TWISTED STEINBERG ALGEBRAS

BECKY ARMSTRONG, LISA ORLOFF CLARK, KRISTIN COURTNEY, YING-FEN LIN, KATHRYN MCCORMICK, AND JACQUI RAMAGGE

Abstract. We introduce twisted Steinberg algebras, which generalise complex Steinberg algebras and are a purely algebraic notion of Renault's twisted groupoid C*-algebras. In particular, for each ample Hausdorff groupoid G and each locally constant 2-cocycle σ on G taking values in the complex unit circle, we study the complex $*$-algebra $A(G,\sigma)$ consisting of locally constant compactly supported functions on G, with convolution and involution twisted by σ. We also introduce a "discretised" analogue of a twist Σ over a Hausdorff étale groupoid G, and we show that there is a one-to-one correspondence between locally constant 2-cocycles on G and discrete twists over G admitting a continuous global section. Given a discrete twist Σ arising from a locally constant 2-cocycle σ on an ample Hausdorff groupoid G, we construct an associated Steinberg algebra $A(G;\Sigma)$, and we show that it coincides with $A(G,\sigma)$. We also prove a graded uniqueness theorem for $A(G,\sigma)$, and under the additional hypothesis that G is effective, we prove a Cuntz–Krieger uniqueness theorem and show that simplicity of $A(G,\sigma)$ is equivalent to minimality of G.

1. Introduction

Steinberg algebras have become a topic of great interest for algebraists and analysts alike since their independent introduction in [28] and [6]. Before Steinberg algebras were specified by name, they appeared in the details of many groupoid C*-algebra constructions, such as those in [9, 13, 14, 22]. Not only have these algebras provided useful insight into the analytic theory of groupoid C*-algebras, they give rise to interesting examples of $*$-algebras; for example, all Leavitt path algebras and Kumjian–Pask algebras can be realised as Steinberg algebras. Moreover, Steinberg algebras have served as a bridge to facilitate the transfer of concepts and techniques between the algebraic and analytic settings; see [3] for one such case.

Thirty years prior to the introduction of Steinberg algebras, Renault [24] initiated the study of twisted groupoid C*-algebras. These are a generalisation of groupoid C*-algebras in which multiplication and involution are twisted by a $\mathbb{T}$-valued 2-cocycle on the groupoid. Twisted groupoid C*-algebras have since proved extremely valuable in the study of structural properties for large classes of C*-algebras. In particular, work of Renault [25], Tu [29], and Barlak and Li [2] has revealed deep connections between twisted groupoid C*-algebras and the UCT problem from the classification program for C*-algebras. For more work on twisted C*-algebras associated to graphs and groupoids, see [1, 11, 12, 15, 16, 17, 18, 19, 27].

Given the success of non-twisted Steinberg algebras and the far-reaching significance of C*-algebraic results relating to twisted groupoid C*-algebras, we expect that a purely algebraic analogue of twisted groupoid C*-algebras will supply several versatile classes of $*$-algebras to the literature, as well as a new avenue to approach important problems in C*-algebras. In this article, we introduce the notion of a *twisted Steinberg algebra* $A(G,\sigma)$ (or $A_{\mathbb{C}}(G,\sigma)$) constructed from an ample Hausdorff groupoid G and a locally constant

Date: October 29, 2019.
2010 *Mathematics Subject Classification.* 16S99 (primary), 22A22 (secondary).
Key words and phrases. Steinberg algebra, topological groupoid, cohomology, graded algebra.

$\mathbb{T}$-valued 2-cocycle σ on G. Our construction generalises the Steinberg algebra $A_{\mathbb{C}}(G)$, and provides a purely algebraic analogue of the twisted groupoid C*-algebra $C^*(G, \sigma)$.

In the non-twisted setting, the Steinberg algebra and the C*-algebra associated to an ample Hausdorff groupoid G are both built from the convolution algebra $C_c(G)$. As a vector space, $C_c(G)$ denotes the set of continuous compactly supported functions from the groupoid to the complex field $\mathbb{C}$, with pointwise operations. The complex Steinberg algebra $A(G)$ of G is the $*$-subalgebra of $C_c(G)$ consisting of locally constant functions, and the full (or reduced) groupoid C*-algebra $C^*(G)$ (or $C^*_r(G)$) is the closure of $C_c(G)$ with respect to the full (or reduced) C*-norm (see [26, Chapter 3]). It turns out (see [6, Proposition 4.2]) that $A(G)$ sits densely inside of both the full and the reduced C*-algebras. Therefore, the definition of a twisted Steinberg algebra should result in the same inclusions; that is, the twisted, complex, involutive Steinberg algebra should sit $*$-algebraically and densely inside the twisted groupoid C*-algebra. However to even make sense of that goal, one must first choose between two methods of constructing a twisted groupoid C*-algebra. The first involves twisting the multiplication on $C^*(G)$ by a continuous $\mathbb{T}$-valued 2-cocycle, whereas the second involves constructing a C*-algebra from a twist over the groupoid itself.

In [24], Renault observed that the structure of a twisted groupoid C*-algebra with multiplication incorporating a 2-cocycle σ could be realised instead by first twisting the groupoid itself, and then constructing an associated C*-algebra. This is achieved by forming a split groupoid extension

$$G^{(0)} \times \mathbb{T} \hookrightarrow G \times_\sigma \mathbb{T} \twoheadrightarrow G,$$

where multiplication and inversion on the groupoid $G \times_\sigma \mathbb{T}$ both incorporate a $\mathbb{T}$-valued 2-cocycle σ on G, and then defining the twisted groupoid C*-algebra to be the completion of the algebra of $\mathbb{T}$-equivariant functions on $C_c(G \times \mathbb{T})$ under a C*-norm. A few years later, while developing a C*-analogue of Feldman–Moore theory, Kumjian [12] observed the need for a more general construction arising from a locally split groupoid extension

$$G^{(0)} \times \mathbb{T} \hookrightarrow \Sigma \twoheadrightarrow G,$$

where Σ is not necessarily homeomorphic to $G \times \mathbb{T}$. It turns out that when G is a second-countable, ample, Hausdorff groupoid, a folklore result (Theorem 4.10) tells us that every twist over G does arise from a $\mathbb{T}$-valued 2-cocycle on G.

Therefore, our first task is to define twisted Steinberg algebras with respect to both notions of a twist, and then to show that they coincide when these twists are constructed using the same 2-cocycle. This is the focus of Sections 3 and 4. In Section 3, we define the twisted Steinberg algebra $A(G, \sigma)$ by taking an ample Hausdorff groupoid G and twisting the multiplication of the classical Steinberg algebra $A(G)$ using a *locally constant* $\mathbb{T}$-valued 2-cocycle σ on G. We then show that $A(G, \sigma)$ sits densely inside the twisted groupoid C*-algebra $C^*(G, \sigma)$. In Section 4.3, we give an alternative construction of a twisted Steinberg algebra built using a twist over G, and then verify that these two definitions of twisted Steinberg algebras agree when the twist over G arises from a 2-cocycle.

