REGULARIZING EFFECTS OF HOMOGENEOUS EVOLUTION EQUATIONS; THE CASE OF HOMOGENEITY ORDER ZERO

DANIEL HAUER AND JOSÉ M. MAZÓN

CONTENTS

ABSTRACT. In this paper, we develop a functional analytical theory for establishing that mild solutions of first-order Cauchy problems involving homogeneous operators of order zero are strong solutions; in particular, the first-order time derivative satisfies a global regularity estimate depending only on the initial value and the positive time. We apply those results to the Cauchy problem associated with the total variational flow operator and the nonlocal fractional 1-Laplace operator.

1. INTRODUCTION

In the pioneering work [7], Bénilan and Crandall showed that for the class of *homogeneous operators A of order $\alpha > 0$ with $\alpha \neq 1$*, defined on a normed space $(X, \|\cdot\|_X)$, every solution of the differential inclusion

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni 0 \tag{1.1}$$

satisfies the global regularity estimate

$$\limsup_{h\to 0+} \frac{\|u(t+h)-u(t)\|_X}{h} \leq 2\,L \frac{\|u_0\|_X}{|\alpha-1|}\frac{1}{t} \qquad \text{for every } t>0. \tag{1.2}$$

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Here, $A \subseteq X \times X$ might be multi-valued and is called *homogeneous of order* α if

$$A(\lambda u) = \lambda^{\alpha} Au \qquad \text{for all } \lambda \geq 0 \text{ and } u \in D(A). \tag{1.3}$$

Moreover, to obtain (1.2), it is assumed that there is a family $\{T_t\}_{t\geq 0}$ associated with A of Lipschitz continuous mappings T_t on X of constant L such that

$$u(t) = T_t u_0 \qquad \text{for every } t \geq 0, \tag{1.4}$$

is (in some given sense) a solution of (1.1) for some initial value $u_0 \in X$. We refer to Definition 3.2 and Definition 3.5 for the different notions of solutions.

Further, if X is equipped with a partial ordering "$\leq$" such that $(X, \leq)$ defines an ordered vector space, and if for this ordering, the family $\{T_t\}_{t\geq 0}$ is *order-preserving* (that is, (2.17) below holds), then every *positive*[1] solution u of (1.1) satisfies the point-wise estimate

$$(\alpha - 1)\frac{\mathrm{d}u}{\mathrm{d}t_+}(t) \geq -\frac{u}{t} \qquad \text{in } \mathcal{D}' \text{ for every } t > 0. \tag{1.5}$$

Estimates of the form (1.2) describe an instantaneous and global *regularizing effect* of solutions u of (1.1), since they imply that the solution u of (1.1) is locally Lipschitz continuous in $t \in (0, +\infty)$. Further (1.5) provides a rate of dissipativity involved in the differential inclusion (1.1).

It is the aim of this paper to extend the theory developed in [7] to the important case $\alpha = 0$; in other words, for the class of *homogeneous operators A of order zero* (see Definition 2.1 below). Important examples of this class of operators include the (negative) *total variational flow operator* $Au = -\Delta_1 u := -\mathrm{div}\left(\frac{Du}{|Du|}\right)$, also known as (negative) 1-*Laplacian*, or the 1-*fractional Laplacian*

$$Au = (-\Delta_1)^s u(x) := PV \int_{\Sigma} \frac{u(y) - u(x)}{|u(y) - u(x)|} \frac{\mathrm{d}y}{|x - y|^{d+s}}, \qquad s \in (0, 1).$$

In our first main result (Theorem 2.3), we establish the global regularity estimate (1.2) for order $\alpha = 0$ and for solutions u of *differential inclusions with a forcing term*:

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) \ni f(t) \qquad \text{on } (0, T), \tag{1.6}$$

where $f : [0, T] \to X$ is an integrable function, and $T > 0$. In Corollary 2.4 and Corollary 2.6, we provide the resulting inequality when $f \equiv 0$ and the right hand-side derivative $\frac{\mathrm{d}u}{\mathrm{d}t_+}(t)$ of u exists at $t > 0$.

In many applications (cf Section 5), X is given by the classical Lebesgue space $(L^r, \|\cdot\|_r)$, $(1 \leq r \leq \infty)$. If $\{T_t\}_{t\geq 0}$ is a semigroup satisfying an *L^q-L^r-regularity estimate*

$$\|T_t u_0\|_r \leq C\, e^{\omega t} \frac{\|u_0\|_q^{\gamma}}{t^{\delta}} \qquad \text{for all } t > 0, \text{ and } u_0 \in L^q, \tag{1.7}$$

for $\omega \in \mathbb{R}$, $\gamma = \gamma(q, r, d)$, $\delta = \delta(q, r, d) > 0$, and some (or for all) $1 \leq q < r$, then we show in Corollary 2.6 that combining (1.2) with (1.7) yields

$$\limsup_{h \to 0+} \frac{\|u(t+h) - u(t)\|_r}{h} \leq C\, L\, 2^{\delta+2}\, e^{\omega\, t} \frac{\|u_0\|_q^{\gamma}}{t^{\delta+1}}. \tag{1.8}$$

[1] Here, we call a measurable function u *positive* if $u \geq 0$ for the given partial ordering "$\leq$".

Regularity estimates similar to (1.7) have been studied recently by many authors (see, for example, [13, 19, 14] covering the linear theory, and [12] the nonlinear one and the references therein).

In Theorem 2.7, Corollary 2.9 and Corollary 2.10 we generalize the pointwise estimate (1.5) to the homogenous order $\alpha = 0$.

We emphasize that the regularizing effect of solutions u of (1.1) remains true with a slightly different inequality (see Corollary 2.12) if the homogeneous operator A is perturbed by a Lipschitz mapping F. This is quite surprising since F might not be homogenous and hence, the operator $A + F$ is also not homogeneous.

In Section 3, we consider the class of *quasi accretive operators* A (see Definition 3.1) and outline how the property that A is homogeneous of order zero is passed on to the *semigroup* $\{T_t\}_{t\geq 0}$ generated by $-A$ (see the paragraph after Definition 3.2). In particular, we discuss when solutions u of (1.1) are differentiable a.e. in $t > 0$.

The fact that every Lipschitz continuous mappings $u : [0, T] \to X$ is differentiable almost everywhere on $(0, T)$ depends on the underlying geometry of the given Banach space X; this property is well-known as the Radon-Nikodým property of a Banach space. The Lebesgue space L^1 has not this property, but alone from the physical point of view, L^1 is for many models not avoidable. In [8], Bénilan and Crandall developed the celebrated theory of *completely accretive* operators A (in L^1). For this class of operators, it is known that for each solutions u of (1.1) in L^1, the derivative $\frac{du}{dt}$ exists in L^1. These results have been extended recently to the notion of *quasi completely accretive* operators in [12]. In Section 4, we study regularity estimates of the form (1.2) for $\alpha = 0$ satisfied by solutions u of (1.1), where A is a quasi completely accretive operator of homogeneous order zero. In fact, the two operators $-\Delta_1$ and $(-\Delta_1)^s$ mentioned above, belong exactly to this class of operators. Thus, our two main examples of differential inclusions discussed in Section 5 are

$$\frac{du}{dt} - \operatorname{div}\left(\frac{Du}{|Du|}\right) + f(\cdot, u) \ni 0 \tag{1.9}$$

$$\frac{du}{dt} + PV \int_{\Sigma} \frac{(u(y) - u(x))}{|u(y) - u(x)|} \frac{dy}{|x-y|^{d+s}} + f(\cdot, u) \ni 0, \tag{1.10}$$

which are equipped, respectively, with some boundary conditions on a domain Σ in $\mathbb{R}^d$, $d \geq 1$. In (1.9) and (1.10), the function f is a Carathéodory function, which is Lipschitz continuous in the second variable with constant $\omega > 0$ uniformly with respect to the first variable (see Section 5 for more details).

Note, if the right hand-side derivative $\frac{du}{dt_+}(t)$ of a solution u of (1.1) exists at every $t \in (0, 1]$, then (1.2) for $\alpha = 0$ becomes

$$\|Au(t)\|_X \leq 2\,L \frac{\|u_0\|_X}{t} \qquad \text{for every } t > 0. \tag{1.11}$$

Here, it is worth mentioning that if the operator A in (1.1) is linear (that is, $\alpha = 1$), then inequality (1.11) means that $-A$ generates an analytic semigroup $\{T_t\}_{t\geq 0}$ (cf [4, 17]). Thus, it is interesting to see that a similar regularity inequality such as (1.11), in particular, holds for certain classes of nonlinear operators.

In addition, if $\|\cdot\|_X$ is the induced norm by an inner product $(\cdot,\cdot)_X$ of a Hilbert space X and A is a sub-differential operators $\partial\varphi$ on X, then inequality (1.11) is also satisfied by solutions of (1.1) (cf [11]). In [2], inequality (1.11) was shown to hold for solutions of (1.9) with $f \equiv 0$ and equipped with Neumann boundary conditions.

2. Main results

Suppose X is a linear vector space and $\|\cdot\|_X$ a semi-norm on X. Then, the main object of this paper is the following class of operators.

Definition 2.1. An operator A on X is said to be *homogeneous of order zero* if for every $u \in D(A)$ and $\lambda \geq 0$, one has that $\lambda u \in D(A)$, and A satisfies (1.3) for $\alpha = 0$.

Remark 2.2. It follows necessarily from (1.3) that for every homogeneous operator A of order $\alpha > 0$, one has that $0 \in A0$. But for homogeneous operators A of order zero, the property $0 \in A0$ does not need to hold.

Now, assume that for the operator A on X and for given $f : [0,T] \to X$ and $u_0 \in X$, the function $u \in C^1([0,T];X)$ is a classical solution of the differential inclusion (1.6) with forcing term f satisfying *initial value* $u(0) = u_0$. If A is homogeneous of order zero, then for $\lambda > 0$, the function

$$v(t) = \lambda^{-1}u(\lambda t), \qquad (t \in [0,T]),$$

satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = \frac{\mathrm{d}u}{\mathrm{d}t}(\lambda t) \in -A(u(\lambda t)) + f(\lambda t) = -A(v(t)) + f(\lambda t)$$

for every $t \in (0,T)$ with initial value $v(0) = \lambda^{-1}u(0) = \lambda^{-1}u_0$. Thus, if for every $t \in [0,T]$, we denote

$$T_t(u_0, f) := u(t) \qquad \text{for every } u_0 \text{ and } f, \tag{2.1}$$

where u is the unique classical solution u of (1.6) with initial value $u(0) = u_0$, then the above reasoning shows that the homogeneity of A is reflected in

$$\lambda^{-1}T_{\lambda t}(u_0, f) = T_t(\lambda^{-1}u_0, f(\lambda\cdot)) \qquad \text{for every } \lambda > 0, \tag{2.2}$$

and all $t \in [0,T]$. Identity (2.2) together with standard growth estimates of the form

$$\begin{aligned} &e^{-\omega t}\|T_t(u_0,f) - T_t(\hat{u}_0,\hat{f})\|_X \\ &\qquad \leq L e^{-\omega s}\|T_s(u_0,f) - T_s(\hat{u}_0,\hat{f})\|_X + L\int_s^t e^{-\omega r}\,\|f(r) - \hat{f}(r)\|_X\,\mathrm{d}r \end{aligned} \tag{2.3}$$

for every $0 \leq s \leq t (\leq T)$, (for some $\omega \in \mathbb{R}$ and $L \geq 1$) are the main ingredients to obtain global regularity estimates of the form (1.2). This leads to our first main result.

