Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: type C

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Abstract

An explicit isomorphism between the R-matrix and Drinfeld presentations of the quantum affine algebra in type A was given by Ding and I. Frenkel (1993). We show that this result can be extended to types B, C and D and give a detailed construction for type C in this paper. In all classical types the Gauss decomposition of the generator matrix in the R-matrix presentation yields the Drinfeld generators. To prove that the resulting map is an isomorphism we follow the work of E. Frenkel and Mukhin (2002) in type A and employ the universal R-matrix to construct the inverse map. A key role in our construction is played by an embedding theorem which allows us to consider the quantum affine algebra of rank $n-1$ in the R-matrix presentation as a subalgebra of the corresponding algebra of rank n of the same type.

Contents

1 Introduction

The quantum affine algebras $U_q(\widehat{\mathfrak{g}})$ were introduced independently by Drinfeld [9] and Jimbo [23] as deformations of the universal enveloping algebras of the affine Lie algebras $\widehat{\mathfrak{g}}$ in the class of Hopf algebras. Another presentation of these algebras was given by Drinfeld [10], which is known as the *new realization* or *Drinfeld presentation*. A detailed construction of the isomorphism between the presentations was given by Beck [2]. Yet another *R-matrix presentation* of the quantum affine algebras was introduced by Reshetikhin and Semenov-Tian-Shansky [32] and further developed by I. Frenkel and Reshetikhin [17].

The algebras $U_q(\widehat{\mathfrak{g}})$ possess a substantive algebraic structure and rich representation theory. Their finite-dimensional irreducible representations were classified by Chari and Pressly in terms of the Drinfeld presentation; see [5, Chapter 12]. A theory of *q-characters* of these representations was developed in [14] and [16]; its connections with classical and quantum integrable systems were reviewed in the expository paper [29].

The R-matrix presentation of the quantum affine algebras can also be used to describe finite-dimensional irreducible representations by following the approach of Tarasov [34]; see also [20]. The role of this presentation in the theory of Knizhnik–Zamolodchikov equations is discussed in detail in the lectures [12].

An explicit isomorphism between the Drinfeld and R-matrix presentations of the algebras $U_q(\widehat{\mathfrak{g}})$ should provide a bridge between the two sides of the theory and widen the spectrum of methods for their investigation. Such an isomorphism was already constructed in the case of simple Lie algebras $\mathfrak{g}$ of type A by Ding and I. Frenkel [8]. We aim to extend this result to the Lie algebras $\mathfrak{g}$ of types B, C and D. The present article is concerned with type C, while types B and D will be dealt with in a forthcoming paper.

Our approach is similar to [8]; it is based on the *Gauss decomposition* of the generator matrices in the R-matrix presentation. The first part of the construction is the verification that the generators arising from the Gauss decomposition do satisfy the required relations of the Drinfeld presentation. The second part is the proof that the resulting homomorphism is injective. We use an argument alternative to [8] and follow the work of E. Frenkel and Mukhin [15] instead, where the map inverse to the Ding–Frenkel isomorphism was constructed. This map relies on the formula for the universal R-matrix corresponding to the algebra $U_q(\widehat{\mathfrak{g}})$ due to Tolstoy and Khoroshkin [35] and Damiani [6]. It turns out to be possible to use this formula in types B, C and D to construct a similar map in those cases.

Similar to the Yangian case in our previous work [26], in this paper we will mainly work with the *extended quantum affine algebra* in type C defined by an R-matrix presentation. We prove an embedding theorem which will allow us to regard the extended algebra of rank $n-1$ as a subalgebra of the corresponding algebra of rank n. We also produce a Drinfeld-type presentation for the extended quantum affine algebra and give explicit formulas for generators of its center. Note that a different approach to the equivalences between Yangian presentations and to the description of the centers of the extended Yangians was developed in [21] and [36] which should also be applicable to quantum affine algebras.

To state our isomorphism theorem, choose simple roots for the symplectic Lie algebra

$\mathfrak{g} = \mathfrak{sp}_{2n}$ in the form

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \quad \text{for} \quad i = 1, \dots, n-1 \qquad \text{and} \qquad \alpha_n = 2\epsilon_n,$$

where $\epsilon_1, \dots, \epsilon_n$ is an orthonormal basis of a Euclidian space with the bilinear form $(\cdot\,,\cdot)$. The Cartan matrix $[A_{ij}]$ is defined by

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}. \tag{1.1}$$

For a variable q we set $q_i = q^{r_i}$ for $i = 1, \dots, n$, where $r_i = (\alpha_i, \alpha_i)/2$, so that $q_i = q$ for $i < n$ and $q_n = q^2$. We will use the standard notation

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} \tag{1.2}$$

for a nonnegative integer k, and

$$[k]_q! = \prod_{s=1}^{k} [s]_q, \qquad \begin{bmatrix} k \\ r \end{bmatrix}_q = \frac{[k]_q!}{[r]_q!\,[k-r]_q!}.$$

The *quantum affine algebra* $U_q(\widehat{\mathfrak{sp}}_{2n})$ *in its Drinfeld presentation* is the associative algebra over $\mathbb{C}(q)$ with generators $x^{\pm}_{i,m}$, $a_{i,l}$, $k_i^{\pm}$ and $q^{\pm c/2}$ for $i = 1, \dots, n$ and $m, l \in \mathbb{Z}$ with $l \neq 0$, subject to the following defining relations: the elements $q^{\pm c/2}$ are central,

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad q^{c/2} q^{-c/2} = q^{-c/2} q^{c/2} = 1,$$

$$k_i k_j = k_j k_i, \qquad k_i\, a_{j,k} = a_{j,k}\, k_i, \qquad k_i\, x^{\pm}_{j,m}\, k_i^{-1} = q_i^{\pm A_{ij}} x^{\pm}_{j,m},$$

$$[a_{i,m}, a_{j,l}] = \delta_{m,-l} \, \frac{[mA_{ij}]_{q_i}}{m} \, \frac{q^{mc} - q^{-mc}}{q_j - q_j^{-1}},$$

$$[a_{i,m}, x^{\pm}_{j,l}] = \pm \frac{[mA_{ij}]_{q_i}}{m} \, q^{\mp |m| c/2} \, x^{\pm}_{j,m+l},$$

$$x^{\pm}_{i,m+1} x^{\pm}_{j,l} - q_i^{\pm A_{ij}} x^{\pm}_{j,l} x^{\pm}_{i,m+1} = q_i^{\pm A_{ij}} x^{\pm}_{i,m} x^{\pm}_{j,l+1} - x^{\pm}_{j,l+1} x^{\pm}_{i,m},$$

$$[x^{+}_{i,m}, x^{-}_{j,l}] = \delta_{ij} \frac{q^{(m-l)\,c/2}\, \psi_{i,m+l} - q^{-(m-l)\,c/2}\, \varphi_{i,m+l}}{q_i - q_i^{-1}},$$

$$\sum_{\pi \in \mathfrak{S}_r} \sum_{l=0}^{r} (-1)^l \begin{bmatrix} r \\ l \end{bmatrix}_{q_i} x^{\pm}_{i,s_{\pi(1)}} \dots x^{\pm}_{i,s_{\pi(l)}} x^{\pm}_{j,m} x^{\pm}_{i,s_{\pi(l+1)}} \dots x^{\pm}_{i,s_{\pi(r)}} = 0, \quad i \neq j,$$

where in the last relation we set $r = 1 - A_{ij}$. The elements $\psi_{i,m}$ and $\varphi_{i,-m}$ with $m \in \mathbb{Z}_+$ are defined by

$$\psi_i(u) := \sum_{m=0}^{\infty} \psi_{i,m} u^{-m} = k_i \exp\Big((q_i - q_i^{-1}) \sum_{s=1}^{\infty} a_{i,s} u^{-s} \Big), \tag{1.3}$$

$$\varphi_i(u) := \sum_{m=0}^{\infty} \varphi_{i,-m} u^{m} = k_i^{-1} \exp\Big(-(q_i - q_i^{-1}) \sum_{s=1}^{\infty} a_{i,-s} u^{s} \Big), \tag{1.4}$$

whereas $\psi_{i,m} = \varphi_{i,-m} = 0$ for $m < 0$.

To introduce the R-matrix presentation of the quantum affine algebra, consider the following elements of the endomorphism algebra $\operatorname{End}(\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}) \cong \operatorname{End}\mathbb{C}^{2n} \otimes \operatorname{End}\mathbb{C}^{2n}$:

$$P = \sum_{i,j=1}^{2n} e_{ij} \otimes e_{ji}, \qquad Q = \sum_{i,j=1}^{2n} q^{\bar{\imath}-\bar{\jmath}}\, \varepsilon_i \varepsilon_j\, e_{i'j'} \otimes e_{ij},$$

and

$$\begin{aligned} R = q \sum_{i=1}^{2n} e_{ii} \otimes e_{ii} + \sum_{i \neq j,j'} e_{ii} \otimes e_{jj} + q^{-1} \sum_{i \neq i'} e_{ii} \otimes e_{i'i'} \\ + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji} - (q - q^{-1}) \sum_{i>j} q^{\bar{\imath}-\bar{\jmath}}\, \varepsilon_i \varepsilon_j\, e_{i'j'} \otimes e_{ij}, \end{aligned}$$

where $e_{ij} \in \operatorname{End}\mathbb{C}^{2n}$ are the matrix units and we used the notation

$$\varepsilon_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n, \\ -1 & \text{for} \quad i = n+1, \dots, 2n, \end{cases}$$

$i' = 2n - i + 1$ and $(\,\overline{1}, \overline{2}, \dots, \overline{2n}\,) = (n, n-1, \dots, 1, -1, \dots, -n)$. Furthermore, consider the formal power series

$$f(u) = 1 + \sum_{k=1}^{\infty} f_k u^k$$

whose coefficients f_k are rational functions in q uniquely determined by the relation

$$f(u)\, f(u\xi) = \frac{1}{(1 - uq^{-2})(1 - uq^{2})(1 - u\xi)(1 - u\xi^{-1})}, \tag{1.5}$$

where $\xi = q^{-2n-2}$. Equivalently, $f(u)$ is given by the infinite product formula

$$f(u) = \prod_{r=0}^{\infty} \frac{(1 - u\,\xi^{2r})(1 - u\,q^{-2}\,\xi^{2r+1})(1 - u\,q^{2}\,\xi^{2r+1})(1 - u\,\xi^{2r+2})}{(1 - u\,\xi^{2r-1})(1 - u\,\xi^{2r+1})(1 - u\,q^{2}\xi^{2r})(1 - u\,q^{-2}\xi^{2r})}. \tag{1.6}$$

Introduce the *R-matrix* $R(u)$ by

$$R(u) = f(u)\Big(q^{-1}(u-1)(u-\xi)R - (q^{-2}-1)(u-\xi)P + (q^{-2}-1)(u-1)\,\xi\, Q\Big). \tag{1.7}$$

This formula goes back to Jimbo [24]; for the significance of the scalar function $f(u)$ see the paper by Frenkel and Reshetikhin [17]. The R-matrix is a solution of the *Yang–Baxter equation*

$$R_{12}(u)\, R_{13}(uv)\, R_{23}(v) = R_{23}(v)\, R_{13}(uv)\, R_{12}(u). \tag{1.8}$$

The associative algebra $U_q^R(\widehat{\mathfrak{sp}}_{2n})$ over $\mathbb{C}(q)$ is generated by an invertible central element $q^{c/2}$ and elements $l_{ij}^{\pm}[\mp m]$ with $1 \leqslant i, j \leqslant 2n$ and $m \in \mathbb{Z}_+$ subject to the following defining relations. We have

$$l_{ij}^{+}[0] = l_{ji}^{-}[0] = 0 \quad \text{for} \quad i > j \qquad \text{and} \qquad l_{ii}^{+}[0]\, l_{ii}^{-}[0] = l_{ii}^{-}[0]\, l_{ii}^{+}[0] = 1,$$

while the remaining relations will be written in terms of the formal power series

$$l_{ij}^{\pm}(u) = \sum_{m=0}^{\infty} l_{ij}^{\pm}[\mp m]\, u^{\pm m} \tag{1.9}$$

which we combine into the respective matrices

$$L^{\pm}(u) = \sum_{i,j=1}^{2n} e_{ij} \otimes l_{ij}^{\pm}(u) \in \operatorname{End} \mathbb{C}^{2n} \otimes U_q^R(\widehat{\mathfrak{sp}}_{2n})[[u, u^{-1}]].$$

Consider the tensor product algebra $\operatorname{End}\mathbb{C}^{2n} \otimes \operatorname{End}\mathbb{C}^{2n} \otimes U_q^R(\widehat{\mathfrak{sp}}_{2n})$ and introduce the series with coefficients in this algebra by

$$L_1^{\pm}(u) = \sum_{i,j=1}^{2n} e_{ij} \otimes 1 \otimes l_{ij}^{\pm}(u) \qquad \text{and} \qquad L_2^{\pm}(u) = \sum_{i,j=1}^{2n} 1 \otimes e_{ij} \otimes l_{ij}^{\pm}(u). \tag{1.10}$$

The defining relations then take the form

$$R(u/v) L_1^{\pm}(u) L_2^{\pm}(v) = L_2^{\pm}(v) L_1^{\pm}(u) R(u/v), \tag{1.11}$$

$$R(uq^{c}/v) L_1^{+}(u) L_2^{-}(v) = L_2^{-}(v) L_1^{+}(u) R(uq^{-c}/v), \tag{1.12}$$

together with the relations

$$L^{\pm}(u) D L^{\pm}(u\,\xi)^{\mathrm{t}} D^{-1} = 1, \tag{1.13}$$

where t denotes the matrix transposition with $e_{ij}^{\mathrm{t}} = \varepsilon_i\, \varepsilon_j\, e_{j',i'}$ and D is the diagonal matrix

$$D = \operatorname{diag}[q^n, \dots, q, q^{-1}, \dots, q^{-n}]. \tag{1.14}$$

Now apply the *Gauss decomposition* to the matrices $L^+(u)$ and $L^-(u)$. There exist unique matrices of the form

$$F^{\pm}(u) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ f_{21}^{\pm}(u) & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{2n\,1}^{\pm}(u) & f_{2n\,2}^{\pm}(u) & \dots & 1 \end{bmatrix}, \qquad E^{\pm}(u) = \begin{bmatrix} 1 & e_{12}^{\pm}(u) & \dots & e_{1\,2n}^{\pm}(u) \\ 0 & 1 & \dots & e_{2\,2n}^{\pm}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

and $H^{\pm}(u) = \operatorname{diag}\big[h_1^{\pm}(u), \dots, h_{2n}^{\pm}(u)\big]$, such that

$$L^{\pm}(u) = F^{\pm}(u) H^{\pm}(u) E^{\pm}(u). \tag{1.15}$$

For $i = 1, \ldots, n$ set

$$X_i^+(u) = e_{i,i+1}^+(uq^{c/2}) - e_{i,i+1}^-(uq^{-c/2}), \qquad X_i^-(u) = f_{i+1,i}^+(uq^{-c/2}) - f_{i+1,i}^-(uq^{c/2}).$$

To state our main result, combine the generators $x_{i,m}^{\pm}$ of the algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ into the series

$$x_i^{\pm}(u) = \sum_{m\in\mathbb{Z}} x_{i,m}^{\pm}\, u^{-m}. \tag{1.16}$$

Main Theorem. *The maps* $q^{c/2} \mapsto q^{c/2}$,

$$\begin{aligned} x_i^{\pm}(u) &\mapsto (q_i - q_i^{-1})^{-1} X_i^{\pm}(uq^i),\\ \psi_i(u) &\mapsto h_{i+1}^-(uq^i)\, h_i^-(uq^i)^{-1},\\ \varphi_i(u) &\mapsto h_{i+1}^+(uq^i)\, h_i^+(uq^i)^{-1}, \end{aligned}$$

for $i = 1, \ldots, n-1$, *and*

$$\begin{aligned} x_n^{\pm}(u) &\mapsto (q_n - q_n^{-1})^{-1} X_n^{\pm}(uq^{n+1}),\\ \psi_n(u) &\mapsto h_{n+1}^-(uq^{n+1})\, h_n^-(uq^{n+1})^{-1},\\ \varphi_n(u) &\mapsto h_{n+1}^+(uq^{n+1})\, h_n^+(uq^{n+1})^{-1}, \end{aligned}$$

define an isomorphism $U_q(\widehat{\mathfrak{sp}}_{2n}) \to U_q^R(\widehat{\mathfrak{sp}}_{2n})$.

For the proof of the Main Theorem we embed $U_q(\widehat{\mathfrak{sp}}_{2n})$ into an extended quantum affine algebra $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$ which is defined by a Drinfeld-type presentation. The next step is to use the Gauss decomposition to construct a homomorphism from the extended quantum affine algebra to the algebra $U(R)$ which is defined by the same presentation as the algebra $U_q^R(\widehat{\mathfrak{sp}}_{2n})$, except that the relation (1.13) is omitted. The expressions on the left hand side of (1.13), considered in the algebra $U(R)$, turn out to be scalar matrices,

$$L^{\pm}(u)DL^{\pm}(u\xi)^{\mathrm{t}}D^{-1} = z^{\pm}(u)\, 1,$$

for certain formal series $z^{\pm}(u)$. Moreover, all coefficients of these series are central in $U(R)$. We will give explicit formulas for $z^{\pm}(u)$, regarded as series with coefficients in the algebra $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$, in terms of its Drinfeld generators. The quantum affine algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ can therefore be considered as the quotient of $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$ by the relations $z^{\pm}(u) = 1$.

As a final step, we construct the inverse map $U(R) \to U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$ by using the universal R-matrix for the quantum affine algebra and producing the associated L-operators corresponding to the vector representation of the algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$.

An immediate consequence of the Main Theorem is the Poincaré–Birkhoff–Witt theorem for the R-matrix presentation $U_q^R(\widehat{\mathfrak{sp}}_{2n})$ of the quantum affine algebra which is implied by the corresponding result of Beck [1] for $U_q(\widehat{\mathfrak{sp}}_{2n})$. As another application, it is straightforward to transfer the classification theorem for finite-dimensional irreducible representations of the algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ to its R-matrix presentation $U_q^R(\widehat{\mathfrak{sp}}_{2n})$; see [5, Chapter 12].

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2 Quantum affine algebras

We start by recalling the original definition of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ as introduced by Drinfeld [9] and Jimbo [23]. We suppose that $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ of rank n and $\widehat{\mathfrak{g}}$ is the corresponding (untwisted) affine Kac–Moody algebra with the affine Cartan matrix $[A_{ij}]_{i,j=0}^n$. We let $\alpha_0, \alpha_1, \dots, \alpha_n$ denote the simple roots and use the notation as in [5, Secs 9.1 and 12.2] so that $q_i = q^{r_i}$ for $r_i = (\alpha_i, \alpha_i)/2$.

2.1 Drinfeld–Jimbo definition and new realization

The *quantum affine algebra* $U_q(\widehat{\mathfrak{g}})$ is a unital associative algebra over $\mathbb{C}(q)$ with generators E_{α_i}, F_{α_i} and $k_i^{\pm 1}$ with $i = 0, 1, \dots, n$, subject to the defining relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i k_j = k_i k_j,$$

$$k_i E_{\alpha_j} k_i^{-1} = q_i^{A_{ij}} E_{\alpha_j}, \qquad k_i F_{\alpha_j} k_i^{-1} = q_i^{-A_{ij}} F_{\alpha_j},$$

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-A_{ij}} (-1)^r \begin{bmatrix} 1 - A_{ij} \\ r \end{bmatrix}_{q_i} (E_{\alpha_i})^r E_{\alpha_j} (E_{\alpha_i})^{1-A_{ij}-r} = 0, \qquad \text{if} \quad i \neq j,$$

$$\sum_{r=0}^{1-A_{ij}} (-1)^r \begin{bmatrix} 1 - A_{ij} \\ r \end{bmatrix}_{q_i} (F_{\alpha_i})^r F_{\alpha_j} (F_{\alpha_i})^{1-A_{ij}-r} = 0, \qquad \text{if} \quad i \neq j.$$

By using the braid group action, the set of generators of the algebra $U_q(\widehat{\mathfrak{g}})$ can be extended to the set of affine root vectors of the form $E_{\alpha+k\delta}$, $F_{\alpha+k\delta}$, $E_{(k\delta,i)}$ and $E_{(k\delta,i)}$, where α runs over the positive roots of $\mathfrak{g}$, and δ is the basic imaginary root; see [2, 3] for details. The root vectors are used in the explicit isomorphism between the Drinfeld–Jimbo presentation of the algebra $U_q(\widehat{\mathfrak{g}})$ and the "new realization" of Drinfeld which goes back to [10], while detailed arguments were given by Beck [2]; see also [3]. In particular, for the Drinfeld presentation of the algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ given in the Introduction, we find that the isomorphism between these presentations is given by

$$\begin{aligned} x_{ik}^+ &\mapsto o(i)^k E_{\alpha_i+k\delta}, & x_{i,-k}^- &\mapsto o(i)^k F_{\alpha_i+k\delta}, & k &\geqslant 0, \\ x_{i,-k}^+ &\mapsto -o(i)^k F_{-\alpha_i+k\delta}\, k_i^{-1} q^{kc}, & x_{i,k}^- &\mapsto -o(i)^k q^{-kc}\, k_i\, E_{-\alpha_i+k\delta}, & k &> 0, \\ a_{i,k} &\mapsto o(i)^k q^{-kc/2} E_{(k\delta,i)}, & a_{i,-k} &\mapsto o(i)^k F_{(k\delta,i)} q^{kc/2}, & k &> 0, \end{aligned}$$

where $o : \{1, 2, \dots, n\} \to \{\pm 1\}$ is a map such that $o(i) = -o(j)$ whenever $A_{ij} < 0$.

2.2 Extended quantum affine algebra

As with the embedding of quantum affine algebras $U_q(\widehat{\mathfrak{sl}}_N) \hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$ considered in [8] and [15], it will be convenient to embed the algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ into an extended quantum affine algebra which we denote by $U_q^{\rm ext}(\widehat{\mathfrak{sp}}_{2n})$.