However, in order to construct a twisted Steinberg algebra using a twist over a groupoid, we are forced to first "discretise" our groupoid extension by replacing the standard topology on $\mathbb{T}$ with the discrete topology. Though this may seem a little artificial to a C*-algebraist, this change is indeed necessary, as we explain in Remarks 4.20. (Nonetheless, this should not come as too much of a surprise, given the purely algebraic nature of Steinberg algebras.) Thus, Section 4.1 is dedicated to introducing these discretised groupoid twists and establishing in this setting the aforementioned folklore result (Theorem 4.10). Then in Section 4.2, we flesh out the relationships between these twists over groupoids and the cohomology theory of groupoids.

Section 5 provides several examples of twisted Steinberg algebras, including a notion of *twisted Kumjian–Pask algebras*. The final two sections of the paper are devoted to proving several important results in Steinberg algebras in the twisted setting. In Section 6 we prove a twisted version of the Cuntz–Krieger uniqueness theorem for effective groupoids (Theorem 6.1), and we show that when G is effective, simplicity of $A(G,\sigma)$ is equivalent to minimality of G (Theorem 6.2). Finally, in Section 7, we show that twisted Steinberg algebras inherit a graded structure from the underlying groupoid, and we prove a graded uniqueness theorem for twisted Steinberg algebras (Theorem 7.2).

In [28], and in much of the related literature, Steinberg algebras are defined more generally by replacing the set $\mathbb{C}$ of scalars with a unital ring R (which may not have an involution). Because our inspiration comes from twisted groupoid C*-algebras, we have chosen to focus on the setting where the ring of scalars for the algebra is $\mathbb{C}$. However, if R is a unital ring with involution $r \mapsto \overline{r} \in R$ such that $\overline{r} = r^{-1}$ for every unit $r \in R^\times \subseteq R$, then we expect that all of the results of Section 3 and much of Section 4 will still hold when $\mathbb{C}$ is replaced by R and $\mathbb{T}$ is replaced by $R^\times$.

2. Preliminaries

In this section we introduce some notation, and we recall relevant background information on topological groupoids, continuous 2-cocycles, and twisted groupoid C*-algebras. Throughout this article, G will always be a locally compact Hausdorff topological groupoid with unit space $G^{(0)}$, composable pairs $G^{(2)} \subseteq G \times G$, and range and source maps $r, s\colon G \to G^{(0)}$. We will refer to such groupoids as *Hausdorff groupoids*. We evaluate composition of groupoid elements from right to left, which means that $\gamma\gamma^{-1} = r(\gamma)$ and $\gamma^{-1}\gamma = s(\gamma)$, for all $\gamma \in G$. We write $G^{(3)}$ for the set of composable triples in G; that is,

$$G^{(3)} \coloneqq \{(\alpha,\beta,\gamma) : (\alpha,\beta),\, (\beta,\gamma) \in G^{(2)}\}.$$

For each $x \in G^{(0)}$, we define

$$G_x \coloneqq s^{-1}(x), \quad G^x \coloneqq r^{-1}(x), \quad \text{and} \quad G_x^x \coloneqq G_x \cap G^x.$$

For any two subsets U and V of a groupoid G, we define

$$U\,{}_s{\times}_r\, V \coloneqq (U \times V) \cap G^{(2)}, \;\; UV \coloneqq \{\alpha\beta : (\alpha,\beta) \in U\,{}_s{\times}_r\, V\}, \;\text{ and }\; U^{-1} \coloneqq \{\alpha^{-1} : \alpha \in U\}.$$

We call a subset B of G a *bisection* if there exists an open subset U of G such that $B \subseteq U$, and $r|_U$ and $s|_U$ are homeomorphisms onto open subsets of G. We say that G is *étale* if r (or, equivalently, s) is a local homeomorphism. If G is étale, then $G^{(0)}$ is open, and both G_x and G^x are discrete in the subspace topology for any $x \in G^{(0)}$. We recall that G is étale if and only if G has a basis of open bisections. We say that G is *ample* if it has a basis of *compact* open bisections. If G is étale, then G is ample if and only if its unit space $G^{(0)}$ is totally disconnected (see [10, Proposition 4.1]).

If B and D are compact open bisections of an ample Hausdorff groupoid, then B^{-1} and BD are also compact open bisections. In fact, the collection of compact open bisections forms an inverse semigroup under these operations (see [22, Proposition 2.2.4]).

The *isotropy* of a groupoid G is the set

$$\mathrm{Iso}(G) \coloneqq \{\gamma \in G : r(\gamma) = s(\gamma)\} = \bigcup_{x \in G^{(0)}} G_x^x.$$

We say that G is *principal* if $\mathrm{Iso}(G) = G^{(0)}$, and that G is *effective* if the topological interior of $\mathrm{Iso}(G)$ is equal to $G^{(0)}$. We say that G is *topologically principal* if the set $\{x \in G^{(0)} : G_x^x = \{x\}\}$ is dense in $G^{(0)}$. Every principal étale groupoid is effective and topologically principal. If G is a Hausdorff étale groupoid, then G is effective if it is topologically principal, and the converse holds if G is additionally second-countable (see

[3, Lemma 3.1]). We will often work with Hausdorff groupoids that are étale, ample, or second-countable, but we will explicitly state these assumptions.

Before we describe algebras of functions defined on a groupoid, a few remarks on preliminary point-set topology and notation are in order. Given topological spaces X and Y, a function $f\colon X \to Y$ is said to be *locally constant* if every element of X has an open neighbourhood U such that $f|_U$ is constant. Every locally constant function is continuous (because the preimage of every singleton set under a locally constant function is open); moreover, if Y has the discrete topology, then every continuous function $f\colon X \to Y$ is locally constant. We write $\mathbb{C}_d$ for the set of complex numbers endowed with the discrete topology, and $\mathbb{T}_d$ for the complex unit circle endowed with the discrete topology. We will frequently view locally constant $\mathbb{C}$-valued (or $\mathbb{T}$-valued) functions as continuous functions taking values in $\mathbb{C}_d$ (or $\mathbb{T}_d$).

Given a complex-valued function f on a topological space X, we define the *support* of f to be the set

$$\operatorname{supp}(f) \coloneqq \{x \in X : f(x) \neq 0\}.$$

If f is continuous, then its support is open, because $\operatorname{supp}(f) = f^{-1}(\mathbb{C}\backslash\{0\})$. If f is locally constant, then its support is clopen, because $\operatorname{supp}(f) = f^{-1}(\mathbb{C}_d\backslash\{0\})$. If $\overline{\operatorname{supp}(f)}$ is compact, then we say that f is *compactly supported*.

