NILPOTENT GROUPS WITH BALANCED PRESENTATIONS

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Abstract. We determine the torsion free nilpotent groups with balanced presentations and which have Hirsch length ≤ 5 or nilpotency class $\leqslant 3$.

A presentation for a finitely generated group G is *balanced* if it has an equal number of generators and relations. (Clearly G has such a presentation if and only if $def(G) \geqslant 0$.) It is well-known that torsion-free nilpotent groups of Hirsch length $h \leqslant 3$ are either free abelian or are central extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$, and have balanced presentations. A finitely generated group G is *homologically balanced* if $\beta_2(N;R) \leqslant \beta_1(N;R)$ for all fields R. If G has a balanced presentation then it is homologically balanced. We shall give short arguments to show that there is one homologically balanced torsion free nilpotent group with Hirsch length $h = 4$ and none with $h = 5$. We show also that any such group with $h = 6$ must be generated by 2 elements.

Theorems 4 and 6 below were originally part of [5], in which it is observed that if both complementary regions of a closed hypersurface in S^4 have nilpotent fundamental groups then these groups have homologically balanced presentations on at most 3 generators. A MathSciNet search for "nilpotent AND Betti number" led to [2], where it is shown that in the parallel world of nilpotent Lie algebras, there are just 3 of dimension $\leqslant 7$ with $\beta_2 = \beta_1 = 2$. One has dimension 4 and the other two have dimension 6. There are direct connections between these aspects of nilpotency. If G is a finitely generated torsion-free nilpotent group let $G_{\mathbb{R}}$ be its Mal'cev completion, which is a connected nilpotent Lie group of dimension $h(G)$ in which G is a lattice. The coset space $M = G_{\mathbb{R}}/G$ is a $K(G,1)$ space, and so $H^q(G;\mathbb{R}) \cong H^q_{DR}(M)$, the de Rham cohomology of M. This in turn is the cohomology of the associated Lie algebra $\mathfrak{G}$ with coefficients in the trivial $\mathfrak{G}$-module $\mathbb{R}$.

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Some information is lost in passing from groups to Lie algebras, since commensurable nilpotent groups have isomorphic completions. Moreover, we do not know whether the 6-dimensional examples arise from nilpotent groups. From dimension 7 onwards, there are uncountably many isomorphism classes of nilpotent Lie algebras, and so most do not derive from discrete groups. Nevertheless, the results of [2] provide alternative proofs of Theorems 6 and 7, and suggest other group theoretic results.

In §1 we set out our notation and prove three lemmas. We consider groups of Hirsch length 4 or 5 in §2. These groups are all nilpotent of class $\leqslant 4$. In §3 we show that there are no other homologically balanced groups which are nilpotent of class $\leqslant 3$, and in the final section we show that if a nilpotent group G of Hirch length 6 is homologically balanced then $\beta_1(G;\mathbb{Q})=2$.

1. PRELIMINARIES

If G is a group ζG, G', G^τ and $G_{[n]}$ shall denote the centre, the commutator subgroup, the preimage in G of the torsion subgroup of G/G' and the nth stage of the lower central series, respectively. . The nilpotency class of G is n if $G_{[n]}\neq 1$ and $G_{[n+1]}=1$. If G is a finitely generated nilpotent group then it has a finite composition series with cyclic factors, and the Hirsch length $h(G)$ is the number of infinite factors in such a series. If G is torsion free and not cyclic then G has nilpotency class $\leqslant h(G)+1-\beta_1(G;\mathbb{Q})$, which is strictly less than $h(G)$.