Beside the scalar function $f(u)$ defined by (1.5) and (1.6) we will also use the function

$$g(u) = f(u)(u - q^{-2})(u - \xi). \tag{2.1}$$

Definition 2.1. The *extended quantum affine algebra* $U_q^{\rm ext}(\widehat{\mathfrak{sp}}_{2n})$ is an associative algebra with generators $X^{\pm}_{i,k}$, $h^{+}_{j,m}$, $h^{-}_{j,-m}$ and $q^{c/2}$, where the subscripts take values $i = 1, \dots, n$ and $k \in \mathbb{Z}$, while $j = 1, \dots, n+1$ and $m \in \mathbb{Z}_+$. The defining relations are written with the use of generating functions in a formal variable u:

$$X^{\pm}_i(u) = \sum_{k\in\mathbb{Z}} X^{\pm}_{i,k}\, u^{-k}, \qquad h^{\pm}_i(u) = \sum_{m=0}^{\infty} h^{\pm}_{i,\pm m}\, u^{\mp m},$$

they take the following form. The element $q^{c/2}$ is central and invertible,

$$h^{+}_{i,0} h^{-}_{i,0} = h^{-}_{i,0} h^{+}_{i,0} = 1, \tag{2.2}$$

for the relations involving $h^{\pm}_i(u)$ we have

$$h^{\pm}_i(u) h^{\pm}_j(v) = h^{\pm}_j(v) h^{\pm}_i(u), \tag{2.3}$$

$$g(uq^{c}/v)\, h^{\pm}_i(u) h^{\mp}_i(v) = g(uq^{-c}/v)\, h^{\mp}_i(v) h^{\pm}_i(u), \tag{2.4}$$

$$g(uq^{c}/v)\, \frac{u_{\pm} - v_{\mp}}{q u_{\pm} - q^{-1} v_{\mp}} h^{\pm}_i(u) h^{\mp}_j(v) = g(uq^{-c}/v)\, \frac{u_{\mp} - v_{\pm}}{q u_{\mp} - q^{-1} v_{\pm}} h^{\mp}_j(v) h^{\pm}_i(u) \tag{2.5}$$

for $i < j$ and $i \neq n$, and

$$g(uq^{c}/v)\, \frac{u_{\pm} - v_{\mp}}{q^2 u_{\pm} - q^{-2} v_{\mp}} h^{\pm}_n(u) h^{\mp}_{n+1}(v) = g(uq^{-c}/v)\, \frac{u_{\mp} - v_{\pm}}{q^2 u_{\mp} - q^{-2} v_{\pm}} h^{\mp}_{n+1}(v) h^{\pm}_n(u), \tag{2.6}$$

where we use the notation $u_{\pm} = u q^{\pm c/2}$. The relations involving $h^{\pm}_i(u)$ and $X^{\pm}_j(v)$ are

$$h^{\pm}_i(u) X^{+}_j(v) = \frac{u - v_{\pm}}{q^{(\epsilon_i,\alpha_j)} u - q^{-(\epsilon_i,\alpha_j)} v_{\pm}} X^{+}_j(v) h^{\pm}_i(u), \tag{2.7}$$

$$h^{\pm}_i(u) X^{-}_j(v) = \frac{q^{(\epsilon_i,\alpha_j)} u_{\pm} - q^{-(\epsilon_i,\alpha_j)} v}{u_{\pm} - v} X^{-}_j(v) h^{\pm}_i(u), \tag{2.8}$$

for $i \neq n+1$, together with

$$h^{\pm}_{n+1}(u) X^{+}_n(v) = \frac{u_{\mp} - v}{q^{-2} u_{\mp} - q^{2} v} X^{+}_n(v) h^{\pm}_{n+1}(u), \tag{2.9}$$

$$h^{\pm}_{n+1}(u) X^{-}_n(v) = \frac{q^{-2} u_{\pm} - q^{2} v}{u_{\pm} - v} X^{-}_n(v) h^{\pm}_{n+1}(u), \tag{2.10}$$

and

$$h^{\pm}_{n+1}(u)^{-1}X^{+}_{n-1}(v)h^{\pm}_{n+1}(u)=\frac{q^{-1}u-qv_{\pm}}{q^{-2}u-q^{2}v_{\pm}}X^{+}_{n-1}(v), \tag{2.11}$$

$$h^{\pm}_{n+1}(u)X^{-}_{n-1}(v)h^{\pm}_{n+1}(u)^{-1}=\frac{q^{-1}u-qv_{\mp}}{q^{-2}u-q^{2}v_{\mp}}X^{-}_{n-1}(v), \tag{2.12}$$

while

$$h^{\pm}_{n+1}(u)X^{+}_{i}(v)=X^{+}_{i}(v)\,h^{\pm}_{n+1}(u), \tag{2.13}$$

$$h^{\pm}_{n+1}(u)X^{-}_{i}(v)=X^{-}_{i}(v)\,h^{\pm}_{n+1}(u), \tag{2.14}$$

for $1\leqslant i\leqslant n-2$. For the relations involving $X^{\pm}_{i}(u)$ we have

$$(uq^{-i}-q^{\pm(\alpha_i,\alpha_j)-j}v)X^{\pm}_{i}(u)X^{\pm}_{j}(v)=(q^{\pm(\alpha_i,\alpha_j)-i}u-q^{-j}v)X^{\pm}_{j}(v)X^{\pm}_{i}(u)$$

and

$$[X^{+}_{i}(u),X^{-}_{j}(v)]=\delta_{ij}(q_i-q_i^{-1})$$
$$\times\Big(\delta(uq^{-c}/v)h^{-}_{i}(v_+)^{-1}h^{-}_{i+1}(v_+)-\delta(uq^{c}/v)h^{+}_{i}(u_+)^{-1}h^{+}_{i+1}(u_+)\Big),$$

together with the *Serre relations*

$$\sum_{\pi\in\mathfrak{S}_r}\sum_{l=0}^{r}(-1)^l\begin{bmatrix}r\\ l\end{bmatrix}_{q_i}X^{\pm}_{i}(u_{\pi(1)})\dots X^{\pm}_{i}(u_{\pi(l)})X^{\pm}_{j}(v)\,X^{\pm}_{i}(u_{\pi(l+1)})\dots X^{\pm}_{i}(u_{\pi(r)})=0, \tag{2.15}$$

which hold for all $i\neq j$ and we set $r=1-A_{ij}$. Here we used the notation

$$\delta(u)=\sum_{r\in\mathbb{Z}}u^r \tag{2.16}$$

for the *formal δ-function.* □

Introduce two formal power series $z^+(u)$ and $z^-(u)$ in u and u^{-1}, respectively, with coefficients in the algebra $U_q^{\mathrm{ext}}(\widehat{\mathfrak{sp}}_{2n})$ by

$$z^{\pm}(u)=\prod_{i=1}^{n-1}h^{\pm}_{i}(u\xi q^{2i})^{-1}\prod_{i=1}^{n}h^{\pm}_{i}(u\xi q^{2i-2})h^{\pm}_{n+1}(u), \tag{2.17}$$

where we keep using the notation $\xi=q^{-2n-2}$. Note that by (2.3) the ordering of the factors in the products is irrelevant.

Proposition 2.2. *The coefficients of $z^{\pm}(u)$ are central elements of $U_q^{\mathrm{ext}}(\widehat{\mathfrak{sp}}_{2n})$.*

Proof. We will outline the arguments for $z^+(u)$; the case of $z^-(u)$ is quite similar. By (2.3) we obviously have $z^+(u)\, h_j^+(v) = h_j^+(v)\, z^+(u)$ for all $j = 1, \ldots, n+1$. It is straightforward to deduce from the defining relations in Definition 2.1 that $z^+(u)\, X_j^\pm(v) = X_j^\pm(v)\, z^+(u)$ for $j = 1, \ldots, n$. We give more details to check that $z^+(u)\, h_{n+1}^-(v) = h_{n+1}^-(v)\, z^+(u)$ as this involves the function (2.1). The remaining calculations are performed in the same way. Applying (2.4) we get

$$\begin{aligned} z^+(u)h_{n+1}^-(v) = {} & g(uq^{-c}/v)\, g(uq^{c}/v)^{-1} \\ & \times \prod_{i=1}^{n-1} h_i^+(u\xi q^{2i})^{-1} \prod_{i=1}^{n-1} h_i^+(u\xi q^{2i-2})\, h_n^+(uq^{-4}) h_{n+1}^-(v) h_{n+1}^+(u). \end{aligned}$$

Furthermore, (2.6) implies

$$\begin{aligned} z^+(u)h_{n+1}^-(v) = {} & g(uq^{-c-4}/v)\, g(uq^{c-4}/v)^{-1}\, g(uq^{-c}/v)\, g(uq^{c}/v)^{-1} \\ & \times \frac{q^{-2}u_- - q^2 v_+}{u_- - v_+} \frac{u_+ - v_-}{q^{-2}u_+ - q^2 v_-} \\ & \times \prod_{i=1}^{n-1} h_i^+(u\xi q^{2i})^{-1} \prod_{i=1}^{n-1} h_i^+(u\xi q^{2i-2}) h_{n+1}^-(v) h_n^+(uq^{-4}) h_{n+1}^+(u). \end{aligned}$$

Due to (2.3), the last product can be rearranged as

$$\prod_{i=1}^{n-2} h_i^+(u\xi q^{2i})^{-1} h_{i+1}^+(u\xi q^{2i}) h_{n-1}^+(uq^{-4})^{-1} h_1^+(u\xi) h_{n+1}^-(v) h_n^+(uq^{-4}) h_{n+1}^+(u).$$

Now, applying (2.5) repeatedly, we come to the relation

$$\begin{aligned} z^+(u)h_{n+1}^-(v) = {} & g(u\xi q^{-c}/v)\, g(u\xi q^{c}/v)^{-1}\, g(uq^{-c}/v)\, g(uq^{c}/v)^{-1} \\ & \times \frac{u_-\xi - v_+}{u_-\xi q - v_+ q^{-1}} \frac{u_+\xi q - v_- q^{-1}}{u_+\xi - v_-} \frac{q^{-1}u_- - q v_+}{u_- - v_+} \frac{u_+ - v_-}{q^{-1}u_+ - q v_-}\, h_{n+1}^-(v) z^+(u). \end{aligned}$$

Replace $g(u)$ by (2.1) to get

$$\begin{aligned} z^+(u)h_{n+1}^-(v) = {} & f(u\xi q^{-c}/v)\, f(u\xi q^{c}/v)^{-1}\, f(uq^{-c}/v)\, f(uq^{c}/v)^{-1} \\ & \times \frac{(u_-/v_+ - q^{-2})(u_-/v_+ - \xi)(u_-/v_+ - q^2)(u_-/v_+\xi - 1)}{(u_+/v_- - q^{-2})(u_+/v_- - \xi)(u_+/v_- - q^2)(u_+/v_-\xi - 1)}\, h_{n+1}^-(v) z^+(u). \end{aligned}$$

Since

$$f(u)f(u\xi) = \frac{1}{(1 - uq^{-2})(1 - uq^2)(1 - u\xi)(1 - u\xi^{-1})},$$

we can conclude that $z^+(u)h_{n+1}^-(v) = h_{n+1}^-(v)z^+(u)$. □

Proposition 2.3. *The maps* $q^{c/2} \mapsto q^{c/2}$,

$$x_i^{\pm}(u) \mapsto (q_i - q_i^{-1})^{-1} X_i^{\pm}(uq^i),$$

$$\psi_i(u) \mapsto h_{i+1}^{-}(uq^i)\, h_i^{-}(uq^i)^{-1},$$

$$\varphi_i(u) \mapsto h_{i+1}^{+}(uq^i)\, h_i^{+}(uq^i)^{-1},$$

for $i = 1, \ldots, n-1$, *and*

$$x_n^{\pm}(u) \mapsto (q_n - q_n^{-1})^{-1} X_n^{\pm}(uq^{n+1}),$$

$$\psi_n(u) \mapsto h_{n+1}^{-}(uq^{n+1})\, h_n^{-}(uq^{n+1})^{-1},$$

$$\varphi_n(u) \mapsto h_{n+1}^{+}(uq^{n+1})\, h_n^{+}(uq^{n+1})^{-1},$$

define an embedding $\varsigma : U_q(\widehat{\mathfrak{sp}}_{2n}) \hookrightarrow U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$.

Proof. Writing the defining relations of the quantum affine algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ in terms of the generating series $x_i^{\pm}(u)$, $\psi_i(u)$ and $\varphi_i(u)$, it is straightforward to check that the maps define a homomorphism. To show that its kernel is zero, we will construct another homomorphism $\varrho : U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n}) \to U_q(\widehat{\mathfrak{sp}}_{2n})$ such that the composition $\varrho \circ \varsigma$ is the identity homomorphism on $U_q(\widehat{\mathfrak{sp}}_{2n})$. We will extend both algebras by adjoining the square roots $k_n^{\pm 1/2}$ and $(t_n t_{n+1})^{\pm 1/2}$ to $U_q(\widehat{\mathfrak{sp}}_{2n})$ and $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$, respectively, where we use the notation $t_i = h_{i,0}^{+}$. We will keep the same notation for thus extended algebras for the rest of the argument. There exist power series $\zeta^{\pm}(u)$ with coefficients in the center of $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$ such that $\zeta^{\pm}(u)\, \zeta^{\pm}(u\xi) = z^{\pm}(u)$; see Proposition 2.2. The mappings

$$X_i^{\pm}(u) \mapsto X_i^{\pm}(u) \qquad \text{and} \qquad h_i^{\pm}(u) \mapsto h_i^{\pm}(u)\, \zeta^{\pm}(u)^{-1}$$

define a homomorphism from the algebra $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$ to itself. The definition of the series $\zeta^{\pm}(u)$ implies that for images of $h_i^{\pm}(u)$ we have the relation

$$h_i^{\pm}(u)\, \zeta^{\pm}(u)^{-1}\, h_i^{\pm}(u\xi)\, \zeta^{\pm}(u\xi)^{-1} = h_i^{\pm}(u)\, h_i^{\pm}(u\xi)\, z^{\pm}(u)^{-1},$$

whose right hand side can be written as a product of series of the form $h_{i+1}^{\pm}(uq^k)\, h_i^{\pm}(uq^k)^{-1}$. Hence the property $\varrho \circ \varsigma = \text{id}$ will be satisfied if we define the map $\varrho : U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n}) \to U_q(\widehat{\mathfrak{sp}}_{2n})$ by

$$X_i^{\pm}(u) \mapsto (q_i - q_i^{-1})\, x_i^{\pm}(uq^{-i}) \qquad \text{for} \quad i = 1, \ldots, n-1,$$

and

$$X_n^{\pm}(u) \mapsto (q_n - q_n^{-1})\, x_i^{\pm}(uq^{-n-1}),$$

while

$$h_i^{\pm}(u) \mapsto \alpha_i^{\pm}(u) \qquad \text{for} \quad i = 1, \ldots, n+1,$$

where the series $\alpha_i^{+}(u)$ are defined by the relations

$$\alpha_i^{+}(u)\, \alpha_i^{+}(u\xi) = \varphi_n(uq^{-n-1})^{-1} \prod_{k=1}^{n-1} \varphi_k(u\xi q^k) \prod_{k=1}^{i-1} \varphi_k(u\xi q^{-k}) \prod_{k=i}^{n-1} \varphi_k(uq^{-k})^{-1}$$

for $i=1,\dots,n$, and

$$\alpha^{+}_{n+1}(u)\,\alpha^{+}_{n+1}(u\xi)=\varphi_n(u\xi q^{-n-1})\prod_{k=1}^{n-1}\varphi_k(u\xi q^{k})\prod_{k=1}^{n-1}\varphi_k(u\xi q^{-k}).$$

The relations defining $\alpha^{-}_i(u)$ are obtained from those above by the respective replacements $\alpha^{+}_i(u)\to\alpha^{-}_i(u)$ and $\varphi_k(u)\to\psi_k(u)$. As with the map ς, a direct calculation verifies that the map ϱ defines a homomorphism. □

By Proposition 2.3, we may regard $U_q(\widehat{\mathfrak{sp}}_{2n})$ as a subalgebra of the extended quantum affine algebra $U^{\rm ext}_q(\widehat{\mathfrak{sp}}_{2n})$. With the notation used in the proof of the proposition, let $\mathcal{C}$ be the subalgebra generated by the coefficients of the series $z^{\pm}(u)$.

Corollary 2.4. *We have the tensor product decomposition*

$$U^{\rm ext}_q(\widehat{\mathfrak{sp}}_{2n})=U_q(\widehat{\mathfrak{sp}}_{2n})\otimes\mathcal{C}.$$

Proof. The decomposition $U^{\rm ext}_q(\widehat{\mathfrak{sp}}_{2n})=U_q(\widehat{\mathfrak{sp}}_{2n})\,\mathcal{C}$ is clear from the proof of Proposition 2.3. The linear independence of elements of $U_q(\widehat{\mathfrak{sp}}_{2n})$ over $\mathcal{C}$ can be verified by the same argument as for an analogous Yangian decomposition; see e.g. [31, Theorem 1.8.2]. □

3 R-matrix presentations

3.1 The algebras $U(R)$ and $U(\overline{R})$

Recall from the Introduction that the algebra $U(R)$ is generated by an invertible central element $q^{c/2}$ and elements $l^{\pm}_{ij}[\mp m]$ with $1\leqslant i,j\leqslant 2n$ and $m\in\mathbb{Z}_+$ such that

$$l^{+}_{ij}[0]=l^{-}_{ji}[0]=0\quad\text{for}\quad i>j\qquad\text{and}\qquad l^{+}_{ii}[0]\,l^{-}_{ii}[0]=l^{-}_{ii}[0]\,l^{+}_{ii}[0]=1,$$

and the remaining relations (1.11) and (1.12) (omitting (1.13)) written in terms of the formal power series (1.9). We will need another algebra $U(\overline{R})$ which is defined in a very similar way, except that it is associated with a different R-matrix $\overline{R}(u)$ instead of (1.7). Namely, the two R-matrices are related by $R(u)=g(u)\overline{R}(u)$ with $g(u)$ defined in (2.1), so that

$$\overline{R}(u)=\frac{u-1}{uq-q^{-1}}\,R+\frac{q-q^{-1}}{uq-q^{-1}}\,P-\frac{(q-q^{-1})(u-1)\xi}{(uq-q^{-1})(u-\xi)}\,Q.\tag{3.1}$$

Note the *unitarity property*

$$\overline{R}_{12}(u)\,\overline{R}_{21}(u^{-1})=1,\tag{3.2}$$

satisfied by this R-matrix, where $\overline{R}_{12}(u)=\overline{R}(u)$ and $\overline{R}_{21}(u)=P\overline{R}(u)P$.

The *algebra* $U(\overline{R})$ *over* $\mathbb{C}(q)$ is generated by an invertible central element $q^{c/2}$ and elements $\ell^{\pm}_{ij}[\mp m]$ with $1\leqslant i,j\leqslant 2n$ and $m\in\mathbb{Z}_+$ such that

$$\ell^{+}_{ij}[0]=\ell^{-}_{ji}[0]=0\quad\text{for}\quad i>j\qquad\text{and}\qquad \ell^{+}_{ii}[0]\,\ell^{-}_{ii}[0]=\ell^{-}_{ii}[0]\,\ell^{+}_{ii}[0]=1.$$

Introduce the formal power series

$$\ell_{ij}^{\pm}(u)=\sum_{m=0}^{\infty}\ell_{ij}^{\pm}[\mp m]\,u^{\pm m} \tag{3.3}$$

which we combine into the respective matrices

$$\mathcal{L}^{\pm}(u)=\sum_{i,j=1}^{2n}e_{ij}\otimes\ell_{ij}^{\pm}(u)\in\operatorname{End}\mathbb{C}^{2n}\otimes U(\overline{R})[[u,u^{-1}]].$$

The remaining defining relations of the algebra $U(\overline{R})$ take the form

$$\overline{R}(u/v)\,\mathcal{L}_1^{\pm}(u)\,\mathcal{L}_2^{\pm}(v)=\mathcal{L}_2^{\pm}(v)\,\mathcal{L}_1^{\pm}(u)\,\overline{R}(u/v), \tag{3.4}$$

$$\overline{R}(uq^{c}/v)\,\mathcal{L}_1^{+}(u)\,\mathcal{L}_2^{-}(v)=\mathcal{L}_2^{-}(v)\,\mathcal{L}_1^{+}(u)\,\overline{R}(uq^{-c}/v), \tag{3.5}$$

where the subscripts have the same meaning as in (1.10). The unitarity property (3.2) implies that relation (3.5) can be written in the equivalent form

$$\overline{R}(uq^{-c}/v)\,\mathcal{L}_1^{-}(u)\,\mathcal{L}_2^{+}(v)=\mathcal{L}_2^{+}(v)\,\mathcal{L}_1^{-}(u)\,\overline{R}(uq^{c}/v). \tag{3.6}$$

Remark 3.1. The defining relations satisfied by the series $\ell_{ij}^{\pm}(u)$ with $1\leqslant i,j\leqslant n$ coincide with those for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$ in [8]. □

Now we will follow [8] to describe a relationship between the algebras $U(R)$ and $U(\overline{R})$. Introduce a Heisenberg algebra $\mathcal{H}_q(n)$ with generators q^c and β_r with $r\in\mathbb{Z}\setminus\{0\}$. The defining relations of $\mathcal{H}_q(n)$ have the form

$$\big[\beta_r,\beta_s\big]=\delta_{r,-s}\,\alpha_r,\qquad r\geqslant 1,$$

and q^c is central and invertible. The elements α_r are defined by the expansion

$$\exp\sum_{r=1}^{\infty}\alpha_r u^r=\frac{g(uq^{-c})}{g(uq^{c})}.$$

So we have the identity

$$g(uq^{c})\exp\sum_{r=1}^{\infty}\beta_r u^r\cdot\,\exp\sum_{s=1}^{\infty}\beta_{-s}v^{-s}=g(uq^{-c})\exp\sum_{s=1}^{\infty}\beta_{-s}v^{-s}\cdot\exp\sum_{r=1}^{\infty}\beta_r u^r.$$

Proposition 3.2. *The mappings*

$$L^{+}(u)\mapsto\exp\sum_{r=1}^{\infty}\beta_r u^r\cdot\mathcal{L}^{+}(u),\qquad L^{-}(u)\mapsto\exp\sum_{r=1}^{\infty}\beta_{-r}u^{-r}\cdot\mathcal{L}^{-}(u), \tag{3.7}$$

define a homomorphism $U(R)\to\mathcal{H}_q(n)\otimes_{\mathbb{C}[q^c,\,q^{-c}]}U(\overline{R})$. □

We will need to apply the matrix transposition defined in (1.13) to certain copies of the endomorphism algebra $\mathrm{End}\,\mathbb{C}^{2n}$ in multiple tensor products. The corresponding partial transposition applied to the a-th copy will be denoted by t_a. We point out the following *crossing symmetry* relations satisfied by the R-matrices:

$$\overline{R}(u)D_1\overline{R}(u\xi)^{\mathrm{t}_1}D_1^{-1} = \frac{(u-q^2)(u\xi-1)}{(1-u)(1-u\xi q^2)}, \tag{3.8}$$

$$R(u)D_1R(u\xi)^{\mathrm{t}_1}D_1^{-1} = \xi^2 q^{-2}, \tag{3.9}$$

where the diagonal matrix D is defined in (1.14) and the meaning of the subscripts is the same as in (1.10).

Proposition 3.3. *In the algebras $U(R)$ and $U(\overline{R})$ we have the relations*

$$DL^{\pm}(u\xi)^{\mathrm{t}}D^{-1}L^{\pm}(u) = L^{\pm}(u)DL^{\pm}(u\xi)^{\mathrm{t}}D^{-1} = z^{\pm}(u)\,1, \tag{3.10}$$

and

$$D\mathcal{L}^{\pm}(u\xi)^{\mathrm{t}}D^{-1}\mathcal{L}^{\pm}(u) = \mathcal{L}^{\pm}(u)D\mathcal{L}^{\pm}(u\xi)^{\mathrm{t}}D^{-1} = \mathfrak{z}^{\pm}(u)\,1, \tag{3.11}$$

for certain series $z^{\pm}(u)$ and $\mathfrak{z}^{\pm}(u)$ with coefficients in the respective algebra.