As motivation for our definition of a twisted Steinberg algebra, it will be helpful to briefly recall the construction of groupoid C*-algebras and Steinberg algebras, and to describe the ways in which twisted groupoid C*-algebras have been defined in the literature.

We begin by describing groupoid C*-algebras, which were introduced by Renault in [24]. In the discussion that follows, it will suffice to restrict our attention to the setting in which the underlying Hausdorff groupoid G is second-countable and étale. Although the étale assumption is not required, this setting is general enough to include a plethora of examples, including the Cuntz–Krieger algebras of all compactly aligned topological higher-rank graphs (see [30, Theorem 3.16]).

Given a second-countable Hausdorff étale groupoid G, the *convolution algebra* $C_c(G)$ is the complex $*$-algebra

$$C_c(G) \coloneqq \{f\colon G \to \mathbb{C} \,:\, f \text{ is continuous and } \overline{\operatorname{supp}(f)} \text{ is compact}\},$$

equipped with multiplication given by the *convolution product*

$$(f * g)(\gamma) \coloneqq \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} f(\alpha)\, g(\beta) = \sum_{\eta\in G^{s(\gamma)}} f(\gamma\eta)\, g(\eta^{-1}),$$

and involution given by $f^*(\gamma) \coloneqq \overline{f(\gamma^{-1})}$. The *full groupoid* C*-*algebra* $C^*(G)$ is defined to be the completion of $C_c(G)$ in the *full* C*-*norm*, and the *reduced groupoid* C*-*algebra* $C_r^*(G)$ is defined to be the completion of $C_c(G)$ in the *reduced* C*-*norm* (see [26, Chapter 3] for the details).

The first conception of a *twisted* groupoid C*-algebra was also introduced by Renault in [24]. In this setting, the "twist" refers to a continuous $\mathbb{T}$-valued 2-cocycle on G, which is incorporated into the definitions of the multiplication and involution of the convolution algebra $C_c(G)$. A 2-*cocycle* is a continuous function $\sigma\colon G^{(2)} \to \mathbb{T}$ that satisfies the 2-*cocycle identity*:

$$\sigma(\alpha,\beta)\,\sigma(\alpha\beta,\gamma) = \sigma(\alpha,\beta\gamma)\,\sigma(\beta,\gamma),$$

for all $(\alpha,\beta,\gamma) \in G^{(3)}$, and is *normalised*, in the sense that

$$\sigma(r(\gamma),\gamma) = 1 = \sigma(\gamma, s(\gamma)),$$

for all $\gamma \in G$. We say that the 2-cocycles $\sigma, \tau\colon G^{(2)} \to \mathbb{T}$ are *cohomologous* if there is a continuous function $b\colon G \to \mathbb{T}$ such that $b(x) = 1$ for all $x \in G^{(0)}$, and

$$\sigma(\alpha,\beta)\,\overline{\tau(\alpha,\beta)} = b(\alpha)\,b(\beta)\,\overline{b(\alpha\beta)},$$

for all $(\alpha,\beta) \in G^{(2)}$. Cohomology of continuous 2-cocycles on G is an equivalence relation. The equivalence class of a continuous 2-cocycle σ under this relation is called its *cohomology class*.

Given a 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}$, the *twisted convolution algebra* $C_c(G,\sigma)$ is the complex $*$-algebra that is equal as a vector space to $C_c(G)$, but has multiplication given by the *twisted convolution product*

$$(f * g)(\gamma) := \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \sigma(\alpha,\beta)\,f(\alpha)\,g(\beta) = \sum_{\eta\in G^{s(\gamma)}} \sigma(\gamma\eta,\eta^{-1})\,f(\gamma\eta)\,g(\eta^{-1}),$$

and involution given by

$$f^*(\gamma) := \overline{\sigma(\gamma,\gamma^{-1})}\,\overline{f(\gamma^{-1})}.$$

The 2-cocycle identity guarantees that the multiplication is associative, and the assumption that the 2-cocycle is normalised implies that the twist is trivial when either multiplying or applying the involution to functions supported on $G^{(0)}$. The *full twisted groupoid* C*-*algebra* $C^*(G,\sigma)$ is defined to be the completion of $C_c(G,\sigma)$ in the *full* C*-*norm*, and the *reduced twisted groupoid* C*-*algebra* $C_r^*(G,\sigma)$ is defined to be the completion of $C_c(G,\sigma)$ in the *reduced* C*-*norm* (see [24, Chapter II.1] for the details). There is also a $*$-algebra norm on $C_c(G,\sigma)$, called the *I-norm*, which is given by

$$\|f\|_{I,\sigma} := \max\Big\{ \sup_{u\in G^{(0)}} \Big\{ \sum_{\gamma\in G^u} |f(\gamma)| \Big\},\ \sup_{u\in G^{(0)}} \Big\{ \sum_{\gamma\in G_u} |f(\gamma)| \Big\} \Big\},$$

for all $f \in C_c(G,\sigma)$. The I-norm dominates the full norm on $C_c(G,\sigma)$.

Renault [24] also introduced an alternative construction of these twisted groupoid C*-algebras involving twisting the groupoid itself, via a split groupoid extension

$$G^{(0)} \times \mathbb{T} \hookrightarrow G \times_\sigma \mathbb{T} \twoheadrightarrow G,$$

called a *twist* over G. In 1986, Kumjian generalised this construction to give twisted groupoid C*-algebras whose twists are not induced by $\mathbb{T}$-valued 2-cocycles. In particular, the extension Σ of G by $G^{(0)} \times \mathbb{T}$ need not admit a continuous global section $P\colon G \to \Sigma$. In Section 4.1 we develop a "discretised" version of this more general notion of a twist. Since our definition is almost identical to Kumjian's (with the difference being the choice of topology on $\mathbb{T}$), we refer the reader to Definition 4.1 for a more precise definition of a twist over a Hausdorff étale groupoid. Given a twist

$$G^{(0)} \times \mathbb{T} \hookrightarrow \Sigma \twoheadrightarrow G,$$

over a Hausdorff étale groupoid G, one defines $C_c(\Sigma)$ with (untwisted) convolution and involution. The completion of the $*$-subalgebra of $C_c(\Sigma)$ consisting of $\mathbb{T}$-equivariant functions with respect to the full (or reduced) C*-norm yields the full (or reduced) twisted groupoid C* algebra $C^*(G,\Sigma)$ (or $C_r^*(G,\Sigma)$). (See [25] or [26, Chapter 5] for more details.)