Let $F(r)$ be the free group of rank r. If $G=F(m)/F(m)_{[k]}$ then $H_2(G;\mathbb{Z})\cong F(m)_{[k]}/F(m)_{[k+1]}$, by the five-term exact sequence of low degree for the homology of G as a quotient of $F(m)$. This abelian group has rank $\frac{1}{k}\Sigma_{d|k}\mu(d)m^{\frac{k}{d}}$, where μ is the Möbius function, by the Witt formulae [8, Theorems 5.11 and 5.12]. Hence $H_2(G;\mathbb{Z})$ has rank $>m$ unless $\beta=1$ or $\beta=2$ and $k\leqslant 3$ or $\beta=3$ and $k=2$. Thus the only relatively free nilpotent groups with $\beta_2(G;\mathbb{Q})\leqslant\beta_1(G;\mathbb{Q})$ are $G\cong\mathbb{Z}^\beta$ with $\beta\leqslant 3$ or $G\cong F(2)/F(2)_{[3]}$. These groups each have balanced presentations and Hirsch length $h(G)\leqslant 3$. Summing the ranks of sections given by the Witt formulae, we see that $h(F(2)/F(2)_{[4]})=5$, $h(F(2)/F(2)_{[5]})=8$, $h(F(3)/F(3)_{[4]})=14$ and $h(F(3)/F(3)_{[5]})=38$.

The following lemma is standard (see [6]), but we include it here for convenience.

Lemma 1. *Let G be a finitely generated nilpotent group. Then G can be generated by d elements, where $d=\max\{\beta_1(G;\mathbb{F}_p)|p\ prime\}$.*

Proof. If G is abelian this is an easy consequence of the structure theorem for finitely generated abelian groups. In general, if G is nilpotent and the image in G/G' of a subset $X \subset G$ generates G/G' then X generates G. (See [8, Lemma 5.9] or [9, 12.1.5].) $\square$

The expression for d is clearly best possible.

A finitely generated nilpotent group G has a maximal finite normal subgroup T and G/T is torsion-free [9, 5.2.7]. The projection of G onto G/T induces isomorphisms on homology with coefficients $\mathbb{Q}$. We may use these observations to extend arguments based on the torsion-free cases to all finitely generated nilpotent groups. We use also without further comment the facts that if $h(G) = h$ then G is an orientable PD_h-group (Poincaré duality group of formal dimension h) over $\mathbb{Q}$, and that $\chi(G) = 0$. See [1, Theorem 9.10].

Lemma 2. *Let G be a finitely generated nilpotent group. Then there is a torsion free nilpotent group $\widehat{G}$ which can be generated by $\beta = \beta_1(G)$ elements, and such that $\beta_i(\widehat{G}) = \beta_i(G)$ for all i.*

Proof. Let $f : F(3) \to G$ be a homomorphism such that $H_1(f)$ is an isomorphism, and let T be the torsion subgroup of G and $J = \mathrm{Im}(f)$. Then T is a finite normal subgroup and $[G : J]$ is finite. Let $\widehat{G} = J/T \cap J$. Let K be the intersection of the conjugates of J in G. Then K is a normal subgroup of finite index in each of G and J, and so the inclusions of K into these groups induce isomorphisms on rational homology. Similarly, the projection of J onto $\widehat{G}$ induces an isomorphism on rational homology. Hence $\beta_i(\widehat{G}) = \beta_i(G)$ for all i. $\square$

The following lemma was prompted by Lemma 2 of [2].

Lemma 3. *Let G be a group with a subgroup $Z \leqslant \zeta G \cap G'$ of rank $z \geqslant 1$, and let $\overline{G} = G/Z$ and R be $\mathbb{Z}$ or a field. Let $\overline{\beta}_i = \beta_1(\overline{G}; R)$. Then $\overline{\beta}_2 \geqslant z$ and*

$$\overline{\beta}_2 - z + \max\{\overline{\beta}_1 z - \overline{\beta}_3, 0\} \leqslant \beta_2(G; R) \leqslant \overline{\beta}_2 - z + \overline{\beta}_1 z + \binom{z}{2}.$$

Proof. The quotient $\overline{G}$ acts trivially on the cohomology of Z, since Z is central in G. Hence the $E^2_{1,1}$ term of the cohomology LHS spectral sequence for G as a central extension of $\overline{G}$ has rank $\overline{\beta}_1 z$. The $E^2_{2,0}$ term has rank $\overline{\beta}_2$, the $E^2_{0,1}$ term has rank z, the $E^2_{3,0}$ term has rank $\overline{\beta}_3$, and the $E^2_{0,2}$ term has rank $\binom{z}{2}$. The differential $d^2_{2,0}$ must be surjective, since $Z \leqslant G'$, and so $\overline{\beta}_2 \geqslant z$. The lemma follows easily. $\square$

