is not finitely generated as an abelian group, while $G/\sqrt{G} \cong \mathbb{Z}$. Hence $\mathbb{Z}[G/\sqrt{G}] \cong \mathbb{Z}[t, t^{-1}]$, and the action of t is multiplication by some $\frac{n}{m} \in \mathbb{Q} \setminus \{0, \pm 1\}$, since $\sqrt{G}$ is torsion-free and of rank 1. After replacing t by t^{-1}, if necessary, we may assume that $\sqrt{G} \cong \mathbb{Z}[t, t^{-1}]/(mt - n)$, for some m, n with $(m, n) = 1$ and $|n| > m > 0$. Hence $G \cong \overline{BS}(m, n)$. Then $c.d.G = 2 \Leftrightarrow G$ is finitely presentable $\Leftrightarrow m = 1$ [5].

If G is finitely generated and $h(\sqrt{G}) = 2$ then $G \cong \mathbb{Z}^2$ or $\pi_1(Kb)$, and so $c.d.G = 2$.

Let $\mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at 2, in which all odd integers are invertible, and let $\mathbb{Z}_{(2)}$ act on $\mathbb{Q}$ through the surjection to $\mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)} \cong \mathbb{Z}^\times = \{\pm 1\}$. Let $\mathbb{Q} \otimes Kb$ be the extension of $\mathbb{Z}_{(2)}$ by $\mathbb{Q}$ with this action. Then if $h = 2$ and G is not finitely generated it is either a subgroup of $\mathbb{Q} \rtimes_{\frac{m}{n}} \mathbb{Z}$, for some nonzero m, n with $(m, n) = 1$ (if $h(\sqrt{G}) = 1$), or is a subgroup of $\mathbb{Q} \otimes Kb$ (if $h(\sqrt{G}) = 2$). Every such group has cohomological dimension 3.

3. Hirsch length 3

Suppose now that $h(G) = 3$. Then $h(\sqrt{G}) = 1$, 2 or 3.

Theorem 2. *Let G be a torsion-free elementary amenable group of Hirsch length 3. If $h(\sqrt{G}) = 1$ then $\sqrt{G}$ is abelian and $G/\sqrt{G} \cong \mathbb{Z}^2$. If $h(\sqrt{G}) = 2$ then $\sqrt{G}$ is abelian and $G/\sqrt{G} \cong \mathbb{Z}$, D_∞ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If $h(\sqrt{G}) = 3$ then G is virtually nilpotent. In all cases, G has derived length at most 3.*

Proof. If $h(\sqrt{G}) = 1$ then $\sqrt{G}$ is isomorphic to a subgroup of $\mathbb{Q}$ and (as in Theorem 1) $G/\sqrt{G}$ has no locally finite normal subgroup. Since $C_G(\sqrt{G})$ is virtually solvable, it follows that $C_G(\sqrt{G}) = \sqrt{G}$ and so $G/\sqrt{G}$ embeds in $Aut(\sqrt{G})$, which is isomorphic to a subgroup of $\mathbb{Q}^\times$. Hence $G/\sqrt{G} \cong \mathbb{Z}^2$, and so G has derived length 2.

If $h(\sqrt{G}) = 2$ then $\sqrt{G}$ is abelian and (as in Theorem 1 again) the maximal locally finite normal subgroup of $G/\sqrt{G}$ has order at most 2. Since $G/\sqrt{G}$ is virtually solvable and $h(G/\sqrt{G}) = 1$, it has an abelian normal subgroup A of rank 1, which we may assume torsion-free and of finite index in $G/\sqrt{G}$. Moreover, $G/\sqrt{G}$ embeds in $Aut(\sqrt{G})$, which is now isomorphic to a subgroup of $GL(2, \mathbb{Q})$. No nontrivial element of A can have both eigenvalues roots of unity, for otherwise $C_G(\sqrt{G}) > \sqrt{G}$. Since the eigenvalues of A have degree $\leqslant 2$ over $\mathbb{Q}$, it follows that no nontrivial element of A can be infinitely divisible in A. Hence $G/\sqrt{G}$ is virtually $\mathbb{Z}$, and so it is either $\mathbb{Z}$ or the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, or an extension of one of these by $\mathbb{Z}/2\mathbb{Z}$.