Theorem 2.3. *For a subset $C \subseteq X$, let $\{T_t\}_{t=0}^T$ be a family of mappings $T_t : C \times L^1(0,T;X) \to C$ satisfying (2.3), (2.2), and $T_t(0,0) \equiv 0$ for all $t \geq 0$. Then for every*

$u_0 \in C$, $f \in L^1(0,T;X)$, and $t \in (0,T]$, $h > 0$, one has that

$$\begin{aligned} &\|T_{t+h}(u_0,f) - T_t(u_0,f)\|_X \\ &\quad \le \tfrac{|h|}{t}\, L\, e^{\omega\, t} \left[2\|u_0\|_X + \left(1+\tfrac{h}{t}\right) \int_0^t e^{-\omega s} \left\| \frac{f(s+\frac{h}{t}s) - f(s)}{\frac{h}{t}} \right\|_X \mathrm{d}s \right. \\ &\qquad\qquad \left. + \int_0^t e^{-\omega s} \|f(s)\|_X \,\mathrm{d}s \right]. \end{aligned} \tag{2.4}$$

In particular, if

$$V(f,t) := \limsup_{\xi\to 0} \int_0^t e^{-\omega s} \left\| \frac{f(s+\xi s) - f(s)}{\xi} \right\|_X \mathrm{d}s,$$

then the family $\{T_t\}_{t\ge 0}$ satisfies

$$\begin{aligned} &\limsup_{h\to 0+} \left\| \frac{T_{t+h}(u_0,f) - T_t(u_0,f)}{h} \right\|_X \\ &\qquad \le \frac{Le^{\omega\, t}}{t} \left[2\|u_0\|_X + V(f,t) + \int_0^t e^{-\omega s}\|f(s)\|_X \,\mathrm{d}s \right]. \end{aligned} \tag{2.5}$$

for every $t > 0$, $u_0 \in C$, $f \in L^1(0,T;X)$, and if f is locally absolutely continuous and differentiable a.e. on $(0,T)$, then

$$\begin{aligned} &\limsup_{h\to 0+} \left\| \frac{T_{t+h}(u_0,f) - T_t(u_0,f)}{h} \right\|_X \\ &\quad \le \frac{Le^{\omega\, t}}{t} \left[2\|u_0\|_X + \int_0^t e^{-\omega s} s\|f'(s)\|_X \,\mathrm{d}s + \int_0^t e^{-\omega s}\|f(s)\|_X \,\mathrm{d}s \right]. \end{aligned} \tag{2.6}$$

Moreover, if the right hand-side derivative $\frac{\mathrm{d}}{\mathrm{d}t_+} T_t(u_0,f)$ exists (in X) at $t > 0$, then

$$\left\| \frac{\mathrm{d}T_t(u_0,f)}{\mathrm{d}t}_+ \right\|_X \le \frac{Le^{\omega\, t}}{t} \left[2\|u_0\|_X + V(t,f) + \int_0^t e^{-\omega s}\|f(s)\|_X \mathrm{d}s \right]. \tag{2.7}$$

Proof. Let $u_0 \in C$, $f \in L^1(0,T;X)$, and for $t > 0$, let $h \neq 0$ satisfying $1 + \frac{h}{t} \ge 0$. Then, choosing $\lambda = 1 + \frac{h}{t}$ in (2.2) gives

$$\begin{aligned} &T_{t+h}(u_0,f) - T_t(u_0,f) \\ &\quad = T_{\lambda t}(u_0,f) - T_t(u_0,f) \\ &\quad = \left(1+\tfrac{h}{t}\right) T_t\left[\left(1+\tfrac{h}{t}\right)^{-1} u_0, f(\cdot + \tfrac{h}{t}\cdot) \right] - T_t(u_0,f) \end{aligned} \tag{2.8}$$

and so,

$$\begin{aligned} &T_{t+h}(u_0,f) - T_t(u_0,f) \\ &\quad = \left(1+\tfrac{h}{t}\right) \left[T_t\left[\left(1+\tfrac{h}{t}\right)^{-1} u_0, f(\cdot + \tfrac{h}{t}\cdot) \right] - T_t(u_0, f(\cdot + \frac{h}{t}\cdot)) \right] \\ &\qquad + \left(1+\tfrac{h}{t}\right) \left[T_t\left[u_0, f(\cdot + \tfrac{h}{t}\cdot) \right] - T_t(u_0,f) \right] \\ &\qquad + \left[\left(1+\tfrac{h}{t}\right) - 1 \right] T_t(u_0,f). \end{aligned} \tag{2.9}$$

Thus, by applying (2.3) and since $T_t(0,0) \equiv 0$, one sees that

$$\begin{aligned}
&\|T_{t+h}(u_0,f) - T_t(u_0,f)\|_X\\
&\quad\le \left(1+\tfrac{h}{t}\right) \left\| T_t\left[\left(1+\tfrac{h}{t}\right)^{-1} u_0, f(\cdot + \tfrac{h}{t}\cdot)\right] - T_t(u_0, f(\cdot + \tfrac{h}{t}\cdot))\right\|_X\\
&\qquad + \left(1+\tfrac{h}{t}\right) \left\| T_t(u_0, f(\cdot + \tfrac{h}{t}\cdot)) - T_t(u_0,f)\right\|_X\\
&\qquad\qquad + \left[\left(1+\tfrac{h}{t}\right) - 1\right] \|T_t(u_0,f)\|_X\\
&\quad\le \left(1+\tfrac{h}{t}\right) L\, e^{\omega t} \left\| \left(1+\tfrac{h}{t}\right)^{-1} u_0 - u_0\right\|_X\\
&\qquad + \left(1+\tfrac{h}{t}\right) L \int_0^t e^{\omega(t-s)} \|f(s+\tfrac{h}{t}s) - f(s)\|_X \,\mathrm{d}s\\
&\qquad\qquad + L\, e^{\omega t} \left|\left(1+\tfrac{h}{t}\right) - 1\right| \left(\|u_0\|_X + \int_0^t e^{-\omega s} \|f(s)\|_X \,\mathrm{d}s\right).
\end{aligned}$$

From this is clear that (2.4)-(2.7) follows. □

In the case $f \equiv 0$, then the mapping T_t given by (2.1) only depends on the initial value u_0, that is,

$$T_t u_0 = T_t(u_0, 0) \qquad \text{for every } u_0 \text{ and } t \ge 0. \tag{2.10}$$

In this case, the estimates in Theorem 2.3 reduce to the following one.

Corollary 2.4. *Let $\{T_t\}_{t\ge 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ satisfying*

$$\|T_t u_0 - T_t \hat{u}_0\|_X \le L\, e^{\omega t}\, \|u_0 - \hat{u}_0\|_X \qquad \text{for all } t \ge 0,\ u,\ \hat{u} \in C, \tag{2.11}$$

$$\lambda^{-1}\, T_{\lambda t} u_0 = T_t[\lambda^{-1} u_0] \qquad \text{for all } \lambda > 0,\ t \ge 0 \text{ and } u_0 \in C, \tag{2.12}$$

and $T_t 0 \equiv 0$ for all $t \ge 0$. Then, for every $u_0 \in C$ and $t, h > 0$, one has that

$$\|T_{t+h} u_0 - T_t u_0\|_X \le 2\, \tfrac{h}{t}\, L\, e^{\omega t} \|u_0\|_X. \tag{2.13}$$

In particular, the family $\{T_t\}_{t\ge 0}$ satisfies

$$\limsup_{h\to 0+} \frac{\|T_{t+h}u_0 - T_t u_0\|_X}{h} \le 2 L e^{\omega t} \frac{\|u_0\|_X}{t} \qquad \text{for every } t > 0,\ u_0 \in C. \tag{2.14}$$

Moreover, if the right hand-side derivative $\frac{\mathrm{d}}{\mathrm{d}t_+} T_t u_0$ exists (in X) at $t > 0$, then

$$\left\| \frac{\mathrm{d} T_t u_0}{\mathrm{d}t_+}\right\|_X \le 2 L e^{\omega t} \frac{\|u_0\|_X}{t}. \tag{2.15}$$

For our next corollary, we recall the following well-known definition.

Definition 2.5. Let C be a subset of X. Then, a family $\{T_t\}_{t\ge 0}$ of mappings $T_t : C \to C$ is called a *semigroup* if $T_{t+s} u = T_t \circ T_s u$ for every $t, s \ge 0$, $u \in C$.

Corollary 2.6. *Let $\{T_t\}_{t\ge 0}$ be a semigroup of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ and suppose, there is a second vector space Y with semi-norm $\|\cdot\|_Y$ such that $\{T_t\}_{t\ge 0}$ satisfies the following Y-X-regularity estimate*

$$\|T_t u_0\|_X \le M\, e^{\hat{\omega} t} \frac{\|u_0\|_Y^{\gamma}}{t^{\delta}} \qquad \text{for every } t > 0 \text{ and } u_0 \in C \tag{2.16}$$

for some $M, \gamma, \delta > 0$ and $\hat{\omega} \in \mathbb{R}$. If for $u_0 \in C$, $\{T_t\}_{t\geq 0}$ satisfies (2.14), then

$$\limsup_{h\to 0+} \frac{\|T_{t+h}u_0 - T_t u_0\|_X}{h} \leq 2^{\delta+2} L\, M\, e^{\frac{1}{2}(\omega+\hat{\omega})t} \frac{\|u_0\|_Y^{\gamma}}{t^{\delta+1}}.$$

Moreover, if the right hand-side derivative $\frac{\mathrm{d}}{\mathrm{d}t_+} T_t u_0$ exists (in X) at $t > 0$, then

$$\left\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+}\right\|_X \leq 2^{\delta+2} L\, M\, e^{\frac{1}{2}(\omega+\hat{\omega})t} \frac{\|u_0\|_Y^{\gamma}}{t^{\delta+1}}.$$

Proof. Since $\{T_t\}_{t\geq 0}$ is a semigroup, one sees by (2.14) and (2.16) that

$$\begin{aligned}
\limsup_{h\to 0+} \frac{\|T_{t+h}u_0 - T_t u_0\|_X}{h} &= \limsup_{h\to 0+} \frac{\|T_{\frac{t}{2}+h}(T_{\frac{t}{2}}u_0) - T_{\frac{t}{2}}(T_{\frac{t}{2}}u_0)\|_X}{h} \\
&\leq 4\, L\, e^{\omega\frac{t}{2}} \frac{\|T_{t/2}u_0\|_X}{t} \\
&\leq 2^{\delta+2} L\, M\, e^{\frac{1}{2}(\omega+\hat{\omega})t} \frac{\|u_0\|_Y^{\gamma}}{t^{\delta+1}}.
\end{aligned}$$

□

Next, suppose that there is a partial ordering "$\leq$" on X such that $(X, \leq)$ is an ordered vector space. Then, we can state the following theorem.

Theorem 2.7. *Let $(X, \leq)$ be an ordered vector space, C be a subset of X, and $\{T_t\}_{t\geq 0}$ be a family of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ satisfying*

(2.17) *for every u_0, $\hat{u}_0 \in C$ satisfying $u_0 \leq \hat{u}_0$, one has $T_t u_0 \leq T_t \hat{u}_0$ for all $t \geq 0$.*

and

(2.18) $$\lambda^{-1}\, T_{\lambda t} u_0 = T_t[\lambda^{-1} u_0] \qquad \text{for all } \lambda > 0,\ t \geq 0 \text{ and } u_0 \in C.$$

Then for every $u_0 \in C$ satisfying $u_0 \geq 0$, one has

(2.19) $$\frac{T_{t+h}u_0 - T_t u_0}{h} \leq \frac{1}{t} T_t u_0 \qquad \text{for every } t, h > 0.$$

Before giving the proof of Theorem 2.7, we state the following definition.

Definition 2.8. If $(X, \leq)$ is an ordered vector space then a family $\{T_t\}_{t\geq 0}$ of mappings $T_t : C \to C$ defined on a subset $C \subseteq X$ is called *order preserving* if $\{T_t\}_{t\geq 0}$ satisfies (2.17).

Proof of Theorem 2.7. Since $\left(1 + \frac{h}{t}\right)^{-1} < 1$, one has that $\left(1 + \frac{h}{t}\right)^{-1} u_0 \leq u_0$. Then, by (2.8) for $f \equiv 0$ and (2.17), one finds

$$\begin{aligned}
T_{t+h}u_0 - T_t u_0 &= \left(1 + \tfrac{h}{t}\right) T_t \left[\left(1 + \tfrac{h}{t}\right)^{-1} u_0\right] - T_t u_0 \\
&= T_t \left[\left(1 + \tfrac{h}{t}\right)^{-1} u_0\right] - T_t u_0 + \tfrac{h}{t} T_t \left[\left(1 + \tfrac{h}{t}\right)^{-1} u_0\right] \\
&\leq \tfrac{h}{t} T_t \left[\left(1 + \tfrac{h}{t}\right)^{-1} u_0\right] \\
&\leq \tfrac{h}{t} T_t u_0,
\end{aligned}$$

from where one sees that (2.19) holds. □

By Theorem 2.7, if the derivative $\frac{\mathrm{d}}{\mathrm{d}t_+}T_t u_0$ exists (in X) at $t>0$, then we can state the following.

Corollary 2.9. *Under the hypotheses of Theorem 2.7, suppose that for $u_0 \in C$ satisfying $u_0 \geq 0$, the right hand-side derivative $\frac{\mathrm{d}}{\mathrm{d}t_+}T_t u_0$ exists (in X) at $t>0$, then*

$$\frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+} \leq \frac{1}{t} T_t u_0.$$

Further, we can conclude from Theorem 2.7 the following result.