We conclude this section with the definition of Steinberg algebras, which were originally introduced in [28, 6], and are a purely algebraic analogue of groupoid C*-algebras. Let G be an *ample* Hausdorff groupoid and let 1_B denote the characteristic function of B from G to $\mathbb{C}$. The *(complex) Steinberg algebra* associated to G is the complex $*$-algebra

$$\begin{aligned} A_{\mathbb{C}}(G) &:= \operatorname{span}\{1_B\colon G \to \mathbb{C} : B \text{ is a compact open bisection of } G\} \\ &= \{f\colon G \to \mathbb{C} : f \text{ is locally constant and } \operatorname{supp}(f) \text{ is compact}\}, \end{aligned}$$

equipped with multiplication given by the *convolution product*

$$(f * g)(\gamma) := \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} f(\alpha)\, g(\beta) = \sum_{\eta\in G^{s(\gamma)}} f(\gamma\eta)\, g(\eta^{-1}),$$

and involution given by $f^*(\gamma) := \overline{f(\gamma^{-1})}$. The Steinberg algebra $A(G) := A_{\mathbb{C}}(G)$ is dense in $C_c(G)$ with respect to the full and reduced C*-norms (as shown in [28, 6]).

3. Twisted Steinberg algebras arising from locally constant 2-cocycles

In this section we introduce the twisted complex Steinberg algebra $A(G,\sigma)$ (or $A_{\mathbb{C}}(G,\sigma)$) associated to an ample Hausdorff groupoid G and a continuous 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}_d$. As a vector space, the twisted Steinberg algebra is identical to the untwisted version defined in Section 2. That is

$$A(G,\sigma) := \operatorname{span}\{1_B\colon G \to \mathbb{C}_d \,:\, B \text{ is a compact open bisection of } G\};$$

we now emphasise that we are viewing $\mathbb{C}$ with the discrete topology.

Lemma 3.1. *Let G be an ample Hausdorff groupoid. Let $C_c(G,\mathbb{C}_d)$ denote the collection of continuous, compactly supported functions $f\colon G \to \mathbb{C}_d$. For any continuous 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}_d$, we have the following:*

(a) *$A(G,\sigma) = C_c(G,\mathbb{C}_d) = \{f \in C_c(G) : f$ is locally constant$\}$ as vector spaces; and*
(b) *for any $f \in A(G,\sigma)$, there exist $\lambda_1,\dots,\lambda_n \in \mathbb{C}\backslash\{0\}$ and mutually disjoint compact open bisections $B_1,\dots,B_n \subseteq G$ such that $f = \sum_{i=1}^n \lambda_i 1_{B_i}$.*

Proof. Part (a) follows from the characterisations of the (untwisted, complex) Steinberg algebra $A(G)$ given in [6, Definition 3.2 and Lemma 3.3], because $A(G,\sigma)$ and $A(G)$ agree as sets. Similarly, part (b) follows from [6, Lemma 3.5]. □

From now on, we will use the characterisations of $A(G,\sigma)$ given in Lemma 3.1 interchangeably with the definition.

As a vector space, $A(G,\sigma)$ is identical to the usual (complex) Steinberg algebra $A(G)$ introduced in [28, 6]. However, we equip $A(G,\sigma)$ with a multiplication and involution that both incorporate the 2-cocycle σ into their definitions, thereby distinguishing $A(G,\sigma)$ from $A(G)$.

Proposition 3.2. *Let G be an ample Hausdorff groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. There is a multiplication (called* (twisted) convolution*) on $A(G,\sigma)$ given by*

$$(f * g)(\gamma) := \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \sigma(\alpha,\beta)\, f(\alpha)\, g(\beta) = \sum_{\eta\in G^{s(\gamma)}} \sigma(\gamma\eta,\eta^{-1})\, f(\gamma\eta)\, g(\eta^{-1}),$$

and an involution given by

$$f^*(\gamma) := \overline{\sigma(\gamma,\gamma^{-1})}\,\overline{f(\gamma^{-1})}.$$

Under these operations, along with pointwise addition and scalar multiplication, $A(G,\sigma)$ is a dense $*$-subalgebra of the twisted convolution algebra $C_c(G,\sigma)$ with respect to the I-norm, and hence also with respect to the full and reduced C-norms.*

We call $A(G,\sigma)$ the twisted Steinberg algebra *associated to the pair (G,σ).*

Remarks 3.3.

(1) If the 2-cocycle σ is trivial (in the sense that $\sigma\big(G^{(2)}\big) = \{1\}$), then $A(G,\sigma)$ is identical to $A(G)$ as a complex $*$-algebra.
(2) We often write fg to denote the convolution product $f * g$ of functions $f,g \in A(G,\sigma)$ if the intended meaning is clear.

(3) If $f, g \in A(G,\sigma)$, then $\operatorname{supp}(fg) \subseteq \operatorname{supp}(f)\operatorname{supp}(g)$. If B and D are compact open bisections of G such that $\operatorname{supp}(f) = B$ and $\operatorname{supp}(g) = D$, then $\operatorname{supp}(fg) = BD$ and $\operatorname{supp}(f^*) = B^{-1}$.

(4) From the 2-cocycle identity, one can readily verify that $\sigma(\gamma, \gamma^{-1}) = \sigma(\gamma^{-1}, \gamma)$ for any $\gamma \in G$.

Proof of Proposition 3.2. Since $A(G,\sigma)$ and $A(G)$ agree as vector spaces, it follows from [22, Proposition 2.2.7] that $A(G,\sigma)$ is dense in $C_c(G,\sigma)$ with respect to the I-norm. We know from [24, Proposition II.1.1] that $C_c(G,\sigma)$ is a $*$-algebra, and so to see that $A(G,\sigma)$ is a $*$-algebra, it suffices to show that $A(G,\sigma)$ is closed under the twisted convolution and involution.