Proof. The proof is the same in both cases so we only consider the algebra $U(\overline{R})$. Multiply both sides of (3.4) by $u/v-\xi$ and set $u/v=\xi$ to get

$$Q\,\mathcal{L}_1^{\pm}(u\xi)\mathcal{L}_2^{\pm}(u) = \mathcal{L}_2^{\pm}(u)\mathcal{L}_1^{\pm}(u\xi)Q. \tag{3.12}$$

By the definition of the element Q, we can write $Q = D_1^{-1}P^{\,\mathrm{t}_1}D_1$. Therefore, (3.12) takes the form

$$P^{\,\mathrm{t}_1}D_1\mathcal{L}_1^{\pm}(u\xi)D_1^{-1}\mathcal{L}_2^{\pm}(u) = \mathcal{L}_2^{\pm}(u)D_1\mathcal{L}_1^{\pm}(u\xi)D_1^{-1}P^{\,\mathrm{t}_1}. \tag{3.13}$$

The image of the operator $P^{\,\mathrm{t}_1}$ in $\mathrm{End}\,(\mathbb{C}^{2n})^{\otimes 2}$ is one-dimensional, so that each side of this equality must be equal to $P^{\,\mathrm{t}_1}$ times a certain series $\mathfrak{z}^{\pm}(u)$ with coefficients in $U(\overline{R})$. Observe that $P^{\,\mathrm{t}_1}D_1 = P^{\,\mathrm{t}_1}D_2^{-1}$ and $P^{\,\mathrm{t}_1}\mathcal{L}_1^{\pm}(u\xi) = P^{\,\mathrm{t}_1}\mathcal{L}_2^{\pm}(u\xi)^{\mathrm{t}}$ and so we get

$$P^{\,\mathrm{t}_1}D_2\mathcal{L}_2^{\pm}(u\xi)^{\mathrm{t}}D_2^{-1}\mathcal{L}_2^{\pm}(u) = \mathcal{L}_2^{\pm}(u)D_2\mathcal{L}_2^{\pm}(u\xi)^{\mathrm{t}}D_2^{-1}P^{\,\mathrm{t}_1} = \mathfrak{z}^{\pm}(u)\,P^{\,\mathrm{t}_1}.$$

The required relations now follow by taking trace of the first copy of $\mathrm{End}\,\mathbb{C}^{2n}$. □

Proposition 3.4. *All coefficients of the series $z^{+}(u)$ and $z^{-}(u)$ belong to the center of the algebra $U(R)$.*

Proof. We will verify that $z^{+}(u)$ commutes with all series $l_{ij}^{-}(v)$; the remaining cases follow by similar or simpler arguments. By the defining relations (1.12) we can write

$$D_1L_1^{+}(u\xi)^{\mathrm{t}}D_1^{-1}L_1^{+}(u)L_2^{-}(v) = D_1L_1^{+}(u\xi)^{\mathrm{t}}D_1^{-1}R(uq^{c}/v)^{-1}L_2^{-}(v)L_1^{+}(u)R(uq^{-c}/v).$$

By (3.9) the right hand side equals

$$\xi^{-2}q^{2}D_1L_1^{+}(u\xi)^{\mathrm{t}}R(u\xi q^{c}/v)^{\mathrm{t}_1}L_2^{-}(v)D_1^{-1}L_1^{+}(u)R(uq^{-c}/v).$$

Applying the patrial transposition t_1 to both sides in (1.12) we get the relation

$$L_1^+(u\xi)^{\mathrm{t}} R(u\xi q^c/v)^{\mathrm{t}_1} L_2^-(v) = L_2^-(v) R(u\xi q^{-c}/v)^{\mathrm{t}_1} L_1^+(u\xi)^{\mathrm{t}}.$$

Hence, using (3.9) and (3.10) we obtain

$$\begin{aligned} z^+(u) L_2^-(v) &= D_1 L_1^+(u\xi)^{\mathrm{t}} D_1^{-1} L_1^+(u) L_2^-(v) \\ &= \xi^{-2} q^2 L_2^-(v) D_1 R(u\xi q^{-c}/v)^{\mathrm{t}_1} D_1^{-1} D_1 L_1^+(u\xi)^{\mathrm{t}} D_1^{-1} L_1^+(u) R(uq^{-c}/v) \\ &= \xi^{-2} q^2 L_2^-(v) D_1 R(u\xi q^{-c}/v)^{\mathrm{t}_1} D_1^{-1} z^+(u) R(uq^{-c}/v) = L_2^-(v) z^+(u), \end{aligned}$$

as required. □

Remark 3.5. The crossing symmetry properties (3.9) of the R-matrix $R(u)$ were essential for Proposition 3.4 to hold. Although the coefficients of the series $\mathfrak{z}^+(u)$ and $\mathfrak{z}^-(u)$ are central in the respective subalgebras of $U(\overline{R})$ generated by the coefficients of the series $\ell_{ij}^+(u)$ and $\ell_{ij}^-(u)$, they are not central in the entire algebra $U(\overline{R})$. □

3.2 Quasideterminants and quantum minors

Let $A = [a_{ij}]$ be an $N \times N$ matrix over a ring with 1. Denote by A^{ij} the matrix obtained from A by deleting the i-th row and j-th column. Suppose that the matrix A^{ij} is invertible. The *ij-th quasideterminant of* A is defined by the formula

$$|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i,$$

where r_i^j is the row matrix obtained from the i-th row of A by deleting the element a_{ij}, and c_j^i is the column matrix obtained from the j-th column of A by deleting the element a_{ij}; see [18], [19]. The quasideterminant $|A|_{ij}$ is also denoted by boxing the entry a_{ij} in the matrix A.

Throughout the rest of this section we will regard elements of the tensor product algebra $\mathrm{End}\,(\mathbb{C}^{2n})^{\otimes m} \otimes U(\overline{R})$ as operators on the space $(\mathbb{C}^{2n})^{\otimes m}$ with coefficients in $U(\overline{R})$. Accordingly, for such an element

$$X = \sum_{a_i, b_i} e_{a_1 b_1} \otimes \ldots \otimes e_{a_m b_m} \otimes X^{a_1 \ldots a_m}_{b_1 \ldots b_m}$$

we will use a standard notation

$$X^{a_1 \ldots a_m}_{b_1 \ldots b_m} = \langle a_1, \ldots, a_m \,|\, X \,|\, b_1, \ldots, b_m \rangle \tag{3.14}$$

and its counterparts $X \,|\, b_1, \ldots, b_m \rangle$ and $\langle a_1, \ldots, a_m \,|\, X$.

Consider the algebra $U(\overline{R})$ and for any $2 \leqslant i, j \leqslant 2'$ introduce the quasideterminant

$$s_{ij}^{\pm}(u) = \begin{vmatrix} \ell_{11}^{\pm}(u) & \ell_{1j}^{\pm}(u) \\ \ell_{i1}^{\pm}(u) & \boxed{\ell_{ij}^{\pm}(u)} \end{vmatrix} = \ell_{ij}^{\pm}(u) - \ell_{i1}^{\pm}(u)\, \ell_{11}^{\pm}(u)^{-1} \ell_{1j}^{\pm}(u).$$

Let the power series $\ell^{\pm}{}^{a_1a_2}_{b_1b_2}(u)$ (*quantum minors*) with coefficients in $U(\overline{R})$ be defined by

$$\ell^{\pm}{}^{a_1a_2}_{b_1b_2}(u) = \langle a_1, a_2 \,|\, \widehat{R}(q^{-2})\, \mathcal{L}^{\pm}_1(u)\, \mathcal{L}^{\pm}_2(uq^2) \,|\, b_1, b_2\rangle, \tag{3.15}$$

where $a_i, b_i \in \{1, \dots, 2n\}$ and we set

$$\widehat{R}(u) = \frac{uq - q^{-1}}{u-1}\, \overline{R}(u). \tag{3.16}$$

The following symmetry properties are straightforward to verify.

Lemma 3.6. *(i) If $a_1 \neq a_2'$ and $a_1 < a_2$ then $\ell^{\pm}{}^{a_1a_2}_{b_1b_2}(u) = -q^{-1}\, \ell^{\pm}{}^{a_2a_1}_{b_1b_2}(u)$.*

(ii) If $b_1 \neq b_2'$ and $b_1 < b_2$ then $\ell^{\pm}{}^{a_1a_2}_{b_1b_2}(u) = -q\, \ell^{\pm}{}^{a_1a_2}_{b_2b_1}(u)$. □

Lemma 3.7. *For any $2 \leqslant i, j \leqslant 2'$ we have*

$$s^{\pm}_{ij}(u) = \ell^{\pm}_{11}(uq^{-2})^{-1}\, \ell^{\pm}{}^{1i}_{1j}(uq^{-2}). \tag{3.17}$$

Moreover,

$$\big[\ell^{\pm}_{11}(u), \ell^{\pm}{}^{1i}_{1j}(v)\big] = 0 \tag{3.18}$$

and

$$\frac{q^{-1}u_{\pm} - q\, v_{\mp}}{u_{\pm} - v_{\mp}}\, \ell^{\pm}_{11}(u)\, \ell^{\mp}{}^{1i}_{1j}(v) = \frac{q^{-1}u_{\mp} - q\, v_{\pm}}{u_{\mp} - v_{\pm}}\, \ell^{\mp}{}^{1i}_{1j}(v)\, \ell^{\pm}_{11}(u). \tag{3.19}$$

Proof. By the definition of quantum minors,

$$\ell^{\pm}{}^{1i}_{1j}(u) = \langle 1, i \,|\, \widehat{R}(q^{-2}) \mathcal{L}^{\pm}_1(u) \mathcal{L}^{\pm}_2(uq^2) \,|\, 1, j\rangle = \ell^{\pm}_{11}(u)\ell^{\pm}_{ij}(uq^2) - q^{-1}\ell^{\pm}_{i1}(u)\ell^{\pm}_{1j}(uq^2). \tag{3.20}$$

The defining relations of the algebra $U(\overline{R})$ give

$$\langle 1, i \,|\, \overline{R}(u/v) \mathcal{L}^{\pm}_1(u) \mathcal{L}^{\pm}_2(v) \,|\, 1, 1\rangle = \langle 1, i \,|\, \mathcal{L}^{\pm}_2(v) \mathcal{L}^{\pm}_1(u) \overline{R}(u/v) \,|\, 1, 1\rangle,$$

and so

$$\Big(\frac{u}{v} - 1\Big) \ell^{\pm}_{11}(u)\ell^{\pm}_{i1}(v) + (q - q^{-1})\frac{u}{v}\, \ell^{\pm}_{i1}(u)\ell^{\pm}_{11}(v) = \Big(\frac{u}{v} q - q^{-1}\Big) \ell^{\pm}_{i1}(v)\ell^{\pm}_{11}(u).$$

In particular,

$$\ell^{\pm}_{11}(uq^{-2})\ell^{\pm}_{i1}(u) = q^{-1}\ell^{\pm}_{i1}(uq^{-2})\ell^{\pm}_{11}(u). \tag{3.21}$$

Relations (3.20) and (3.21) imply

$$\begin{aligned}
\ell^{\pm}{}^{1i}_{1j}(uq^{-2}) &= \ell^{\pm}_{11}(uq^{-2})\ell^{\pm}_{ij}(u) - q^{-1}\ell^{\pm}_{i1}(uq^{-2})\ell^{\pm}_{1j}(u) \\
&= \ell^{\pm}_{11}(uq^{-2})\ell^{\pm}_{ij}(u) - q^{-1}\ell^{\pm}_{i1}(uq^{-2})\ell^{\pm}_{11}(u)\ell^{\pm}_{11}(u)^{-1}\ell^{\pm}_{1j}(u) \\
&= \ell^{\pm}_{11}(uq^{-2})\ell^{\pm}_{ij}(u) - \ell^{\pm}_{11}(uq^{-2})\ell^{\pm}_{i1}(u)\ell^{\pm}_{11}(u)^{-1}\ell^{\pm}_{1j}(u) = \ell^{\pm}_{11}(uq^{-2})s^{\pm}_{ij}(u),
\end{aligned}$$

thus proving (3.17). To verify (3.19), note that by the Yang–Baxter equation (1.8) and relations (3.5) and (3.6) we have

$$\langle 1, 1, i \,|\, \overline{R}_{01}(u_{\pm}/v_{\mp})\overline{R}_{02}(u_{\pm}\, q^{-2}/v_{\mp})\mathcal{L}^{\pm}_0(u)\widehat{R}_{12}(q^{-2})\mathcal{L}^{\mp}_1(v)\mathcal{L}^{\mp}_2(vq^2) \,|\, 1, 1, j\rangle =$$

$$\langle 1, 1, i \,|\, \widehat{R}_{12}(q^{-2})\mathcal{L}^{\mp}_1(v)\mathcal{L}^{\mp}_2(vq^2)\mathcal{L}^{\pm}_0(u)\overline{R}_{02}(u_{\mp}\, q^{-2}/v_{\pm})\overline{R}_{01}(u_{\pm}/v_{\mp}) \,|\, 1, 1, j\rangle$$

which gives (3.19). The calculation for (3.18) is quite similar. □

We point out the following consequences of Lemma 3.7: for $2 \leqslant i, j \leqslant 2'$ we have

$$[\ell_{11}^{\pm}(u), s_{ij}^{\pm}(v)] = 0 \tag{3.22}$$

and

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell_{11}^{\pm}(u)s_{ij}^{\mp}(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}s_{ij}^{\mp}(v)\,\ell_{11}^{\pm}(u). \tag{3.23}$$

3.3 Homomorphism theorems

Now we aim to make a connection between the algebras $U(\overline{R})$ associated with the Lie algebras $\mathfrak{sp}_{2n-2}$ and $\mathfrak{sp}_{2n}$. Since the rank n will vary, we will indicate the dependence on n by adding a subscript $[n]$ to the R-matrices. Consider the algebra $U(\overline{R}^{[n-1]})$ and let the indices of the generators $\ell_{ij}^{\pm}[\mp m]$ range over the sets $2 \leqslant i, j \leqslant 2'$ and $m = 0, 1, \dots,$ where $i' = 2n - i + 1$, as before.

Theorem 3.8. *The mappings* $q^{\pm c/2} \mapsto q^{\pm c/2}$ *and*

$$\ell_{ij}^{\pm}(u) \mapsto \begin{vmatrix} \ell_{11}^{\pm}(u) & \ell_{1j}^{\pm}(u) \\ \ell_{i1}^{\pm}(u) & \boxed{\ell_{ij}^{\pm}(u)} \end{vmatrix}, \qquad 2 \leqslant i, j \leqslant 2', \tag{3.24}$$

define a homomorphism $U(\overline{R}^{[n-1]}) \to U(\overline{R}^{[n]})$.

Proof. Consider the tensor product algebra $\mathrm{End}\,(\mathbb{C}^{2n})^{\otimes 4} \otimes U(\overline{R}^{[n]})$. We begin with calculations of certain matrix elements of operators which are straightforward from the definition of the R-matrix (3.1). We will use notation (3.16) and suppose that $2 \leqslant i, j \leqslant 2'$. Then

$$\overline{R}_{13}^{[n]}(a)\overline{R}_{23}^{[n]}(aq^2)\,|\,1, i, 1, j\rangle = \frac{aq - q^{-1}}{aq^2 - q^{-2}}\,|\,1, i, 1, j\rangle + \frac{(q - q^{-1})(a-1)aq^2}{(aq^2-1)(aq^2 - q^{-2})}\,|\,1, 1, i, j\rangle + \frac{(q - q^{-1})^2 aq^2}{(aq^2-1)(aq^2 - q^{-2})}\,|\,i, 1, 1, j\rangle$$

and

$$\widehat{R}_{12}^{[n]}(q^{-2})\overline{R}_{13}^{[n]}(a)\overline{R}_{23}^{[n]}(aq^2)\,|\,1, i, 1, j\rangle = \frac{a-1}{aq - q^{-1}}\widehat{R}_{12}^{[n]}(q^{-2})\,|\,1, i, 1, j\rangle. \tag{3.25}$$

Furthermore, we have

$$\widehat{R}_{34}^{[n]}(q^{-2})\overline{R}_{14}^{[n]}(aq^{-2})\,|\,1, i, 1, j\rangle = \frac{aq^{-1} - q}{a-1}\widehat{R}_{34}^{[n]}(q^{-2})\,|\,1, i, 1, j\rangle, \tag{3.26}$$

$$\widehat{R}_{12}^{[n]}(q^{-2})\widehat{R}_{34}^{[n]}(q^{-2})\overline{R}_{14}^{[n]}(aq^{-2})\,|\,1, 1', 1, 1\rangle = 0 \tag{3.27}$$

and

$$\widehat{R}_{12}^{[n]}(q^{-2})\widehat{R}_{34}^{[n]}(q^{-2})\overline{R}_{14}^{[n]}(aq^{-2})\,|\,1, 1, 1, 1'\rangle = -\frac{(q^{-2}-1)(aq^{-1}-q)}{(a-1)(a\xi^{-1}q^{-2}-1)}\widehat{R}_{12}^{[n]}(q^{-2})\widehat{R}_{34}^{[n]}(q^{-2}) \times \sum_{a=2}^{2'} \varepsilon_{1'}\varepsilon_a q^{\bar{a}-\bar{1}'}\,|\,1, a', 1, a\rangle. \tag{3.28}$$

These observations together with the formula for matrix elements of $\overline{R}^{[n]}_{24}(a)$ given by

$$\overline{R}^{[n]}_{24}(a)\,|\,1,i,1,j\rangle = \frac{a-1}{aq-q^{-1}}\big\{\big(\delta_{ij}q^{1-\delta_{ij'}} + (1-\delta_{ij})(1-\delta_{ij'}) + (1-\delta_{i,i'})q^{-1}\delta_{ij'}\big)\,|\,1,i,1,j\rangle$$
$$+\,\delta_{i>j}(q-q^{-1})\,|\,1,j,1,i\rangle - (q-q^{-1})\delta_{ij'}\sum_{a>j}\varepsilon_a\varepsilon_j q^{\bar{a}-\bar{\jmath}}\,|\,1,a',1,a\rangle$$
$$+\,\frac{q-q^{-1}}{a-1}\,|\,1,j,1,i\rangle + \frac{q^{-1}-q}{a\xi^{-1}-1}\delta_{ij'}\sum_{a=2}^{2'}\varepsilon_a\varepsilon_j q^{\bar{a}-\bar{\jmath}}\,|\,1,a',1,a\rangle$$
$$+\,\frac{q^{-1}-q}{a\xi^{-1}-1}\delta_{ij'}\varepsilon_1\varepsilon_j q^{\bar{1}-\bar{\jmath}}\,|\,1,1',1,1\rangle + \frac{(q^{-1}-q)a\xi^{-1}}{a\xi^{-1}-1}\delta_{ij'}\varepsilon_{1'}\varepsilon_j q^{\bar{1}'-\bar{\jmath}}\,|\,1,1,1,1'\rangle\big\}$$

lead to the relation

$$\widehat{R}^{[n]}_{12}(q^{-2})\widehat{R}^{[n]}_{34}(q^{-2})\overline{R}^{[n]}_{14}(aq^{-2})\overline{R}^{[n]}_{24}(a)\,|\,1,i,1,j\rangle$$
$$= \frac{aq^{-1}-q}{a-1}\widehat{R}^{[n]}_{12}(q^{-2})\widehat{R}^{[n]}_{34}(q^{-2})\overline{R}^{[n-1]}_{24}(a)\,|\,1,i,1,j\rangle. \qquad (3.29)$$

Now applying the Yang–Baxter equation (1.8) and relations (3.25) and (3.29) we deduce the following matrix element formulas:

$$\widehat{R}^{[n]}_{12}(q^{-2})\widehat{R}^{[n]}_{34}(q^{-2})\overline{R}^{[n]}_{14}(aq^{-2})\overline{R}^{[n]}_{24}(a)\overline{R}^{[n]}_{13}(a)\overline{R}^{[n]}_{23}(aq^{2})\,|\,1,i,1,j\rangle$$
$$= \frac{aq^{-1}-q}{aq-q^{-1}}\,\widehat{R}^{[n]}_{12}(q^{-2})\widehat{R}^{[n]}_{34}(q^{-2})\overline{R}^{[n-1]}_{24}(a)\,|\,1,i,1,j\rangle \qquad (3.30)$$

and

$$\langle 1,i,1,j\,|\,\overline{R}^{[n]}_{23}(aq^{2})\overline{R}^{[n]}_{13}(a)\overline{R}^{[n]}_{24}(a)\overline{R}^{[n]}_{14}(aq^{-2})\widehat{R}^{[n]}_{12}(q^{-2})\widehat{R}^{[n]}_{34}(q^{-2})$$
$$= \frac{aq^{-1}-q}{aq-q^{-1}}\,\langle 1,i,1,j\,|\,\overline{R}^{[n-1]}_{24}(a)\widehat{R}^{[n]}_{12}(q^{-2})\widehat{R}^{[n]}_{34}(q^{-2}). \qquad (3.31)$$

To complete the proof of the theorem, introduce the matrices

$$\Gamma^{\pm}(u) = \sum_{i,j=2}^{2'} e_{ij}\otimes \ell^{\pm}{}^{1i}_{1j}(u) \in \operatorname{End}\mathbb{C}^{2n}\otimes U(\overline{R}^{[n]}).$$

Our next step is to verify that the following relations hold in the algebra $U(\overline{R}^{[n]})$:

$$\overline{R}^{[n-1]}(u/v)\Gamma^{\pm}_1(u)\Gamma^{\pm}_2(v) = \Gamma^{\pm}_2(v)\Gamma^{\pm}_1(u)\overline{R}^{[n-1]}(u/v),$$
$$\frac{q^{-1}u_+ - qv_-}{qu_+ - q^{-1}v_-}\,\overline{R}^{[n-1]}(uq^{c}/v)\Gamma^{+}_1(u)\Gamma^{-}_2(v) = \frac{q^{-1}u_- - qv_+}{qu_- - q^{-1}v_+}\,\Gamma^{-}_2(v)\Gamma^{+}_1(u)\overline{R}^{[n-1]}(uq^{-c}/v).$$

The calculations are quite similar in both cases so we only give details for the first relation. The Yang–Baxter equation and the defining relations for the algebra $U(\overline{R}^{[n]})$ give

$$\overline{R}^{[n]}_{23}\Big(\frac{uq^2}{v}\Big)\overline{R}^{[n]}_{13}\Big(\frac{u}{v}\Big)\overline{R}^{[n]}_{24}\Big(\frac{u}{v}\Big)\overline{R}^{[n]}_{14}\Big(\frac{u}{vq^2}\Big)\widehat{R}^{[n]}_{12}(q^{-2})\mathcal{L}^{\pm}_1(u)\,\mathcal{L}^{\pm}_2(uq^2)\widehat{R}^{[n]}_{34}(q^{-2})\mathcal{L}^{\pm}_3(v)\,\mathcal{L}^{\pm}_4(vq^2)$$

$$=\widehat{R}^{[n]}_{34}(q^{-2})\mathcal{L}^{\pm}_3(v)\,\mathcal{L}^{\pm}_4(vq^2)\widehat{R}^{[n]}_{12}(q^{-2})\mathcal{L}^{\pm}_1(u)\,\mathcal{L}^{\pm}_2(uq^2)\overline{R}^{[n]}_{14}\Big(\frac{u}{vq^2}\Big)\overline{R}^{[n]}_{24}\Big(\frac{u}{v}\Big)\overline{R}^{[n]}_{13}\Big(\frac{u}{v}\Big)\overline{R}^{[n]}_{23}\Big(\frac{uq^2}{v}\Big).$$

Hence, assuming that $2\leqslant i,j,k,l\leqslant 2'$ and applying (3.30) and (3.31) we get

$$\langle 1,k,1,l\,|\,\overline{R}^{[n-1]}_{24}\Big(\frac{u}{v}\Big)\widehat{R}^{[n]}_{12}(q^{-2})\mathcal{L}^{\pm}_1(u)\,\mathcal{L}^{\pm}_2\big(uq^2\big)\widehat{R}^{[n]}_{34}(q^{-2})\mathcal{L}^{\pm}_3(v)\,\mathcal{L}^{\pm}_4(vq^2)\,|\,1,i,1,j\rangle$$
$$=\langle 1,k,1,l\,|\,\widehat{R}^{[n]}_{34}(q^{-2})\mathcal{L}^{\pm}_3(v)\,\mathcal{L}^{\pm}_4(vq^2)\widehat{R}^{[n]}_{12}(q^{-2})\mathcal{L}^{\pm}_1(u)\,\mathcal{L}^{\pm}_2(uq^2)\overline{R}^{[n]}_{24}\Big(\frac{u}{v}\Big)\,|\,1,i,1,j\rangle,$$

which is equivalent to

$$\overline{R}^{[n-1]}_{24}(u/v)\Gamma^{\pm}_2(u)\Gamma^{\pm}_4(v)=\Gamma^{\pm}_4(v)\Gamma^{\pm}_2(u)\overline{R}^{[n-1]}_{24}(u/v),$$

as required. Finally, set

$$S^{\pm}(u)=\sum_{2\leqslant i,j\leqslant 2'}e_{ij}\otimes s^{\pm}_{ij}(u).$$

By Lemma 3.7,

$$S^{\pm}(u)=\ell^{\pm}_{11}(uq^{-2})^{-1}\Gamma^{\pm}(uq^{-2})$$

and

$$\frac{q^{-1}u_{\pm}-qv_{\mp}}{u_{\pm}-v_{\mp}}\,\ell^{\pm}_{11}(u)\,\Gamma^{\mp}(v)=\frac{q^{-1}u_{\mp}-qv_{\pm}}{u_{\mp}-v_{\pm}}\,\Gamma^{\mp}(v)\,\ell^{\pm}_{11}(u).$$

The above relations for the matrices $\Gamma^{\pm}(u)$ imply

$$\overline{R}^{[n-1]}(u/v)S^{\pm}_1(u)S^{\pm}_2(v)=S^{\pm}_2(v)S^{\pm}_1(u)\overline{R}^{[n-1]}(u/v),$$
$$\overline{R}^{[n-1]}(uq^{\pm c}/v)S^{\pm}_1(u)S^{\mp}_2(v)=S^{\mp}_2(v)S^{\pm}_1(u)\overline{R}^{[n-1]}(uq^{\mp c}/v),$$

thus completing the proof. □

The following is a generalization of Theorem 3.8 which is immediate from the Sylvester theorem for quasideterminants [18], [28]; cf. the proof of its Yangian counterpart given in [26, Thm 3.7]. Fix a positive integer m such that $m<n$. Suppose that the generators $\ell^{\pm}_{ij}(u)$ of the algebra $U(\overline{R}^{[n-m]})$ are labelled by the indices $m+1\leqslant i,j\leqslant(m+1)'$ with $i'=2n-i+1$ as before.