If G has a normal subgroup H such that $H/\sqrt{G} \cong \mathbb{Z}/2\mathbb{Z}$ then conjugation in G must preserve the filtration $0 < H' < \sqrt{G}$ of $\sqrt{G}$. Therefore elements of G' act nilpotently on $\sqrt{G}$, and so G/H cannot be D_∞. Thus if $h(\sqrt{G}) = 2$ then $G/\sqrt{G} \cong \mathbb{Z}$, D_∞ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and G has derived length 2, 3 or 2, respectively.

If $h(\sqrt{G}) = 3$ then $h(G/\sqrt{G}) = 0$, and so G is virtually nilpotent. Since iterated commutators live in finitely generated subgroups, the derived length of G is the maximum of the derived lengths of its finitely generated subgroups. Finitely generated torsion-free virtually nilpotent groups of Hirsch length 3 are polycyclic, and are fundamental groups of $\mathbb{N}il^3$-manifolds. These are Seifert fibred over flat 2-orbifolds without reflector curves, and so these groups have derived length $\leqslant 3$. Hence G has derived length $\leqslant 3$. $\square$

Corollary 3. *If G is finitely generated then it is a minimax group.*

Proof. If $h(\sqrt{G}) = 1$ and G is finitely generated then $\sqrt{G}$ is finitely generated as a $\mathbb{Z}[\mathbb{Z}^2]$-module. Since it is also torsion-free and of rank 1 as an abelian group, it is in fact a cyclic $\mathbb{Z}[\mathbb{Z}^2]$-module. Hence $\sqrt{G} \cong \mathbb{Z}[\frac{1}{D}]$ for some $D > 0$.

If $h(\sqrt{G}) = 2$ then G has a subgroup K of index $\leqslant 2$ such that $K/\sqrt{G} \cong \mathbb{Z}$. If G is finitely generated then K is also finitely generated. Then $\sqrt{G}$ is again finitely generated as a Λ-module, and is torsion-free and of rank 2 as an abelian group. Hence it is isomorphic as a group to a subgroup of $\mathbb{Z}[\frac{1}{m}]^2$, for some $m > 0$.

If G is finitely generated and $h(\sqrt{G}) = 3$ then G is polycyclic. In all cases it is clear that G is a minimax group. $\square$

We shall consider more closely the cases with $h(\sqrt{G}) = 1$ or 2.

Lemma 4. *If G is finitely generated and $h(\sqrt{G}) = 1$ then G is a semidirect product $\overline{BS}(m,n) \rtimes \mathbb{Z}$, where mn has at least 2 distinct prime factors.*

Proof. If $h(\sqrt{G}) = 1$ then G has a presentation

$$\langle a, t, u \mid ta^m t^{-1} = a^n,\ ua^p u^{-1} = a^q,\ utu^{-1} = ta^e,\ \langle\langle a \rangle\rangle'\ \rangle.$$

for some nonzero m, n, p, q with $(m,n) = (p,q) = 1$ and some $e \in \mathbb{Z}[\frac{1}{D}]$, where D is the product of the prime factors of $mnpq$. Hence $\sqrt{G} \cong \mathbb{Z}[\frac{1}{D}]$. After a change of basis for $G/\sqrt{G}$, if necessary, we may assume that mn has a prime factor which does not divide pq. We may further arrange that p divides m and q divides n, after replacing t by tu^N or tu^{-N} for N large enough, if necessary. Hence D is the product of the prime factors of mn. It must have at least 2 prime factors, since $G/\sqrt{G} \cong \mathbb{Z}^2$ maps injectively to $Aut(\sqrt{G}) \cong \mathbb{Z}[\frac{1}{D}]^\times$.