When $Z \cong \mathbb{Z}$ the spectral sequence reduces to the Gysin sequence associated to G as an extension of $\overline{G}$ by $\mathbb{Z}$, and the bounds given by

Lemma 3 simplify to

$$\beta_2(\overline{G};R)-1\leqslant\beta_2(G;R)\leqslant\beta_2(\overline{G};R)-1+\overline{\beta}_1.$$

Note also that if $h=h(\overline{G})\leqslant 6$ (and $R=\mathbb{Q}$) then $\beta_3(\overline{G})$ is determined by $\beta_1(\overline{G})$ and $\beta_2(\overline{G})$, via Poincaré duality and the condition $\chi(\overline{G})=0$.

If $G=\widetilde{F}/Z$ where $\widetilde{F}$ is a relatively free nilpotent group and $Z\leqslant\zeta\widetilde{F}\cap\widetilde{F}'$ then the right hand inequality gives a lower bound for $\beta_2(G)$.

2. HIRSCH LENGTH $\leqslant 5$

Lubotzky has extended the Golod-Shafarevich argument to show that if a nilpotent group G can be generated by d elements and p is a prime such that $d=\beta_1(G;\mathbb{F}_p)$ (as in Lemma 1) then either $\beta_2(G;\mathbb{F}_p)>\frac{d^2}{4}$ or $G\cong\mathbb{Z}$ or $\mathbb{Z}^2$. Hence homologically balanced nilpotent groups can be generated by 3 elements [6].

Theorem 4. *There is just one torsion-free nilpotent group of Hirsch length* 4 *which is homologically balanced.*

Proof. Let N be a torsion-free nilpotent group of Hirsch length 4. If R is a field then $\beta_1(N;R)\geqslant 2$, since N is not virtually cyclic, and $\beta_2(N;R)=2(\beta_1(N;R)-1)$, since N is an orientable PD_4-group and $\chi(N)=0$. Hence if $\beta_2(N;R)\leqslant\beta_1(N;R)$ then $\beta_1(N;R)=2$. If N is homologically balanced this holds for all fields R, and so $N/N'\cong\mathbb{Z}^2$. Therefore N can be generated by 2 elements. Since N' is nilpotent and $h(N')=2$, we see that $N'\cong\mathbb{Z}^2$ also.

The central quotient $N/\zeta N$ is torsion-free [9, 5.2.19], and not cyclic, for otherwise N would be abelian. Hence $\zeta N\leqslant N'$. Since N is a quotient of $F(2)$ and $h(N)=4$, we must have $N/N_{3]}\cong F(2)/F(2)_{[3]}$, and so $N_{[3]}\cong\mathbb{Z}$ and N has nilpotency class 3. Moreover, $\zeta N=N_{[3]}<N'$. (See also [3, Theorem 1.5].) Since N is a central extension of $F(2)/F(2)_{[3]}$ by $\mathbb{Z}$, it has a presentation

$$\langle x,y,u,z\mid u=[x,y],\ [x,u]=z^a,\ [y,u]=z^b,\ xz=zx,\ yz=zy\rangle,$$

in which x,y represent a basis for $N/N_{[3]}$ and z represents a generator for $N_{[3]}$. Since $N/N'\cong\mathbb{Z}^2$ we must have $(a,b)=1$, and after a change of basis for $F(2)$ we may assume that $a=1$ and $b=0$. The relation $yz=zy$ is then a consequence of the others, and so the presentation simplifies to $\langle x,y\mid[x,[x,[x,y]]]=[y,[x,y]]=1\rangle$. □

Is every homologically balanced nilpotent group of Hirsch length 4 torsion-free?