Theorem 3.9. *The mapping*

$$\ell^{\pm}_{ij}(u) \mapsto \begin{vmatrix} \ell^{\pm}_{11}(u) & \dots & \ell^{\pm}_{1m}(u) & \ell^{\pm}_{1j}(u) \\ \dots & \dots & \dots & \dots \\ \ell^{\pm}_{m1}(u) & \dots & \ell^{\pm}_{mm}(u) & \ell^{\pm}_{mj}(u) \\ \ell^{\pm}_{i1}(u) & \dots & \ell^{\pm}_{im}(u) & \boxed{\ell^{\pm}_{ij}(u)} \end{vmatrix}, \qquad m+1 \leqslant i, j \leqslant (m+1)', \tag{3.32}$$

defines a homomorphism $\psi_m : U(\overline{R}^{[n-m]}) \to U(\overline{R}^{[n]})$. □

As another application of the Sylvester theorem for quasideterminants, we get a consistence property of the homomorphisms (3.32); cf. [26, Prop. 3.8] and [31, eq. (1.85)] for its Yangian counterparts. We will write $\psi_m = \psi_m^{(n)}$ to indicate the dependence of n. For a parameter l we have the corresponding homomorphism

$$\psi_m^{(n-l)} : U(\overline{R}^{[n-l-m]}) \to U(\overline{R}^{[n-l]})$$

provided by (3.32). Then we have the equality of maps

$$\psi_l^{(n)} \circ \psi_m^{(n-l)} = \psi_{l+m}^{(n)}. \tag{3.33}$$

Suppose that $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are subsets of $\{1, \dots, 2n\}$, assuming that $a_1 < a_2 < \dots < a_k$ and $b_1 < b_2 < \dots < b_k$, such that $a_i \neq a_j'$ and $b_i \neq b_j'$ for all i, j. Introduce the corresponding *type A quantum minors* as the matrix elements (3.14):

$$\ell^{\pm}\,{}^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u) = \langle a_1, \dots, a_k \,|\, \overline{R}_{k-1,k}(\overline{R}_{k-2,k}\overline{R}_{k-2,k-1}) \dots (\overline{R}_{1,k} \dots \overline{R}_{1,2}) \\ \times \mathcal{L}^{\pm}_1(u)\mathcal{L}^{\pm}_2(uq^2) \dots \mathcal{L}^{\pm}_k(uq^{2k-2}) \,|\, b_1, \dots, b_k\rangle,$$

where $\overline{R}_{ij} = \overline{R}_{ij}(q^{2(i-j)})$. They are given by the following formulas:

$$\ell^{\pm}\,{}^{a_1,\dots,a_k}_{b_1,\dots,b_k}(u) = \sum_{\sigma \in \mathfrak{S}_k} (-q)^{-l(\sigma)} \ell^{\pm}_{a_{\sigma(1)}b_1}(u) \dots \ell^{\pm}_{a_{\sigma(k)}b_k}(uq^{2k-2}) \\ = \sum_{\sigma \in \mathfrak{S}_k} (-q)^{l(\sigma)} \ell^{\pm}_{a_k b_{\sigma(k)}}(uq^{2k-2}) \dots \ell^{\pm}_{a_1 b_{\sigma(1)}}(u),$$

where $l(\sigma)$ denotes the number of inversions of the permutation $\sigma \in \mathfrak{S}_k$. The assumptions on the indices a_i and b_i imply that certain relations for these quantum minors take the same form as those for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$. Such relations for the latter can be deduced by applying R-matrix calculations which are quite analogous to the Yangian case; cf. [22], [28] and [31, Ch. 1]. In particular, for $1 \leqslant i, j \leqslant k$ we have

$$[\ell^{\pm}_{a_i b_j}(u), \ell^{\pm}\,{}^{a_1,a_2,\dots,a_k}_{b_1,b_2,\dots,b_k}(v)] = 0,$$

$$\prod_{a=1}^{k-1} \frac{u_{\pm}q^{-k} - v_{\mp}q^{k}}{u_{\pm}q^{1-k} - v_{\mp}q^{k-1}} \ell^{\pm}_{a_i b_j}(u) \ell^{\mp}\,{}^{a_1,a_2,\dots,a_k}_{b_1,b_2,\dots,b_k}(v) = \prod_{a=1}^{k-1} \frac{u_{\mp}q^{-k} - v_{\pm}q^{k}}{u_{\mp}q^{1-k} - v_{\pm}q^{k-1}} \ell^{\mp}\,{}^{a_1,a_2,\dots,a_k}_{b_1,b_2,\dots,b_k}(v) \ell^{\pm}_{a_i b_j}(u).$$

Corollary 3.10. *Under the assumptions of Theorem 3.9 we have*

$$\big[\ell^{\pm}_{ab}(u),\psi_m(\ell^{\pm}_{ij}(v))\big]=0,$$

$$\frac{u_{\pm}-v_{\mp}}{qu_{\pm}-q^{-1}v_{\mp}}\ell^{\pm}_{ab}(u)\psi_m(\ell^{\mp}_{ij}(v))=\frac{u_{\mp}-v_{\pm}}{qu_{\mp}-q^{-1}v_{\pm}}\psi_m(\ell^{\mp}_{ij}(v))\ell^{\pm}_{ab}(u),$$

for all $1\leqslant a,b\leqslant m$ *and* $m+1\leqslant i,j\leqslant(m+1)'$.

Proof. Both formulas are verified with the use of the relations between the quasideterminants and quantum minors:

$$\begin{vmatrix}\ell^{\pm}_{11}(u)&\dots&\ell^{\pm}_{1m}(u)&\ell^{\pm}_{1j}(u)\\ \dots&\dots&\dots&\dots\\ \ell^{\pm}_{m1}(u)&\dots&\ell^{\pm}_{mm}(u)&\ell^{\pm}_{mj}(u)\\ \ell^{\pm}_{i1}(u)&\dots&\ell^{\pm}_{im}(u)&\boxed{\ell^{\pm}_{ij}(u)}\end{vmatrix}=\ell^{\pm\,1\dots m}_{\;\;1\dots m}(uq^{-2m})^{-1}\cdot\ell^{\pm\,1\dots m\,i}_{\;\;1\dots m\,j}(uq^{-2m}).$$

They are consequences of the type A relations; cf. the Yangian case in [26, Sec. 3]. □

4 Gauss decomposition

We will apply the Gauss decompositions (1.15) to the generators matrices $L^{\pm}(u)$ and $\mathcal{L}^{\pm}(u)$ for the respective algebras $U(R^{[n]})$ and $U(\overline{R}^{[n]})$. Each of these algebras is generated by the coefficients of the matrix elements of the triangular and diagonal matrices which we will refer to as the *Gaussian generators*. Our goal in this section is to produce necessary relations satisfied by these generators to be able to get presentations of the R-matrix algebras $U(R^{[n]})$ and $U(\overline{R}^{[n]})$.

4.1 Gaussian generators

The entries of the matrices $F^{\pm}(u)$, $H^{\pm}(u)$ and $E^{\pm}(u)$ which occur in the decompositions (1.15) can be described by the universal quasideterminant formulas as follows [18], [19]:

$$h^{\pm}_i(u)=\begin{vmatrix}l^{\pm}_{11}(u)&\dots&l^{\pm}_{1\,i-1}(u)&l^{\pm}_{1\,i}(u)\\ \vdots&\ddots&\vdots&\vdots\\ l^{\pm}_{i-1\,1}(u)&\dots&l^{\pm}_{i-1\,i-1}(u)&l^{\pm}_{i-1\,i}(u)\\ l^{\pm}_{i\,1}(u)&\dots&l^{\pm}_{i\,i-1}(u)&\boxed{l^{\pm}_{i\,i}(u)}\end{vmatrix},\qquad i=1,\dots,2n,\tag{4.1}$$

whereas

$$e^{\pm}_{ij}(u)=h^{\pm}_i(u)^{-1}\begin{vmatrix}l^{\pm}_{11}(u)&\dots&l^{\pm}_{1\,i-1}(u)&l^{\pm}_{1\,j}(u)\\ \vdots&\ddots&\vdots&\vdots\\ l^{\pm}_{i-1\,1}(u)&\dots&l^{\pm}_{i-1\,i-1}(u)&l^{\pm}_{i-1\,j}(u)\\ l^{\pm}_{i\,1}(u)&\dots&l^{\pm}_{i\,i-1}(u)&\boxed{l^{\pm}_{i\,j}(u)}\end{vmatrix}\tag{4.2}$$

and

$$f_{ji}^{\pm}(u) = \begin{vmatrix} l_{11}^{\pm}(u) & \dots & l_{1\,i-1}^{\pm}(u) & l_{1\,i}^{\pm}(u) \\ \vdots & \ddots & \vdots & \vdots \\ l_{i-1\,1}^{\pm}(u) & \dots & l_{i-1\,i-1}^{\pm}(u) & l_{i-1\,i}^{\pm}(u) \\ l_{j\,1}^{\pm}(u) & \dots & l_{j\,i-1}^{\pm}(u) & \boxed{l_{j\,i}^{\pm}(u)} \end{vmatrix} h_i^{\pm}(u)^{-1} \tag{4.3}$$

for $1 \leqslant i < j \leqslant 2n$. The same formulas hold for the expressions of the entries of the respective triangular matrices $\mathcal{F}^{\pm}(u)$ and $\mathcal{E}^{\pm}(u)$ and the diagonal matrices $\mathcal{H}^{\pm}(u) = \text{diag}\,[\mathfrak{h}_i^{\pm}(u)]$ in terms of the formal series $\ell_{ij}^{\pm}(u)$, which arise from the Gauss decomposition

$$\mathcal{L}^{\pm}(u) = \mathcal{F}^{\pm}(u)\,\mathcal{H}^{\pm}(u)\,\mathcal{E}^{\pm}(u)$$

for the algebra $U(\overline{R}^{[n]})$. We will denote by $\mathfrak{e}_{ij}(u)$ and $\mathfrak{f}_{ji}(u)$ the entries of the respective matrices $\mathcal{E}^{\pm}(u)$ and $\mathcal{F}^{\pm}(u)$ for $i < j$.

The following Laurent series with coefficients in the respective algebras $U(R^{[n]})$ and $U(\overline{R}^{[n]})$ will be used frequently:

$$X_i^{+}(u) = e_{ii+1}^{+}(u_+) - e_{ii+1}^{-}(u_-), \qquad X_i^{-}(u) = f_{i+1,i}^{+}(u_-) - f_{i+1,i}^{-}(u_+), \tag{4.4}$$

$$\mathcal{X}_i^{+}(u) = \mathfrak{e}_{ii+1}^{+}(u_+) - \mathfrak{e}_{ii+1}^{-}(u_-), \qquad \mathcal{X}_i^{-}(u) = \mathfrak{f}_{i+1,i}^{+}(u_-) - \mathfrak{f}_{i+1,i}^{-}(u_+). \tag{4.5}$$

Proposition 4.1. *Under the homomorphism $U(R) \to \mathcal{H}_q(n) \otimes_{\mathbb{C}[q^c,\,q^{-c}]} U(\overline{R})$ provided by Proposition 3.2 we have*

$$\begin{aligned} e_{ij}^{\pm}(u) &\mapsto \mathfrak{e}_{ij}^{\pm}(u), \\ f_{ij}^{\pm}(u) &\mapsto \mathfrak{f}_{ij}^{\pm}(u), \\ h_i^{\pm}(u) &\mapsto \exp \sum_{k=1}^{\infty} \beta_{\pm k} u^{\pm k} \cdot \mathfrak{h}_i^{\pm}(u). \end{aligned}$$

Proof. This is immediate from the formulas for the Gaussian generators. □

4.2 Images of the generators under the homomorphism ψ_m

Suppose that $0 \leqslant m < n$. We will use the superscript $[n-m]$ to indicate square submatrices corresponding to rows and columns labelled by $m+1, m+2, \dots, (m+1)'$. In particular, we set

$$\mathcal{F}^{\pm[n-m]}(u) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \mathfrak{f}_{m+2\,m+1}^{\pm}(u) & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \mathfrak{f}_{(m+1)'\,m+1}^{\pm}(u) & \dots & \mathfrak{f}_{(m+1)'\,(m+2)'}^{\pm}(u) & 1 \end{bmatrix},$$

$$\mathcal{E}^{\pm[n-m]}(u) = \begin{bmatrix} 1 & \mathfrak{e}_{m+1\,m+2}^{\pm}(u) & \dots & \mathfrak{e}_{m+1\,(m+1)'}^{\pm}(u) \\ 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathfrak{e}_{(m+2)'\,(m+1)'}^{\pm}(u) \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

and $\mathcal{H}^{\pm[n-m]}(u) = \mathrm{diag}\,\big[\mathfrak{h}^{\pm}_{m+1}(u),\dots,\mathfrak{h}^{\pm}_{(m+1)'}(u)\big]$. Furthermore, introduce the products of these matrices by

$$\mathcal{L}^{\pm[n-m]}(u) = \mathcal{F}^{\pm[n-m]}(u)\,\mathcal{H}^{\pm[n-m]}(u)\,\mathcal{E}^{\pm[n-m]}(u).$$

The entries of $\mathcal{L}^{\pm[n-m]}(u)$ will be denoted by $\ell^{\pm[n-m]}_{ij}(u)$.

Proposition 4.2. *The series $\ell^{\pm[n-m]}_{ij}(u)$ coincides with the image of the generator series $\ell^{\pm}_{ij}(u)$ of the extended quantum affine algebra $U(\overline{R}^{[n-m]})$ under the homomorphism (3.32),*

$$\ell^{\pm[n-m]}_{ij}(u) = \psi_m\big(\ell^{\pm}_{ij}(u)\big), \qquad m+1 \leqslant i,j \leqslant (m+1)'.$$

Proof. This follows by the same argument as for the Yangian case; see [26, Prop. 4.1]. □

Corollary 4.3. *The following relations hold in $U(\overline{R}^{[n]})$:*

$$\overline{R}^{[n-m]}_{12}(u/v)\,\mathcal{L}^{\pm[n-m]}_1(u)\,\mathcal{L}^{\pm[n-m]}_2(v) = \mathcal{L}^{\pm[n-m]}_2(v)\,\mathcal{L}^{\pm[n-m]}_1(u)\,\overline{R}^{[n-m]}_{12}(u/v), \tag{4.6}$$

$$\overline{R}^{[n-m]}_{12}(u_+/v_-)\,\mathcal{L}^{+[n-m]}_1(u)\,\mathcal{L}^{-[n-m]}_2(v) = \mathcal{L}^{-[n-m]}_2(v)\,\mathcal{L}^{+[n-m]}_1(u)\,\overline{R}^{[n-m]}_{12}(u_-/v_+). \tag{4.7}$$

Proof. This is immediate from Proposition 4.2. □

Proposition 4.4. *Suppose that $m+1 \leqslant j,k,l \leqslant (m+1)'$ and $j \neq l'$. Then the following relations hold in $U(\overline{R}^{[n]})$: if $j = l$ then*

$$\mathfrak{e}^{\pm}_{mj}(u)\ell^{\mp[n-m]}_{kl}(v) = \frac{qu_{\mp} - q^{-1}v_{\pm}}{u_{\mp} - v_{\pm}}\ell^{\mp[n-m]}_{kj}(v)\mathfrak{e}^{\pm}_{ml}(u) - \frac{(q-q^{-1})u_{\mp}}{u_{\mp} - v_{\pm}}\ell^{\mp[n-m]}_{kj}(v)\mathfrak{e}^{\mp}_{mj}(v), \tag{4.8}$$

$$\mathfrak{e}^{\pm}_{mj}(u)\ell^{\pm[n-m]}_{kl}(v) = \frac{qu - q^{-1}v}{u - v}\ell^{\pm[n-m]}_{kj}(v)\mathfrak{e}^{\pm}_{ml}(u) - \frac{(q-q^{-1})u}{u - v}\ell^{\pm[n-m]}_{kj}(v)\mathfrak{e}^{\pm}_{mj}(v);$$

if $j < l$ then

$$[\mathfrak{e}^{\pm}_{mj}(u), \ell^{\mp[n-m]}_{kl}(v)] = \frac{(q-q^{-1})v_{\pm}}{u_{\mp} - v_{\pm}}\ell^{\mp[n-m]}_{kj}(v)\mathfrak{e}^{\pm}_{ml}(u) - \frac{(q-q^{-1})u_{\mp}}{u_{\mp} - v_{\pm}}\ell^{\mp[n-m]}_{kj}(v)\mathfrak{e}^{\mp}_{ml}(v), \tag{4.9}$$

$$[\mathfrak{e}^{\pm}_{mj}(u), \ell^{\pm[n-m]}_{kl}(v)] = \frac{(q-q^{-1})v}{u - v}\ell^{\pm[n-m]}_{kj}(v)\mathfrak{e}^{\pm}_{ml}(u) - \frac{(q-q^{-1})u}{u - v}\ell^{\pm[n-m]}_{kj}(v)\mathfrak{e}^{\pm}_{ml}(v);$$

if $j > l$ then

$$[\mathfrak{e}^{\pm}_{mj}(u), \ell^{\mp[n-m]}_{kl}(v)] = \frac{(q-q^{-1})u_{\mp}}{u_{\mp} - v_{\pm}}\ell^{\mp[n-m]}_{kj}(v)(\mathfrak{e}^{\pm}_{ml}(u) - \mathfrak{e}^{\mp}_{ml}(v)),$$

$$[\mathfrak{e}^{\pm}_{mj}(u), \ell^{\pm[n-m]}_{kl}(v)] = \frac{(q-q^{-1})u}{u - v}\ell^{\pm[n-m]}_{kj}(v)(\mathfrak{e}^{\pm}_{ml}(u) - \mathfrak{e}^{\pm}_{ml}(v)).$$

Proof. It is sufficient to verify the relations for $m = 1$; the general case will then follow by the application of the homomorphism ψ_m. The calculations are similar for all the relations so we only verify (4.9). By the defining relations,

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{1j}(u)\ell^\mp_{kl}(v) + \frac{(q-q^{-1})u_\pm}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{kj}(u)\ell^\mp_{1l}(v)$$
$$= \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{kl}(v)\ell^\pm_{1j}(u) + \frac{(q-q^{-1})v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{kj}(v)\ell^\pm_{1l}(u). \quad (4.10)$$

Since $\ell^\mp_{kl}(v) = \ell^{\mp[n-1]}_{kl}(v) + \mathfrak{f}^\mp_{k1}(v)\,\mathfrak{h}^\mp_1(v)\,\mathfrak{e}^\mp_{1l}(v)$, we can write the left hand side of (4.10) as

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{1j}(u)\ell^{\mp[n-1]}_{kl}(v) + \frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{1j}(u)\mathfrak{f}^\mp_{k1}(v)\,\mathfrak{h}^\mp_1(v)\,\mathfrak{e}^\mp_{1l}(v)$$
$$+ \frac{(q-q^{-1})v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{kj}(u)\ell^\mp_{1l}(v).$$

By the defining relations, we have

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{1j}(u)\ell^\mp_{k1}(v) + \frac{(q-q^{-1})u_\pm}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{kj}(u)\ell^\mp_{11}(v)$$
$$= \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{k1}(v)\ell^\pm_{1j}(u) + \frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{kj}(v)\ell^\pm_{11}(u).$$

Hence, the left hand side of (4.10) equals

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{1j}(u)\ell^{\mp[n-1]}_{kl}(v) + \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}^\mp_{k1}(v)\ell^\mp_{11}(v)\ell^\pm_{1j}(u)\mathfrak{e}^\mp_{1l}(v)$$
$$+ \frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{kj}(v)\ell^\pm_{11}(u)\mathfrak{e}^\mp_{1l}(v).$$

Furthermore, using the relation

$$\ell^\pm_{1j}(u)\ell^\mp_{11}(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{11}(v)\ell^\pm_{1j}(u) + \frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{1j}(v)\ell^\pm_{11}(u),$$

we can bring the left hand side of (4.10) to the form

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{1j}(u)\ell^{\mp[n-1]}_{kl}(v) + \mathfrak{f}^\mp_{k1}(v)\ell^\pm_{1j}(u)\ell^\mp_{1l}(v) + \frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\ell^{\mp[n-1]}_{kj}(v)\ell^\pm_{11}(u)\mathfrak{e}^\mp_{1l}(v).$$

For $j < l$ we have

$$\ell^\pm_{1j}(u)\ell^\mp_{1l}(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{1l}(v)\ell^\pm_{1j}(u) + \frac{(q-q^{-1})v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{1j}(v)\ell^\pm_{1l}(u),$$

so that the left hand side of (4.10) becomes

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^{\pm}_{1j}(u)\ell^{\mp[n-1]}_{kl}(v) - \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}vu_\pm}\ell^{\mp[n-1]}_{kl}(v)\ell^{\pm}_{1j}(u)$$
$$= \frac{(q-q^{-1})v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^{\mp[n-1]}_{kj}(v)\ell^{\pm}_{11}(u)\mathfrak{e}^{\pm}_{1l}(u) - \frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\ell^{\mp[n-1]}_{kj}(v)\ell^{\pm}_{11}(u)\mathfrak{e}^{\mp}_{1l}(v).$$

Finally, Corollary 3.10 implies

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^{\pm}_{11}(u)\ell^{\mp[n-1]}_{kl}(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^{\mp[n-1]}_{kl}(v)\ell^{\pm}_{11}(u),$$
$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^{\pm}_{11}(u)\ell^{\mp[n-1]}_{kj}(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^{\mp[n-1]}_{kj}(v)\ell^{\pm}_{11}(u),$$

thus completing the proof of (4.9). □

Quite similar arguments prove the following counterpart of Proposition 4.4 involving the generator series $\mathfrak{f}^{\pm}_{ji}(u)$.