Fix $f, g \in A(G,\sigma)$. By Lemma 3.1(b), there exist mutually disjoint compact open bisections $B_1, \dots, B_m, C_1, \dots, C_n \subseteq G$ and scalars $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n \in \mathbb{C}\backslash\{0\}$ such that

$$f = \sum_{i=1}^{m} \lambda_i 1_{B_i} \quad \text{and} \quad g = \sum_{j=1}^{n} \mu_j 1_{C_j}.$$

We claim that $fg \in A(G,\sigma)$. Since $0 \notin \sigma\big(G^{(2)}\big)$, [26, Proposition 3.1.1] implies that for each $\gamma \in G$, the set

$$\big\{(\alpha,\beta) \in G^{(2)} : \alpha\beta = \gamma \text{ and } \sigma(\alpha,\beta)\, f(\alpha)\, g(\beta) \neq 0\big\}$$

is finite. Since σ is locally constant, we can assume that for all $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, there exists $\nu_{i,j} \in \mathbb{T}_d$ such that $\sigma(\alpha,\beta) = \nu_{i,j}$ for all $(\alpha,\beta) \in (B_i)_s{\times}_r(C_j)$ (because otherwise we can further refine the bisections to ensure that this is true). Thus, for all $\gamma \in G$, we have

$$\begin{aligned}
(fg)(\gamma) &= \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \sigma(\alpha,\beta)\, f(\alpha)\, g(\beta) \\
&= \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \sigma(\alpha,\beta) \left(\sum_{i=1}^{m} \lambda_i 1_{B_i}(\alpha)\right)\left(\sum_{j=1}^{n} \mu_j 1_{C_j}(\beta)\right) \\
&= \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \sum_{i=1}^{m}\sum_{j=1}^{n} \nu_{i,j}\, \lambda_i\, \mu_j\, 1_{B_i}(\alpha)\, 1_{C_j}(\beta) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \nu_{i,j}\, \lambda_i\, \mu_j\, 1_{B_iC_j}(\gamma).
\end{aligned}$$

Hence $fg \in A(G,\sigma)$.

We now show that $f^* \in A(G,\sigma)$. Since σ is locally constant, we can assume that for all $i \in \{1,\dots,m\}$, there exists $\kappa_i \in \mathbb{T}_d$ such that $\sigma(\gamma,\gamma^{-1}) = \kappa_i$ for all $\gamma \in B_i$ (because otherwise we can further refine the bisections to ensure that this is true). Thus, for all $\gamma \in G$, we have

$$f^*(\gamma) \;=\; \overline{\sigma(\gamma,\gamma^{-1})}\,\overline{f(\gamma^{-1})} \;=\; \overline{\sigma(\gamma,\gamma^{-1})}\,\overline{\left(\sum_{i=1}^{m} \lambda_i\, 1_{B_i}(\gamma^{-1})\right)} \;=\; \sum_{i=1}^{m} \overline{\kappa_i}\,\overline{\lambda_i}\, 1_{B_i^{-1}}(\gamma).$$

Hence $f^* \in A(G,\sigma)$. □

Note that we used that σ is locally constant in order to show that $A(G,\sigma)$ is closed under the twisted convolution and involution.

In the untwisted Steinberg algebra setting, given compact open bisections B and D of G, we have $1_B 1_D = 1_{BD}$. This is not the case in the twisted setting, due to the presence of the 2-cocycle in the convolution formula. Instead, we have the following properties concerning the generators 1_B of the twisted Steinberg algebra $A(G,\sigma)$.

Lemma 3.4. *Let G be an ample Hausdorff groupoid, $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle, and B and D be compact open bisections of G.*

(a) *For all $(\alpha,\beta) \in B\,{}_s\!\times_r D$, we have*

$$(1_B 1_D)(\alpha\beta) = \sigma(\alpha,\beta)\,1_B(\alpha)\,1_D(\beta) = \sigma(\alpha,\beta)\,1_{BD}(\alpha\beta) = \sigma(\alpha,\beta).$$

(b) *If $B \subseteq G^{(0)}$ or $D \subseteq G^{(0)}$, then $1_B 1_D = 1_{BD}$.*

(c) *For all $\gamma \in G$, we have $1_B^*(\gamma) = \overline{\sigma(\gamma,\gamma^{-1})}\,1_{B^{-1}}(\gamma)$.*

(d) *We have $1_B 1_B^* = 1_{r(B)}$ and $1_B^* 1_B = 1_{s(B)}$.*

(e) *We have $1_B 1_B^* 1_B = 1_B$ and $1_B^* 1_B 1_B^* = 1_B^*$.*

Proof. (a) This follows immediately from the definition of the twisted convolution product because B and D are bisections.

(b) Suppose that $B \subseteq G^{(0)}$ or $D \subseteq G^{(0)}$, and fix $\gamma \in G$. If $\gamma \in BD$, then $\gamma = \alpha\beta$ for some pair $(\alpha,\beta) \in B\,{}_s\!\times_r D$. Since σ is normalised, we have $\sigma(\alpha,\beta) = 1$, and so

$$(1_B 1_D)(\gamma) = \sigma(\alpha,\beta)\,1_B(\alpha)\,1_D(\beta) = 1_B(\alpha)\,1_D(\beta) = 1_{BD}(\gamma).$$

If $\gamma \notin BD$, then $(1_B 1_D)(\gamma) = 0 = 1_{BD}(\gamma)$. Thus $1_B 1_D = 1_{BD}$.

(c) If $\gamma \in B^{-1}$, then we have

$$1_B^*(\gamma) = \overline{\sigma(\gamma,\gamma^{-1})}\,\overline{1_B(\gamma^{-1})} = \overline{\sigma(\gamma,\gamma^{-1})}\,1_{B^{-1}}(\gamma).$$

If $\gamma \notin B^{-1} = \operatorname{supp}(1_B^*)$, then

$$1_B^*(\gamma) = 0 = 1_{B^{-1}}(\gamma) = \overline{\sigma(\gamma,\gamma^{-1})}\,1_{B^{-1}}(\gamma).$$

(d) We know that $\operatorname{supp}(1_B 1_B^*) = BB^{-1} = r(B)$, and for all $\gamma \in B$, we have

$$\begin{aligned}(1_B 1_B^*)\big(r(\gamma)\big) &= (1_B 1_B^*)(\gamma\gamma^{-1})\\ &= \sigma(\gamma,\gamma^{-1})\,1_B(\gamma)\,1_B^*(\gamma^{-1})\\ &= \sigma(\gamma,\gamma^{-1})\,1_B(\gamma)\,\overline{\sigma(\gamma^{-1},\gamma)}\,1_{B^{-1}}(\gamma^{-1}) \qquad \text{(using part (c))}\\ &= 1\\ &= 1_{r(B)}\big(r(\gamma)\big).\end{aligned}$$

Similarly, we have $\operatorname{supp}(1_B^* 1_B) = B^{-1}B = s(B)$, and so for all $\gamma \in B$, we have

$$\begin{aligned}(1_B^* 1_B)\big(s(\gamma)\big) &= (1_B^* 1_B)(\gamma^{-1}\gamma)\\ &= \sigma(\gamma^{-1},\gamma)\,1_B^*(\gamma^{-1})\,1_B(\gamma)\\ &= \sigma(\gamma^{-1},\gamma)\,\overline{\sigma(\gamma^{-1},\gamma)}\,1_{B^{-1}}(\gamma^{-1})\,1_B(\gamma) \qquad \text{(using part (c))}\\ &= 1\\ &= 1_{s(B)}\big(s(\gamma)\big).\end{aligned}$$

(e) Parts (b) and (d) imply that

$$1_B 1_B^* 1_B = 1_{r(B)} 1_B = 1_{r(B)B} = 1_B, \quad \text{and} \quad 1_B^* 1_B 1_B^* = 1_{s(B)} 1_B^*.$$

Hence $\operatorname{supp}(1_B^* 1_B 1_B^*) = s(B)B^{-1} = B^{-1}$. For all $\gamma \in B$, we have

$$(1_B^* 1_B 1_B^*)(\gamma^{-1}) = \sigma(s(\gamma),\gamma^{-1})\,1_{s(B)}\big(s(\gamma)\big)\,1_B^*(\gamma^{-1}) = 1_B^*(\gamma^{-1}),$$

and so $1_B^* 1_B 1_B^* = 1_B^*$. □

The proof of the following result is inspired by the proof of [24, Proposition II.1.2].