Corollary 2.10. *In addition to the hypotheses of Theorem 2.7, suppose that there is a linear functional $\Lambda : X \to \mathbb{R}$ satisfying*

$$\Lambda x \geq 0 \qquad \text{for every } x \in X \text{ satisfying } x \geq 0 \tag{2.20}$$

$$\Lambda T_t u_0 = \Lambda u_0 \text{ for every } t \geq 0 \text{ and } u_0 \in X \text{ satisfying } u_0 \geq 0. \tag{2.21}$$

Then, the following estimate holds for each $\nu \in \{+,-\}$,

$$\Lambda[T_{t+h}u_0 - T_t u_0]^{\nu} \leq \frac{h}{t}\,\Lambda x \qquad \text{for all } t, h > 0,\ u_0 \in C \text{ with } u_0 \geq 0. \tag{2.22}$$

Example 2.11. If $X = L^q(\Sigma,\mu)$ for some Σ-measure space (Σ,μ) and $1 \leq q \leq \infty$, then an example for Λ satisfying (2.20) and (2.21) is given by

$$\Lambda x = \int_\Sigma x \,\mathrm{d}\mu \qquad \text{for every } x \in X.$$

Proof of Corollary 2.10. Let $u_0 \in C$ with $u_0 \geq 0$, and $t, h > 0$. Then we note first that by (2.21),

$$0 = \Lambda T_{t+h}u_0 - \Lambda T_t u_0 = \Lambda(T_{t+h}u_0 - T_t u_0)$$

and since

$$\Lambda(T_{t+h}u_0 - T_t u_0) = \Lambda\left[T_{t+h}u_0 - T_t u_0\right]^+ - \Lambda\left[T_{t+h}u_0 - T_t u_0\right]^-,$$

one has that

$$\Lambda\left[T_{t+h}u_0 - T_t u_0\right]^+ = \Lambda\left[T_{t+h}u_0 - T_t u_0\right]^-. \tag{2.23}$$

Further, by Theorem 2.7 and since $T_t u_0 \geq 0$, it follows from the definition of $[x]^+ = \max\{x,0\}$, $(x \in \mathbb{R})$, that

$$[T_{t+h}u_0 - T_t u_0]^+ \leq \frac{h}{t} T_t u_0. \tag{2.24}$$

By the linearity of Λ and by (2.20), one has that $x \leq y$ yields $\Lambda x \leq \Lambda y$. Thus applying Λ to (2.24) leads to (2.22) for $\nu =$ "$+$". Moreover, by (2.23), inequality (2.22) also holds for $\nu =$ "$-$". This completes the proof of this corollary. $\square$

For the last result of this section, we consider the following differential inclusion

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A(u(t)) + F(u(t)) \ni 0 \qquad \text{on } (0,+\infty), \tag{2.25}$$

for some operator $A \subseteq X \times X$ and a Lipschitz-continuous mapping $F : X \to X$ with Lipschitz constant $\omega \geq 0$ and satisfying $F(0) = 0$. As for the differential inclusion (1.6) and the case $f \equiv 0$, suppose, there is a subset $C \subseteq X$ and a family $\{T_t\}_{t\geq 0}$ of mappings $T_t : C \to C$ associated with A through the relation that for every given $u_0 \in C$, the function u defined by (1.4) is the unique solution of (2.25) with initial value $u(0) = u_0$. On the other hand, setting

$$f(t) := -F(u(t)), \qquad (t \geq 0), \tag{2.26}$$

one has that

$$T_t(u_0, f) = u(t) = T_t u_0 \qquad \text{for every } t \geq 0,\, u_0 \in C. \tag{2.27}$$

Thus, by Theorem 2.3 for $T = +\infty$ we have the following estimates.

Corollary 2.12. *Let $F : X \to X$ be a Lipschitz continuous mapping with Lipschitz-constant $\omega > 0$ and satisfying $F(0) = 0$. Suppose, there is a subset $C \subseteq X$, and a family $\{T_t\}_{t\geq 0}$ of mappings $T_t : C \to C$ satisfying*

$$\|T_t u_0\|_X \leq e^{\omega t}\, \|u_0\|_X \qquad \textit{for all } t \geq 0,\, u_0 \in C, \tag{2.28}$$

and in relation with (2.27), suppose that $\{T_t\}_{t\geq 0}$ satisfies (2.2) and (2.3) for f given by (2.26). Then for every $u_0 \in C$, and $t, h > 0$ such that $|h|/t < 1$, one has that

$$\left\| \frac{T_{t+h}u_0 - T_t u_0}{h} \right\|_X \leq \left[2e^{L2\omega \int_0^t e^{-\omega s} \mathrm{d}s} + \omega \int_0^t e^{L2\omega \int_s^t e^{-\omega r} \mathrm{d}r} \mathrm{d}s \right] \frac{e^{\omega t} L \|u_0\|_X}{t}. \tag{2.29}$$

Moreover, if the derivative $\frac{\mathrm{d}}{\mathrm{d}t} T_t u_0$ exists (in X) for a.e. $t > 0$, then

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t} \right\|_X \leq e^{\omega t} L \left[2e^{L\omega \int_0^t e^{-\omega s} s \,\mathrm{d}s} + \omega \int_0^t e^{L\omega \int_s^t e^{-\omega r} r \,\mathrm{d}r} \,\mathrm{d}s \right] \frac{\|u_0\|_X}{t} \tag{2.30}$$

for a.e. $t > 0$.

For the proof of this corollary, we will employ the following version of Gronwall's lemma.

Lemma 2.13. *Let $a \in L^1(0,T)$, $B : [0,T] \to \mathbb{R}$ be an absolutely continuous function, and $v \in L^\infty(0,T)$ satisfy*

$$v(t) \leq \int_0^t a(s) v(s) \,\mathrm{d}s + B(t) \qquad \textit{for a.e. } t \in (0,T).$$

Then,

$$v(t) \leq B(0)\, e^{\int_0^t a(s) \,\mathrm{d}s} + \int_0^t e^{\int_s^t a(r) \,\mathrm{d}r}\, B'(s) \,\mathrm{d}s \qquad \textit{for a.e. } t \in (0,T).$$

We now give the proof of Corollary 2.12.

Proof. Let $u_0 \in C$, and $t, h > 0$ such that $|h|/t < 1$. Then, by the hypotheses of this corollary, we are in the position to apply Theorem 2.3 to $T_t(u_0, f)$ for f given by (2.26). Then by (2.4), one finds

$$\begin{aligned} \left\| \frac{T_{t+h}u_0 - T_t u_0}{\frac{h}{t}} \right\|_X &\leq L\, e^{\omega t} \Big[2\|u_0\|_X + \int_0^t e^{-\omega s} \|F(T_s u_0)\|_X \mathrm{d}s + \\ &\qquad + \left(1 + \tfrac{h}{t}\right) \int_0^t e^{-\omega s} \left\| \frac{F(T_{s+\frac{h}{t}s} u_0) - F(T_s u_0)}{\frac{h}{t}} \right\|_X \mathrm{d}s \Big]. \end{aligned}$$

Since F is globally Lipschitz continuous with constant $\omega > 0$, $F(0) = 0$ and by (2.28), it follows that

$$\left\|\frac{T_{t+h}u_0 - T_t u_0}{\frac{h}{t}}\right\|_X \le L\, e^{\omega\, t}\Big[(2+\omega t)\|u_0\|_X + \\ + \left(1+\tfrac{h}{t}\right)\,\omega\int_0^t e^{-\omega s}\left\|\frac{T_{s+\frac{h}{t}s}u_0 - T_s u_0}{\frac{h}{t}}\right\|_X \,\mathrm{d}s\Big].$$

Since $|h|/t < 1$,

$$e^{-\omega\, t}\left\|\frac{T_{t+h}u_0 - T_t u_0}{\frac{h}{t}}\right\|_X \le L\,\Big[(2+\omega t)\|u_0\|_X + \\ +2\,\omega\int_0^t e^{-\omega s}\left\|\frac{T_{s+\frac{h}{t}s}u_0 - T_s u_0}{\frac{h}{t}}\right\|_X \,\mathrm{d}s\Big]. \tag{2.31}$$

Due to (2.31), we can apply Gronwall's lemma to

$$B(t) = L(2+\omega t)\,\|u_0\|_X \quad \text{and} \quad a(t) = L2\omega\, e^{-\omega t}.$$

Then, one sees that (2.29) holds. Now, suppose that the derivative $\frac{\mathrm{d}}{\mathrm{d}t}T_t u_0$ exists (in X) for a.e. $t > 0$, then by (2.7), the Lipschitz continuity of F and by (2.28), one one has that

$$e^{-\omega t}\, t\left\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}\right\|_X \le \left[2\|u_0\|_X + \omega\int_0^t e^{-\omega s} s\left\|\frac{\mathrm{d}T_s u_0}{\mathrm{d}s}\right\|_X \mathrm{d}s + \omega t\,\|u_0\|_X\right]$$

for a.e. $t > 0$. Now, applying Gronwall's lemma to

$$B(t) = L(2+\omega t)\,\|u_0\|_X \quad \text{and} \quad a(t) = L\omega\, e^{-\omega t}t,$$

leads to (2.30). This completes the proof of this corollary. □

3. ACCRETIVE OPERATORS OF HOMOGENEOUS ORDER ZERO

Suppose X is Banach space with norm $\|\cdot\|_X$. Then, we begin this section with the following definition.

Definition 3.1. For $\omega \in \mathbb{R}$, an operator A on X is called *ω-quasi m-accretive operator* on X if $A + \omega I$ is *accretive*, that is, for every (u,v), $(\hat{u},\hat{v}) \in A$ and every $\lambda \ge 0$,

$$\|u - \hat{u}\|_X \le \|u - \hat{u} + \lambda(\omega(u-\hat{u}) + v - \hat{v})\|_X.$$

and if for A the *range condition*

$$Rg(I + \lambda A) = X \qquad \text{for some (or equivalently, for all) } \lambda > 0,\, \lambda\,\omega < 1, \tag{3.1}$$

holds.

If A is ω-quasi m-accretive operator, then the classical existence theorem [9, Theorem 6.5] (cf [6, Corollary 4.2]), for every $u_0 \in \overline{D(A)}^X$ and $f \in L^1(0,T;X)$, there is a unique *mild* solution $u \in C([0,T];X)$ of (1.6).

Definition 3.2. For given $u_0 \in \overline{D(A)}^X$ and $f \in L^1(0,T;X)$, a function $u \in C([0,T];X)$ is called a *mild solution* of the inhomogeneous differential inclusion (1.6) with initial value u_0 if $u(0) = u_0$ and for every $\varepsilon > 0$, there is a *partition* $\tau_\varepsilon : 0 = t_0 < t_1 < \cdots < t_N = T$ and a *step function*

$$u_{\varepsilon,N}(t) = u_0\,\mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} u_i\,\mathbb{1}_{(t_{i-1},t_i]}(t) \qquad \text{for every } t \in [0,T]$$

satisfying

$$\begin{aligned} &t_i - t_{i-1} < \varepsilon \qquad \text{for all } i = 1,\dots,N,\\ &\sum_{N=1}^{N}\int_{t_{i-1}}^{t_i} \|f(t) - \overline{f}_i\|\,\mathrm{d}t < \varepsilon \qquad \text{where } \overline{f}_i := \frac{1}{t_i - t_{i-1}}\int_{t_{i-1}}^{t_i} f(t)\,\mathrm{d}t,\\ &\frac{u_i - u_{i-1}}{t_i - t_{i-1}} + Au_i \ni \overline{f}_i \qquad \text{for all } i = 1,\dots,N, \end{aligned}$$

and

$$\sup_{t\in[0,T]} \|u(t) - u_{\varepsilon,N}(t)\|_X < \varepsilon.$$

In particular, if A is ω-quasi m-accretive, and if for given $u_0 \in \overline{D(A)}^X$, $f \in L^1(0,T;X)$, the function $u : [0,T] \to X$ is the unique mild solution of (1.6) with initial value $u(0) = u_0$, then by (2.1) the family $\{T_t\}_{t=0}^T$ defines a *semigroup* of *ω-quasi contractions* $T_t : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ for $C = \overline{D(A)}^X$; that is, $\{T_t\}_{t=0}^T$ satisfies

- (*semigroup property*) $T_{t+s} = T_t \circ T_s$ for every $t, s \in [0,T]$;
- (*strong continuity*) for every $(u_0, f) \in \overline{D(A)}^X \times L^1(0,T;X)$, $t \mapsto T_t(u_0, f)$ belongs to $C([0,T];X)$;
- (*ω-quasi contractivity*) T_t satisfies (2.3)

Furthermore, keeping $f \equiv 0$ and only varying $u_0 \in \overline{D(A)}^X$, shows that by

$$T_t u_0 = T_t(u_0, 0) \qquad \text{for every } t \geq 0. \tag{2.10}$$

defines a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ of ω-quasi contractions $T_t : \overline{D(A)}^X \to \overline{D(A)}^X$. For the family $\{T_t\}_{t\geq 0}$ on $\overline{D(A)}^X$, the operator

$$A_0 := \left\{ (u_0, v) \in X \times X \;\middle|\; \lim_{h\downarrow 0} \frac{T_h(u_0,0) - u_0}{h} = v \text{ in } X \right\}$$

is an ω-quasi accretive well-defined mapping $A_0 : D(A_0) \to X$ and called the *infinitessimal generator* of $\{T_t\}_{t\geq 0}$. Under additional conditions on the geometry of the Banach space X (see Definition 3.7), one has that $A_0 \subseteq A$. Thus, we say (ignoring the abuse of details) that both families $\{T_t\}_{t=0}^T$ on $\overline{D(A)}^X \times L^1(0,T;X)$ and $\{T_t\}_{t\geq 0}$ on $\overline{D(A)}^X$ are *generated by* $-A$.

In application, usually X is given by the Lebesgue space $L^\infty(\Sigma,\mu)$ (or $L^r(\Sigma,\mu)$ for $1 \leq r < \infty$) and Y is given by $L^1(\Sigma,\mu)$ (or $L^r(\Sigma,\mu)$ for some $1 \leq q < r$) for some σ-finite measure space (Σ,μ). Then, L^1-L^∞-decay estimates are intimately connected with abstract Sobolev inequalities satisfied by the infinitesimal generator $-A$ of the semigroup $\{T_t\}_{t\geq 0}$. For more details to the linear semigroup theory we refer to the monograph [19] and to [12] for the nonlinear semigroup theory.