Thus $G \cong \overline{BS}(m,n) \rtimes_\theta \mathbb{Z}$, for some automorphism θ of $\overline{BS}(m,n)$. $\square$

Theorem 5. *A finitely generated torsion-free elementary amenable group G of Hirsch length 3 is coherent if and only if it is FP_2 and $h(\sqrt{G}) \geqslant 2$.*

Proof. If G is coherent then it is finitely presentable and hence FP_2.

Suppose that $h(\sqrt{G}) = 1$. Then $\sqrt{G} \cong \mathbb{Z}[\frac{1}{D}]$ for some $D > 1$, and the image of $G/\sqrt{G}$ in $Aut(\sqrt{G}) \cong \mathbb{Z}[\frac{1}{D}]^\times$ has rank 2. Hence it contains a proper fraction $\frac{p}{q}$ with $p, q \neq \pm 1$, and so G has a subgroup isomorphic to $\overline{BS}(p,q)$. Since this subgroup is not even FP_2 [2], G is not coherent.

If $h(\sqrt{G}) = 2$ then we may assume that $G/\sqrt{G} \cong \mathbb{Z}$. If, moreover, G is FP_2 then G is an HNN extension with base a finitely generated subgroup of $\sqrt{G}$ [2], and the HNN extension is ascending, since G is solvable. Any finitely generated subgroup of G is either a subgroup of the base or is itself an ascending HNN extension with finitely generated base, and so is finitely presentable.

If $h(\sqrt{G}) = 3$ then G is polycyclic, and every subgroup is finitely presentable. $\square$

It remains an open question whether an FP_2 torsion-free solvable group G with $h(G) = 3$ and $h(\sqrt{G}) = 1$ must be finitely presentable. Note also that the argument shows that G is *almost coherent* (finitely generated subgroups are FP_2) if and only if it is coherent.

We shall assume next that $h(\sqrt{G})=2$ and that $G/\sqrt{G}\cong\mathbb{Z}$. Since $\mathbb{Q}\otimes\sqrt{G}\cong\mathbb{Q}^2$, the action of $G/\sqrt{G}$ on $\sqrt{G}$ by conjugation in G determines a conjugacy class of matrices M in $GL(2,\mathbb{Q})$. Hence $G\cong\sqrt{G}\rtimes_M\mathbb{Z}$.

Lemma 6. *A matrix $M\in GL(2,\mathbb{Q})$ is conjugate to an integral matrix if and only if $\det M$ and $tr\,M\in\mathbb{Z}$.*

Proof. These conditions are clearly necessary. If they hold then the characteristic polynomial is a monic polynomial with $\mathbb{Z}$ coefficents. If $x\in\mathbb{Q}^2$ is not an eigenvector for M then the subgroup generated by x and Mx is a lattice. Since M preserves this lattice, by the Cayley-Hamilton Theorem, it is conjugate to an integral matrix. □

If G is finitely generated then $\sqrt{G}$ is finitely generated as a $\mathbb{Z}[G/\sqrt{G}]$-module. It is finitely generated as an abelian group (and so G is polycyclic) $\Leftrightarrow$ M is conjugate to a matrix in $GL(2,\mathbb{Z})$ $\Leftrightarrow$ $\det M=\pm1$ and $tr\,M\in\mathbb{Z}$.

If G is FP_2 then G is an ascending HNN extension with base $\mathbb{Z}^2$ (as in Theorem 5 above). Hence M (or M^{-1}) must be conjugate to an integral matrix, and G is finitely presentable. On the other hand, if $G\cong\sqrt{G}\rtimes_M\mathbb{Z}$ and neither M nor M^{-1} is conjugate to an integral matrix then G cannot be FP_2.