An alternative argument uses the observation that the image of N in $Aut(N')$ preserves the flag $1<N_{[3]}<N'$. Since N is nilpotent and

$N/N' \cong \mathbb{Z}^2$ this image is infinite cyclic, and so $N \cong C \rtimes \mathbb{Z}$, where C is the centralizer of N' in N. Moreover, $C \cong \mathbb{Z}^3$, since $C/N' \cong \mathbb{Z}$. Hence $N \cong \mathbb{Z}^3 \rtimes_A \mathbb{Z}$, where $A \in SL(3,\mathbb{Z})$ is a triangular matrix. It is not hard to see that since N/N' is torsion free of rank 2 there is a basis for C such that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in SL(3,\mathbb{Z}).$$

Hence N has the 2-generator balanced presentation

$$\langle t, u \mid [t,[t,[t,u]]] = [u,[t,u]] = 1\rangle,$$

in which u corresponds to the column vector $(1,0,0)^{tr}$ in $\mathbb{Z}^3$.

Corollary 5. *This group is the only torsion-free nilpotent group of Hirsch length* 4 *which can be generated by* 2 *elements.*

Proof. If G is a nilpotent group which can be generated by 2 elements then either $G/G' \cong \mathbb{Z}^2$ or $h(G) \leqslant 1$. Hence if G is torsion-free of Hirsch length 4 then $\beta_1(G;R) = 2$ for all fields R. Since G is an orientable PD_4-group and $\chi(G) = 0$ it follows that G is homologically balanced, and so $G \cong N$. □

This group is a quotient of $\widetilde{F} = F(2)/F(2)_{[4]}$ by a maximal infinite cyclic subgroup of $\zeta\widetilde{F}$. The quotient of $\widetilde{F}$ by the subgroup generated by the pth power of a nontrivial central element is nilpotent of Hirsch length 4, and has a finite normal subgroup of order p. Thus the condition on torsion is necessary for this corollary.

As most of our results other than Theorem 4 involve showing that $\beta_2(G;\mathbb{Q}) > \beta_1(G;\mathbb{Q})$, for various groups G, we shall simplify the notation henceforth, so that $H_i(G)$, $H^i(G)$ and $\beta_i(G)$ denote homology, cohomology and Betti numbers with coefficients $\mathbb{Q}$, respectively.

Theorem 6. *Let N be a finitely generated nilpotent group of Hirsch length $h(N) = 5$. Then N is not homologically balanced.*

Proof. It shall suffice to show that $\beta_2(N) > \beta = \beta_1(N)$.

If G is any finitely generated group then the kernel of the homomorphism $\psi_G : \wedge^2 H^1(G) \to H^2(G)$ induced by cup product is isomorphic to $Hom(G^\tau/[G,G^\tau],\mathbb{Q})$ [4]. If G is solvable then $G^\tau/[G,G^\tau]$ has rank $\leqslant h(G^\tau) = h(G) - \beta_1(G)$. Hence

$$\beta_2(G) - \beta_1(G) \geqslant \binom{\beta_1(G}{2} - h(G).$$

Thus we may assume that $\beta \leqslant 3$. Since N is nilpotent and $h(N) > 1$ we must then have $\beta = 2$ or 3. The quotient of N by its maximal finite normal subgroup is torsion free [9, 5.2.7], and has the same rational Betti numbers as N, so we may also assume that N is torsion-free.

The intersection $N' \cap \zeta N$ is nontrivial [9, 5.2.1], and so we may choose a maximal infinite cyclic subgroup $A \leqslant N' \cap \zeta N$. Let $\overline{N} = N/A$. Then $h(\overline{N}) = 4$ and $\overline{N}$ is also torsion-free, since the preimage of any finite subgroup in N is torsion-free and virtually $\mathbb{Z}$. Hence $\overline{N}$ is an orientable PD_4-group. Moreover, $\beta_1(\overline{N}) = \beta < 4$, since $A \leqslant N'$, and so $\overline{N}^\tau/[\overline{N}, \overline{N}^\tau] \neq 1$.