Moreover (cf [9, Chapter 4.3]), for given $u_0 \in \overline{D(A)}^X$ and any step function $f = \sum_{i=1}^N f_i \,\mathbb{1}_{(t_{i-1},t_i]} \in L^1(0,T;X)$, let $u : [0,T] \to X$ given by

$$u(t) = u_0 \,\mathbb{1}_{\{t=0\}}(t) + \sum_{i=1}^{N} u_i(t)\mathbb{1}_{(t_{i-1},t_i]}(t) \tag{3.2}$$

is the unique mild solution of (1.6), where u_i is the unique mild solution of

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + A(u_i(t)) \ni f_i \quad \text{on } (t_{i-1},t_i), \text{ and} \quad u_i(t_{i-1}) = u_{i-1}(t_{i-1}). \tag{3.3}$$

Then for every $i = 1,\dots,N$, the semigroup $\{T_t\}_{t=0}^T$ is obtained by the *exponential formula*

$$T_t(u(t_{i-1}), f_i) = u_i(t) = \lim_{n\to\infty} \left[J^{A_i}_{\frac{t-t_{i-1}}{n}}\right]^n u(t_{i-1}) \qquad \text{in } C([t_{i-1},t_i];X) \tag{3.4}$$

for every $i = 1,\dots,N$, where for $\mu > 0$, $J^{A_i}_\mu = (I+\mu A_i)^{-1}$ is the *resolvent operator* of the operator A_i and $A_i := \{(x, y - f_i) \,:\, (x,y) \in A\}$.

As for classical solutions, the fact that A is homogeneous of order zero, is also reflected in the notion of mild solution and so in $\{T_t\}_{t\geq 0}$. This is shown in our next lemma.

Lemma 3.3. *Let A be a ω-quasi m-accretive and $\{T_t\}_{t\geq 0}$ be the semigroup on $\overline{D(A)}^X \times L^1_{loc}([0,+\infty);X)$ generated by $-A$. If A is homogeneous of order zero, then $\{T_t\}_{t\geq 0}$ satisfies (2.2) for every $(u_0, f) \in \overline{D(A)}^X \times L^1(0,T;X)$.*

Proof. For every $\mu > 0$, $v \in X$, and $\lambda > 0$, one has that

$$J^{A_i}_\mu \left[\lambda^{-1} v\right] = u \qquad \text{if and only if} \qquad u + \mu A_i u \ni \lambda^{-1} v,$$

which if A is *homogeneous of order zero*, is equivalent to

$$\lambda u + \lambda\mu A_i(\lambda u) \ni v \qquad \text{or} \qquad J^{A_i}_{\lambda\mu} v = \lambda u.$$

Therefore,

$$\lambda^{-1} J^{A_i}_{\lambda\mu} v = J^{A_i}_\mu \left[\lambda^{-1} v\right] \qquad \text{for all } \lambda, \mu > 0, v \in X. \tag{3.5}$$

Now, for $u_0 \in \overline{D(A)}^X$ and a partition

$$\pi : 0 = t_0 < t_1 < \cdots < t_N = T \quad \text{ of } [0,T]$$

let $f = \sum_{i=1}^N f_i \mathbb{1}_{(t_{i-1},t_i]} \in L^1(0,T;X)$ be a step function and u be the unique mild solution of (1.6) for f. Then u is given by (3.2), were on each subinterval $(t_{i-1}, t_i]$, u_i is the unique mild solution of (3.3). For $t > 0$, $n \in \mathbb{N}$, and $\lambda \in (0,1]$, apply (3.5) to

$$\mu = \frac{t}{n} \qquad \text{and} \qquad v = J^{A_1}_{\lambda\frac{t}{n}}[\lambda^{-1} u_0].$$

Then,

$$\left[J^{A_1}_{\frac{t}{n}}\right]^2 [\lambda^{-1} u_0] = J^{A_1}_{\frac{t}{n}} \left[\lambda^{-1} J^{A_1}_{\lambda\frac{t}{n}} u_0\right] = \lambda^{-1} \left[J^{A_1}_{\lambda\frac{t}{n}}\right]^2 u_0.$$

Iterating this equation n-times, one finds that

$$\lambda^{-1}\left[J^{A_1}_{\lambda\frac{t}{n}}\right]^n u_0 = \left[J^{A_1}_{\frac{t}{n}}\right]^n\left[\lambda^{-1}u_0\right] \tag{3.6}$$

and so, by (3.4) sending $n \to +\infty$ in the latter equation, yields on the one site

$$\lim_{n\to+\infty}\lambda^{-1}\left[J^{A_1}_{\lambda\frac{t}{n}}\right]^n u_0 = \lambda^{-1}u_1(\lambda t) = \lambda^{-1}u(\lambda t)$$

for every $t \in [0, \frac{t_1}{\lambda}]$, and on the other side

$$\lim_{n\to+\infty}\left[J^{A_1}_{\frac{t}{n}}\right]^n\left[\lambda^{-1}u_0\right] = v(t)$$

for every $t \in [0, \frac{t_1}{\lambda}]$, where v is the unique mild solution of (3.3) for $i = 1$ on $(0, \frac{t_1}{\lambda})$ with initial value $v(0) = \lambda^{-1}u_0$. By uniqueness of the two limits, we have thereby shown that

$$\lambda^{-1}T_{\lambda t}(u_0, f_1) = T_t(\lambda^{-1}u_0, f_1\mathbb{1}_{(0,\frac{t_1}{\lambda}]}) \qquad \text{for every } t \in \left[0, \frac{t_1}{\lambda}\right].$$

Similarly, for every $i = 2, 3, \dots, N$, replacing in (3.6) u_0 by $u(t_{i-1})$ (where $u(t_{i-1}) = u(\lambda\frac{t_{i-1}}{\lambda}) = v(\frac{t_{i-1}}{\lambda})$), A_1 by A_i, and $\frac{t}{n}$ by $\frac{t-\frac{t_{i-1}}{\lambda}}{n}$ gives

$$\lambda^{-1}\left[J^{A_i}_{\lambda\frac{t-\frac{t_{i-1}}{\lambda}}{n}}\right]^n u(t_{i-1}) = \left[J^{A_i}_{\frac{t-\frac{t_{i-1}}{\lambda}}{n}}\right]^n\left[\lambda^{-1}v(\frac{t_{i-1}}{\lambda})\right]$$

and by sending $n \to +\infty$, limit (3.4) leads one one side to

$$\lim_{n\to+\infty}\lambda^{-1}\left[J^{A_i}_{\lambda\frac{t-\frac{t_{i-1}}{\lambda}}{n}}\right]^n u(t_{i-1}) = \lambda^{-1}u(\lambda t)$$

and on the other side,

$$\lim_{n\to+\infty}\left[J^{A_i}_{\frac{t-\frac{t_{i-1}}{\lambda}}{n}}\right]^n\left[\lambda^{-1}v(\frac{t_{i-1}}{\lambda})\right] = v(t)$$

for every $t \in \left[\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda}\right]$, where v is the unique mild solution of (3.3) for i on $\left(\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda}\right)$ with initial value $v(\frac{t_{i-1}}{\lambda}) = \lambda^{-1}v(\frac{t_{i-1}}{\lambda}) = \lambda^{-1}u(t_{i-1})$. Therefore, and since u is given by (3.2), we have shown that

$$\lambda^{-1}T_{\lambda t}(u(t_{i-1}), f_i) = T_t(\lambda^{-1}u(t_{i-1}), f_i\mathbb{1}_{(\frac{t_{i-1}}{\lambda},\frac{t_i}{\lambda}]}) \qquad \text{for } t \in \left[\frac{t_{i-1}}{\lambda}, \frac{t_i}{\lambda}\right].$$

Since for every step function f on a partition π of $[0, T]$, u is given by (3.2), we have thereby shown that (2.2) holds if f is a step function. Now, by (2.3), an approximation argument shows that if A is homogeneous of order zero, then the semigroup $\{T_t\}_{t\geq 0}$ on $\overline{D(A)}^X \times L^1(0, T; X)$ generated by $-A$ satisfies (2.2). $\square$

By the above Lemma and Theorem 2.3, we can now state the following.

Corollary 3.4. *For $\omega \in \mathbb{R}$, suppose A is an ω-quasi m-accretive operator on a Banach space X, and A is homogeneous of order zero satisfying $0 \in A0$. Then, for every $(u_0, f) \in \overline{D(A)}^X \times L^1(0,T;X)$, the semigroup $\{T_t\}_{t\geq 0}$ of mapping $T_t : \overline{D(A)}^X \times L^1(0,T;X) \to \overline{D(A)}^X$ generated by $-A$ satisfies* (2.4)-(2.7).

For having that regularity estimate (2.7) (respectively, (2.15)) is satisfied by the semigroup $\{T_t\}_{t\geq 0}$, one requires that each mild solution u of (1.6) (respectively, of (1.1)) is *differentiable* and a *stronger* notion of solutions of (1.6). The next definition is taken from [9, Definition 1.2] (cf [6, Chapter 4]).

Definition 3.5. A locally absolutely continuous function $u[0,T] :\to X$ is called a *strong solution* of differential inclusion (1.1) if u is differentiable a.e. on $(0,T)$, and for a.e. $t \in (0,T)$, $u(t) \in D(A)$ and $f(t) - \frac{\mathrm{d}u}{\mathrm{d}t}(t) \in A(u(t))$.

The next characterization of strong solutions of (1.1) highlights the important point of *a.e. differentiability*.

Proposition 3.6 ([9, Theorem 7.1])**.** *Let X be a Banach space, $f \in L^1(0,T;X)$ and for $\omega \in \mathbb{R}$, A be ω-quasi m-accretive in X. Then u is a strong solution of the differential inclusion* (1.6) *on $[0,T]$ if and only if u is a mild solution on $[0,T]$ and u is "absolutely continuous" on $[0,T]$ and differentiable a.e. on $(0,T)$.*

Of course, every strong solution u of (1.6) is a mild solution of (1.6), absolutely continuous and differentiable a.e. on $[0,T]$. Moreover, the differential inclusion (1.6) admits mild and Lipschitz continuous solutions if A is ω-quasi m-accretive in X (cf [9, Lemma 7.8]). But absolutely continuous vector-valued functions $u : [0,T] \to X$ are not, in general, differentiable a.e. on $(0,T)$. However, if one assumes additional geometric properties on X, then the latter implication holds true. Our next definition is taken from [9, Definition 7.6] (cf [5, Chapter 1]).

Definition 3.7. A Banach space X is said to have the *Radon-Nikodým property* if every absolutely continuous function $F : [a,b] \to X$, (a, $b \in \mathbb{R}$, $a < b$), is differentiable almost everywhere on (a,b).

Known examples of Banach spaces X admitting the Radon-Nikodým property are:

- **(Dunford-Pettis)** if $X = Y^*$ is separable, where Y^* is the dual space of a Banach space Y;
- if X is *reflexive*.

We emphasize that $X_1 = L^1(\Sigma,\mu)$, $X_2 = L^\infty(\Sigma,\mu)$, or $X_3 = C(\mathcal{M})$ for a σ-finite measure space (Σ,μ), or respectively, for a compact metric space $(\mathcal{M},d)$ don't have, in general, the Radon-Nikodým property (cf [5]). Thus, it is quite surprising that there is a class of operators A (namely, the class of *completely accretive operators*, see Section 4 below), for which the differential inclusion (1.6) nevertheless admits strong solutions (with values in $L^1(\Sigma,\mu)$ or $L^\infty(\Sigma,\mu)$).

Now, by Corollary 3.4 and Proposition 3.6, we can conclude the following results. We emphasize that one crucial point in the statement of Corollary 3.8 below is that due to the uniform estimate (2.7), one has that for all initial values $u_0 \in \overline{D(A)}^X$, the unique mild solution u of (1.6) satisfying $u(0) = u_0$ is a strong solution, and not only for $u_0 \in D(A)$.

Corollary 3.8. *For $\omega \in \mathbb{R}$, suppose A is an ω-quasi m-accretive operator on a Banach space X admitting the Radon-Nikodým property, and $\{T_t\}_{t\geq 0}$ is the semigroup on $\overline{D(A)}^X \times L^1(0,T;X)$ generated by $-A$. If A is homogeneous of order zero satisfying $0 \in A0$, then for every $u_0 \in \overline{D(A)}^X$ and $f \in BV(0,T;X)$, the unique mild solution u of (1.6) satisfying $u(0) = u_0$ is a strong solution and satisfies (2.7) for every $t > 0$.*

Now by Corollary 2.12 and Proposition 3.6, we obtain the following result when A is perturbed by a Lipschitz mapping.

Corollary 3.9. *Suppose X is a Banach space with the Radon-Nikodým property, $F : X \to X$ be a Lipschitz continuous mapping with Lipschitz-constant $\omega > 0$ satisfying $F(0) = 0$, A an m-accretive operator on X, and $\{T_t\}_{t\geq 0}$ is the semigroup on $\overline{D(A)}^X$ generated by $-(A+F)$. If A is homogeneous of order zero satisfying $0 \in A0$, then (2.30) holds for every $u_0 \in \overline{D(A)}^X$ and a.e. $t > 0$.*

If the Banach space X and its dual space X^* are *uniformly convex*, then (cf [6, Theorem 4.6]) for every $u_0 \in D(A)$, $f \in W^{1,1}(0,T;X)$, the mild solution $u(t) = T_t(u_0, f)$, $(t \geq 0)$, of (1.6) is a strong solution of (1.6), u is everywhere differentiable from the right, $\frac{\mathrm{d}u}{\mathrm{d}t_+}$ is right continuous, and

$$\frac{\mathrm{d}u}{\mathrm{d}t_+}(t) + (A - f(t))^\circ u(t) = 0 \qquad \text{for every } t \geq 0,$$

where for every $t \in [0,T]$, $(A - f(t))^\circ$ denotes the *minimal selection* of $A - f(t)$ defined by

$$(A - f(t))^\circ := \Big\{(u,v) \in A - f(t) \,\Big|\, \|v\|_X = \inf_{\hat{v} \in Au - f(t)} \|\hat{v}\|_X \Big\}.$$

Thus, under those assumptions on X and by Proposition 3.6, we can state the following three corollaries. We begin by stating the inhomogeneous case.