We conclude this section by giving some examples realizing the other possibilities for $G/\sqrt{G}$ allowed for by Theorem 2. Torsion-free polycyclic groups G with $h(\sqrt{G})=2$ are $\mathbb{S}ol^3$-manifold groups. There are such groups with $G/\sqrt{G}\cong\mathbb{Z}$, D_∞ or $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. (The examples with $G/\sqrt{G}\cong D_\infty$ are fundamental groups of the unions of two twisted I-bundles over a torus along their boundaries.)

For instance, the group G with presentation

$$\langle u,v,y\mid uyu^{-1}=y^{-1},\ vyv^{-1}=v^{-2}y^{-1},\ v^2=u^2y\rangle$$

is a generalized free product with amalgamation $A*_C B$ where $A=\langle u,y\rangle\cong B=\langle v,u^2y\rangle\cong\pi_1(Kb)$ and $C=\langle u^2,y\rangle\cong\mathbb{Z}^2$. It is clear that $G/C\cong D_\infty$, and it is easy to check that $C=\sqrt{G}$.

If G is the group with presentation

$$\langle t,x,y\mid tx=xt,\ tyt^{-1}=y^n,\ xyx^{-1}=y^{-1}\rangle$$

then $\sqrt{G}$ is normally generated by x^2 and y, so $h(\sqrt{G})=2$ and $G/\sqrt{G}\cong\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

If $G/\sqrt{G}\cong D_\infty$ then G is generated by $\sqrt{G}$ and two elements u, v with squares in $\sqrt{G}$. The matrices in $GL(2,\mathbb{Q})$ corresponding to the actions of u and v have determinant -1. Hence $t=uv$ corresponds to a matrix with determinant 1. There are finitely generated examples of this type which are not polycyclic. For instance, let F be the group with presentation

$$\langle u, v, x, y \mid u^2 = x,\ uyu^{-1} = y^{-1},\ v^2 = xy,\ vy^3v^{-1} = x^2y^{-1}\rangle,$$

and let K be the normal closure of the image of $\{x, y\}$ in F. Then $F/K \cong D_\infty$ and $K/K' \cong \mathbb{Z}[\frac{1}{3}]^2$, and F/K' is torsion-free, solvable and $h(F/K') = 3$.

However, if such a group G is FP_2 then so is the subgroup generated by $\sqrt{G}$ and t. Hence this subgroup is an ascending HNN extension with finitely generated base $H \leqslant \sqrt{G}$ [2]. Since t maps $H \cong \mathbb{Z}^2$ into itself and has determinant 1 it must be an automorphism of H, and so G is polycyclic.

4. FINITELY PRESENTABLE IMPLIES CONSTRUCTIBLE

In this section we shall show that if a torsion-free solvable group G of Hirsch length 3 is finitely presentable then it is in fact constructible, and we shall describe all such groups.

If G is FP_2 and G/G' is infinite then G is an HNN extension $H*_\varphi$ with finitely generated base H [2], and the extension is ascending since G is solvable. Clearly $h(H) = h(G) - 1 \leqslant 2$, and $c.d.G \leqslant c.d.H + 1$. In fact $h(H)$ must be 2, for otherwise $H \cong \mathbb{Z}$ and $c.d.G = 2$. In our next theorem we shall need the stronger hypothesis that G be finitely presentable. (Homological methods do not seem to be useful here; a spectral sequence argument shows that $H_i(G;\mathbb{Z})$ is finite for all $i > 1$.)

Theorem 7. *Let G be a torsion-free solvable group of Hirsch length 3. Then G is finitely presentable if and only if it is constructible.*

Proof. If G is constructible then it is finitely presentable. Assume that G is finitely presentable. If $\sqrt{G}$ has rank 1 then G has a presentation

$$\langle a, t, u \mid ta^mt^{-1} = a^n,\ ua^pu^{-1} = a^q,\ utu^{-1}t^{-1} = C(a, t, u),\ R\ \rangle,$$

for some nonzero m, n, p, q with $(m, n) = (p, q) = 1$ and word $C(a, t, u)$ of weight 0 in each of t and u, and some finite set of relators R. Let D be the product of the prime factors of $mnpq$. Then $\sqrt{G} \cong \mathbb{Z}[\frac{1}{D}]$, and contains the image of c in G. As observed after Corollary 3, we may assume that p and q divide m and n, respectively and that mn has a prime factor which does not divide pq.