Let $e \in H^2(\overline{N}; \mathbb{Z})$ classify the extension

$$0 \to A \to N \to \overline{N} \to 1.$$

There is an associated "Gysin" exact sequence [7, Example 5C]:

$$0 \to \mathbb{Q}e \to H^2(\overline{N}) \to H^2(N) \to H^1(\overline{N}) \xrightarrow{\cup e} H^3(\overline{N}) \to H^3(N) \to \dots$$

Suppose first that $\beta = 2$. Then $\beta_2(\overline{N}) = 2$ also, since $\chi(\overline{N}) = 0$. Hence $\psi_{\overline{N}} = 0$, since $\overline{N}^\tau/[\overline{N}, \overline{N}^\tau] \neq 1$ and $\binom{\beta}{2} = 1$. Since the cup product of $H^3(\overline{N})$ with $H^1(\overline{N})$ is a non-singular pairing, it follows that $\alpha \cup e = 0$ for all $\alpha \in H^1(\overline{N})$. Hence $\beta_2(N) = 2 - 1 + 2 = 3 > \beta$.

When $\beta = 3$ we must look a little more closely at the consequences of Poincaré duality. We note first that $\beta_2(\overline{N}) = 2(\beta - 1) = 4$. Since cup-product of odd-dimensional cohomology classes is skew-symmetric, for any $e \in H^2(\overline{N})$ the homomorphism $-\cup e$ from $H^1(\overline{N})$ to $H^3(\overline{N})$ has a skew-symmetric matrix, if these cohomology groups are given bases which are Kronecker dual with respect to the cup-product pairing into $H^4(\overline{N}) \cong \mathbb{Q}$. Since $\beta = 3$ is odd, $\det(- \cup e) = 0$, and so we see that $\beta_2(N) \geqslant \beta_2(\overline{N}) - 1 + 1 = 4 > \beta$ again. □

3. nilpotency class $\leqslant 3$

In this section we shall use the fact that nilpotent groups of class $\leqslant 3$ are closely related to central extensions of the corresponding free nilpotent groups to estimate the Betti numbers of low degree.

Theorem 7. *Let G be a finitely generated nilpotent group of class 2. If G is homologically balanced then $h(G) \leqslant 3$.*

Proof. Let $\beta = \beta_1(G)$ and $h = h(G)$. We may assume that G is infinite, and can be generated by at most 3 elements, and so $1 \leqslant \beta \leqslant 3$. Then G is a quotient of $F(3)/F(3)_{[3]}$, and so $h(G) \leqslant 6$. If $h = 4$ then $\beta = 3$, for otherwise G must have class > 2 [3, Theorem 1.5]. But then G is an orientable PD_4-group over $\mathbb{Q}$, and so $\beta_2(G) = 2(\beta - 1) = 4 > \beta$. We cannot have $h = 5$, by Theorem 6. If $h = 6$ then the epimorphism from

$F(3)/F(3)_{[3]}$ to $\overline{G}$ is an isomorphism, and so $\beta_2(G) = 8 > \beta$. Therefore $h \leqslant 3$. □

If G is an infinite nilpotent group which is homologically balanced and has nontrivial torsion must it be one of the groups $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$?

Theorem 8. *Let G be a finitely generated nilpotent group of class 3. If $h(G) > 4$ then $\beta_2(G) > \beta_1(G)$.*

Proof. We may assume that $h(G) \geqslant 6$, by Theorem 6, and that G is torsion free and is generated by $\beta \leqslant 3$ elements, by Lemma 2 and Lubotzky's result [6]. Two-generator nilpotent groups of class 3 are covered by Theorems 4 and 6, since $h(F(2)/F(2)_{[4]}) = 5$, and so we may also assume that $\beta = 3$. Then $h(G) \leqslant h(F(3)/F(3)_{[4]}) = 14$. Let $Z = G_{[3]}$ and $\overline{G} = G/Z$. Then Z is central in G and $Z < G'$ [9, 5.1.9].

If $\overline{G} \cong F(3)/F(3)_{[3]}$ then $\overline{\beta}_1 = 3$ and $\overline{\beta}_2 = 8$. Hence $\overline{\beta}_3 = 12$, since $\overline{G}$ is an orientable PD_6-group and $\chi(\overline{G}) = 0$. If $z = \operatorname{rk} Z \geqslant 4$ then Lemma 3 gives $\beta_2(G) \geqslant 8 + 2z - 12 \geqslant 4$, while if $z \leqslant 3$ then it gives $\beta_2(G) \geqslant 5$.