Corollary 3.10. *Suppose X and its dual space X^* are uniformly convex, for $\omega \in \mathbb{R}$, A is an ω-quasi m-accretive operator on X, and $\{T_t\}_{t\geq 0}$ is the semigroup on $\overline{D(A)}^X \times L^1(0,T;X)$ generated by $-A$. If A is homogeneous of order zero satisfying $0 \in A0$, then for every $u_0 \in \overline{D(A)}^X$ and $f \in W^{1,1}(0,T;X)$,*

$$\begin{aligned}
&\|(A - f(t))^\circ T_t(u_0, f)\|_X \\
&\qquad \leq \frac{e^{\omega t}}{t}\left[2\|u_0\|_X + \int_0^t e^{-\omega s} s \|f'(s)\|_X \,\mathrm{d}s + \int_0^t e^{-\omega s}\|f(s)\|_X \mathrm{d}s\right]
\end{aligned}$$

for every $t > 0$.

The following corollary states the homogeneous case.

Corollary 3.11. *Suppose X and its dual space X^* are uniformly convex, for $\omega \in \mathbb{R}$, A is an ω-quasi m-accretive operator on X, and $\{T_t\}_{t\geq 0}$ is the semigroup on $\overline{D(A)}^X$ generated by $-A$. If A is homogeneous of order zero satisfying $0 \in A0$, then*

$$\|A^\circ T_t u_0\|_X \leq 2e^{\omega t}\frac{\|u_0\|_X}{t} \qquad \textit{for every } t > 0 \textit{ and } u_0 \in \overline{D(A)}^X.$$

The last corollary states the case when A is perturbed by a Lipschitz mapping. This follows from [6, Theorem 4.6] and Corollary 2.12.

Corollary 3.12. *Suppose X and its dual space X^* are uniformly convex, $F : X \to X$ be a Lipschitz continuous mapping with Lipschitz-constant $\omega > 0$ satisfying $F(0) = 0$, A an m-accretive operator on X, and $\{T_t\}_{t\geq 0}$ is the semigroup on $\overline{D(A)}^X$ generated by $-(A+F)$. If A is homogeneous of order zero satisfying $0 \in A0$, then for every $u_0 \in \overline{D(A)}^X$,*

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right\|_X \leq e^{\omega t} \left[2e^{L\omega \int_0^t e^{-\omega s} s \, \mathrm{d}s} + \omega \int_0^t e^{L\omega \int_s^t e^{-\omega r} r \, \mathrm{d}r} \, \mathrm{d}s \right] \frac{\|u_0\|_X}{t}$$

for every $t > 0$.

4. Completely accretive operators of homogeneous order zero

In [8], Bénilan and Crandall introduced the celebrated class of *completely accretive* operators A and showed: even though the underlying Banach spaces does not admit the Radon-Nikodým property, but if A is completely accretive and homogeneous of order $\alpha > 0$ with $\alpha \neq 1$, then the mild solutions of differential inclusion (1.1) involving A are strong. In this section we will see that this also happen for completely accretive operators of homogeneous order zero.

4.1. **General framework.**

We begin by outlining our framework and then provide a brief introduction to the class of completely accretive operators.

For the rest of this paper, suppose $(\Sigma, \mathcal{B}, \mu)$ is a σ-finite measure space, and $M(\Sigma, \mu)$ the space of μ-a.e. equivalent classes of measurable functions $u : \Sigma \to \mathbb{R}$. For $u \in M(\Sigma, \mu)$, we write $[u]^+$ to denote $\max\{u, 0\}$ and $[u]^- = -\min\{u, 0\}$. We denote by $L^q(\Sigma, \mu)$, $1 \leq q \leq \infty$, the corresponding standard Lebesgue space with norm

$$\|\cdot\|_q = \begin{cases} \left(\int_\Sigma |u|^q \, \mathrm{d}\mu \right)^{1/q} & \text{if } 1 \leq q < \infty, \\ \inf \left\{ k \in [0, +\infty] \;\middle|\; |u| \leq k \ \mu\text{-a.e. on } \Sigma \right\} & \text{if } q = \infty. \end{cases}$$

For $1 \leq q < \infty$, we identify the dual space $(L^q(\Sigma, \mu))'$ with $L^{q'}(\Sigma, \mu)$, where q' is the conjugate exponent of q given by $1 = \frac{1}{q} + \frac{1}{q'}$.

Next, we first briefly recall the notion of *Orlicz spaces* (cf [18, Chapter 3]). A continuous function $\psi : [0, +\infty) \to [0, +\infty)$ is an *N-function* if it is convex, $\psi(s) = 0$ if and only if $s = 0$, $\lim_{s\to 0+} \psi(s)/s = 0$, and $\lim_{s\to\infty} \psi(s)/s = \infty$. Given an N-function ψ, the *Orlicz space* is defined as follows

$$L^\psi(\Sigma, \mu) := \left\{ u \in M(\Sigma, \mu) \;\middle|\; \int_\Sigma \psi\left(\frac{|u|}{\alpha} \right) \, \mathrm{d}\mu < \infty \text{ for some } \alpha > 0 \right\}$$

and equipped with the *Orlicz-Minkowski norm*

$$\|u\|_\psi := \inf \left\{ \alpha > 0 \;\middle|\; \int_\Sigma \psi\left(\frac{|u|}{\alpha} \right) \, \mathrm{d}\mu \leq 1 \right\}.$$

With these preliminaries in mind, we are now in the position to recall the notation of *completely accretive* operators introduced in [8] and further developed to the ω-*quasi* case in [12].

Let J_0 be the set given by

$$J_0 = \Big\{ j : \mathbb{R} \to [0, +\infty] \;\Big|\; j \text{ is convex, lower semicontinuous, } j(0) = 0 \Big\}.$$

Then, for every $u, v \in M(\Sigma, \mu)$, we write

$$u \ll v \quad \text{if and only if} \quad \int_\Sigma j(u)\,\mathrm{d}\mu \le \int_\Sigma j(v)\,\mathrm{d}\mu \quad \text{for all } j \in J_0.$$

Remark 4.1. Due to the interpolation result [8, Proposition 1.2], for given u, $v \in M(\Sigma, \mu)$, the relation $u \ll v$ is equivalent to the two conditions

$$\begin{cases} \int_\Sigma [u-k]^+\,\mathrm{d}\mu & \le \int_\Sigma [v-k]^+\,\mathrm{d}\mu \qquad \text{for all } k > 0 \text{ and} \\ \int_\Sigma [u+k]^-\,\mathrm{d}\mu & \le \int_\Sigma [v+k]^-\,\mathrm{d}\mu \qquad \text{for all } k > 0. \end{cases}$$

Thus, the relation $\ll$ is closely related to the theory of rearrangement-invariant function spaces (cf [10]). Another, useful characterization of relation " $\ll$" is the following (cf [8, Remark 1.5]): for u, $v \in M(\Sigma, \mu)$, $u \ll v$ if and only if $u^+ \ll v^+$ and $u^- \ll v^-$.

Further, the relation $\ll$ on $M(\Sigma, \mu)$ has the following properties. We omit the easy proof of this proposition.

Proposition 4.2. *For every u, v, $w \in M(\Sigma, \mu)$, one has that*

(1) $u^+ \ll u$, $u^- \ll -u$;
(2) $u \ll v$ if and only if $u^+ \ll v^+$ and $u^- \ll v^-$;
(3) (positive homogeneity) *if $u \ll v$ then $\alpha u \ll \alpha v$ for all $\alpha > 0$;*
(4) (transitivity) *if $u \ll v$ and $v \ll w$ then $u \ll w$;*
(5) if $u \ll v$ then $|u| \ll |v|$;
(6) (convexity) *for every $u \in M(\Sigma, \mu)$, the set $\{w \mid w \ll u\}$ is convex.*

With these preliminaries in mind, we can now state the following definitions.

Definition 4.3. A mapping $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ is called a *complete contraction* if

$$Su - S\hat{u} \ll u - \hat{u} \qquad \text{for every } u, \hat{u} \in D(S).$$

More generally, for $L > 0$, we call S to be an *L-complete contraction* if

$$L^{-1}Su - L^{-1}S\hat{u} \ll u - \hat{u} \qquad \text{for every } u, \hat{u} \in D(S),$$

or for $L = e^{\omega t}$ with $\omega \in \mathbb{R}$ and $t \ge 0$, S is then also called an *ω-quasi complete contraction*.

Remark 4.4. Note, for every $1 \le q < \infty$, $j_q(\cdot) = |[\cdot]^+|^q \in \mathcal{J}_0$, $j_\infty(\cdot) = [[\cdot]^+ - k]^+ \in \mathcal{J}_0$ for every $k \ge 0$ (and for large enough $k > 0$ if $q = \infty$), and for every N-function ψ and $\alpha > 0$, $j_{\psi,\alpha}(\cdot) = \psi(\frac{[\cdot]^+}{\alpha}) \in \mathcal{J}_0$. This shows that for every L-complete contraction $S : D(S) \to M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$, the mapping $L^{-1}S$ is order-preserving and contractive respectively for every L^q-norm ($1 \le q \le \infty$), and every L^ψ-norm with N-function ψ.

Now, we can state the definition of completely accretive operators.

Definition 4.5. An operator A on $M(\Sigma,\mu)$ is called *completely accretive* if for every $\lambda>0$, the resolvent operator J_λ of A is a complete contraction, or equivalently, if for every (u_1,v_1), $(u_2,v_2)\in A$ and $\lambda>0$, one has that

$$u_1-u_2 \ll u_1-u_2+\lambda(v_1-v_2).$$

If X is a linear subspace of $M(\Sigma,\mu)$ and A an operator on X, then A is *m-completely accretive on* X if A is completely accretive and satisfies the *range condition* (3.1). Further, for $\omega\in\mathbb{R}$, an operator A on a linear subspace $X\subseteq M(\Sigma,\mu)$ is called *ω-quasi (m)-completely accretive* in X if $A+\omega I$ is (m)-completely accretive in X.

Before stating a useful characterization of completely accretive operators, we first need to introduce the following function spaces. Let

$$L^{1+\infty}(\Sigma,\mu):=L^1(\Sigma,\mu)+L^\infty(\Sigma,\mu)\quad\text{and}\quad L^{1\cap\infty}(\Sigma,\mu):=L^1(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)$$

be the *sum* and the *intersection space* of $L^1(\Sigma,\mu)$ and $L^\infty(\Sigma,\mu)$, which are equipped, respectively, with the norms

$$\|u\|_{1+\infty}:=\inf\left\{\|u_1\|_1+\|u_2\|_\infty\middle|u=u_1+u_2,\ u_1\in L^1(\Sigma,\mu),u_2\in L^\infty(\Sigma,\mu)\right\},$$
$$\|u\|_{1\cap\infty}:=\max\left\{\|u\|_1,\|u\|_\infty\right\}$$

are Banach spaces. In fact, $L^{1+\infty}(\Sigma,\mu)$ and and $L^{1\cap\infty}(\Sigma,\mu)$ are respectively the largest and the smallest of the rearrangement-invariant Banach function spaces (cf [10, Chapter 3.1]). If $\mu(\Sigma)$ is finite, then $L^{1+\infty}(\Sigma,\mu)=L^1(\Sigma,\mu)$ with equivalent norms, but if $\mu(\Sigma)=\infty$ then $L^{1+\infty}(\Sigma,\mu)$ contains $\bigcup_{1\le q\le\infty}L^q(\Sigma,\mu)$. Further, we will employ the space

$$L_0(\Sigma,\mu):=\left\{u\in M(\Sigma,\mu)\;\middle|\;\int_\Sigma\left[|u|-k\right]^+\mathrm{d}\mu<\infty\text{ for all }k>0\right\},$$

which equipped with the $L^{1+\infty}$-norm is a closed subspace of $L^{1+\infty}(\Sigma,\mu)$. In fact, one has (cf [8]) that $L_0(\Sigma,\mu)=\overline{L^1(\Sigma,\mu)\cap L^\infty(\Sigma,\mu)}^{1+\infty}$. Since for every $k\ge 0$, $T_k(s):=[|s|-k]^+$ is a Lipschitz mapping $T_k:\mathbb{R}\to\mathbb{R}$ and by Chebyshev's inequality, one sees that $L^q(\Sigma,\mu)\hookrightarrow L_0(\Sigma,\mu)$ for every $1\le q<\infty$ (and $q=\infty$ if $\mu(\Sigma)<+\infty$), and $L^\psi(\Sigma,\mu)\hookrightarrow L_0(\Sigma,\mu)$ for every N-function ψ.

Proposition 4.6 ([12]). *Let P_0 denote the set of all functions $T\in C^\infty(\mathbb{R})$ satisfying $0\le T'\le 1$ such that T' is compactly supported, and $x=0$ is not contained in the support $supp(T)$ of T. Then for $\omega\in\mathbb{R}$, an operator $A\subseteq L_0(\Sigma,\mu)\times L_0(\Sigma,\mu)$ is ω-quasi completely accretive if and only if*

$$\int_\Sigma T(u-\hat{u})(v-\hat{v})\,\mathrm{d}\mu+\omega\int_\Sigma T(u-\hat{u})(u-\hat{u})\,\mathrm{d}\mu\ge 0$$

for every $T\in P_0$ and every (u,v), $(\hat{u},\hat{v})\in A$.