We may assume that each of the relations in R has weight 0 in each of t and u. Then we may write $C(a, t, u)$ and each relator in R as a product of conjugates $b_{i,j} = t^iu^jau^{-j}t^{-i}$ of a. Since R is finite the exponents i, j involved lie in a finite range $[-L, L]$, for some $L \geqslant 0$.

The relations imply that the normal closure of the image of a in G is $\sqrt{G} \cong \mathbb{Z}[\frac{1}{D}]$. Hence the images of the $b_{i,j}$s in G commute, and are powers of an element α represented by a word $w = W(a,t,u)$ which is a product of powers of (some of) the $b_{i,j}$s. In particular, $a = \alpha^N$ and $b_{i,j} = \alpha^{e(i,j)}$, for some exponents N and $e(i,j)$. Clearly $N = e(0,0)$.

It follows also that $t\alpha^m t^{-1} = \alpha^n$ and $u\alpha^p u^{-1} = \alpha^q$. Hence adjoining a new generator α and new relations

(1) $a = \alpha^N$;
(2) $t\alpha^m t^{-1} = \alpha^n$;
(3) $u\alpha^p u^{-1} = \alpha^q$;
(4) $\alpha = W(a,t,u)$; and
(5) $t^i u^j a u^{-j} t^{-i} = \alpha^{e(i,j)}$, for all $i,j \in [-L,L]$.

gives an equivalent presentation.

We may use the first relation to eliminate the generator a. Since the image of α in G generates an infinite cyclic subgroup, the relations R must be consequences of these, and so we may delete the relations in R. Moreover the relation $\alpha = W(a,t,u)$ collapses to a tautology, and so may also be deleted, and we may use the final set of relations to write $C(a,t,u)$ as a power of α. Since $tb_{i,j}t^{-1} = b_{i+1,j}$ and $ub_{i,j}u^{-1} = b_{i,j+1}$, we see that $e(i,j) = (\frac{n}{m})^i(\frac{q}{p})^j e(0,0)$, for all $i,j \in [-L,L]$. Since α generates the subgroup spanned by the $b_{i,j}$s it follows that $N = (mnpq)^L$ and $e(i,j) = N(\frac{n}{m})^i(\frac{q}{p})^j$ for $i,j \in [-L,L]$. Hence the final set of relations are consequences of the second and third relations.

Thus G has the finite presentation

$$\langle t,u,\alpha \mid t\alpha^m t^{-1} = \alpha^n,\ u\alpha^p u^{-1} = \alpha^q,\ utu^{-1}t^{-1} = \alpha^c\rangle,$$

for some $c \in \mathbb{Z}$. Since the subgroup generated by the images of t and α is isomorphic to $BS(m,n)$ and is solvable, either m or $n = 1$ [2].

If $h(\sqrt{G}) = 2$ then G has a subgroup J of index $\leqslant 2$ which is an ascending HNN extension with finitely generated base $H \leqslant \sqrt{G}$. Since $h(H) = 2$, we have $H \cong \mathbb{Z}^2$. Hence J is constructible, and G is also constructible.

If $h(\sqrt{G}) = 3$ then G is virtually nilpotent, and so is again constructible. ∎

Theorem 8. *Let G be a torsion-free elementary amenable group of Hirsch length 3. Then G is constructible if and only if either*

(1) $G \cong BS(1,n) \rtimes_\theta \mathbb{Z}$ for some $n \neq 0$ or ± 1 and some $\theta \in Aut(BS(1,n))$;
(2) $G \cong H*_\varphi$ is a properly ascending HNN extension with base $H \cong \mathbb{Z}^2$ or $\pi_1(Kb)$; or

(3) G is polycyclic.