Lemma 3.5. *Let G be an ample Hausdorff groupoid, and $\sigma, \tau\colon G^{(2)} \to \mathbb{T}_d$ be two continuous 2-cocycles whose cohomology classes coincide. Then $A(G,\sigma)$ is $*$-isomorphic to $A(G,\tau)$.*

Proof. For this proof, we will use $*$ to denote convolution, in order to distinguish it from the pointwise product.

Since σ is cohomologous to τ, there is a continuous function $b\colon G \to \mathbb{T}_d$ such that $b(x) = 1$ for all $x \in G^{(0)}$, and

$$\sigma(\alpha,\beta)\,\overline{\tau(\alpha,\beta)} = b(\alpha)\,b(\beta)\,\overline{b(\alpha\beta)}, \tag{3.1}$$

for all $(\alpha,\beta) \in G^{(2)}$.

For each $f \in A(G,\sigma) = C_c(G,\mathbb{C}_d)$, let $\phi(f)$ denote the pointwise product bf. Since $bf\colon G \to \mathbb{C}_d$ is continuous and satisfies $\operatorname{supp}(bf) = \operatorname{supp}(f)$, we have $bf \in C_c(G,\mathbb{C}_d) = A(G,\tau)$. We claim that $\phi\colon A(G,\sigma) \to A(G,\tau)$ is a $*$-isomorphism. It is clear that ϕ is linear, so we must show that it respects the twisted convolution and involution.

For all $\gamma \in G$, letting $\alpha = \gamma$ and $\beta = \gamma^{-1}$ in Equation (3.1) gives

$$\sigma(\gamma,\gamma^{-1})\,\overline{\tau(\gamma,\gamma^{-1})} = b(\gamma)\,b(\gamma^{-1})\,\overline{b(\gamma\gamma^{-1})} = b(\gamma)\,b(\gamma^{-1}),$$

and hence

$$b(\gamma)\,\overline{\sigma(\gamma,\gamma^{-1})} = \overline{\tau(\gamma,\gamma^{-1})}\,\overline{b(\gamma^{-1})}. \tag{3.2}$$

Thus, for all $f \in A(G,\sigma)$ and $\gamma \in G$, we have

$$\begin{aligned}
\phi(f^*)(\gamma) &= b(\gamma)\,f^*(\gamma)\\
&= b(\gamma)\,\overline{\sigma(\gamma,\gamma^{-1})}\,\overline{f(\gamma^{-1})}\\
&= \overline{\tau(\gamma,\gamma^{-1})}\,\overline{b(\gamma^{-1})}\,\overline{f(\gamma^{-1})} \qquad \text{(using Equation (3.2))}\\
&= (bf)^*(\gamma)\\
&= \phi(f)^*(\gamma).
\end{aligned}$$

For all $(\alpha,\beta) \in G^{(2)}$, Equation (3.1) implies that

$$\sigma(\alpha,\beta)\,b(\alpha\beta) = \tau(\alpha,\beta)\,b(\alpha)\,b(\beta). \tag{3.3}$$

Hence, for all $f, g \in A(G,\sigma)$ and $\gamma \in G$, we have

$$\begin{aligned}
\big(\phi(f) * \phi(g)\big)(\gamma) &= \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \tau(\alpha,\beta)\,\phi(f)(\alpha)\,\phi(g)(\beta)\\
&= \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \tau(\alpha,\beta)\,b(\alpha) f(\alpha)\,b(\beta)\,g(\beta)\\
&= \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \sigma(\alpha,\beta)\,b(\alpha\beta)\,f(\alpha)\,g(\beta) \qquad \text{(using Equation (3.3))}\\
&= b(\gamma) \sum_{\substack{(\alpha,\beta)\in G^{(2)},\\ \alpha\beta=\gamma}} \sigma(\alpha,\beta)\,f(\alpha)\,g(\beta)\\
&= \big(b(f*g)\big)(\gamma)\\
&= \phi(f*g)(\gamma).
\end{aligned}$$

Therefore, ϕ is a $*$-homomorphism.

We now show that ϕ is a bijection. For each $h \in A(G, \tau)$, we have $\bar{b}h \in A(G, \sigma)$, and so $\phi(\bar{b}h) = b\bar{b}h = h$. Hence ϕ is surjective. To see that ϕ is injective, suppose that $f, g \in A(G, \sigma)$ satisfy $\phi(f) = \phi(g)$. Then $f = \bar{b}bf = \bar{b}\phi(f) = \bar{b}\phi(g) = \bar{b}bg = g$. Therefore, ϕ is a $*$-isomorphism. □

Proposition 3.6. *Let G be an ample Hausdorff groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. The set*

$$\{1_B\colon G \to \mathbb{C}_d : B \text{ is a nonempty compact open subset of } G^{(0)}\}$$

forms a local unit for $A(G, \sigma)$. That is, for any finite collection $f_1, \dots, f_n \in A(G, \sigma)$, there exists a compact open subset E of $G^{(0)}$ such that

$$1_E\, f_i = f_i = f_i\, 1_E,$$

for each $i \in \{1, \dots, n\}$.

Proof. Since multiplication by 1_E for $E \subseteq G^{(0)}$ is not affected by the 2-cocycle, this follows from the analogous non-twisted result [5, Lemma 2.6]. □

4. Twisted Steinberg algebras arising from discrete twists

There is another (often more general) notion of a twisted groupoid C*-algebra which is constructed from a "twist" over the groupoid itself; that is, an algebra built from a locally split groupoid extension of an ample Hausdorff groupoid G by $G^{(0)} \times \mathbb{T}$. In this section, we define a discretised analogue of this twist and its associated twisted Steinberg algebra. The primary modification is to replace the standard topology on $\mathbb{T}$ with the discrete topology. Many of the results in Section 4.1 and Section 4.2 have roots or inspiration in Kumjian's study of groupoid C*-algebras built from groupoid extensions in [12].