Remark 4.7. For convenience, we denote the unique extension of $\{T_t\}_{t\ge0}$ on $L^\psi(\Sigma,\mu)$ or $L^1(\Sigma,\mu)$ again by $\{T_t\}_{t\ge0}$.

Definition 4.8. A Banach space $X\subseteq M(\Sigma,\mu)$ with norm $\|\cdot\|_X$ is called *normal* if the norm $\|\cdot\|_X$ has the following property:

$$\begin{cases}\text{for every } u\in X, v\in M(\Sigma,\mu)\text{ satisfying }\quad v\ll u,\\ \text{one has that}\quad v\in X\quad\text{and}\quad \|v\|_X\le\|u\|_X.\end{cases}$$

Typical examples of normal Banach spaces $X \subseteq M(\Sigma,\mu)$ are Orlicz-spaces $L^{\psi}(\Sigma,\mu)$ for every N-function ψ, $L^q(\Sigma,\mu)$, $(1 \le q \le \infty)$, $L^{1\cap\infty}(\Sigma,\mu)$, $L_0(\Sigma,\mu)$, and $L^{1+\infty}(\Sigma,\mu)$.

Remark 4.9. It is important to point out that if X is a normal Banach space, then for every $u \in X$, one always has that u^+, u^- and $|u| \in X$. To see this, recall that by (1) Proposition 4.2, if $u \in X$, then $u^+ \ll u$ and $u^- \ll -u$. Thus, u^+ and $u^- \in X$ and since $|u| = u^+ + u^-$, one also has that $|u| \in X$.

The dual space $(L_0(\Sigma,\mu))'$ of $L_0(\Sigma,\mu)$ is isometrically isomorphic to the space $L^{1\cap\infty}(\Sigma,\mu)$. Thus, a sequence $(u_n)_{n\ge 1}$ in $L_0(\Sigma,\mu)$ is said to be *weakly convergent* to u in $L_0(\Sigma,\mu)$ if

$$\langle v, u_n\rangle := \int_\Sigma v\, u_n \,\mathrm{d}\mu \to \int_\Sigma v\, u \,\mathrm{d}\mu \qquad \text{for every } v \in L^{1\cap\infty}(\Sigma,\mu).$$

For the rest of this paper, we write $\sigma(L_0, L^{1\cap\infty})$ to denote the *weak topology* on $L_0(\Sigma,\mu)$. For this weak topology, we have the following compactness result.

Proposition 4.10 ([8, Proposition 2.11])**.** *Let $u \in L_0(\Sigma,\mu)$. Then, the following statements hold.*

(1) The set $\Big\{v \in M(\Sigma,\mu) \;\Big|\; v \ll u\Big\}$ is $\sigma(L_0, L^{1\cap\infty})$-sequentially compact in $L_0(\Sigma,\mu)$;

(2) Let $X \subseteq M(\Sigma,\mu)$ be a normal Banach space satisfying $X \subseteq L_0(\Sigma,\mu)$ and

$$\begin{cases} \text{for every } u \in X,\ (u_n)_{n\ge1} \subseteq M(\Sigma,\mu) \text{ with } u_n \ll u \text{ for all } n \ge 1 \\ \text{and } \lim_{n\to+\infty} u_n(x) = u(x)\ \mu\text{-a.e. on } \Sigma, \text{ yields } \quad \lim_{n\to+\infty} u_n = u \text{ in } X. \end{cases} \tag{4.1}$$

Then for every $u \in X$ and sequence $(u_n)_{n\ge1} \subseteq M(\Sigma,\mu)$ satisfying

$$u_n \ll u \text{ for all } n \ge 1 \quad \text{and} \quad \lim_{n\to+\infty} u_n = u\ \sigma(L_0, L^{1\cap\infty})\text{-weakly in } X,$$

one has that

$$\lim_{n\to+\infty} u_n = u \qquad \text{in } X.$$

Note, examples of normal Banach spaces $X \subseteq L_0(\Sigma,\mu)$ satisfying (4.1) are $X = L^p(\Sigma,\mu)$ for $1 \le p < \infty$ and $L_0(\Sigma,\mu)$.

To complete this section we state the following Proposition summarizing statements from [12, Proposition 2.9 & Proposition 2.10], which we will need in the sequel (cf [8] for the case $\omega = 0$).

Proposition 4.11. *For $\omega \in \mathbb{R}$, let A be ω-quasi completely accretive in $L_0(\Sigma,\mu)$.*

(1.) If there is a $\lambda_0 > 0$ such that $Rg(I+\lambda A)$ is dense in $L_0(\Sigma,\mu)$, then for the closure $\overline{A}$ of A in $L_0(\Sigma,\mu)$ and every normal Banach space with $X \subseteq L_0(\Sigma,\mu)$, the restriction $\overline{A}_X := \overline{A} \cap (X\times X)$ of A on X is the unique ω-quasi m-completely accretive extension of the part $A_X = A \cap (X \times X)$ of A in X.

(2.) For a given normal Banach space $X \subseteq L_0(\Sigma,\mu)$, and $\omega \in \mathbb{R}$, suppose A is ω-quasi m-completely accretive in X, and $\{T_t\}_{t\ge0}$ be the semigroup generated by $-A$ on $\overline{D(A)}^X$. Further, let $\{S_t\}_{t\ge0}$ be the semigroup generated by $-\overline{A}$, where $\overline{A}$ denotes the closure of A in $\overline{X}^{L^{1+\infty}}$. Then, the following statements hold.

(a) *The semigroup* $\{S_t\}_{t\ge 0}$ *is* ω*-quasi completely contractive on* $\overline{D(A)}^{L^{1+\infty}}$, T_t *is the restriction of* S_t *on* $\overline{D(A)}^X$, S_t *is the closure of* T_t *in* $L^{1+\infty}(\Sigma,\mu)$, *and*

$$S_t u_0 = L^{1+\infty}(\Sigma,\mu) - \lim_{n\to+\infty}\left(I+\frac{t}{n}A\right)^{-n} u_0 \quad \text{for all } u_0 \in \overline{D(A)}^{L^{1+\infty}} \cap X; \tag{4.2}$$

(b) *If there exists* $u \in L^{1\cap\infty}(\Sigma,\mu)$ *such that the orbit* $\{T_t u \,|\, t \ge 0\}$ *is locally bounded on* $\mathbb{R}_+$ *with values in* $L^{1\cap\infty}(\Sigma,\mu)$, *then, for every N-function* ψ, *the semigroup* $\{T_t\}_{t\ge 0}$ *can be extrapolated to a strongly continuous, order-preserving semigroup of* ω*-quasi contractions on* $\overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\psi}$ *(respectively, on* $\overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^1}$*), and to an order-preserving semigroup of* ω*-quasi contractions on* $\overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\infty}$. *We denote each extension of* T_t *on on those spaces again by* T_t.

(c) *The restriction* $A_X := \overline{A} \cap (X\times X)$ *of* $\overline{A}$ *on* X *is the unique* ω*-quasi m-complete extension of* A *in* X*; that is,* $A = A_X$.

(d) *The operator* A *is sequentially closed in* $X\times X$ *equipped with the relative* $(L_0(\Sigma,\mu)\times(X,\sigma(L_0,L^{1\cap\infty})))$*-topology.*

(e) *The domain of* A *is characterized by*

$$D(A) = \left\{ u \in \overline{D(A)}^{L^{1+\infty}} \cap X \,\middle|\, \exists\, v \in X \text{ s.t. } e^{-\omega t}\frac{S_t u - u}{t} \ll v \text{ for small } t > 0 \right\};$$

(f) *For every* $u \in D(A)$, *one has that*

$$\lim_{t\to 0+}\frac{S_t u - u}{t} = -A^\circ u \qquad \text{strongly in } L_0(\Sigma,\mu). \tag{4.3}$$

4.2. **The subclass of homogeneous operators of order zero.** As mentioned in Section 3, the Banach spaces $X_1 = L^1(\Sigma,\mu)$ and $X_2 = L^\infty(\Sigma,\mu)$ don't have the Radon-Nikodým property. But for the class of quasi m-completely accretive operators A defined on a normal Banach space $X \subseteq M(\Sigma,\mu)$, for semigroup $\{T_t\}_{t\ge 0}$ generated by $-A$, the time-derivative $\frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+}$ exists in X at every $t > 0$ for every $u_0 \in \overline{D(A)}^X$. This fact follows from the following compactness result. Here, the partial ordering "$\le$" is the standard one defined by $u \le v$ for u, $v \in M(\Sigma,\mu)$ if $u(x) \le v(x)$ for μ-a.e. $x \in \Sigma$, and we use the symbol $\hookrightarrow$ for indicating continuous embeddings.

Lemma 4.12. *Let* $X \subseteq L_0(\Sigma,\mu)$ *be a normal Banach space satisfying* (4.1). *For* $\omega \in \mathbb{R}$, *let* $\{T_t\}_{t\ge 0}$ *be a family of mappings* $T_t : C \to C$ *defined on a subset* $C \subseteq X$ *of* ω*-quasi complete contractions satisfying* (2.18) *and* $T_t 0 = 0$ *for all* $t \ge 0$. *Then, for every* $u_0 \in C$ *and* $t > 0$, *the set*

$$\left\{ \frac{T_{t+h}u_0 - T_t u_0}{h} \,\middle|\, h \neq 0, t+h > 0 \right\} \tag{4.4}$$

is $\sigma(L_0, L^{1\cap\infty})$*-weakly sequentially compact in* $L_0(\Sigma,\mu)$.

Proof. Let $u_0 \in C$, $t > 0$, and $h \neq 0$ such that $t+h > 0$. Then by taking $\lambda = 1 + \frac{h}{t}$ in (2.18), one sees that

$$|T_{t+h}u_0 - T_t u_0| = |\lambda\, T_t\left[\lambda^{-1}u_0\right] - T_t u_0|$$

$$\leq \lambda \left|T_t\left[\lambda^{-1}u_0\right]-T_tu_0\right|+|1-\lambda|\,|T_tu_0|.$$

Since T_t is an ω-quasi complete contraction, by (3) of Proposition 4.2, and since $T_t0=0$, $(t\geq 0)$, one has that

$$\lambda\,e^{-\omega t}\left|T_t\left[\lambda^{-1}u_0\right]-T_tu_0\right|\ll |1-\lambda|\,|u_0|$$

and

$$|1-\lambda|\,e^{-\omega t}|T_tu_0|\ll |1-\lambda|\,|u_0|.$$

Since the set $\{w\,|\,w\ll |1-\lambda|\,|u_0|\}$ is convex (see (6) of Proposition 4.2), we can conclude that

$$\frac{1}{2}e^{-\omega t}|T_{t+h}u_0-T_tu_0|\ll\;|1-\lambda|\,|u_0|=\frac{|h|}{t}\,|u_0|$$

and hence, by (3) of Proposition 4.2,

$$\frac{|T_{t+h}u_0-T_tu_0|}{|h|}\ll 2\,e^{\omega t}\,\frac{|u_0|}{t}. \tag{4.5}$$

Since for every $u\in M(\Sigma,\mu)$, one always has that $u^+\ll|u|$, the transitivity of "$\ll$" (see (4) of Proposition 4.2) implies for

$$f_h:=\frac{T_{t+h}u_0-T_tu_0}{|h|},\qquad\text{one has that}\qquad f_h^+\ll 2\,e^{\omega t}\,\frac{|u_0|}{t}$$

Therefore, by (1) of Proposition 4.10, the two sets $\{f_h^+|\,h\neq 0,t+h>0\}$ and $\{|f_h|\,|\,h\neq 0,t+h>0\}$ are $\sigma(L_0,L^{1\cap\infty})$- weakly sequentially compact in $L_0(\Sigma,\mu)$, and since $f_h^-=|f_h|-f_h^+$ and $f_h=f_h^+-f_h^-$, we have thereby shown that the claim of this lemma holds. □

With these preliminaries in mind, we can now state the regularization effect of the semigroup $\{T_t\}_{t\geq 0}$ generated by a ω-quasi m-completely accretive operator of homogeneous order zero.

Theorem 4.13. *Let $X\subseteq L_0(\Sigma,\mu)$ be a normal Banach space satisfying (4.1). For $\omega\in\mathbb{R}$, let A be ω-quasi m-completely accretive in X, and $\{T_t\}_{t\geq 0}$ be the semigroup generated by $-A$ on $\overline{D(A)}^X$. If $(0,0)\in A$ and A is homogeneous of order zero, then for every $u_0\in\overline{D(A)}^X$ and $t>0$, $\frac{\mathrm{d}T_tu_0}{\mathrm{d}t}$ exists in X and*

$$|A^\circ T_tu_0|\leq 2e^{\omega t}\,\frac{|u_0|}{t}\qquad \mu\text{-a.e. on }\Sigma. \tag{4.6}$$

In particular,

$$\left\|\frac{\mathrm{d}T_tu_0}{\mathrm{d}t}\right\|\leq 2e^{\omega t}\,\frac{\|u_0\|}{t}\qquad\text{for every }t>0, \tag{4.7}$$

and

$$\frac{\mathrm{d}T_tu_0}{\mathrm{d}t}\leq\frac{T_tu_0}{t}\qquad \mu\text{-a.e. on }\Sigma\text{ for every }t>0\text{ if }u_0\geq 0, \tag{4.8}$$

for every $u_0\in\overline{D(A)}^X$ (then $\|\cdot\|$ denotes the norm on X), respectively, for every $u_0\in\overline{\overline{D(A)}^X\cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\psi}$ (then $\|\cdot\|$ is the L^ψ-norm) for every N-function ψ or for every $1\leq\psi\equiv p<\infty$, and for every $u_0\in\overline{D(A)}^X\cap L^\infty(\Sigma,\mu)$ (where then $\|\cdot\|$ is the L^∞-norm).