Proof. It shall suffice to show that if G is constructible then it is one of the groups listed here, as they are all clearly constructible. We may also assume that G is not polycyclic, and so $h(\sqrt{G}) = 1$ or 2.

Since G is constructible it has a subgroup J of finite index which is an ascending HNN extension with base a constructible solvable group of Hirsch length 2. Since G is not polycyclic, we may assume that $J = G$, by Theorem 2 (when $h(\sqrt{G}) = 1$) and by Theorem 2 with the observations towards the end of §3 (when $h(\sqrt{G}) = 2$). Constructible solvable groups of Hirsch length 2 are in turn Baumslag-Solitar groups $BS(1,m)$ with $m \neq 0$.

If $h(\sqrt{G}) = 1$ then $|m| > 1$ and $G \cong BS(1,m)*_\varphi$, for some injective endomorphism of $BS(1,m)$. We shall use the presentation for $BS(1,m)$ given in §2. After replacing a by $t^{-k}at^k$, if necessary, we may assume that $\varphi(a) = a^q$ and $\varphi(t) = ta^r$, for some $q \neq 0$ and r in $\mathbb{Z}$. Then G has a presentation

$$\langle a, t, u \mid tat^{-1} = a^m,\ uau^{-1} = a^q,\ utu^{-1} = ta^r\rangle.$$

Let $s = tu$ and $n = mq$. Then $sas^{-1} = a^n$, and the subgroup $H \cong BS(1,n)$ generated by a and s is normal in G. Conjugation by u generates an automorphism θ of H, since q is invertible in $\mathbb{Z}[\frac{1}{n}]$. Hence $G \cong BS(1,n) \rtimes_\theta \mathbb{Z}$, and so G is of type (1).

If $h(\sqrt{G}) = 2$ then $m = \pm 1$, and so $H \cong \mathbb{Z}^2$ or $\pi_1(Kb)$. Since the HNN extension is properly ascending, G is not polycyclic, and so G is of type (2). □

We have allowed an overlap between classes (1) and (2) in Theorem 8, for simplicity of formulation. Polycyclic groups of Hirsch length 3 are virtually semidirect products $\mathbb{Z}^2 \rtimes \mathbb{Z}$, and hence are ascending HNN extensions, but the extensions are not properly ascending, and so classes (2) and (3) are disjoint.

Taking into account the fact that solvable groups G with $c.d.G = h(G)$ are constructible [7], we may summarize the above two theorems as follows.

Corollary 9. *If G is a torsion-free elementary amenable group of Hirsch length 3 then $c.d.G = 3 \Leftrightarrow G$ is constructible $\Leftrightarrow G$ is finitely presentable $\Leftrightarrow G$ is one of the groups listed in Theorem 8 above.* □

References

[1] Baumslag, G. and Bieri, R. Constructable solvable groups, Math. Z. 151 (1976), 249–257.
[2] Bieri, R. and Strebel, R. Almost finitely presentable soluble groups, Comment. Math. Helv. 53 (1978), 258–278.
[3] Čarin, V. S. On soluble groups of type A_4, Mat. Sbornik 94 (1960), 895–914.
[4] Davis, J. F. and Hillman, J. A. Aspherical 4-manifolds with elementary amenable fundamental group, in preparation.
[5] Gildenhuys, D. Classification of soluble groups of cohomological dimension two, Math. Z. 166 (1979), 21–25.
[6] Hillman, J. A. and Linnell, P. A. Elementary amenable groups of finite Hirsch length are locally-finite by virtually solvable, J. Aust. Math. Soc. 52 (1992), 237–241.
[7] Kropholler, P. H. Cohomological dimension of soluble groups, J. Pure Appl. Alg. 43 (1986), 281–287.
[8] Robinson, D. J. S. *A Course in the Theory of Groups*, GTM 80, Springer-Verlag, Berlin – Heidelberg – New York (1982).

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
Email address: jonathan.hillman@sydney.edu.au