The results in Section 4.1 and Section 4.2 also hold in the non-discrete setting with the same proofs. Replacing $\mathbb{T}_d$ with $\mathbb{T}$ will not change any of the algebraic arguments therein, and the topological arguments carry through mutatis mutandis. As our ultimate focus is algebraic, we present all of our results in terms of $\mathbb{T}_d$.

4.1. Discrete twists over Hausdorff étale groupoids.

The definition of a *twist* over a Hausdorff étale groupoid, which we refer to as a *classical twist*, can be found in [26, Definition 5.1.1]. The following is our discretised version.

Definition 4.1. Let G be a Hausdorff étale groupoid. A *discrete twist* over G is a sequence

$$G^{(0)} \times \mathbb{T}_d \overset{i}{\hookrightarrow} \Sigma \overset{q}{\twoheadrightarrow} G,$$

where the groupoid $G^{(0)} \times \mathbb{T}_d$ is regarded as a trivial group bundle with fibres $\mathbb{T}_d$, Σ is a Hausdorff groupoid with $\Sigma^{(0)} = i\big(G^{(0)} \times \{1\}\big)$, and i and q are continuous groupoid homomorphisms that restrict to homeomorphisms of unit spaces, such that the following conditions hold.

(a) The sequence is exact, in the sense that $i(\{x\} \times \mathbb{T}_d) = q^{-1}(x)$ for every $x \in G^{(0)}$, i is injective, and q is surjective.[1]

(b) The groupoid Σ is a locally trivial G-bundle, in the sense that for each $\alpha \in G$, there is an open bisection B_α of G containing α, and a continuous map $P_\alpha\colon B_\alpha \to \Sigma$ such that
  (i) $q \circ P_\alpha = \mathrm{id}_{B_\alpha}$;
  (ii) $P_\alpha(G^{(0)} \cap B_\alpha) \subseteq \Sigma^{(0)}$; and

[1] Although it is not explicitly stated in [26, Definition 5.1.1] that the groupoid homomorphism $q\colon \Sigma \to G$ is surjective and satisfies $q(i(x, z)) = x$ for every $(x, z) \in G^{(0)} \times \mathbb{T}_d$, it is implicitly assumed.

(iii) the map $(\beta, z) \mapsto i(r(\beta), z)\, P_\alpha(\beta)$ is a homeomorphism from $B_\alpha \times \mathbb{T}_d$ to $q^{-1}(B_\alpha)$.

(c) The image of i is *central* in Σ, in the sense that $i(r(\epsilon), z)\, \epsilon = \epsilon\, i(s(\epsilon), z)$ for all $\epsilon \in \Sigma$ and $z \in \mathbb{T}_d$.

We denote a discrete twist over G either by (Σ, i, q), or simply by Σ. We identify $\Sigma^{(0)}$ with $G^{(0)}$ via i. A continuous map $P_\alpha \colon B_\alpha \to \Sigma$ is called a *(continuous) local section* if it satisfies parts (i) and (ii) of condition (b). A *(classical) twist* over G has the same definition as above, with the exception that $\mathbb{T}_d$ is replaced by $\mathbb{T}$.

In brief, we think of a twist over G as a locally split extension Σ of G by $G^{(0)} \times \mathbb{T}_d$, where the image of $G^{(0)} \times \mathbb{T}_d$ is central in Σ. If G is ample, then the open bisections from condition (b) can be chosen to be compact.

Example 4.2. If G is a discrete group, then a twist over G as defined above is a central extension of G.

The following result is an immediate consequence of Definition 4.1.

Lemma 4.3. *Let G be a Hausdorff étale groupoid, and (Σ, i, q) be a discrete twist over G. Then i is a homeomorphism onto its image.*

Proof. Since i is injective and continuous by definition, we need only show that i is an open map. Fix open sets $U \subseteq G^{(0)}$ and $W \subseteq \mathbb{T}_d$. For each $x \in U$, condition (b)(iii) of Definition 4.1 implies that there is an open bisection B_x of G containing x, and a homeomorphism $\psi_x \colon B_x \times \mathbb{T}_d \to q^{-1}(B_x)$ given by $\psi_x(\beta, z) \coloneqq i(r(\beta), z)\, P_x(\beta)$. In particular, for each $y \in B_x \cap G^{(0)}$ and $z \in \mathbb{T}_d$, we have $\psi_x(y, z) = i(y, z)$, since $P_x(y) \in \Sigma^{(0)}$. Therefore,

$$i(U \times W) = \bigcup_{x \in U} \psi_x\big((B_x \cap U) \times W\big),$$

which is an open subset of Σ, because each ψ_x is a homeomorphism onto the open set $q^{-1}(B_x)$, and each $B_x \cap U$ is open. □

We define a notion of an isomorphism of discrete twists in an analogous way to the non-discrete version.

Definition 4.4. We say that two twists (Σ, i, q) and (Σ', i', q') over a Hausdorff étale groupoid G are *isomorphic* if there exists a groupoid isomorphism[2] $\phi \colon \Sigma \to \Sigma'$ that is equivariant for i' and q'; or, equivalently, if the following diagram commutes.

$$\begin{array}{ccccc} G^{(0)} \times \mathbb{T}_d & \xrightarrow{\ i\ } & \Sigma & \xrightarrow{\ q\ } & G \\ \| & & \downarrow{\scriptstyle \phi} & & \| \\ G^{(0)} \times \mathbb{T}_d & \xrightarrow{\ i'\ } & \Sigma' & \xrightarrow{\ q'\ } & G \end{array}$$

It is natural to ask whether there is a correspondence between twists over a groupoid and locally constant 2-cocycles which can be used to "twist" the multiplication in Steinberg algebras, given the shared terminology. As one familiar with the literature would expect, we can readily build a twist over a Hausdorff étale groupoid from a locally constant 2-cocycle. To demonstrate this, we adapt the construction outlined in [26, Example 5.1.5] to the setting where the continuous 2-cocycle maps into $\mathbb{T}_d$ (rather than $\mathbb{T}$), which is equivalent to insisting that the 2-cocycle is locally constant.

[2]We say that $\phi \colon \Sigma \to \Sigma'$ is a *groupoid isomorphism* if it is a homeomorphism such that $\phi(\delta\epsilon) = \phi(\delta)\phi(\epsilon)$ for all $(\delta, \epsilon) \in \Sigma^{(2)}$.