Proof. Let $u_0 \in \overline{D(A)}^X$, $t > 0$, and $(h_n)_{n\geq 1} \subseteq \mathbb{R}$ be a zero sequence such that $t + h_n > 0$ for all $n \geq 1$. Due to Lemma 3.3, we can apply Lemma 4.12. Thus, there is a $z \in L_0(\Sigma, \mu)$ and a subsequence $(h_{k_n})_{n\geq 1}$ of $(h_n)_{n\geq 1}$ such that

$$\lim_{n\to+\infty} \frac{T_{t+h_{k_n}}u_0 - T_t u_0}{h_{k_n}} = z \qquad \text{weakly in } L_0(\Sigma, \mu). \tag{4.9}$$

Moreover, by (2e) of Proposition 4.11, one has that $(T_t u_0, -z) \in A$. Thus (2f) of the same proposition 4.11 yields that $z = -A^{\circ}T_t u_0$ and

$$\lim_{n\to 0} \frac{T_{t+h_{k_n}}u_0 - T_t u_0}{h_{k_n}} = -A^{\circ}u_0 \qquad \text{strongly in } L_0(\Sigma, \mu). \tag{4.10}$$

After possibly passing to another subsequence, we have that limit (4.10) holds also μ-a.e. on Σ. Since $2e^{-\omega t}\,\frac{|u_0|}{t} \in X$ and $X \subseteq L_0(\Sigma, \mu)$, (2) of Proposition 4.10 implies that

$$\lim_{h\to 0} \frac{T_{t+h}u_0 - T_t u_0}{h} = -A^{\circ}T_t u_0 \qquad \text{exists in } X \text{ and } \mu\text{-a.e. on } \Sigma. \tag{4.11}$$

Thus and since by (4.5),

$$\frac{|T_{t+h_{k_n}}u_0 - T_t u_0|}{|h_{k_n}|} \leq 2\,e^{-\omega t}\,\frac{|u_0|}{t} \qquad \text{for all } n \geq 1,$$

sending $n \to +\infty$ in the last inequality, gives (4.6). In particular, by Corollary 2.4, one has that (4.7) holds for the norm $\|\cdot\|_X$ on X and by Theorem 2.7, it follows that (4.8) holds. Moreover, we have that $-A^{\circ}T_t u_0 = \frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+}$ μ-a.e. on Σ for every $t > 0$. Thus, sending $h \to 0+$ in (4.5) shows that (4.6) holds. Further, by the μ-a.e.-limit (4.11), applying Fatou's lemma to (4.5) yields that (4.7) holds for the L^{ψ}-norm for every N-function ψ and the L^p-norm $1 \leq p < \infty$. Since (4.7) holds for all $p < \infty$, sending $1 \leq p \to +\infty$ completes the proof of this theorem. □

Remark 4.14 (Open problem). We emphasize that the crucial point in the previous proof is that due to the zero-order homogeneity of A, the set (4.4) is $\sigma(L_0, L^{1\cap\infty})$-weakly sequentially compact in $L_0(\Sigma, \mu)$ and hence, for every $t > 0$, $T_t u_0 \in D(A)$ and

$$\frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+} = \lim_{h\to 0+} \frac{T_{t+h}u_0 - T_t u_0}{h} = -A^{\circ}T_t u_0 \qquad \text{exists in } X.$$

We believe that this remains true if the infinitesimal generator of the semigroup $\{T_t\}_{t\geq 0}$ is of the form $A + F$ where A is homogeneous of order zero and F is Lipschitz-continuous. But so far, we are not able to show this result.

As a final result of this section, we state the following decay estimates for semigroups generated by the perturbed operator $A + F$. Here, we write $L^{q_0\cap\infty}$ for the intersection space $L^{q_0}(\Sigma, \mu) \cap L^{\infty}(\Sigma, \mu)$.

Theorem 4.15. *Let $F : M(\Sigma, \mu) \to M(\Sigma, \mu)$ be a mapping such that for every N-function ψ and for $\psi \equiv 1$ and $\psi \equiv +\infty$, the restriction $F_{|L^{\psi}} : L^{\psi}(\Sigma, \mu) \to L^{\psi}(\Sigma, \mu)$ is Lipschitz continuous with constant Lipschitz $\omega > 0$ and $F(0) = 0$. Let A be an m-completely accretive operator on normal Banach space $X \subseteq L_0(\Sigma, \mu)$, and $\{T_t\}_{t\geq 0}$ the* *semigroup generated by* $-(A+F)$ *on* $\overline{D(A)}^X$. *If* $(0,0) \in A$ *and* A *is homogeneous of order zero, then*

$$\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+} \right\|_\psi \le \left[2e^{\omega \int_0^t e^{-\omega s} \mathrm{d}s} + \omega \int_0^t e^{\omega \int_s^t e^{-\omega r} \mathrm{d}r} \mathrm{d}s \right] \frac{e^{\omega t} \|u_0\|_\psi}{t} \tag{4.12}$$

for every $t > 0$, *and every* $u_0 \in \overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\psi}$ *for every N-function* ψ, *and every* $1 \le \psi \equiv p < \infty$, *and for every* $u_0 \in \overline{D(A)}^X \cap L^{q_0\cap\infty}(\Sigma,\mu)$ *for* $q = \infty$ *(where then* $\|\cdot\|_\psi$ *is the* L^∞*-norm).*

Proof. Since $(0,0) \in (A+F)$, $T_t 0 = 0$ for all $t \ge 0$. Thus, $u \equiv 0 \in L^{1\cap\infty}(\Sigma,\mu)$ such that $\{T_t u \,|\, t \ge 0\}$ is locally bounded in $\mathbb{R}_+$. Thus, by Proposition 4.11, for every N-function ψ (respectively, for $\psi \equiv 1$ and $\psi \equiv \infty$) each T_t admits a unique extension (which we denote again by T_t) of an ω-quasi contractions on $\overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\psi}$ with respect to the L^ψ-norm. In addition, the family $\{T_t\}_{t\ge0}$ remains a semigroup satisfying (2.28) and in relation with (2.27), $\{T_t\}_{t\ge0}$ satisfies (2.2) and (2.3) for f given by (2.26). Further, for $1 < \psi \equiv q < \infty$, $L^q(\Sigma,\mu)$ and its dual space $L^{q'}(\Sigma,\mu)$ are uniformly convex. Therefore, by Corollary 2.12 and Proposition 3.6, for every $u_0 \in \overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^q}$, for every $t > 0$, $\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+$ exists in $L^q(\Sigma,\mu)$,

$$\frac{\mathrm{d}T_t u_0}{\mathrm{d}t_+} = \lim_{h\to0+} \frac{T_{t+h}u_0 - T_t u_0}{h} \quad \text{exists } \mu\text{-a.e. on } \Sigma, \tag{4.13}$$

and (4.12) holds. Moreover, by Corollary 2.12, one has that (2.29) holds for the L^ψ-norm and every $u_0 \in \overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^\psi}$ and every N-function ψ, respectively for the L^1-norm and every $u_0 \in \overline{\overline{D(A)}^X \cap L^{1\cap\infty}(\Sigma,\mu)}^{L^1}$. Thus and by (4.13), sending $h \to 0+$ in (2.29) one obtains that (4.12) holds for all N-function ψ and $q = 1$.

Next, let $u_0 \in \overline{D(A)}^X \cap L^{q_0\cap\infty}(\Sigma,\mu)$ for some $1 \le q_0 < +\infty$ and $t > 0$. We assume $\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\|_\infty > 0$ (otherwise, there is nothing to show). Then, for every $s \in (0, \|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\|_\infty)$ and every $q_0 \le q < \infty$, Chebyshev's inequality yields

$$\mu\left(\left\{ \left| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right| \ge s \right\}\right)^{1/q} \le \frac{\left\| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right\|_q}{s}$$

and so, by (4.12),

$$s\,\mu\left(\left\{ \left| \frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+ \right| \ge s \right\}\right)^{1/q} \le \left[2e^{\omega \int_0^t e^{-\omega s} \mathrm{d}s} + \omega \int_0^t e^{\omega \int_s^t e^{-\omega r} \mathrm{d}r} \mathrm{d}s \right] \frac{e^{\omega t} \|u_0\|_q}{t}.$$

Thus and since $\lim_{q\to\infty} \|u_0\|_q = \|u_0\|_\infty$, sending $q \to +\infty$ in the last inequality, yields

$$s \le \left[2e^{\omega \int_0^t e^{-\omega s} \mathrm{d}s} + \omega \int_0^t e^{\omega \int_s^t e^{-\omega r} \mathrm{d}r} \mathrm{d}s \right] \frac{e^{\omega t} \|u_0\|_\infty}{t}$$

and since $s \in (0, \left\|\frac{\mathrm{d}T_t u_0}{\mathrm{d}t}_+\right\|_\infty)$ was arbitrary, we have thereby shown that (4.12) also holds for $q = \infty$.

□

5. Application

Throughout this section, let Σ be an open set of $\mathbb{R}^d$ and the Lebesgue space $L^q(\Sigma)$ is equipped with the classical Lebesgue measure. Suppose $f : \Sigma \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous *Carathéodory* function, that is, f satisfies the following three properties:

(5.1) • $f(\cdot, u) : \Sigma \to \mathbb{R}$ is measurable on Σ for every $u \in \mathbb{R}$,

(5.2) • $f(x, 0) = 0$ for a.e. $x \in \Sigma$, and

• there is a constant $\omega \geq 0$ such that

$$|f(x,u) - f(x,\hat{u})| \leq \omega\, |u - \hat{u}| \quad \text{for all } u, \hat{u} \in \mathbb{R}, \text{ a.e. } x \in \Sigma. \tag{5.3}$$

Then, for every $1 \leq q \leq \infty$, $F : L^q(\Sigma) \to L^q(\Sigma)$ defined by

$$F(u)(x) := f(x, u(x)) \qquad \text{for every } u \in L^q(\Sigma)$$

is the associated *Nemytskii operator* on $L^q(\Sigma)$. Moreover, by (5.3), F is globally Lipschitz continuous on $L^q(\Sigma)$ with constant $\omega > 0$ and $F(0)(x) = 0$ for a.e. $x \in \Sigma$.

5.1. **Decay estimates of the total variational flow.** In this subsection, we consider the perturbed total variational flow operator (1-Laplace operator) given by

$$Au := -\Delta_1 u + f(x,u) \qquad \text{with} \qquad \Delta_1 u = \operatorname{div}\left(\frac{Du}{|Du|}\right),$$

equipped with either Neumann or Dirichlet boundary conditions on a bounded domain Σ in $\mathbb{R}^d$, $d \geq 1$.

Here, we use the following notation. A function $u \in L^1(\Sigma)$ is said to be a *function of bounded variation in* Σ, if the distributional partial derivatives $D_1 u := \frac{\partial u}{\partial x_1}, \dots, D_d u := \frac{\partial u}{\partial x_d}$ are finite Radon measures in Σ, that is, if

$$\int_\Omega u\, D_i\varphi \,\mathrm{d}x = -\int_\Omega \varphi \,\mathrm{d}D_i u$$

for all $\varphi \in C_c^\infty(\Sigma)$, $i = 1, \dots, d$. The linear vector space of functions $u \in L^1(\Sigma)$ of bounded variation in Σ is denoted by $BV(\Sigma)$. Further, we set $Du = (D_1 u, \dots, D_d u)$ for the *distributional gradient* of u. Then, Du belongs to the class $M^b(\Omega, \mathbb{R}^d)$ of $\mathbb{R}^d$-valued bounded Radon measure on Ω, and we either write $|Du|(\Sigma)$ or $\int_\Sigma |Du|$ to denote the *total variation measure* of Du. The space $BV(\Sigma)$ equipped with the norm $\|u\|_{BV(\Sigma)} := \|u\|_{L^1(\Sigma)} + |Du|(\Sigma)$ forms a Banach space. Further, let

$$X_1(\Sigma) = \left\{ z \in L^\infty(\Sigma, \mathbb{R}^d) \,\middle|\, \operatorname{div}(z) := \sum_{i=1}^d D_i z \in L^1(\Sigma) \right\},$$

$\operatorname{sign}_0(s)$, $(s \in \mathbb{R})$, is the classical sign function with the additional property that $\operatorname{sign}_0(0) = 0$, and for every $k > 0$, $T_k(s) := [k - [k - |s|]^+ \operatorname{sign}_0(s)$, $(s \in \mathbb{R})$.