Proposition 4.5. *Suppose that $m+1 \leqslant j,k,l \leqslant (m+1)'$ and $j \neq k'$. Then the following relations hold in $U(\overline{R}^{[n]})$: if $j = k$ then*

$$\mathfrak{f}^{\pm}_{jm}(u)\ell^{\mp[n-m]}_{jl}(v) = \frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^{\mp[n-m]}_{jl}(v)\mathfrak{f}^{\pm}_{jm}(u) + \frac{(q-q^{-1})v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{f}^{\mp}_{jm}(v)\ell^{\mp[n-m]}_{jl}(v),$$
$$\mathfrak{f}^{\pm}_{jm}(u)\ell^{\pm[n-m]}_{jl}(v) = \frac{u-v}{qu - q^{-1}v}\ell^{\pm[n-m]}_{jl}(uv)\,\mathfrak{f}^{\pm}_{jm}(u) + \frac{(q-q^{-1})v}{qu - q^{-1}v}\mathfrak{f}^{\pm}_{jm}(v)\ell^{\pm[n-m]}_{jl}(v);$$

if $j < k$ then

$$[\mathfrak{f}^{\pm}_{jm}(u), \ell^{\mp[n-m]}_{kl}(v)] = \frac{(q-q^{-1})v_\mp}{u_\pm - v_\mp}\mathfrak{f}^{\mp}_{km}(v)\ell^{\mp[n-m]}_{jl}(v) - \frac{(q-q^{-1})u_\pm}{u_\pm - v_\mp}\mathfrak{f}^{\pm}_{km}(u)\ell^{\mp[n-m]}_{jl}(v),$$
$$[\mathfrak{f}^{\pm}_{jm}(u), \ell^{\pm[n-m]}_{kl}(v)] = \frac{(q-q^{-1})v}{u-v}\mathfrak{f}^{\pm}_{km}(v)\ell^{\pm[n-m]}_{jl}(v) - \frac{(q-q^{-1})u}{u-v}\mathfrak{f}^{\pm}_{km}(u)\ell^{\pm[n-m]}_{jl}(v);$$

if $j > k$ then

$$[\mathfrak{f}^{\pm}_{jm}(u), \ell^{\mp[n-m]}_{kl}(v)] = \frac{(q-q^{-1})v_\mp}{u_\pm - v_\mp}(\mathfrak{f}^{\mp}_{km}(v) - \mathfrak{f}^{\pm}_{km}(u))\ell^{\mp[n-m]}_{jl}(v),$$
$$[\mathfrak{f}^{\pm}_{jm}(u), \ell^{\pm[n-m]}_{kl}(v)] = \frac{(q-q^{-1})v}{u-v}(\mathfrak{f}^{\pm}_{km}(v) - \mathfrak{f}^{\pm}_{km}(u))\ell^{\pm[n-m]}_{jl}(v).$$ □

4.3 Type A relations

Due to the observation made in Remark 3.1 and the quasideterminant formulas (4.1), (4.2) and (4.3), some of the relations between the Gaussian generators will follow from those for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$; see [8]. To reproduce them, set

$$\mathcal{L}^{A\pm}(u) = \sum_{i,j=1}^{n} e_{ij} \otimes \ell^{\pm}_{ij}(u)$$

and consider the R-matrix used in [8] which is given by

$$R_A(u) = \sum_{i=1}^{n} e_{ii}\otimes e_{ii} + \frac{u-1}{qu-q^{-1}}\sum_{i\neq j} e_{ii}\otimes e_{jj} + \frac{q-q^{-1}}{qu-q^{-1}}\sum_{i>j} e_{ij}\otimes e_{ji} + \frac{(q-q^{-1})u}{qu-q^{-1}}\sum_{i<j} e_{ij}\otimes e_{ji}. \quad (4.11)$$

By comparing it with the R-matrix (3.1), we come to the relations in the algebra $U(\overline{R}^{[n]})$:

$$R_A(u/w)\mathcal{L}_1^{A\pm}(u)\mathcal{L}_2^{A\pm}(v) = \mathcal{L}_2^{A\pm}(v)\mathcal{L}_1^{A\pm}(u)R_A(u/v),$$
$$R_A(uq^c/v)\mathcal{L}_1^{A+}(u)\mathcal{L}_2^{A-}(v) = \mathcal{L}_2^{A-}(v)\mathcal{L}_1^{A+}(u)R_A(uq^{-c}/v).$$

Hence we get the following relations for the Gaussian generators which were verified in [8], where we use notation (4.5).

Proposition 4.6. *For any $1\leqslant i,j\leqslant n$ in the algebra $U(\overline{R}^{[n]})$ we have*

$$\mathfrak{h}_i^{\pm}(u)\mathfrak{h}_j^{\pm}(v) = \mathfrak{h}_j^{\pm}(v)\mathfrak{h}_i^{\pm}(u), \qquad \mathfrak{h}_i^{\pm}(u)\mathfrak{h}_i^{\mp}(v) = \mathfrak{h}_i^{\mp}(v)\mathfrak{h}_i^{\pm}(u),$$
$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^{\pm}(u)\mathfrak{h}_j^{\mp}(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}_j^{\mp}(v)\mathfrak{h}_i^{\pm}(u), \quad \text{for} \quad i<j.$$

Moreover,

$$\mathfrak{h}_i^{\pm}(u)\mathcal{X}_j^{+}(v) = \frac{u_\mp - v}{q^{(\epsilon_i,\alpha_j)}u_\mp - q^{-(\epsilon_i,\alpha_j)}v}\mathcal{X}_j^{+}(v)\mathfrak{h}_i^{\pm}(u),$$
$$\mathfrak{h}_i^{\pm}(u)\mathcal{X}_j^{-}(v) = \frac{q^{(\epsilon_i,\alpha_j)}u_\pm - q^{-(\epsilon_i,\alpha_j)}v}{u_\pm - v}\mathcal{X}_j^{-}(v)\mathfrak{h}_i^{\pm}(u),$$

and

$$(uq^{-i} - q^{\pm(\alpha_i,\alpha_j)-j}v)\mathcal{X}_i^{\pm}(u)\mathcal{X}_j^{\pm}(v) = (q^{\pm(\alpha_i,\alpha_j)-i}u - q^{-j}v)\mathcal{X}_j^{\pm}(v)\mathcal{X}_i^{\pm}(u),$$
$$[\mathcal{X}_i^{+}(u),\mathcal{X}_j^{-}(v)] = \delta_{ij}(q-q^{-1})\Big(\delta\big(\frac{uq^c}{v}\big)\,\mathfrak{h}_i^{+}(u_+)^{-1}\mathfrak{h}_{i+1}^{+}(u_+) - \delta\big(\frac{u}{vq^c}\big)\,\mathfrak{h}_i^{-}(v_+)^{-1}\mathfrak{h}_{i+1}^{-}(v_+)\Big),$$

together with the Serre relations (2.15) for the series $\mathcal{X}_i^{\pm}(u)$. □

Remark 4.7. Consider the inverse matrices $\mathcal{L}^{\pm}(u)^{-1} = [\ell_{ij}^{\pm}(u)']_{i,j=1,\dots,2n}$. By the defining relations (3.4) and (3.5), we have

$$\mathcal{L}_1^{\pm}(u)^{-1}\mathcal{L}_2^{\pm}(v)^{-1}\overline{R}^{[n]}(u/v) = \overline{R}^{[n]}(u/v)\mathcal{L}_2^{\pm}(v)^{-1}\mathcal{L}_1^{\pm}(u)^{-1},$$
$$\mathcal{L}_2^{-}(v)^{-1}\mathcal{L}_1^{+}(u)^{-1}\overline{R}^{[n]}(uq^c/v) = \overline{R}^{[n]}(uq^{-c}/v)\mathcal{L}_2^{-}(v)^{-1}\mathcal{L}_1^{+}(u)^{-1}.$$

So we can get another family of generators of the algebra $U(\overline{R}^{[n]})$ which satisfy the defining relations of $U_q(\widehat{\mathfrak{gl}}_n)$. Namely, these relations are satisfied by the coefficients of the series $\ell_{ij}^{\pm}(u)'$ with $i,j=n',\dots,1'$. In particular, by taking the inverse matrices, we get a Gauss decomposition for the matrix $[\ell_{ij}^{\pm}(u)']_{i,j=n',\dots,1'}$ from the Gauss decomposition of the matrix $\mathcal{L}^{\pm}(u)$. □

4.4 Relations for the long root generators

In the particular case $n = 1$ the R-matrix (3.1) takes the form

$$\overline{R}^{[1]}(u) = \sum_{i=1}^{2} e_{ii} \otimes e_{ii} + \frac{u-1}{q^2u - q^{-2}} \sum_{i \neq j} e_{ii} \otimes e_{jj} + \frac{(q^2 - q^{-2})u}{q^2u - q^{-2}} e_{12} \otimes e_{21} + \frac{q^2 - q^{-2}}{q^2u - q^{-2}} e_{21} \otimes e_{12}$$

and so it coincides with the R-matrix associated with $U_{q^2}(\widehat{\mathfrak{gl}}_2)$; cf. (4.11). Therefore, a set of relations involving the long root generators are implied by Corollary 4.3 and Proposition 4.6.

Proposition 4.8. *The following relations hold in the algebra* $U(\overline{R}^{[n]})$*:*

$$\mathfrak{h}_i^{\pm}(u)\mathfrak{h}_j^{\pm}(v) = \mathfrak{h}_j^{\pm}(v)\mathfrak{h}_i^{\pm}(u), \qquad i, j = n, n+1,$$
$$\mathfrak{h}_i^{\pm}(u)\mathfrak{h}_i^{\mp}(v) = \mathfrak{h}_i^{\mp}(v)\mathfrak{h}_i^{\pm}(u), \qquad i = n, n+1,$$

$$\frac{u_{\pm} - v_{\mp}}{q^2u_{\pm} - q^{-2}v_{\mp}} \mathfrak{h}_n^{\pm}(u)\mathfrak{h}_{n+1}^{\mp}(v) = \frac{u_{\mp} - v_{\pm}}{q^2u_{\mp} - q^{-2}v_{\pm}} \mathfrak{h}_{n+1}^{\mp}(v)\mathfrak{h}_n^{\pm}(u).$$

Moreover,

$$\mathfrak{h}_n^{\pm}(u)\mathcal{X}_n^{+}(v) = \frac{u_{\mp} - v}{q^2u_{\mp} - q^{-2}v}\mathcal{X}_n^{+}(v)\mathfrak{h}_n^{\pm}(u),$$
$$\mathfrak{h}_{n+1}^{\pm}(u)\mathcal{X}_n^{+}(v) = \frac{u_{\mp} - v}{q^{-2}u_{\mp} - q^{2}v}\mathcal{X}_n^{+}(v)\mathfrak{h}_{n+1}^{\pm}(u),$$
$$\mathfrak{h}_n^{\pm}(u)\mathcal{X}_n^{-}(v) = \frac{q^2u_{\pm} - q^{-2}v}{u_{\pm} - v}\mathcal{X}_n^{-}(v)\mathfrak{h}_n^{\pm}(u),$$
$$\mathfrak{h}_{n+1}^{\pm}(u)\mathcal{X}_n^{-}(v) = \frac{q^{-2}u_{\pm} - q^{2}v}{u_{\pm} - v}\mathcal{X}_n^{-}(v)\mathfrak{h}_{n+1}^{\pm}(u),$$

and

$$(u - q^{\pm(\alpha_n,\alpha_n)}v)\mathcal{X}_n^{\pm}(u)\mathcal{X}_n^{\pm}(v) = (q^{\pm(\alpha_n,\alpha_n)}u - v)\mathcal{X}_n^{\pm}(v)\mathcal{X}_n^{\pm}(u),$$

$$[\mathcal{X}_n^{+}(u), \mathcal{X}_n^{-}(v)] = (q^2 - q^{-2})\Big(\delta\big(\frac{uq^c}{v}\big)\, \mathfrak{h}_n^{+}(u_+)^{-1}\mathfrak{h}_{n+1}^{+}(u_+) - \delta\big(\frac{u}{vq^c}\big)\, \mathfrak{h}_n^{-}(v_+)^{-1}\mathfrak{h}_{n+1}^{-}(v_+)\Big).$$

4.5 Formulas for the series $z^{\pm}(u)$ and $\mathfrak{z}^{\pm}(u)$

Recall that the series $z^{\pm}(u)$ and $\mathfrak{z}^{\pm}(u)$ were defined in Proposition 3.3. We will now indicate the dependence on n by adding the corresponding superscript. Write relation (3.11) in the form

$$D\mathcal{L}^{\pm}(u\xi)^{\mathrm{t}}D^{-1} = \mathcal{L}^{\pm}(u)^{-1}\mathfrak{z}^{\pm\,[n]}(u). \tag{4.12}$$

Using the Gauss decomposition for $\mathcal{L}^{\pm}(u)$ and taking the $(2n, 2n)$-entry on both sides of (4.12) we get

$$\mathfrak{h}_1^{\pm}(u\xi) = \mathfrak{h}_{1'}^{\pm}(u)^{-1}\mathfrak{z}^{\pm\,[n]}(u). \tag{4.13}$$

By Proposition 4.2, in the subalgebra generated by the coefficients of the series $\ell_{ij}^{\pm[1]}(u)$ we have

$$\mathfrak{z}^{\pm\,[1]}(u) = \mathfrak{h}_1^{\pm}(uq^{-4})\,\mathfrak{h}_{1'}^{\pm}(u).$$

Lemma 4.9. *The following relations hold in the algebra* $U(\overline{R}^{[n]})$:

$$\mathfrak{e}^{\pm}_{(i+1)'\,i'}(u) = -\mathfrak{e}^{\pm}_{i,i+1}(u\xi q^{2i}) \qquad \textit{and} \qquad \mathfrak{f}^{\pm}_{i'\,(i+1)'}(u) = -\mathfrak{f}^{\pm}_{i+1,i}(u\xi q^{2i}). \tag{4.14}$$

Proof. By Propositions 3.3 and 4.2, for any $1 \leqslant i \leqslant n-1$ we have

$$\mathcal{L}^{\pm\,[n-i+1]}(u)^{-1}\mathfrak{z}^{\pm\,[n-i+1]}(u) = D^{[n-i+1]}\mathcal{L}^{\pm\,[n-i+1]}(u\xi q^{2i-2})'(D^{[n-i+1]})^{-1}, \tag{4.15}$$

where

$$D^{[n-i+1]} = \mathrm{diag}[\,q^{n-i+1}, \dots, q, q^{-1}, \dots, q^{-n+i-1}\,].$$

By taking the (i', i') and $((i+1)', i')$ entries on both sides of (4.15) we get

$$\mathfrak{h}_i^{\pm}(u\xi q^{2i-2}) = \mathfrak{h}_{i'}^{\pm}(u)^{-1}\mathfrak{z}^{\pm\,[n-i+1]}(u) \tag{4.16}$$

and

$$-\mathfrak{e}^{\pm}_{(i+1)',i'}(u)\,\mathfrak{h}_{i'}^{\pm}(u)^{-1}\,\mathfrak{z}^{\pm\,[n-i+1]}(u) = q\,\mathfrak{h}_i^{\pm}(u\xi q^{2i-2})\mathfrak{e}^{\pm}_{i,i+1}(u\xi q^{2i-2}).$$

Due to (4.16), this formula can written as

$$-\,\mathfrak{e}^{\pm}_{(i+1)',i'}(u)\,\mathfrak{h}_i^{\pm}(u\xi q^{2i-2}) = q\,\mathfrak{h}_i^{\pm}(u\xi q^{2i-2})\,\mathfrak{e}^{\pm}_{i,i+1}(u\xi q^{2i-2}). \tag{4.17}$$

Furthermore, by the results of [8],

$$q\,\mathfrak{h}_i^{\pm}(u)\,\mathfrak{e}^{\pm}_{i,i+1}(u) = \mathfrak{e}^{\pm}_{i,i+1}(uq^2)\,\mathfrak{h}_i^{\pm}(u),$$

so that (4.17) is equivalent to

$$\mathfrak{e}^{\pm}_{(i+1)',i'}(u)\,\mathfrak{h}_i^{\pm}(u\xi q^{2i-2}) = \mathfrak{e}^{\pm}_{i,i+1}(u\xi q^{2i})\,\mathfrak{h}_i^{\pm}(u\xi q^{2i-2}), \tag{4.18}$$

thus proving the first relation in (4.14). The second relation is verified in a similar way. □

Proposition 4.10. *In the algebras* $U(\overline{R}^{[n]})$ *and* $U(R^{[n]})$ *we have the respective formulas:*

$$\mathfrak{z}^{\pm\,[n]}(u) = \prod_{i=1}^{n-1}\mathfrak{h}_i^{\pm}(u\xi q^{2i})^{-1}\prod_{i=1}^{n}\mathfrak{h}_i^{\pm}(u\xi q^{2i-2})\,\mathfrak{h}_{n+1}^{\pm}(u),$$

$$z^{\pm\,[n]}(u) = \prod_{i=1}^{n-1}h_i^{\pm}(u\xi q^{2i})^{-1}\prod_{i=1}^{n}h_i^{\pm}(u\xi q^{2i-2})\,h_{n+1}^{\pm}(u).$$

Proof. The arguments for both formulas are quite similar so we only give the proof of the first one. Taking the $(2',2')$-entry on both sides of (4.15) and expressing the entries of the matrices $\mathcal{L}^{\pm\,[n]}(u)^{-1}$ and $\mathcal{L}^{\pm\,[n]}(u\xi)^{\mathrm{t}}$ in terms of the Gauss generators, we get

$$\mathfrak{h}_2^{\pm}(u\xi)+\mathfrak{f}_{21}^{\pm}(u\xi)\,\mathfrak{h}_1^{\pm}(u\xi)\,\mathfrak{e}_{12}^{\pm}(u\xi)=\big(\mathfrak{h}_{2'}^{\pm}(u)^{-1}+\mathfrak{e}_{2',1'}^{\pm}(u)\mathfrak{h}_{1'}^{\pm}(u)^{-1}\mathfrak{f}_{1',2'}^{\pm}(u)^{-1}\big)\mathfrak{z}^{\pm\,[n]}(u).$$

As we pointed out in Remark 3.5, the coefficients of the series $\mathfrak{z}^{\pm\,[n]}(u)$ are central in the respective subalgebras generated by the coefficients of $\ell_{ij}^{\pm\,[n]}(u)$. Therefore, using (4.13), we can rewrite the above relation as

$$\mathfrak{h}_{2'}^{\pm}(u)^{-1}\mathfrak{z}^{\pm\,[n]}(u)=\mathfrak{h}_2^{\pm}(u\xi)+\mathfrak{f}_{21}^{\pm}(u\xi)\,\mathfrak{h}_1^{\pm}(u\xi)\,\mathfrak{e}_{12}^{\pm}(u\xi)-\mathfrak{e}_{2',1'}^{\pm}(u)\,\mathfrak{h}_1^{\pm}(u\xi)\,\mathfrak{f}_{1',2'}^{\pm}(u).$$

Now apply Lemma 4.9 to obtain

$$\mathfrak{h}_{2'}^{\pm}(u)^{-1}\mathfrak{z}^{\pm\,[n]}(u)=\mathfrak{h}_2^{\pm}(u\xi)+\mathfrak{f}_{21}^{\pm}(u\xi)\,\mathfrak{h}_1^{\pm}(u\xi)\,\mathfrak{e}_{12}^{\pm}(u\xi)-\mathfrak{e}_{12}^{\pm}(u\xi q^2)\,\mathfrak{h}_1^{\pm}(u\xi)\,\mathfrak{f}_{21}^{\pm}(u\xi q^2).$$

On the other hand, by the results of [8] we have

$$\mathfrak{h}_1^{\pm}(u)\,\mathfrak{e}_{12}^{\pm}(u)=q^{-1}\mathfrak{e}_{12}^{\pm}(uq^2)\,\mathfrak{h}_1^{\pm}(u),\qquad \mathfrak{h}_1^{\pm}(u)\,\mathfrak{f}_{21}^{\pm}(uq^2)=q^{-1}\mathfrak{f}_{21}^{\pm}(u)\,\mathfrak{h}_1^{\pm}(u),$$

and

$$[\mathfrak{e}_{12}^{\pm}(u),\mathfrak{f}_{21}^{\pm}(v)]=\frac{u(q-q^{-1})}{u-v}\,(\mathfrak{h}_2^{\pm}(v)\mathfrak{h}_1^{\pm}(v)^{-1}-\mathfrak{h}_2^{\pm}(u)\,\mathfrak{h}_1^{\pm}(u)^{-1}).$$

This leads to the expression

$$\mathfrak{h}_{2'}^{\pm}(u)^{-1}\mathfrak{z}^{\pm\,[n]}(u)=\mathfrak{h}_2^{\pm}(u\xi q^2)\,\mathfrak{h}_1^{\pm}(u\xi q^2)^{-1}\mathfrak{h}_1^{\pm}(u\xi).$$

Since $\mathfrak{z}^{\pm\,[n-1]}(u)=\mathfrak{h}_{2'}^{\pm}(u)\mathfrak{h}_2^{\pm}(u\xi q^2)$, we get a recurrence formula

$$\mathfrak{z}^{\pm\,[n]}(u)=\mathfrak{h}_1^{\pm}(u\xi q^2)^{-1}\mathfrak{h}_1^{\pm}(u\xi)\,\mathfrak{z}^{\pm\,[n-1]}(u)$$

thus completing the proof. □

4.6 Drinfeld-type relations in the algebras $U(\overline{R}^{[n]})$ and $U(R^{[n]})$

We will now extend the sets of relations produced in Secs 4.3 and 4.4 to obtain all necessary relations in the algebras $U(\overline{R}^{[n]})$ and $U(R^{[n]})$ to be able to prove the Main Theorem.