If $h(\overline{G}) = 5$ then $\overline{\beta}_2 \geqslant \beta_2(F(3)/F(3)_{[3]}) - 2 = 6$, by Lemma 3 (applied to $G = F(3)/F(3)_{[3]}$ and $Z \cong \mathbb{Z}$). Since $\overline{\beta}_3 = \overline{\beta}_2$, by Poincaré duality, it then follows that $\overline{\beta}_2 \leqslant \beta_2(F(3)/F(3)_{[3]}) + 1 = 9$. In this case we find that if $z \geqslant 3$ then $\beta_2(G) \geqslant 2z \geqslant 6$, while if $z < 3$ then $\beta_2(G) \geqslant 6 - 2 = 4$.

Finally, if $h(\overline{G}) = 4$ then $\overline{\beta}_3 = \overline{\beta}_1$, by Poincaré duality, and $\overline{\beta}_2 = 2(\overline{\beta}_1 - 1) = 4$, since $\chi(\overline{G}) = 0$. Since $h(G) \geqslant 6$, we have $z > 1$. Hence $\beta z > \overline{\beta}_3$, and so $\beta_2(G) \geqslant 2z + 1 \geqslant 5$, by Lemma 3.

In all cases $\beta_2(G) > \beta$. □

If G is any group then $G'' = [G', G'] \leqslant G_{[4]}$, and so all nilpotent groups of class 3 are metabelian. If G can be generated by two elements then we can improve this slightly. Let $\{x, y\}$ be a basis for $F(2)$. Then $F(2)'$ is the normal closure of $[x, y]$, so $F(2)''$ is generated by (conjugates of) elements of the form $[[x, y], \lambda[x, y]\lambda^{-1}]$, for $\lambda \in F(2)$. Since $[[x, y], \lambda[x, y]\lambda^{-1}] = [[x, y], [\lambda, [x, y]]]$ is in $F(2)_{[5]}$, for all such λ, the quotient $F(2)/F(2)_{[5]}$ is metabelian.

We may use this observation to push a little further.

Theorem 9. *Let G be a finitely generated nilpotent group of class 4 which can be generated by 2 elements. If $h = h(G) > 4$ then $\beta_2(G) > \beta_1(G)$.*

Proof. Since G is a finitely generated nilpotent group of class 4 which can be generated by 2 elements it is a quotient of $F(2)/F(2)_{[5]}$, and so

$h \leqslant 8$. Moreover $G/G' \cong \mathbb{Z}^2$, since $h > 1$, and G' is abelian (of rank $\leqslant 6$), since $F(2)/F(2)_{[5]}$ is metabelian.

We may assume that $h(G) \neq 5$, by Theorem 6. Suppose that $h = 6$. Then $A = G'$ has rank 4. We shall use the LHS spectral sequence

$$H_p(\mathbb{Z}^2; H_q(A)) \Rightarrow H_{p+q}(G)$$

for G as an extension of $\mathbb{Z}^2$ by A to estimate $\beta_2(G)$. In this case there are no nonzero differentials beginning or ending at the $(1,1)$ position, and so it is enough to estimate the rank of $H_1(\mathbb{Z}^2; \mathbb{Q}\otimes A)$. The homology groups $H_i(\mathbb{Z}^2; \mathbb{Q} \otimes A)$ may be computed from the complex

$$0 \to \mathbb{Q} \otimes A \to \mathbb{Q} \otimes A^2 \to \mathbb{Q} \otimes A \to 0$$

arising from the standard resolution of the augmentation $\mathbb{Z}[\mathbb{Z}^2]$-module. Let $b_i = \mathrm{rk}\, H_i(\mathbb{Z}^2; \mathbb{Q} \otimes A)$, for $i \geqslant 0$. Then $b_0 - b_1 + b_2 = 0$, since the above complex clearly has Euler characteristic 0. We have $b_0 = 1$, since $h(G_{[3]}) = 3$, and $b_2 = 2$, since $G_{[4]} < A$ is central and of rank 2. Therefore $\beta_2(G) \geqslant b_1 = 3$.