The Neumann total variational flow operator. In [2] (see also [3]), the negative *total variational flow operator* (1-*Laplace operator*) $-\Delta_1^N$ in $L^1(\Sigma)$ equipped with Neumann boundary conditions was introduced by

$$-\Delta_1^N = \left\{ (u,v) \in L^1(\Sigma) \times L^1(\Sigma) \;\middle|\; \begin{array}{l} T_k(u) \in BV(\Sigma)\,\forall\, k > 0 \;\&\; \exists\, z \in X_1(\Sigma) \\ \text{such that } \|z\|_\infty \le 1 \;\&\; (5.4) \text{ holds} \end{array} \right\},$$

where

$$\begin{cases} v = -\mathrm{div}(z) & \text{in } \mathcal{D}'(\Sigma), \text{ and} \\ \displaystyle\int_\Sigma (\xi - T_k(u)\, v \,\mathrm{d}x \le \int_\Sigma z \cdot D\xi \,\mathrm{d}x - \int_\Sigma |DT_k(u)| \end{cases} \tag{5.4}$$

for every $\xi \in W^{1,1}(\Sigma) \cap L^\infty$ and all $k > 0$. Moreover, the negative Neumann 1-Laplace operator $-\Delta_1^N$ is m-completely accretive in $L^1(\Sigma)$ with dense domain. Therefore, under the hypotheses (5.1)–(5.3), the operator $-\Delta_1^N + F$ is ω-quasi m-completely accretive on $L^1(\Sigma)$ (cf [12]). Now, it is not difficult to see that $-\Delta_1^N$ is homogeneous of order zero and $0 \in -\Delta_1^N 0$. Thus, by Theorem 4.13 and Theorem 4.15, we can state the following regularity result.

Corollary 5.1. *For every $1 \le q < \infty$ and $u_0 \in L^\psi(\Sigma)$ (respectively $u_0 \in L^{1\cap\infty}(\Sigma)$ if $q = \infty$), the unique mild solution u of problem*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} - \mathit{div}\left(\dfrac{Du}{|Du|}\right) + f(x,u) = 0 & \text{on } \Sigma \times (0,+\infty), \\ D_\nu u = 0 & \text{on } \partial\Sigma \times (0,+\infty), \\ u(0) = u_0 & \text{on } \Sigma \times \{t = 0\}, \end{cases} \tag{5.5}$$

is a strong solution satisfying (4.12). Moreover, if $f \equiv 0$, then either for every $1 \le \psi \equiv p \le \infty$ or N-function ψ and every $u_0 \in L^\psi(\Sigma)$, the unique mild solution u of problem (5.5) satisfies (4.7) and (4.8).

Next, for $d \ge 2$, the following *Sobolev-Poincaré inequality* (cf [3, formula (B.2), p.302])

$$\|u - \bar{u}\|_{\frac{d}{d-1}} \le C\,|Du|(\Sigma) \qquad \text{for every } u \in BV(\Sigma),$$

holds for a constant $C = C(d) > 0$, where $\bar{u} := \frac{1}{|\Sigma|}\int_\Omega u(x)\,\mathrm{d}x$ denotes the mean-value of an integrable function u. Thus, we have that the negative Neumann 1-Laplace operator $A = -\Delta_1^N$ satisfies the following abstract Sobolev inequality

$$\|u - \bar{u}\|_r^\sigma \le C\,[u,v]_2 \qquad \text{for every } (u,v) \in A$$

with parameters $r = \frac{d}{d-1} > 1$ and $\sigma = 1$, where $[\cdot,\cdot]_2$ denote the L^2-inner product. Now, it follows from [12, Theorem 1.2], the semigroup $\{T_t\}_{t\ge 0}$ generated by $\Delta_1^N - F$ satisfies the following L^2-$L^{\frac{d}{d-1}}$*-regularity estimate*

$$\|T_t u - \bar{u}\|_{\frac{d}{d-1}} \le \frac{C}{2}\, t^{-1}\, e^{3\omega t} \|u - \bar{u}\|_2^2 \qquad \text{for every } t > 0$$

and $u \in L^2(\Sigma)$. Further, for every $q > d-1$, one has

$$\|T_t u\|_\infty \le \tilde{C}\, t^{-\frac{1}{(\frac{d}{d-1}-1)q-1}}\, e^{\left(\frac{1}{(\frac{d}{d-1}-1)q-1}+1\right)\omega t}\, \|u\|_{\frac{dq}{d-1}}^{\frac{(\frac{d}{d-1}-1)q}{(\frac{d}{d-1}-1)q-1}} \qquad \text{for every } t > 0$$

and $u \in L^{\frac{dq}{d-1}}(\Sigma)$. Thus, by Corollary 2.6 and inequality (4.12) for $q = \infty$, we also have the following estimate.

Corollary 5.2. *Let $d \geq 2$ and $q > d-1$. Then for every $u_0 \in L^{\frac{dq}{d-1}}(\Sigma)$, the unique solution u of (5.5) satisfies*

$$\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_\infty \leq \tilde{C}\, C_\omega(t/2) 2^{\frac{1}{(\frac{d}{d-1}-1)q-1}+2} e^{\frac{\omega}{2}\left(\frac{1}{(\frac{d}{d-1}-1)q-1}+1\right)t} \frac{\|u_0\|_{dq/(d-1)}^{\frac{(\frac{d}{d-1}-1)q}{(\frac{d}{d-1}-1)q-1}}}{t^{\frac{1}{(\frac{d}{d-1}-1)q-1}+1}}$$

for every $t > 0$, where $C_\omega(t)$ is the constant in (4.12).

Remark 5.3 (The Dirichlet boundary case). In [1] (cf [3]), existence and uniqueness of the the parabolic initial boundary-value problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} - \mathrm{div}\left(\dfrac{Du}{|Du|}\right) = 0 & \text{on } \Sigma \times (0,+\infty), \\ u = \varphi & \text{on } \partial\Sigma \times (0,+\infty), \\ u(0) = u_0 & \text{on } \Sigma \times \{t = 0\}, \end{cases} \tag{5.6}$$

associated with the total variational flow equipped with (inhomogeneous) Dirichlet boundary conditions was established. For every boundary term $\varphi \in L^1(\Sigma)$, the negative *Dirichlet total variational flow operator (1-Laplace operator)* $\Delta_1^D u := \mathrm{div}\left(\frac{Du}{|Du|}\right)$ is m-completely accretive in $L^1(\Sigma)$. But only in the homogeneous case $\varphi \equiv 0$, the operator Δ_1^D is homogeneous of order zero. Thus, the same statement as given in Corollary 5.1 holds in the Dirichlet case with $\varphi \equiv 0$.

5.2. **Decay estimates of the nonlocal total variational Flow.** In this very last section, we consider for $0 < s < 1$, the perturbed fractional 1-Laplace operator

$$Au := \mathrm{PV}\int_\Sigma \frac{(u(y)-u(x))}{|u(y)-u(x)|}\frac{\mathrm{d}y}{|x-y|^{d+s}} + f(x,u)$$

equipped with either Dirichlet on a domain Σ in $\mathbb{R}^d$ or with or vanishing conditions if $\Sigma = \mathbb{R}^d$, $d \geq 1$.

For $0 < s < 1$, let $\mathcal{W}_0^{s,1}(\Sigma)$ be the Banach space given by

$$\mathcal{W}_0^{s,1}(\Sigma) = \left\{u \in L^1(\Sigma)\;\middle|\; [u]_{s,1} < \infty \text{ and } u = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Sigma\right\}$$

equipped with the norm $\|\cdot\|_{\mathcal{W}_0^{s,1}} := \|\cdot\|_1 + [\cdot]_{s,1}$, where

$$[u]_{s,1} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|}{|x-y|^{d+s}}\mathrm{d}y\mathrm{d}x \qquad \text{for every } u \in \mathcal{W}_0^{s,1}(\Sigma).$$

Further, let $B_{L^\infty_{as}}$ denote the closed unit ball of all *anti-symmetric* $\eta \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, that is,

$$\eta(x,y) = -\eta(y,x) \qquad \text{for a.e. } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \quad \text{and } \|\eta\|_\infty \leq 1.$$

Then, it was shown in [16, Section 3.] that the *fractional Dirichlet 1-Laplace operator* $(-\Delta_1^D)^s$ in $L^2(\Sigma)$ can be realized by (the graph)

$$(-\Delta_1^D)^s = \left\{(u,v) \in L^2(\Sigma)\times L^2(\Sigma)\;\middle|\; u \in \mathcal{W}_0^{s,1}(\Sigma)\;\&\;\exists\,\eta \in B_{L^\infty_{as}} \text{ s.t. (5.7) holds}\right\}$$

where

$$
\begin{cases}
\eta(x,y) \in \operatorname{sign}(u(x)-u(y)) & \text{for a.e. } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \text{ and} \\
\displaystyle\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\eta(x,y)(\xi(x)-\xi(y))}{|y-x|^{d+s}}\,\mathrm{d}y\mathrm{d}x = \int_\Sigma v(x)\,\xi(x)\,\mathrm{d}x & \\
& \text{for all } \xi \in \mathcal{W}_0^{s,1}(\Sigma)\cap L^2(\Sigma),
\end{cases}
\tag{5.7}
$$

and $(-\Delta_1^D)^s$ is m-completely accretive in $L^2(\Sigma)$ with dense domain $D((-\Delta_1^D)^s)$ in $L^2(\Sigma)$. One immediately sees that $(-\Delta_1^D)^s$ is homogeneous of order zero and $0 \in (-\Delta_1^D)^s 0$. Moreover, under the hypotheses (5.1)–(5.3), the operator $(-\Delta_1^D)^s + F$ is ω-quasi m-completely accretive on $L^2(\Sigma)$. Thus, by Theorem 4.15, we have the following regularity result.

Corollary 5.4. *For every $1 \le q < \infty$ and $u_0 \in L^\psi(\Sigma)$ (respectively $u_0 \in L^{1\cap\infty}(\Sigma)$ if $q = \infty$), the unique mild solution u of problem*

$$
\begin{cases}
\displaystyle\frac{\mathrm{d}u}{\mathrm{d}t} + PV\int_\Sigma \frac{u(x)-u(y)}{|u(y)-u(x)|}\frac{\mathrm{d}y}{|x-y|^{d+s}} + f(x,u) = 0 & \text{on } \Sigma\times(0,+\infty), \\
u = 0 & \text{on } \partial\Sigma\times(0,+\infty), \\
u(0) = u_0 & \text{on } \Sigma\times\{t=0\},
\end{cases}
\tag{5.8}
$$

is a strong solution satisfying (4.12). Moreover, if $f \equiv 0$, then either for every $1 \le \psi \equiv p \le \infty$ or N-function ψ and every $u_0 \in L^\psi(\Sigma)$, the unique mild solution u of problem (5.5) satisfies (4.7) and (4.8).

Further, since for every $0 < s < 1$ and $d \ge 1$, the following *(fractional) Sobolev inequality* (cf [15, Theorem 14.29])

$$
\|u\|_{\frac{d}{d-s}} \le C\,[u]_{s,1} \qquad \text{for every } u \in \mathcal{W}_0^{s,1}(\Sigma),
$$

holds for a constant $C = C(d,s) > 0$, we have that the fractional Dirichlet 1-Laplace operator $A = (-\Delta_1^D)^s$ satisfies the following abstract Sobolev inequality

$$
\|u\|_r^\sigma \le C\,[u,v]_2 \qquad \text{for every } (u,v) \in A
$$

with parameters $r = \frac{d}{d-s} > 1$ and $\sigma = 1$, where $[\cdot,\cdot]_2$ denote the L^2-inner product. Thus, by [12, Theorem 1.2], the semigroup $\{T_t\}_{t\ge0}$ generated by $-((-\Delta_1^D)^s + F)$ satisfies the following L^2-$L^{\frac{d}{d-s}}$*-regularity estimate*

$$
\|T_t u\|_{\frac{d}{d-s}} \le \frac{C}{2}\,t^{-1}\,e^{3\omega t}\|u\|_2^2 \qquad \text{for every } t > 0
$$

and $u \in L^2(\Sigma)$. Further, for every $q > \frac{d-s}{s}$, one has that

$$
\|T_t u\|_\infty \le \tilde{C}\,t^{-\frac{d-s}{s(q+1)-d}}\,e^{\left(\frac{d-s}{s(q+1)-d}+1\right)\omega t}\,\|u\|_{\frac{dq}{d-s}}^{\frac{d-s}{s(q+1)-d}\frac{sq}{d-s}} \qquad \text{for every } t > 0
$$

and $u \in L^{\frac{dq}{d-s}}(\Sigma)$. Thus, by Corollary 2.6 and inequality (4.12) for $q = \infty$, we also have the following estimate.

Corollary 5.5. *Let $d \geq 1$, $0 < s < 1$ and $q > \frac{d-s}{s}$. Then for every $u_0 \in L^{\frac{dq}{d-s}}(\Sigma)$, the unique solution u of (5.8) satisfies*

$$\left\| \frac{\mathrm{d}u}{\mathrm{d}t} \right\|_\infty \leq \tilde{C}\, C_\omega(t/2) 2^{\frac{d-s}{s(q+1)-d}+2} e^{\frac{\omega}{2}\left(\frac{d-s}{s(q+1)-d}+1\right)t} \frac{\|u_0\|_{dq/(d-s)}^{\frac{d-s}{s(q+1)-d}\frac{sq}{d-s}}}{t^{\frac{d-s}{s(q+1)-d}+1}} \qquad \text{for every } t > 0,$$

where $C_\omega(t)$ is the constant in (4.12).

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(Daniel Hauer) SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF SYDNEY, NSW 2006, AUSTRALIA
E-mail address: daniel.hauer@sydney.edu.au

(José M. Mazón) DEPARTAMENT D'ANÀLISI MATEMÀTICA, UNIVERSITAT DE VALÈNCIA, VALENCIA, SPAIN
E-mail address: mazon@uv.es