Theorem 4.11. *The following relations between the Gaussian generators hold in the algebra $U(\overline{R}^{[n]})$. For the relations involving $\mathfrak{h}_i^{\pm}(u)$ we have*

$$\mathfrak{h}_i^{\pm}(u)\mathfrak{h}_j^{\pm}(v)=\mathfrak{h}_j^{\pm}(v)\mathfrak{h}_i^{\pm}(u),$$

$$\mathfrak{h}_i^{\pm}(u)\mathfrak{h}_i^{\mp}(v)=\mathfrak{h}_i^{\mp}(v)\mathfrak{h}_i^{\pm}(u),$$

$$\frac{u_{\pm}-v_{\mp}}{qu_{\pm}-q^{-1}v_{\mp}}\mathfrak{h}_i^{\pm}(u)\mathfrak{h}_j^{\mp}(v)=\frac{u_{\mp}-v_{\pm}}{qu_{\mp}-q^{-1}v_{\pm}}\mathfrak{h}_j^{\mp}(v)\mathfrak{h}_i^{\pm}(u) \tag{4.19}$$

for $i<j$ and $i\neq n$, and

$$\frac{u_\pm - v_\mp}{q^2u_\pm - q^{-2}v_\mp}\mathfrak{h}^\pm_n(u)\mathfrak{h}^\mp_{n+1}(v) = \frac{u_\mp - v_\pm}{q^2u_\mp - q^{-2}v_\pm}\mathfrak{h}^\mp_{n+1}(v)\mathfrak{h}^\pm_n(u).$$

The relations involving $\mathfrak{h}^\pm_i(u)$ and $\mathcal{X}^\pm_j(v)$ are

$$\mathfrak{h}^\pm_i(u)\mathcal{X}^+_j(v) = \frac{u - v_\pm}{q^{(\epsilon_i,\alpha_j)}u - q^{-(\epsilon_i,\alpha_j)}v_\pm}\mathcal{X}^+_j(v)\mathfrak{h}^\pm_i(u),$$

$$\mathfrak{h}^\pm_i(u)\mathcal{X}^-_j(v) = \frac{q^{-(\epsilon_i,\alpha_j)}u_\pm - q^{(\epsilon_i,\alpha_j)}v}{u_\pm - v}\mathcal{X}^-_j(v)\mathfrak{h}^\pm_i(u)$$

for $i\neq n+1$, together with

$$\mathfrak{h}^\pm_{n+1}(u)\mathcal{X}^+_n(v) = \frac{u_\mp - v}{q^{-2}u_\mp - q^2v}\mathcal{X}^+_n(v)\mathfrak{h}^\pm_{n+1}(u),$$

$$\mathfrak{h}^\pm_{n+1}(u)\mathcal{X}^-_n(v) = \frac{q^{-2}u_\pm - q^2v}{u_\pm - v}\mathcal{X}^-_n(v)\mathfrak{h}^\pm_{n+1}(u),$$

and

$$\mathfrak{h}^\pm_{n+1}(u)^{-1}\mathcal{X}^+_{n-1}(v)\mathfrak{h}^\pm_{n+1}(u) = \frac{q^{-1}u - qv_\pm}{q^{-2}u - q^2v_\pm}\mathcal{X}^+_{n-1}(v),$$

$$\mathfrak{h}^\pm_{n+1}(u)\mathcal{X}^-_{n-1}(v)\mathfrak{h}^\pm_{n+1}(u)^{-1} = \frac{q^{-1}u - qv_\mp}{q^{-2}u - q^2v_\mp}\mathcal{X}^-_{n-1}(v),$$

while

$$\mathfrak{h}^\pm_{n+1}(u)\mathcal{X}^+_i(v) = \mathcal{X}^+_i(v)\mathfrak{h}^\pm_{n+1}(u),$$

$$\mathfrak{h}^\pm_{n+1}(u)\mathcal{X}^-_i(v) = \mathcal{X}^-_i(v)\mathfrak{h}^\pm_{n+1}(u),$$

for $1\leqslant i\leqslant n-2$. For the relations involving $\mathcal{X}^\pm_i(u)$ we have

$$(uq^{-i} - q^{\pm(\alpha_i,\alpha_j)-j}v)\mathcal{X}^\pm_i(u)\mathcal{X}^\pm_j(v) = (q^{\pm(\alpha_i,\alpha_j)-i}u - q^{-j}v)\mathcal{X}^\pm_j(v)\mathcal{X}^\pm_i(u)$$

and

$$[\mathcal{X}^+_i(u),\mathcal{X}^-_j(v)] = \delta_{ij}(q_i - q_i^{-1})\Big(\delta\big(\frac{u}{vq^c}\big)\mathfrak{h}^-_i(v_+)^{-1}\mathfrak{h}^-_{i+1}(v_+) - \delta\big(\frac{uq^c}{v}\big)\mathfrak{h}^+_i(u_+)^{-1}\mathfrak{h}^+_{i+1}(u_+)\Big)$$

together with the Serre relations

$$\sum_{\pi\in\mathfrak{S}_r}\sum_{l=0}^{r}(-1)^l\begin{bmatrix} r\\ l\end{bmatrix}_{q_i}\mathcal{X}^\pm_i(u_{\pi(1)})\dots\mathcal{X}^\pm_i(u_{\pi(l)})\mathcal{X}^\pm_j(v)\,\mathcal{X}^\pm_i(u_{\pi(l+1)})\dots\mathcal{X}^\pm_i(u_{\pi(r)}) = 0, \qquad (4.20)$$

which hold for all $i\neq j$ and we set $r = 1 - A_{ij}$.

Proof. We only need to verify the relations complementary to those produced in Secs 4.3 and 4.4. The additional relations are verified by very similar arguments to those used in [8] and so we give only a few details illustrating the calculations which are more specific to type C. We start with (4.19) and take $j = n+1$. By using Corollary 3.10, for $i < n$ we deduce

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\big(\mathfrak{h}_{n+1}^\mp(v) + \mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)\big)$$
$$= \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\big(\mathfrak{h}_{n+1}^\pm(v) + \mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)\big)\mathfrak{h}_i^\pm(u) \quad (4.21)$$

and

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{h}_i^\pm(u).$$

Hence, the left hand side of (4.21) equals

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{h}_{n+1}^\mp(v) + \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{h}_i^\pm(u)\mathfrak{e}_{n,n+1}^\mp(v)). \quad (4.22)$$

By Corollary 3.10,

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{h}_n^\mp(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}_n^\mp(v)\mathfrak{h}_i^\pm(u),$$

so that (4.22) can be written as

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{h}_{n+1}^\mp(v) + \frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_i^\pm(u)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)).$$

Using Corollary 3.10 once again, we find

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)\mathfrak{h}_i^\pm(u)$$

for $i = 1, 2, \dots, n-1$, and so the left hand side of (4.21) takes the form

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{h}_{n+1}^\mp(v) + \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v))\mathfrak{h}_i^\pm(u),$$

which implies (4.19) with $j = n+1$.

The relations involving $\mathfrak{h}_{n+1}^\pm(v)$ and $\mathcal{X}_i^\pm(u)$ with $i = 1, 2, \dots, n-2$ are implied by the following:

$$\mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{h}_{n+1}^\pm(v) = \mathfrak{h}_{n+1}^\pm(v)\mathfrak{e}_{i,i+1}^\pm(u), \qquad \mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{h}_{n+1}^\mp(v) = \mathfrak{h}_{n+1}^\mp(v)\mathfrak{e}_{i,i+1}^\pm(u), \quad (4.23)$$
$$\mathfrak{f}_{i+1,i}^\pm(u)\mathfrak{h}_{n+1}^\pm(v) = \mathfrak{h}_{n+1}^\pm(v)\mathfrak{f}_{i+1,i}^\pm(u), \qquad \mathfrak{f}_{i+1,i}^\pm(u)\mathfrak{h}_{n+1}^\mp(v) = \mathfrak{h}_{n+1}^\mp(v)\mathfrak{f}_{i+1,i}^\pm(u).$$

We will verify the second relation in (4.23). Corollary 3.10 implies

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u)\big(\mathfrak{h}_{n+1}^\mp(v) + \mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)\big)$$
$$= \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\big(\mathfrak{h}_{n+1}^\mp(v) + \mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)\big)\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u). \quad (4.24)$$

Moreover, by Corollary 3.10, we have the following relations:

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u),$$
$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{h}_n^\mp(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}_n^\mp(v)\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u),$$
$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u).$$

Thus, the left hand side of (4.24) equals

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_{n+1}^\mp(v) + \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{h}_n^\mp(v)\mathfrak{e}_{n,n+1}^\mp(v)\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u)$$

and so,

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{h}_{n+1}^\mp(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}_{n+1}^\mp(v)\mathfrak{h}_i^\pm(u)\mathfrak{e}_{i,i+1}^\pm(u).$$

Using now (4.19) with $j = n+1$ we get the second relation in (4.23).

The remaining cases of the type C-specific relations involving $\mathcal{X}_i^\pm(u)$ and $\mathfrak{h}_j^\pm(v)$ are deduced with the use of Remark 4.7, Lemma 4.9 and Corollary 3.10. In particular, Remark 4.7 and the corresponding type A relations in [8] imply

$$\mathfrak{h}_{n'}^\pm(u)^{-1}\mathfrak{e}_{n',(n-1)'}^\pm(v)\mathfrak{h}_{n'}^\pm(u) = \frac{qu - q^{-1}v}{u - v}\mathfrak{e}_{n',(n-1)'}^\pm(v) - \frac{(q - q^{-1})v}{u - v}\mathfrak{e}_{n',(n-1)'}^\pm(u).$$

By Lemma 4.9, we can write this relation as

$$\mathfrak{h}_{n'}^\pm(u)^{-1}\mathfrak{e}_{n-1,n}^\pm(vq^{-4})\mathfrak{h}_{n'}^\pm(u) = \frac{qu - q^{-1}v}{u - v}\mathfrak{e}_{n-1,n}^\pm(vq^{-4}) - \frac{(q - q^{-1})v}{u - v}\mathfrak{e}_{n-1,n}^\pm(uq^{-4}),$$

which leads to the relations involving $\mathcal{X}_{n-1}^\pm(u)$ and $\mathfrak{h}_{n+1}^\pm(v)$.

Now turn to the relations between the series $\mathcal{X}_i^\pm(u)$. For $i < n-1$ we have

$$\mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{e}_{n,n+1}^\pm(v) = \mathfrak{e}_{n,n+1}^\pm(v)\mathfrak{e}_{i,i+1}^\pm(u), \qquad \mathfrak{e}_{i,i+1}^\pm(u)\mathfrak{e}_{n,n+1}^\mp(v) = \mathfrak{e}_{n,n+1}^\mp(v)\mathfrak{e}_{i,i+1}^\pm(u),$$
$$\mathfrak{f}_{i+1,i}^\pm(u)\mathfrak{f}_{n+1,n}^\pm(v) = \mathfrak{f}_{n+1,n}^\pm(v)\mathfrak{f}_{i+1,i}^\pm(u), \qquad \mathfrak{f}_{i+1,i}^\pm(u)\mathfrak{f}_{n+1,n}^\mp(v) = \mathfrak{f}_{n+1,n}^\mp(v)\mathfrak{f}_{i+1,i}^\pm(u).$$

This is verified with the same use of Corollary 3.10 as above. Therefore, for all $i < n-1$ we get $\mathcal{X}_i^\pm(u)\mathcal{X}_n^\pm(v) = \mathcal{X}_n^\pm(v)\mathcal{X}_i^\pm(u)$. For the relation involving $\mathcal{X}_{n-1}^\pm(u)$ and $\mathcal{X}_n^\pm(v)$ it will be sufficient to consider the case $n = 2$ and then apply Theorem 3.9. We find from the defining relations (3.5) and (3.6) that the expression

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{12}(u)\ell^\mp_{23}(v) + \frac{(q-q^{-1})u_\pm}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{22}(u)\ell^\mp_{13}(v) \tag{4.25}$$

equals

$$\frac{(u_\mp - v_\pm)q^{-1}(u_\mp - q^{-4}v_\pm)}{(qu_\mp - q^{-1}v_\pm)(u_\mp - q^{-6}v_\pm)}\ell^\mp_{23}(v)\ell^\pm_{12}(u) + \frac{(q-q^{-1})v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{22}(v)\ell^\pm_{13}(u)$$
$$+\frac{(q-q^{-1})(u_\mp - v_\pm)v_\pm q^{-3}}{(qu_\mp - q^{-1}v_\pm)(u_\mp - q^{-6}v_\pm)}\ell^\mp_{21}(v)\ell^\pm_{14}(u) + \frac{(q-q^{-1})(u_\mp - v_\pm)v_\pm q^{-4}}{(qu_\mp - q^{-1}v_\pm)(u_\mp - q^{-6}v_\pm)}\ell^\mp_{22}(v)\ell^\pm_{13}(u)$$
$$-\frac{(q-q^{-1})(u_\mp - v_\pm)q^{-1}u_\mp}{(qu_\mp - q^{-1}v_\pm)(u_\mp - q^{-6}v_\pm)}\ell^\mp_{24}(v)\ell^\pm_{11}(u).$$

On the other hand, applying the formula for $\ell^\mp_{23}(v)$ arising from the Gauss decomposition, we can write (4.25) as

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{12}(u)\mathfrak{h}^\mp_2(v)\mathfrak{e}^\mp_{23}(v) + \frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{12}(u)\mathfrak{f}^\mp_{21}(v)\mathfrak{h}^\mp_2(v)\mathfrak{e}^\mp_{13}(v)$$
$$+\frac{(q-q^{-1})u_\pm}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{22}(u)\ell^\mp_{13}(v).$$

By using the defining relations between the series $\ell^+_{12}(u)$ and $\ell^-_{21}(v)$ we can bring (4.25) to the form

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{12}(u)\mathfrak{h}^\mp_2(v)\mathfrak{e}^\mp_{23}(v) + \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{21}(v)\ell^\pm_{12}(u)\mathfrak{e}^\mp_{13}(v)$$
$$+\frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\ell^\mp_{22}(v)\ell^\pm_{11}(u)\mathfrak{e}^\mp_{13}(v).$$

Further, by using the defining relations between $\ell^\pm_{12}(u)$ and $\ell^\mp_{11}(v)$ we can write (4.25) as

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{12}(u)\mathfrak{h}^\mp_2(v)\mathfrak{e}^\mp_{23}(v) + \mathfrak{f}^\mp_{21}(v)\ell^\pm_{12}(u)\ell^\mp_{13}(v) + \frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}^\mp_2(v)\ell^\pm_{11}(u)\mathfrak{e}^\mp_{13}(v).$$

As a next step, apply the relations between $\ell^+_{12}(u)$ and $\ell^-_{13}(v)$ to write the sum

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\ell^\pm_{12}(u)\mathfrak{h}^\mp_2(v)\mathfrak{e}^\mp_{23}(v) + \frac{(q-q^{-1})u_\mp}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}^\mp_2(v)\ell^\pm_{11}(u)\mathfrak{e}^\mp_{13}(v)$$

as

$$\frac{(u_\mp - v_\pm)q^{-1}(u_\mp - q^{-4}v_\pm)}{(qu_\mp - q^{-1}v_\pm)(u_\mp - q^{-6}v_\pm)}\mathfrak{h}^\mp_2(v)\mathfrak{e}^\mp_{23}(v)\ell^\pm_{12}(u) + \frac{(q-q^{-1})v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{h}^\mp_2(v)\ell^\pm_{13}(u)$$
$$+\frac{(q-q^{-1})(u_\mp - v_\pm)v\pm q^{-4}}{(qu_\mp - q^{-1}v_\pm)(u_\mp - q^{-6}v_\pm)}\mathfrak{h}^\mp_2(v)\ell^\pm_{13}(u)$$
$$-\frac{(q-q^{-1})(u_\mp - v_\pm)q^{-1}u_\mp}{(qu_\mp - q^{-1}v_\pm)(u_\mp - q^{-6}v_\pm)}\mathfrak{h}^\mp_2(v)\mathfrak{e}_{24}(v)\ell^\pm_{11}(u).$$

Using (4.8), we get the relations:

$$\begin{aligned}\ell_{12}^{\pm}(u)\mathfrak{h}_2^{\mp}(v) &= \frac{qu_{\mp}-q^{-1}v_{\pm}}{u_{\mp}-v_{\pm}}\mathfrak{h}_1^{\pm}(u)\mathfrak{h}_2^{\mp}(v)\mathfrak{e}_{12}^{\pm}(u) + \frac{(q-q^{-1})u_{\mp}}{u_{\mp}-v_{\pm}}\mathfrak{h}_1^{\pm}(u)\mathfrak{h}_2^{\mp}(v)\mathfrak{e}_{12}^{\mp}(v)\\ &= \frac{qu_{\pm}-q^{-1}v_{\mp}}{u_{\pm}-v_{\mp}}\mathfrak{h}_2^{\mp}(v)\mathfrak{h}_1^{\pm}(u)\mathfrak{e}_{12}^{\pm}(u) + \frac{(q-q^{-1})u_{\mp}}{qu_{\mp}-q^{-1}v_{\pm}}\frac{qu_{\pm}-q^{-1}v_{\mp}}{u_{\pm}-v_{\mp}}\mathfrak{h}_2^{\mp}(v)\mathfrak{h}_1^{\pm}(u)\mathfrak{e}_{12}^{\mp}(v).\end{aligned}$$

Hence, since the series $\mathfrak{h}_1^{\pm}(u)$ and $\mathfrak{h}_2^{\mp}(v)$ are invertible, and $[\mathfrak{h}_1^{\pm}(u), \mathfrak{e}_{23}^{\mp}(v)] = 0$, we come to the relation

$$\begin{aligned}&\mathfrak{e}_{12}^{\pm}(u)\mathfrak{e}_{23}^{\mp}(v) + \frac{(q-q^{-1})u_{\mp}}{qu_{\mp}-q^{-1}v_{\pm}}\mathfrak{e}_{13}^{\mp}(v) + \frac{(q-q^{-1})u_{\mp}}{qu_{\mp}-q^{-1}v_{\pm}}\mathfrak{e}_{12}^{\mp}(v)\mathfrak{e}_{23}^{\mp}(v) \qquad (4.26)\\ &= \frac{(u_{\mp}-v_{\pm})q^{-1}(u_{\mp}-q^{-4}v_{\pm})}{(qu_{\mp}-q^{-1}v_{\pm})(u_{\mp}-q^{-6}v_{\pm})}\mathfrak{e}_{23}^{\mp}(v)\mathfrak{e}_{12}^{\pm}(u) + \frac{(q-q^{-1})v_{\pm}}{qu_{\mp}-q^{-1}v_{\pm}}\mathfrak{e}_{13}^{\pm}(u)\\ &+ \frac{(q-q^{-1})(u_{\mp}-v_{\pm})v_{\pm}q^{-4}}{(qu_{\mp}-q^{-1}v_{\pm})(u_{\mp}-q^{-6}v_{\pm})}\mathfrak{e}_{13}^{\pm}(u) - \frac{(q-q^{-1})(u_{\mp}-v_{\pm})q^{-1}u_{\mp}}{(qu_{\mp}-q^{-1}v_{\pm})(u_{\mp}-q^{-6}v_{\pm})}\mathfrak{h}_1^{\pm}(u)^{-1}\mathfrak{e}_{24}^{\mp}(v)\mathfrak{h}_1^{\pm}(u).\end{aligned}$$

Similar arguments imply the relations

$$\begin{aligned}\frac{u_{\pm}-v_{\mp}}{qu_{\pm}-q^{-1}v_{\mp}}\mathfrak{h}_1^{\pm}(u)\mathfrak{h}_2^{\mp}(v)\mathfrak{e}_{24}^{\mp}(v) &= \frac{(u_{\mp}-v_{\pm})q^{-1}(u_{\mp}-v_{\pm}q^{-4})}{(qu_{\mp}-q^{-1}v_{\pm})(u_{\mp}-q^{-6}v_{\pm})}\mathfrak{h}_2^{\mp}(v)\mathfrak{e}_{24}^{\mp}(v)\mathfrak{h}_1^{\pm}(u)\\ &+ \frac{(q-q^{-1})(u_{\mp}-v_{\pm})v_{\pm}q^{-3}}{(qu_{\mp}-q^{-1}v_{\pm})(u_{\mp}-q^{-6}v_{\pm})}\mathfrak{h}_2^{\mp}(v)\mathfrak{h}_1^{\pm}(u)\mathfrak{e}_{13}^{\pm}(u)\\ &- \frac{(q-q^{-1})(u_{\mp}-v_{\pm})v_{\pm}q^{-5}}{(qu_{\mp}-q^{-1}v_{\pm})(u_{\mp}-q^{-6}v_{\pm})}\mathfrak{h}_2^{\mp}(v)\mathfrak{e}_{23}(v)\mathfrak{h}_1^{\pm}(u)\mathfrak{e}_{12}^{\pm}(u).\end{aligned}$$

Due to the relation

$$\frac{u_{\pm}-v_{\mp}}{qu_{\pm}-q^{-1}v_{\mp}}\mathfrak{h}_1^{\pm}(u)\mathfrak{h}_2^{\mp}(v) = \frac{u_{\mp}-v_{\pm}}{qu_{\mp}-q^{-1}v_{\pm}}\mathfrak{h}_2^{\mp}(v)\mathfrak{h}_1^{\pm}(u),$$

we come to

$$\begin{aligned}\mathfrak{h}_1^{\pm}(u)\mathfrak{e}_{24}^{\mp}(v) &= \frac{q^{-1}(u_{\mp}-v_{\pm}q^{-4})}{u_{\mp}-q^{-6}v_{\pm}}\mathfrak{e}_{24}^{\mp}(v)\mathfrak{h}_1^{\pm}(u)\\ &+ \frac{(q-q^{-1})v_{\pm}q^{-3}}{u_{\mp}-q^{-6}v_{\pm}}\mathfrak{h}_1^{\pm}(u)\mathfrak{e}_{13}^{\pm}(u) - \frac{(q-q^{-1})v_{\pm}q^{-5}}{u_{\mp}-q^{-6}v_{\pm}}\mathfrak{e}_{23}^{\mp}(v)\mathfrak{h}_1^{\pm}(u)\mathfrak{e}_{12}^{\pm}(u).\end{aligned}$$

Together with the expression in (4.26), this gives

$$\begin{aligned}\mathfrak{e}_{12}^{\pm}(u)\mathfrak{e}_{23}^{\mp}(v) &= \frac{1}{q^2u_{\mp}-q^{-2}v_{\pm}}\big((q^2-q^{-2})v_{\pm}\mathfrak{e}_{13}^{\pm}(u) + (u_{\mp}-v_{\pm})\mathfrak{e}_{23}^{\mp}(v)\mathfrak{e}_{12}^{\pm}(u)\\ &- (q^2-q^{-2})u_{\mp}\mathfrak{e}_{12}^{\mp}(v)\mathfrak{e}_{23}^{\mp}(v) - (q^2-q^{-2})u_{\mp}\mathfrak{e}_{13}^{\mp}(v)\big)\\ &+ \frac{(q-q^{-1})u_{\mp}(u_{\mp}-v_{\pm})}{(q^2u_{\mp}-q^{-2}v_{\pm})(qu_{\mp}-q^{-1}v_{\pm})}\big(\mathfrak{e}_{12}^{\mp}(v)\mathfrak{e}_{23}^{\mp}(v) + \mathfrak{e}_{13}^{\mp}(v) - q^2\mathfrak{e}_{24}^{\mp}(v)\big).\end{aligned}$$

Multiply both sides by $qu_\mp - q^{-1}v_\pm$ and set $qu_\mp = q^{-1}v_\pm$ to see that the second summand vanishes. Finally, by applying Theorem 3.9, we come to the relation

$$\begin{aligned}(q^2u_\mp - q^{-2}v_\pm)\mathfrak{e}^\pm_{n-1,n}(u)\mathfrak{e}^\mp_{n,n+1}(v)&\\ &= (u_\mp - v_\pm)\mathfrak{e}^\mp_{n,n+1}(v)\mathfrak{e}^\pm_{n-1,n}(u) + (q^2-q^{-2})v_\pm\mathfrak{e}^\pm_{n-1,n+1}(u)\\ &- (q^2-q^{-2})u_\mp\mathfrak{e}^\mp_{n-1,n}(v)\mathfrak{e}^\mp_{n,n+1}(v) - (q^2-q^{-2})u_\mp\mathfrak{e}^\mp_{n-1,n+1}(v).\end{aligned}$$