Thus either $h(G) = 7$ or 8. If $h(G/G_{[4]}) = 4$ then $\beta_2(G/G_{[4]}) = 2$, and so $h(G_{[4]}) \leqslant 2$, by Lemma 3. But then $h(G) \leqslant 6$. Therefore $G/G_{[4]} \cong F(2)/F(2)_{[4]}$ and $G_{[4]}$ has rank 2 or 3. The estimates of Lemma 3 then show that $\beta_2(G) > 2$. Hence $\beta_2(G) > \beta_1(G)$ in all cases. $\square$

Can we improve these results to cover all metabelian nilpotent groups?

4. Hirsch length 6

The Lie algebra results of [2] provide alternative confirmations of Theorems 6 and 7, and imply that if a nilpotent group G of Hirsch length 6 is homologically balanced then $\beta_1(G; \mathbb{Q}) = 2$. We shall give a group-theoretic argument for the latter claim, based the fact that if $G = F(3)/F(3)_{[3]}$ then $\beta_2(G) = 8$.

Theorem 10. *Let G be a finitely generated nilpotent group of Hirsch length 6. If $\beta = \beta_1(G) = 3$ then $\beta_2(G) > \beta$, and so G is not homologically balanced.*

Proof. We may assume that G is torsion-free and there is an epimorphism $f : F(3) \to G$, by Lemma 2. Since $\beta = 3$ and $h(G) = 6$, the quotient $G_{[2]}/G_{[3]}$ must be infinite, and so $h(G_{[3]}) \leqslant 2$.

We may assume that $h(G_{[3]}) = 2$ by Theorem 8. Then $G_{[3]} \cong \mathbb{Z}^2$ and $G_{[2]}/G_{[3]}$ has rank 1. Hence cup product from $\wedge^2 H^1(G)$ into $H^2(G)$ has rank 2 [4]. Let Z be an infinite cyclic subgroup of $\zeta G \cap G_{[3]}$, and let $\overline{G} = G/Z$. Then $\beta_2(\overline{G}) > \beta_1(\overline{G})$, by Theorem 6.

If $\beta_2(G)=\beta$ then $\beta_2(\overline{G})=4$, by Lemma 3, and then cup product with e maps $H^1(\overline{G})$ injectively to $H^3(\overline{G})$, by the exactness of the Gysin sequence for the extension. (See Theorem 6.) Let a_1,a_2,a_3 be a basis for $H^1(\overline{G})$. Then the cup product of $\langle a_1\cup e, a_2\cup e, a_3\cup e\rangle$ with the image of $\wedge^2H^1(\overline{G})$ in $H^2(\overline{G})$ is trivial. But this contradicts the fact that the cup-product pairing of $H^2(\overline{G})$ with $H^3(\overline{G})$ into $H^5(\overline{G})\cong\mathbb{Q}$ is non-degenerate, since $\overline{G}$ is an orientable PD_5-group over $\mathbb{Q}$. Therefore we must again have $\beta_2(G)>\beta$. □

Corollary 11. *If $h(G)=6$ and $\beta_2(G)=\beta_1(G)$ then $\beta_i(G)=2$ for $0<i<6$.*

Proof. Since $\beta_1(G)\leqslant 3$ [6], we must have $\beta_2(G)=\beta_1(G)=2$, by the theorem, and the corollary then follows since G is an orientable PD_6 group over $\mathbb{Q}$ and $\chi(G)=0$. □

An argument similar to that of Theorem 10 shows that if $h(G)=6$ and $\beta_2(G)=\beta_1(G)=2$ then $h(G_{[3]})=3$ and $h(G_{[4]})=2$. Since G cannot have class 4, by Theorem 9, $h(G_{[5]})=1$. We may assume without loss of generality that G is a torsion-free quotient of $F(2)$, by Lemma 2, and so $G_{[5]}\cong\mathbb{Z}$. In this case $b_2=b_0=1$, and so $b_1=2$, in the notation of Theorem 9. Are there any metabelian examples?

Beyond this, are there any examples at all of homologically balanced nilpotent groups of Hirsch length >6?

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