Quite similar calculations lead to its counterparts:

$$\begin{aligned}(q^2u - q^{-2}v)\mathfrak{e}^\pm_{n-1,n}(u)\mathfrak{e}^\pm_{n,n+1}(v)&\\ &= (u - v)\mathfrak{e}^\pm_{n,n+1}(v)\mathfrak{e}^\pm_{n-1,n}(u) + (q^2-q^{-2})v\mathfrak{e}^\pm_{n-1,n+1}(u)\\ &- (q^2-q^{-2})u\mathfrak{e}^\pm_{n-1,n}(v)\mathfrak{e}^\pm_{n,n+1}(v) - (q^2-q^{-2})u\mathfrak{e}^\pm_{n-1,n+1}(v),\end{aligned}$$

$$\begin{aligned}(u_\pm - v_\mp)\mathfrak{f}^\pm_{n,n-1}(u)\mathfrak{f}^\mp_{n+1,n}(v)&\\ &= (q^2u_\pm - q^{-2}v_\mp)\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{f}^\pm_{n,n-1}(u) + (q^2-q^{-2})v_\mp\mathfrak{f}^\mp_{n+1,n-1}(v)\\ &- (q^2-q^{-2})v_\mp\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{f}^\mp_{n,n-1}(v) - (q^2-q^{-2})u_\pm\mathfrak{f}^\pm_{n+1,n-1}(u)\end{aligned}$$

and

$$\begin{aligned}(u - v)\mathfrak{f}^\pm_{n,n-1}(u)\mathfrak{f}^\pm_{n+1,n}(v) &= (q^2u - q^{-2}v)\mathfrak{f}^\pm_{n+1,n}(v)\mathfrak{f}^\pm_{n,n-1}(u) + (q^2-q^{-2})v\mathfrak{f}^\pm_{n+1,n-1}(v)\\ &- (q^2-q^{-2})v\mathfrak{f}^\pm_{n+1,n}(v)\mathfrak{f}^\pm_{n,n-1}(v) - (q^2-q^{-2})u\mathfrak{f}^\pm_{n+1,n-1}(u).\end{aligned}$$

As a consequence, we have thus verified the relations

$$(uq^2 - q^{-2}v)^{\pm 1}\mathcal{X}^\pm_{n-1}(u)\mathcal{X}^\pm_n(v) = (u-v)^{\pm 1}\mathcal{X}^\pm_n(v)\mathcal{X}^\pm_{n-1}(u).$$

Now suppose that $i \leqslant n-1$ and verify the relations

$$\begin{aligned}&\mathfrak{e}^\pm_{i,i+1}(u)\mathfrak{f}^\pm_{n+1,n}(v) = \mathfrak{f}^\pm_{n+1,n}(v)\mathfrak{e}^\pm_{i,i+1}(u), \qquad \mathfrak{e}^\pm_{i,i+1}(u)\mathfrak{f}^\mp_{n+1,n}(v) = \mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{e}^\pm_{i,i+1}(u),\\ &\mathfrak{f}^\pm_{i+1,i}(u)\mathfrak{e}^\pm_{n,n+1}(v) = \mathfrak{e}^\pm_{n,n+1}(v)\mathfrak{f}^\pm_{i+1,i}(u), \qquad \mathfrak{f}^\pm_{i+1,i}(u)\mathfrak{e}^\mp_{n,n+1}(v) = \mathfrak{e}^\mp_{n,n+1}(v)\mathfrak{f}^\pm_{i+1,i}(u).\end{aligned} \tag{4.27}$$

We only do the second relation in (4.27) as the arguments are quite similar. If $i \leqslant n-2$ then Corollary 3.10 gives

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}^\pm_i(u)\mathfrak{e}^\pm_{i,i+1}(u)\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{h}^\mp_n(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{h}^\mp_n(v)\mathfrak{h}^\pm_i(u)\mathfrak{e}^\pm_{i,i+1}(u)$$

and

$$\frac{u_\pm - v_\mp}{qu_\pm - q^{-1}v_\mp}\mathfrak{h}^\pm_i(u)\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{h}^\mp_n(v) = \frac{u_\mp - v_\pm}{qu_\mp - q^{-1}v_\pm}\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{h}^\mp_n(v)\mathfrak{h}^\pm_i(u).$$

Therefore,

$$\mathfrak{h}^\pm_i(u)\mathfrak{e}^\pm_{i,i+1}(u)\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{h}^\mp_n(v) = \mathfrak{h}^\pm_i(u)\mathfrak{f}^\mp_{n+1,n}(v)\mathfrak{h}^\mp_n(v)\mathfrak{e}^\pm_{i,i+1}(u).$$

Since $[\mathfrak{h}^{\mp}_n(v), \mathfrak{e}^{\pm}_{i,i+1}(u)] = 0$, the required relation follows. Now let $i = n-1$. Due to (4.8), we have

$$\mathfrak{e}^{\pm}_{n-1,n}(u)\mathfrak{f}^{\mp}_{n+1,n}(v)\mathfrak{h}^{\mp}_n(v) = \frac{qu_{\mp} - q^{-1}v_{\pm}}{u_{\mp} - v_{\pm}}\mathfrak{f}^{\mp}_{n+1,n}(v)\mathfrak{h}^{\mp}_n(v)\mathfrak{e}^{\pm}_{n-1,n}(u) + \frac{(q-q^{-1})v_{\pm}}{u_{\mp} - v_{\pm}}\mathfrak{f}^{\mp}_{n+1,n}(v)\mathfrak{h}^{\mp}_n(v)\mathfrak{e}^{\mp}_{n-1,n}(v)$$

and

$$\mathfrak{e}^{\pm}_{n-1,n}(u)\mathfrak{h}^{\mp}_n(v) = \frac{qu_{\mp} - q^{-1}v_{\pm}}{u_{\mp} - v_{\pm}}\mathfrak{h}^{\mp}_n(v)\mathfrak{e}^{\pm}_{n-1,n}(u) + \frac{(q-q^{-1})v_{\pm}}{u_{\mp} - v_{\pm}}\mathfrak{h}^{\mp}_n(v)\mathfrak{e}^{\mp}_{n-1,n}(v).$$

Hence

$$\mathfrak{e}^{\pm}_{n-1,n}(u)\mathfrak{f}^{\mp}_{n+1,n}(v)\mathfrak{h}^{\mp}_n(v) = \mathfrak{f}^{\mp}_{n+1,n}(v)\mathfrak{e}^{\pm}_{n-1,n}(u)\mathfrak{h}^{\mp}_n(v),$$

so that the second relation in (4.27) is verified. Thus, by applying Proposition 4.8 we thus derive all cases for the commutator formula for the series $\mathcal{X}^{+}_i(u)$ and $\mathcal{X}^{-}_j(v)$.

To complete the proof of the theorem, we will now verify the Serre relations (4.20). By Proposition 4.1, these relations have the same form for the algebras $U(R)$ and $U(\overline{R})$. We will work with the algebra $U(R)$ and introduce its elements $x^{\pm}_{i,m}$ and $a_{i,l}$ for $i = 1, \dots, n$ and $m, l \in \mathbb{Z}$ with $l \neq 0$ by the formulas

$$x^{\pm}_i(u) = (q_i - q_i^{-1})^{-1} X^{\pm}_i(uq^i),$$
$$\psi_i(u) = h^{-}_{i+1}(uq^i)\, h^{-}_i(uq^i)^{-1},$$
$$\varphi_i(u) = h^{+}_{i+1}(uq^i)\, h^{+}_i(uq^i)^{-1},$$

for $i = 1, \dots, n-1$, and

$$x^{\pm}_n(u) = (q_n - q_n^{-1})^{-1} X^{\pm}_n(uq^{n+1}),$$
$$\psi_n(u) = h^{-}_{n+1}(uq^{n+1})\, h^{-}_n(uq^{n+1})^{-1},$$
$$\varphi_n(u) = h^{+}_{n+1}(uq^{n+1})\, h^{+}_n(uq^{n+1})^{-1},$$

and the expansions (1.3), (1.4) and (1.16), where $k_i = h^{-}_{i0}\, h^{+}_{i+1\,0}$ and $h^{\pm}_{i0}$ denotes the constant term of the series (4.1). In terms of the elements $x^{\pm}_{i,m}$ the Serre relations take the form

$$\sum_{\pi \in \mathfrak{S}_r} \sum_{l=0}^{r} (-1)^l \begin{bmatrix} r \\ l \end{bmatrix}_{q_i} x^{\pm}_{i,k_{\pi(1)}} \dots x^{\pm}_{i,k_{\pi(l)}} x^{\pm}_{j,s} x^{\pm}_{i,k_{\pi(l+1)}} \dots x^{\pm}_{i,k_{\pi(r)}} = 0, \qquad (4.28)$$

for any integers $k_1, \dots, k_r, s$. We will keep the indices $i \neq j$ fixed and denote the left hand side in (4.28) by $x^{\pm}(k_1, \dots, k_r; s)$. We will adapt an argument used in the Yangian context by Levendorski [30] to the quantum affine algebra case. We will prove the relation $x^{\pm}(k_1, \dots, k_r; s) = 0$ by using an induction argument on the number of nonzero entries among the entries of the tuples $(k_1, \dots, k_r; s)$. The induction base is the relation

$x^{\pm}(0,\dots,0;0)=0$. It holds because of the well-known equivalence between the Drinfeld–Jimbo definition of the the quantized enveloping algebra $U_q(\mathfrak{sp}_{2n})$ and its R-matrix presentation; see [33]. In our notation, the algebra $U_q(\mathfrak{sp}_{2n})$ can be identified with the subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ obtained by restricting the range of the indices of the generators to the set $\{1,\dots,n\}$, as defined in Section 2.1, whereas its R-matrix presentation is the subalgebra of $U(R)$ generated by the zero mode elements $l^{\pm}_{ij}[0]$ with $1 \leqslant i,j \leqslant 2n$; see Section 3.

The induction step will be based on the identities in the algebra $U(R)$ which are implied by the previously verified relations,

$$\varphi_i(u)\,x^{\pm}_j(v)=\Big[\frac{u\,q^{(\alpha_i,\alpha_j)\mp c/2}-v}{u\,q^{\mp c/2}-v\,q^{(\alpha_i,\alpha_j)}}\Big]^{\pm 1}x^{\pm}_j(v)\,\varphi_i(u)$$

and

$$\psi_i(u)\,x^{\pm}_j(v)=\Big[\frac{v\,q^{(\alpha_i,\alpha_j)\mp c/2}-u}{v\,q^{\mp c/2}-u\,q^{(\alpha_i,\alpha_j)}}\Big]^{\mp 1}x^{\pm}_j(v)\,\psi_i(u).$$

By taking the coefficients of powers of u and v we derive that

$$[a_{i,k},x^{\pm}_{j,m}]=\pm\frac{[kA_{ij}]_{q_i}}{k}\,q^{\mp|k|\,c/2}x^{\pm}_{j,k+m}.$$

The rest of the argument is quite similar to [30]; it amounts to calculating the commutators

$$\big[a_{i,k},x^{\pm}(k_1,\dots,k_p,0,\dots,0;s)\big]\qquad\text{and}\qquad\big[a_{j,k},x^{\pm}(k_1,\dots,k_p,0,\dots,0;s)\big]$$

for a given $0\leqslant p<r$. By the induction hypothesis, both commutators are zero which leads to a system of two linear equations with a nonzero determinant. Therefore, all elements of the form $x^{\pm}(k_1,\dots,k_{p+1},0,\dots,0;s)$ are also equal to zero. This proves that $x^{\pm}(k_1,\dots,k_r;s)=0$, as required. □

Now recall the extended quantum affine algebra $U^{\rm ext}_q(\widehat{\mathfrak{sp}}_{2n})$ as introduced in Definition 2.1. By using Theorem 4.11 and Proposition 4.1 connecting the Gaussian generators of the algebras $U(R)$ and $U(\overline{R})$, we come to the following homomorphism theorem.

Theorem 4.12. *The mapping*

$$\begin{aligned}
X^{+}_i(u)&\mapsto e^{+}_{i\,i+1}(u_+)-e^{-}_{i\,i+1}(u_-), &&\textit{for}\quad i=1,\dots,n,\\
X^{-}_i(u)&\mapsto f^{+}_{i+1,i}(u_-)-f^{-}_{i+1,i}(u_+), &&\textit{for}\quad i=1,\dots,n,\\
h^{\pm}_j(u)&\mapsto h^{\pm}_j(u), &&\textit{for}\quad j=1,\dots,n+1.
\end{aligned}$$

defines a homomorphism $DR: U^{\rm ext}_q(\widehat{\mathfrak{sp}}_{2n})\to U(R)$.

We will show in the next section that the homomorphism DR provided by Theorem 4.12 is in fact an isomorphism. To this end, we will construct an inverse map by employing the universal R-matrix for the algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$ in a way similar to the type A case; see [15].

5 The universal R-matrix and inverse map

We will use explicit formulas for the universal R-matrix for the algebra $U_q(\widehat{\mathfrak{g}})$ obtained by Khoroshkin and Tolstoy [27] and Damiani [6, 7].

Recall the Cartan matrix for $\mathfrak{g} = \mathfrak{sp}_{2n}$ defined in (1.1) and consider the diagonal matrix $C = \mathrm{diag}[r_1, r_2, \dots, r_n]$ with $r_i = (\alpha_i, \alpha_i)/2$. Then the matrix $B = [B_{ij}] := CA$ is symmetric with $B_{ij} = (\alpha_i, \alpha_j)$. We will use the notation $\tilde{B} = [\tilde{B}_{ij}]$ for the inverse matrix B^{-1}. We will also need the q-deformed matrix $B(q) = [B_{ij}(q)]$ with $B_{ij}(q) = [B_{ij}]_q$ and its inverse $\tilde{B}(q) = [\tilde{B}_{ij}(q)]$; see (1.2). It is clear that both matrices $\tilde{B}$ and $\tilde{B}(q)$ are symmetric. The entries of $\tilde{B}$ are given by

$$\tilde{B}_{ij} = \begin{cases} n/4 & \text{for} \quad i = j = n, \\ j/2 & \text{for} \quad i = n > j, \\ j & \text{for} \quad n > i \geqslant j, \end{cases} \tag{5.1}$$

while for any integer k we have

$$\tilde{B}_{ij}(q^k) = \begin{cases} \dfrac{[n]_{q^k}}{[2]_{q^k}[2]_{q^{k(n+1)}}} & \text{for} \quad i = j = n, \\ \dfrac{[j]_{q^k}}{[2]_{q^{k(n+1)}}} & \text{for} \quad i = n > j, \\ \dfrac{[2]_{q^{k(n+1-i)}}[j]_{q^k}}{[2]_{q^{k(n+1)}}} & \text{for} \quad n > i \geqslant j. \end{cases} \tag{5.2}$$

With the presentation of the algebra $U_q(\widehat{\mathfrak{g}})$ used in Section 2.1, consider the extended algebra $U_q(\tilde{\mathfrak{g}})$ which is obtained by adjoining an additional element d with the relations

$$[d, k_i] = 0, \qquad [d, E_{\alpha_i}] = \delta_{i,0} E_{\alpha_i}, \qquad [d, F_{\alpha_i}] = -\delta_{i,0} F_{\alpha_i}.$$

For a formal variable u define an automorphism D_u of the algebra $U_q(\tilde{\mathfrak{g}}) \otimes \mathbb{C}[u, u^{-1}]$ by

$$D_u(E_{\alpha_i}) = u^{\delta_{i,0}} E_{\alpha_i}, \qquad D_u(F_{\alpha_i}) = u^{-\delta_{i,0}} F_{\alpha_i}, \qquad D_u(k_i) = k_i, \qquad D_u(d) = d.$$

The *universal R-matrix* is an element $\mathfrak{R} \in U_q(\tilde{\mathfrak{g}}) \,\hat{\otimes}\, U_q(\tilde{\mathfrak{g}})$ of a completed tensor product satisfying certain conditions; see Drinfeld [11]. The conditions imply that this element is a solution of the Yang–Baxter equation

$$\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12}.$$

The explicit formula for $\mathfrak{R}$ uses the $\hbar$-adic settings so we will regard the quantum affine algebra over $\mathbb{C}[[\hbar]]$ and set $q = \exp(\hbar) \in \mathbb{C}[[\hbar]]$. Introduce elements $h_1, \dots, h_n$ by setting $k_i = \exp(\hbar h_i)$. According to [7], the universal R-matrix admits a triangular decomposition

$$\mathfrak{R} = \mathfrak{R}^{>0}\,\mathfrak{R}^{0}\,\mathfrak{R}^{<0}\,\mathcal{K}, \tag{5.3}$$

where

$$\mathfrak{R}^{>0} = \prod_{\alpha\in\Delta_+} \prod_{k\geqslant 0} \exp_{q_i}\big((q_i^{-1} - q_i) E_{\alpha+k\delta} \otimes F_{\alpha+k\delta}\big),$$

$$\mathfrak{R}^{<0} = \prod_{\alpha\in\Delta_+} \prod_{k>0} \exp_{q_i}\big((q_i^{-1} - q_i) E_{-\alpha+k\delta} \otimes F_{-\alpha+k\delta}\big),$$

and

$$\mathcal{K} = T\, q^{-(c\otimes d + d\otimes c)}, \qquad T = \exp(-\hbar \tilde{B}_{ij}\, h_i \otimes h_j).$$

We will work with the parameter-dependent R-matrix defined by

$$\mathcal{R}(u) = (D_u \otimes \mathrm{id})\, \mathfrak{R}\, q^{c\otimes d + d\otimes c}.$$

It satisfies the Yang–Baxter equation in the form

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(uvq^{-c_2})\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(uvq^{c_2})\mathcal{R}_{12}(u) \tag{5.4}$$

where $c_2 = 1 \otimes c \otimes 1$; cf. [17].

A straightforward calculation verifies the following formulas for the vector representation of the quantum affine algebra. As before, we denote by $e_{ij} \in \mathrm{End}\,\mathbb{C}^{2n}$ the standard matrix units.

Proposition 5.1. *The mappings* $q^{\pm c/2} \mapsto 1$,

$$\begin{aligned}
x^+_{ik} &\mapsto q^{-ik} e_{i+1,i} - q^{-(2n+2-i)k} e_{i',(i+1)'},\\
x^-_{ik} &\mapsto q^{-ik} e_{i,i+1} - q^{-(2n+2-i)k} e_{(i+1)',i'},\\
a_{ik} &\mapsto \frac{[k]_{q_i}}{k} \big(q^{-ik} (q^{-k} e_{i+1,i+1} - q^k e_{ii}) + q^{-(2n+2-i)k} (q^{-k} e_{i'i'} - q^k e_{(i+1)'(i+1)'})\big)
\end{aligned}$$

for $i = 1, \ldots, n-1$, *and*

$$\begin{aligned}
x^+_{nk} &\mapsto q^{-(n+1)k} e_{n+1,n},\\
x^-_{nk} &\mapsto q^{-(n+1)k} e_{n,n+1},\\
a_{nk} &\mapsto \frac{[k]_{q_n}}{k} \big(q^{-(n+1)k} (q^{-2k} e_{n+1,n+1} - q^{2k} e_{nn})\big)
\end{aligned}$$

define a representation $\pi_V : U_q(\widehat{\mathfrak{sp}}_{2n}) \to \mathrm{End}\, V$ *of the algebra* $U_q(\widehat{\mathfrak{sp}}_{2n})$ *on the vector space* $V = \mathbb{C}^{2n}$. □

It follows from the results of [17] that the R-matrix defined in (1.7) coincides with the image of the universal R-matrix:

$$R(u) = (\pi_V \otimes \pi_V)\, \mathcal{R}(u).$$

Introduce the *L-operators* in $U_q(\widehat{\mathfrak{sp}}_{2n})$ by the formulas

$$\begin{aligned}
\tilde{L}^+(u) &= (\mathrm{id} \otimes \pi_V)\, \mathcal{R}_{21}(uq^{c/2}),\\
\tilde{L}^-(u) &= (\mathrm{id} \otimes \pi_V)\, \mathcal{R}_{12}(u^{-1}q^{-c/2})^{-1}.
\end{aligned}$$

Recall the series $z^{\pm}(u)$ defined in (2.17). Their coefficients are central in the algebra $U_q^{\rm ext}(\widehat{\mathfrak{sp}}_{2n})$; see Proposition 2.2. Therefore, the Yang–Baxter equation (5.4) implies the relations for the L-operators:

$$R(u/v)L_1^{\pm}(u)L_2^{\pm}(v) = L_2^{\pm}(v)L_1^{\pm}(u)R(u/v),$$
$$R(u_+/v_-)L_1^{\pm}(u)L_2^{\mp}(v) = L_2^{\mp}(v)L_1^{\pm}(u)R(u_-/v_+),$$

where we set

$$L^+(u) = \tilde{L}^+(u)\prod_{m=0}^{\infty} z^+(u\xi^{-2m-1})\, z^+(u\xi^{-2m-2})^{-1}, \tag{5.5}$$

$$L^-(u) = \tilde{L}^-(u)\prod_{m=0}^{\infty} z^-(u\xi^{-2m-1})\, z^-(u\xi^{-2m-2})^{-1}. \tag{5.6}$$

Note that although these formulas for the entries of the matrices $L^{\pm}(u)$ involve a completion of the center of the algebra $U_q^{\rm ext}(\widehat{\mathfrak{sp}}_{2n})$, it will turn out that the coefficients of the series in $u^{\pm 1}$ actually belong to $U_q^{\rm ext}(\widehat{\mathfrak{sp}}_{2n})$. Thus, we may conclude that the mapping

$$RD : L^{\pm}(u) \mapsto L^{\pm}(u) \tag{5.7}$$

defines a homomorphism RD from the algebra $U(R)$ to a completed algebra $U_q^{\rm ext}(\widehat{\mathfrak{sp}}_{2n})$, where we use the same notation for the corresponding elements of the algebras.

Returning to the universal R-matrix, observe that formula (5.3) implies the corresponding decomposition of the matrix $\mathcal{R}(u)$:

$$\mathcal{R}(u) = \mathcal{R}^{>0}(u)\mathcal{R}^{0}(u)\mathcal{R}^{<0}(u), \tag{5.8}$$

where

$$\mathcal{R}^{>0}(u) = \prod_{\alpha\in\Delta_+}\prod_{k\geqslant 0} \exp_{q_i}\big((q_i^{-1}-q_i)u^k E_{\alpha+k\delta}\otimes F_{\alpha+k\delta}\big),$$

$$\mathcal{R}^{<0}(u) = T^{-1}\prod_{\alpha\in\Delta_+}\prod_{k>0} \exp_{q_i}\big((q_i^{-1}-q_i)u^k E_{-\alpha+k\delta}\otimes F_{-\alpha+k\delta}\big)\, T$$

and

$$\mathcal{R}^{0}(u) = \exp\Big(\sum_{k>0}\sum_{i,j=1}^{n} \frac{(q_i^{-1}-q_i)(q_j^{-1}-q_j)}{q^{-1}-q}\,\frac{k}{[k]_q}\tilde{B}_{ij}(q^k)u^k q^{kc/2} a_{i,k}\otimes a_{j,-k}q^{-kc/2}\Big)\, T.$$

By using the vector representation π_V defined in Proposition 5.1, introduce the matrices $F^+(u)$, $E^+(u)$ and $H^+(u)$ by setting

$$\begin{aligned} F^+(u) &= (\mathrm{id}\otimes\pi_V)\,\mathcal{R}_{21}^{>0}(uq^{c/2}) \\ &= (\mathrm{id}\otimes\pi_V)\prod_{\alpha\in\Delta_+}\prod_{k\geqslant 0} \exp_{q_i}\big((q_i^{-1}-q_i)u^k q^{kc/2} F_{\alpha+k\delta}\otimes E_{\alpha+k\delta}\big), \end{aligned}$$

$$\begin{aligned}E^+(u) &= (\mathrm{id}\otimes\pi_V)\,\mathcal{R}_{21}^{<0}(uq^{c/2})\\ &= (\mathrm{id}\otimes\pi_V)\Big(T_{21}^{-1}\prod_{\alpha\in\Delta_+}\prod_{k>0}\exp_{q_i}\big((q_i^{-1}-q_i)u^kq^{kc/2}F_{-\alpha+k\delta}\otimes E_{-\alpha+k\delta}\big)\,T_{21}\Big)\end{aligned}$$

and

$$\begin{aligned}H^+(u) &= (\mathrm{id}\otimes\pi_V)\\ &\Big(\exp\Big(\sum_{k>0}\sum_{i,j=1}^{n}\frac{(q_i^{-1}-q_i)(q_j^{-1}-q_j)}{q^{-1}-q}\frac{k}{[k]_q}\tilde{B}_{ij}(q^k)u^kq^{kc/2}a_{j,-k}q^{-kc/2}\otimes q^{kc/2}a_{i,k}\Big)T_{21}\Big)\\ &\qquad\times\prod_{m=0}^{\infty}z^+(u\xi^{-2m-1})z^+(u\xi^{-2m-2})^{-1}.\end{aligned}$$

The decomposition (5.8) implies the corresponding decomposition for the matrix $L^+(u)$:

$$L^+(u) = F^+(u)H^+(u)E^+(u).$$

Recall the Drinfeld generators $x_{i,k}^{\pm}$ of the algebra $U_q(\widehat{\mathfrak{sp}}_{2n})$, as defined in the Introduction, and combine them into the formal series

$$\begin{aligned}x_i^-(u)^{\geqslant 0} &= \sum_{k\geqslant 0}x_{i,-k}^-u^k, & x_i^+(u)^{>0} &= \sum_{k>0}x_{i,-k}^+u^k,\\ x_i^-(u)^{<0} &= \sum_{k>0}x_{i,k}^-u^{-k}, & x_i^+(u)^{\leqslant 0} &= \sum_{k\geqslant 0}x_{i,k}^+u^{-k}.\end{aligned}$$

Furthermore, set

$$\begin{aligned}f_i^+(u) &= (q_i^{-1}-q_i)x_i^-(u_+q^{-i})^{\geqslant 0}, & e_i^+(u) &= (q_i^{-1}-q_i)x_i^+(u_-q^{-i})^{>0},\\ f_i^-(u) &= (q_i^{-1}-q_i)x_i^-(u_-q^{-i})^{<0}, & e_i^-(u) &= (q_i^{-1}-q_i)x_i^+(u_-q^{-i})^{\leqslant 0}\end{aligned}$$

for $i = 1,\dots,n-1$, and

$$\begin{aligned}f_n^+(u) &= (q_n^{-1}-q_n)x_n^-(u_+q^{-(n+1)})^{\geqslant 0}, & e_n^+(u) &= (q_n^{-1}-q_n)x_n^+(u_-q^{-(n+1)})^{>0},\\ f_n^-(u) &= (q_n-q_n^{-1})x_n^-(u_-q^{-(n+1)})^{<0}, & e_n^-(u) &= (q_n^{-1}-q_n)x_n^+(u_-q^{-(n+1)})^{\leqslant 0}.\end{aligned}$$

Proposition 5.2. *The matrix $F^+(u)$ is lower unitriangular and has the form*

$$F^+(u) = \begin{bmatrix} 1 & & & & & & \\ f_1^+(u) & 1 & & & & & \bigcirc \\ & \ddots & \ddots & & & & \\ & & f_n^+(u) & 1 & & & \\ & & & -f_{n-1}^+(u\xi q^{2(n-1)}) & 1 & & \\ & \bigstar & & & \ddots & \ddots & \\ & & & & & -f_1^+(u\xi q^2) & 1 \end{bmatrix}.$$

Proof. By the construction of the root vectors $E_{\alpha+k\delta}$ and the formulas for the representation π_V, we only need to evaluate the image of the product

$$\prod_{k\geqslant 0} \exp_{q_i}\big((q_i^{-1}-q_i)(uq^{c/2})^k F_{\alpha_i+k\delta}\otimes E_{\alpha_i+k\delta}\big)$$

for simple roots α_i with $i=1,\dots,n$. Due to the isomorphism of Sec. 2.1, we can rewrite it in terms of Drinfeld generators as

$$\prod_{k\geqslant 0} \exp_{q_i}\big((q_i^{-1}-q_i)(uq^{c/2})^k x^-_{i,-k}\otimes x^+_{i,k}\big).$$

Suppose first that $i\leqslant n-1$. Using the formulas for the action of the generators $x^+_{i,k}$ from Proposition 5.1, we get

$$\begin{aligned}(\mathrm{id}\otimes\pi_V)&\prod_{k\geqslant 0} \exp_{q_i}\big((q_i^{-1}-q_i)(uq^{c/2})^k x^-_{i,-k}\otimes x^+_{i,k}\big) \qquad (5.9)\\ =\prod_{k\geqslant 0} &\exp_{q_i}\big((q_i^{-1}-q_i)(u_+q^{-i})^k x^-_{i,-k}\otimes e_{i+1,i}-(q_i^{-1}-q_i)(u_+q^{-(2n+2-i)})^k x^-_{i,-k}\otimes e_{i',(i+1)'}\big).\end{aligned}$$

Expanding the q-exponent, we can write this expression in the form

$$\begin{aligned}1&+(q_i^{-1}-q_i)\sum_{k\geqslant 0} x^-_{i,-k}(u_+q^{-i})^k\otimes e_{i+1,i}-(q_i^{-1}-q_i)\sum_{k\geqslant 0} x^-_{i,-k}(u_+q^{-(2n+2-i)})^k\otimes e_{i',(i+1)'}\\ &=1+(q_i^{-1}-q_i)x^-_i(u_+q^{-i})^{\geqslant 0}\otimes e_{i+1,i}-(q_i^{-1}-q_i)x^-_i(u_+q^{-(2n+2-i)})^{\geqslant 0}\otimes e_{i',(i+1)'}\end{aligned}$$

which coincides with $1+f^+_i(u)\otimes e_{i+1,i}-f^+_i(u\xi q^{2i})\otimes e_{i',(i+1)'}$, as required. A similar calculation shows that expression (5.9) with $i=n$ simplifies to $1+f^+_n(u)\otimes e_{n+1,n}$. □

As in Sec. 2.2, we will assume that the algebra $U_q^{\mathrm{ext}}(\widehat{\mathfrak{sp}}_{2n})$ is extended by adjoining the square roots $k_n^{\pm 1/2}$.

Lemma 5.3. *The image* $(\mathrm{id}\otimes\pi_V)\big(T_{21}\big)$ *is the diagonal matrix*

$$\begin{aligned}\mathrm{diag}\Big[k_1\dots k_{n-1}k_n^{1/2},\quad k_2\dots k_{n-1}k_n^{1/2},\quad \dots,\quad k_n^{1/2},\\ k_n^{-1/2},\quad k_{n-1}^{-1}k_n^{-1/2},\quad \dots,\quad k_1^{-1}\dots k_{n-1}^{-1}k_n^{-1/2}\Big].\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}(\mathrm{id}\otimes\pi_V)\big(T_{21}\big)&=\exp\Big(-\hbar\sum_{b=1}^{n}\sum_{a=1}^{n}\tilde{B}_{ab}h_b\otimes\pi_V(h_a)\Big)\\ &=\exp\Big(-\hbar\sum_{b=1}^{n}\sum_{a=1}^{n-1}\tilde{B}_{ab}h_b\otimes\big(e_{a+1,a+1}-e_{a,a}-e_{(a+1)',(a+1)'}+e_{a',a'}\big)\Big)\\ &\qquad\times\exp\Big(-2\hbar\sum_{b=1}^{n}\tilde{B}_{nb}h_b\otimes\big(e_{n+1,n+1}-e_{n,n}\big)\Big)\end{aligned}$$

which equals

$$\exp\Big(\hbar\sum_{b=1}^{n}\sum_{a=2}^{n-1}(\tilde{B}_{ab}-\tilde{B}_{a-1,b})h_j\otimes e_{a,a}+\hbar\sum_{b=1}^{n}\tilde{B}_{1b}h_b\otimes e_{1,1}$$
$$-\hbar\sum_{b=1}^{n}\sum_{a=2}^{n-1}(\tilde{B}_{ab}-\tilde{B}_{a-1,b})h_b\otimes e_{a',a'}+\hbar\sum_{b=1}^{n}\tilde{B}_{1b}h_b\otimes e_{1',1'}$$
$$+\hbar\sum_{b=1}^{n}(2\tilde{B}_{nb}-\tilde{B}_{n-1,b})h_j\otimes e_{n,n}-\hbar\sum_{b=1}^{n}(2\tilde{B}_{nb}-\tilde{B}_{n-1,b})h_j\otimes e_{n',n'}\Big).$$

The claim now follows by applying formula (5.1) for the entries $\tilde{B}_{ij}$. For instance, the $(1,1)$-entry of the diagonal matrix is found by

$$\exp\Big(\hbar\sum_{b=1}^{n}\tilde{B}_{1b}h_b\Big)=\exp\Big(\hbar\sum_{b=1}^{n-1}h_b+\hbar\frac{1}{2}h_n)\Big)=\prod_{b=1}^{n-1}k_b\,k_n^{1/2},$$

and the remaining entries are obtained by the same calculation. □

Proposition 5.4. *The matrix $E^+(u)$ is upper unitriangular and has the form*

$$E^+(u)=\begin{bmatrix}1 & e_1^+(u) & & & & & & & \\ & 1 & e_2^+(u) & & & & \bigstar & & \\ & & \ddots & \ddots & & & & & \\ & & & 1 & e_n^+(u) & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & 1 & -e_2^+(u\xi q^4) & & \\ & \bigcirc & & & & & 1 & -e_1^+(u\xi q^2) \\ & & & & & & & 1\end{bmatrix}.$$

Proof. It is sufficient to evaluate the image of the product

$$T_{21}^{-1}\prod_{k>0}\exp_{q_i}\big((q_i^{-1}-q_i)z^kq^{kc/2}F_{-\alpha_i+k\delta}\otimes E_{-\alpha_i+k\delta}\big)\,T_{21}$$

with respect to $\mathrm{id}\otimes\pi_V$ for simple roots α_i with $i=1,\dots,n$. Using the isomorphism of Sec. 2.1, we can rewrite the internal product in terms of Drinfeld generators as

$$\prod_{k>0}\exp_{q_i}\big((q_i^{-1}-q_i)(zq^{c/2})^kq^{-kc}x^+_{i,-k}k_i\otimes q^{kc}k_i^{-1}x^-_{i,k}\big).$$

The remaining calculation is performed in the same way as in the proof of Proposition 5.2 with the use of Proposition 5.1, Lemma 5.3 and the relations $k_ix^{\pm}_{j,k}k_i^{-1}=q_i^{\pm A_{ij}}x^{\pm}_{j,k}$. □

In the next proposition we use the series $z^{\pm}(u)$ introduced in (2.17). Their coefficients belong to the center of the algebra $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$; see Proposition 2.2. For a positive integer m with $m<n$ we will denote by $z^{\pm\,[n-m]}(u)$ the respective series for the subalgebra of $U_q^{\text{ext}}(\widehat{\mathfrak{sp}}_{2n})$, whose generators are all elements $X^{\pm}_{i,k}$, $h^{\pm}_{j,k}$ and $q^{c/2}$ such that $i,j\geqslant m+1$; see Definition 2.1. We also denote by $\xi^{[n-m]}$ the parameter ξ for this subalgebra so that $\xi^{[n-m]}=q^{-2n+2m-2}$.

Proposition 5.5. *The matrix $H^+(u)$ is diagonal and has the form*

$$H^+(u)=\operatorname{diag}\big[h_1^+(u),\dots,h_n^+(u),z^{+\,[2]}(u)\,h^+_{n-1}(u\xi^{[2]})^{-1},\dots,z^{+\,[n]}(u)\,h^+_1(u\xi^{[n]})^{-1}\big].$$

Proof. By definition,

$$H^+(u)=\exp\Big(\sum_{k>0}\sum_{i,j=1}^{n}\frac{(q_i^{-1}-q_i)(q_j^{-1}-q_j)}{q^{-1}-q}\,\frac{k}{[k]_q}\tilde{B}_{ij}(q^k)u^kq^{kc/2}a_{j,-k}q^{-kc/2}\otimes q^{kc/2}\pi_V(a_{i,k})\Big)$$
$$\times\,(\mathrm{id}\otimes\pi_V)(T_{21})\prod_{m=0}^{\infty}z^+(u\xi^{-2m-1})z^+(u\xi^{-2m-2})^{-1}.$$

Using the formulas for $\pi_V(a_{i,k})$ from Proposition 5.1, we can write the first factor as the exponent of the expression

$$\sum_{k>0}\sum_{j=1}^{n}\sum_{i=1}^{n-1}(q_j-q_j^{-1})\tilde{B}_{ij}(q^k)u^ka_{j,-k}$$
$$\otimes\big(q^{-(i-1)k}e_{i,i}-q^{-(i+1)k}e_{i+1,i+1}-\xi^kq^{(i-1)k}e_{i',i'}+\xi^kq^{(i+1)k}e_{(i+1)',(i+1)'}\big)$$
$$+\sum_{k>0}\sum_{j=1}^{n}(q_j-q_j^{-1})\tilde{B}_{nj}(q^k)(q^k+q^{-k})u^ka_{j,-k}\otimes\big(q^{-(n-1)k}e_{n,n}-q^{-(n+3)k}e_{n+1,n+1}\big).$$

Consider the $(1,1)$-entry (the coefficient of $e_{1,1}$) in the first factor in the formula for $H^+(u)$. Using formula (5.2) for $\tilde{B}_{1,j}(q^k)$ we get

$$\exp\Big(\sum_{k>0}\sum_{j=1}^{n}(q_j-q_j^{-1})\tilde{B}_{1,j}(q^k)u^ka_{j,-k}\Big)$$
$$=\exp\Big(\sum_{k>0}\sum_{j=1}^{n-1}(q-q^{-1})\frac{q^{jk}+\xi^{-k}q^{-jk}}{1+\xi^{-k}}u^ka_{j,-k}\Big)\exp\Big(\sum_{k>0}(q_n-q_n^{-1})\frac{q^{(n+1)k}}{1+\xi^{-k}}u^ka_{n,-k}\Big).$$

By expanding the fractions into power series, we can write this expression as

$$\exp\Big(\sum_{k>0}\sum_{j=1}^{n-1}\sum_{m=0}^{\infty}(q-q^{-1})(-1)^m\big(\xi^{-mk}q^{jk}+\xi^{-mk-k}q^{-jk}\big)u^ka_{j,-k}\Big)$$
$$\times\exp\Big(\sum_{k>0}\sum_{m=0}^{\infty}(q_n-q_n^{-1})(-1)^m\xi^{-mk}q^{-(n+1)k}u^ka_{n,-k}\Big).$$

Using the definition (1.4) of the series $\varphi_i(u)$ and setting $\tilde{\varphi}_i(u) = k_i\varphi_i(u)$, we can bring the expression to the form

$$\prod_{m=0}^{\infty}\prod_{j=1}^{n-1}\tilde{\varphi}_j(u\xi^{-2m}q^j)^{-1}\tilde{\varphi}_j(u\xi^{-2m-1}q^j)\tilde{\varphi}_j(u\xi^{-2m-1}q^{-j})^{-1}\tilde{\varphi}_j(u\xi^{-2m-2}q^{-j})$$
$$\times\prod_{m=0}^{\infty}\tilde{\varphi}_n(u\xi^{-2m}q^{n+1})^{-1}\tilde{\varphi}_n(u\xi^{-2m-1}q^{n+1}).$$

Setting $\tilde{h}_i^+(u) = t_i^{-1}h_i^+(u)$ with $t_i = h_{i,0}^+$ and applying Proposition 2.3, we can write this as

$$\prod_{m=0}^{\infty}\prod_{j=1}^{n-1}\tilde{h}_j^+(u\xi^{-2m}q^{2j})\tilde{h}_j^+(u\xi^{-2m-1}q^{2j})^{-1}\times\prod_{m=0}^{\infty}\prod_{j=1}^{n}\tilde{h}_j^+(u\xi^{-2m}q^{2j-2})^{-1}\tilde{h}_j^+(u\xi^{-2m-1}q^{2j-2})$$
$$\times\prod_{m=0}^{\infty}\tilde{h}_{n+1}^+(u\xi^{-2m-1})^{-1}\tilde{h}_{n+1}^+(u\xi^{-2m-2})\times\tilde{h}_1^+(u).$$

Now use definition (2.17) of the series $z^{\pm}(u)$ to conclude that

$$\exp\Big(\sum_{k>0}\sum_{j=1}^{n}(q_j-q_j^{-1})\tilde{B}_{1,j}(q^k)u^k a_{j,-k}\Big)=\prod_{m=0}^{\infty}z^+(u\xi^{-2m-1})^{-1}z^+(u\xi^{-2m-2})\times\tilde{h}_1^+(u).$$

Furthermore, Lemma 5.3 implies that the $(1,1)$-entry of the matrix $(\mathrm{id}\otimes\pi_V)(T_{21})$ equals $\prod_{j=1}^{n-1}k_j\times k_n^{1/2}=t_1^{-1}$. This proves that the $(1,1)$-entry of the matrix $H^+(u)$ is $h_1^+(u)$.

It is clear that the matrix $H^+(u)$ is diagonal, and we perform quite similar calculations to evaluate the (i,i)-entries for $i=2,\dots,2n$. For instance, if $i=2,\dots,n-1$ then formula (5.2) for $\tilde{B}_{ij}(q^k)$ implies that the exponent

$$\exp\Big(\sum_{k>0}\sum_{j=1}^{n}(q_j-q_j^{-1})\big(q^{-(i-1)k}\tilde{B}_{ij}(q^k)-q^{-ik}\tilde{B}_{i-1,j}(q^k)\big)u^k a_{j,-k}\Big)$$

can be written in terms of the series $\tilde{\varphi}_i(u)$ as

$$\prod_{m=0}^{\infty}\prod_{j=1}^{i-1}\tilde{\varphi}_j(u\xi^{-2m}q^j)^{-1}\tilde{\varphi}_j(u\xi^{-2m}q^{-j})\tilde{\varphi}_j(u\xi^{-2m-1}q^j)u\tilde{\varphi}_j(\xi^{-2m-1}q^{-j})^{-1}$$
$$\times\prod_{m=0}^{\infty}\prod_{j=i}^{n-1}\tilde{\varphi}_j(u\xi^{-2m}q^j)^{-1}\tilde{\varphi}_j(u\xi^{-2m-1}q^{-j})\tilde{\varphi}_j(u\xi^{-2m-1}q^j)\tilde{\varphi}_j(u\xi^{-2m-2}q^{-j})^{-1}$$
$$\times\prod_{m=0}^{\infty}\tilde{\varphi}_n(u\xi^{-2m}q^{n+1})^{-1}\tilde{\varphi}_n(u\xi^{-2m-1}q^{n+1}).$$

Using the relations $\tilde{\varphi}_j(u) = \tilde{h}_j^+(uq^j)^{-1}\tilde{h}_{j+1}^+(uq^j)^{-1}$ and $\tilde{\varphi}_n(u) = \tilde{h}_n^+(uq^{n+1})^{-1}\tilde{h}_{n+1}^+(uq^{n+1})^{-1}$, we can write this expression in the form

$$\prod_{m=0}^{\infty}\Big(\prod_{j=1}^{n-1}\tilde{h}_j^+(u\xi^{-2m}q^{2j})\prod_{j=1}^{n}\tilde{h}_j^+(u\xi^{-2m}q^{2j-2})^{-1}\times\tilde{h}_n^+(u\xi^{-2m-1})^{-1}\Big)$$
$$\times\prod_{m=0}^{\infty}\Big(\prod_{j=1}^{n-1}\tilde{h}_j^+(u\xi^{-2m-1}q^{2j})^{-1}\prod_{j=1}^{n}\tilde{h}_j^+(u\xi^{-2m-1}q^{2j-2})\times\tilde{h}_{n+1}^+(u\xi^{-2m-2})\Big)\times\tilde{h}_i^+(u)$$

which equals

$$\prod_{m=0}^{\infty}z^+(u\xi^{-2m-1})^{-1}z^+(u\xi^{-2m-2})\times\tilde{h}_i^+(u).$$

Furthermore, by Lemma 5.3, the coefficient of $e_{i,i}$ in the image $(\mathrm{id}\otimes\pi_V)(T_{21})$ coincides with $\prod_{j=i}^{n-1}k_ik_n^{1/2}=t_i^{-1}$. This shows that for $i=2,\dots,n-1$ the (i,i)-entry of the matrix $H_i^+(u)$ is $h_i^+(u)$. We omit the calculations in the remaining cases which are quite similar. □

Now turn to the matrix $L^-(u)$. By definition (5.6) of $L^-(u)$, we have

$$L^-(u)=F^-(u)H^-(u)E^-(u),$$

where

$$E^-(u)=(\mathrm{id}\otimes\pi_V)\mathcal{R}^{>0}(u_+^{-1})^{-1},$$
$$F^-(u)=(\mathrm{id}\otimes\pi_V)\mathcal{R}^{<0}(u_+^{-1})^{-1},$$

and

$$H^-(u)=(\mathrm{id}\otimes\pi_V)(\mathcal{R}^0(u_+^{-1}))^{-1}\prod_{m=0}^{\infty}z^-(u\xi^{2m-1})^{-1}z^-(u\xi^{2m-2}).$$

Proposition 5.6. *The matrix $E^-(u)$ is upper unitriangular and has the form*

$$E^-(u)=\begin{bmatrix}1 & e_1^-(u) & & & & & & \\ & 1 & e_2^-(u) & & & & \bigstar & \\ & & \ddots & \ddots & & & & \\ & & & 1 & e_n^-(u) & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & -e_2^-(u\xi q^4) & \\ & \bigcirc & & & & & 1 & -e_1^-(u\xi q^2)\\ & & & & & & & 1\end{bmatrix}.$$

The matrix $F^-(u)$ is lower unitriangular and has the form

$$F^-(u) = \begin{bmatrix} 1 & & & & & & & & \\ f_1^-(u) & 1 & & & & & \bigcirc & & \\ & \ddots & \ddots & & & & & & \\ & & f_{n-1}^-(u) & 1 & & & & & \\ & & & f_n^-(u) & 1 & & & & \\ & & & & -f_{n-1}^-(u\xi q^{2(n-1)}) & 1 & & & \\ & & \bigstar & & & \ddots & \ddots & & \\ & & & & & & -f_1^-(u\xi q^2) & 1 \end{bmatrix}.$$

The matrix $H^-(u)$ is diagonal and has the form

$$H^-(u) = \operatorname{diag}\big[h_1^-(u), \dots, h_n^-(u), z^{-[2]}(u)\, h_{n-1}^-(u\xi^{[2]})^{-1}, \dots, z^{-[n]}(u)\, h_1^-(u\xi^{[n]})^{-1}\big].$$

Proof. As a first step, use the same arguments as in the proofs of Propositions 5.2, 5.4 and 5.5 to evaluate the matrices

$$E^-(u)^{-1} = (\mathrm{id} \otimes \pi_V)\mathcal{R}^+(u_+^{-1}), \qquad F^-(u)^{-1} = (\mathrm{id} \otimes \pi_V)\mathcal{R}^-(u_+^{-1}),$$

and

$$H^-(u)^{-1} = (\mathrm{id} \otimes \pi_V)(\mathcal{R}^0(u_+^{-1})) \prod_{m=0}^{\infty} z^-(u\xi^{2m-1})\, z^-(u\xi^{2m-2})^{-1}.$$

The required expressions are then obtained by inverting the respective matrices. □

Taking into account Propositions 5.2, 5.4, 5.5 and 5.6 we arrive at the following result.

Corollary 5.7. *The homomorphism*

$$RD : U(R) \to U_q^{\mathrm{ext}}(\widehat{\mathfrak{sp}}_{2n})$$

defined in (5.7) is the inverse map to the homomorphism DR defined in Theorem 4.12. Hence the algebra $U(R)$ is isomorphic to $U_q^{\mathrm{ext}}(\widehat{\mathfrak{sp}}_{2n})$. □

Corollary 5.7 together with the results of Secs 2.2 and 4.5 complete the proof of the Main Theorem.

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