Example 4.5. Let G be a Hausdorff étale groupoid, and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. Let $G \times_\sigma \mathbb{T}_d$ be the set $G \times \mathbb{T}_d$ endowed with the product topology, with multiplication given by

$$(\alpha, z)(\beta, w) := (\alpha\beta,\, \sigma(\alpha, \beta)\, zw),$$

and inversion given by

$$(\alpha, z)^{-1} := (\alpha^{-1},\, \overline{\sigma(\alpha, \alpha^{-1})}\, \overline{z}) = (\alpha^{-1},\, \overline{\sigma(\alpha^{-1}, \alpha)}\, \overline{z}),$$

for all $(\alpha, \beta) \in G^{(2)}$ and $z, w \in \mathbb{T}_d$. Then $G \times_\sigma \mathbb{T}_d$ is a Hausdorff groupoid. In fact, unlike in the classical setting, G being étale implies that $G \times_\sigma \mathbb{T}_d$ is étale, because for each $z \in \mathbb{T}_d$ and bisection U of G, $r|_{U\times\{z\}}$ is a homeomorphism onto $r(U) \times \{1\}$. Define $i\colon G^{(0)} \times \mathbb{T}_d \to G \times_\sigma \mathbb{T}_d$ by $i(x, z) := (x, z)$, and $q\colon G \times_\sigma \mathbb{T}_d \to G$ by $q(\gamma, z) := \gamma$. Then q is easily verified to be a surjective groupoid homomorphism, and since σ is normalised, i is an injective groupoid homomorphism. Just as in [26, Example 5.1.5], it is routine to then check that $(G \times_\sigma \mathbb{T}_d, i, q)$ is a discrete twist over G.

Example 4.5 shows that any locally constant 2-cocycle on a Hausdorff étale groupoid G gives rise to a discrete twist over G. According to folklore, the converse is true when G is additionally second-countable and ample. The proof of this fact and its consequences will be the focus of the remainder of this subsection.

Before we proceed, we need two technical results regarding the left and right group actions of $\mathbb{T}_d$ on Σ that are induced by the map $i\colon G^{(0)} \times \mathbb{T}_d \to \Sigma$. Identifying $\Sigma^{(0)}$ with $G^{(0)}$, these actions are given by

$$z \cdot \epsilon := i(r(\epsilon), z)\, \epsilon \quad \text{ and } \quad \epsilon \cdot z := \epsilon\, i(s(\epsilon), z),$$

for each $z \in \mathbb{T}_d$ and $\epsilon \in \Sigma$. Since the image of i is central in Σ, we have $z \cdot \epsilon = \epsilon \cdot z$, and $(z \cdot \epsilon)(w \cdot \delta) = (zw) \cdot (\epsilon\delta)$ for all $(\epsilon, \delta) \in \Sigma^{(2)}$ and $z, w \in \mathbb{T}_d$.

Lemma 4.6. *Let G be a Hausdorff étale groupoid. Suppose that (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) are discrete twists over G, and $\phi\colon \Sigma_1 \to \Sigma_2$ is an isomorphism of twists, as defined in Definition 4.4. Then ϕ respects the action of $\mathbb{T}_d$, in the sense that $\phi(z \cdot \epsilon) = z \cdot \phi(\epsilon)$, for all $z \in \mathbb{T}_d$ and $\epsilon \in \Sigma_1$.*

Proof. Since $\phi\colon \Sigma_1 \to \Sigma_2$ is an isomorphism of twists, we have $i_2 = \phi \circ i_1$. Thus, for all $z \in \mathbb{T}_d$ and $\epsilon \in \Sigma_1$, we have

$$\phi(z \cdot \epsilon) = \phi\big(i_1(r(\epsilon), z)\, \epsilon\big) = i_2(r(\epsilon), z)\, \phi(\epsilon) = z \cdot \phi(\epsilon). \qquad \square$$

The following result is inspired by [26, Proposition 5.1.3].

Lemma 4.7. *Let G be a Hausdorff étale groupoid. Suppose that (Σ, i, q) is a discrete twist over G, and $\delta, \epsilon \in \Sigma$ satisfy $q(\delta) = q(\epsilon)$. Then $r(\delta) = r(\epsilon)$, and there is a unique $z \in \mathbb{T}_d$ such that $\epsilon = z \cdot \delta$.*

Proof. Fix $\delta, \epsilon \in \Sigma$ such that $q(\delta) = q(\epsilon)$. Then $q(r(\delta)) = r(q(\delta)) = r(q(\epsilon)) = q(r(\epsilon))$, and hence $r(\delta) = r(\epsilon)$, because q restricts to a homeomorphism of unit spaces. Thus $q(\epsilon\delta^{-1}) = q(\epsilon)q(\epsilon)^{-1} = r(q(\epsilon)) \in G^{(0)}$, so there is a unique element $z \in \mathbb{T}_d$ such that $\epsilon\delta^{-1} = i\big(r(q(\epsilon)), z\big)$. By identifying $\Sigma^{(0)}$ with $G^{(0)}$, we obtain $\epsilon = i(r(\epsilon), z)\, \delta = z \cdot \delta$. $\square$

Notice that in the case where Σ is the twist $G \times_\sigma \mathbb{T}_d$ described in Example 4.5, we can check Lemma 4.7 directly. Identifying $\Sigma^{(0)} = G^{(0)} \times \{1\}$ with $G^{(0)}$, we have

$$z \cdot (\alpha, w) = i(r(\alpha), z)(\alpha, w) = (r(\alpha), z)(\alpha, w) = (\alpha, zw),$$

for all $z \in \mathbb{T}_d$ and $(\alpha, w) \in \Sigma$. If $q(\delta) = q(\epsilon)$ for some $\delta, \epsilon \in \Sigma$, then $\delta = (\alpha, w_1)$ and $\epsilon = (\alpha, w_2)$ for some $\alpha \in G$ and unique $w_1, w_2 \in \mathbb{T}_d$. Clearly there is a unique $z \in \mathbb{T}_d$ such that $zw_1 = w_2$, and hence $z \cdot \delta = (\alpha, zw_1) = \epsilon$.

Our key tool in what follows will be a *(continuous) global section*; that is, a continuous map $P\colon G \to \Sigma$, such that $q \circ P = \mathrm{id}_G$ and $P(G^{(0)}) \subseteq \Sigma^{(0)} = i(G^{(0)} \times \{1\})$. Our next result shows that every discrete twist admitting a continuous global section is isomorphic to a twist coming from a locally constant 2-cocycle, as described in Example 4.5. Parts of this result are inspired by the analogous non-discrete versions in [12, Section 4] and [26, Chapter 5].

Proposition 4.8. *Let G be a Hausdorff étale groupoid, and (Σ, i, q) be a discrete twist over G. Suppose that Σ is* topologically trivial, *in the sense that it admits a continuous global section $P\colon G \to \Sigma$. Then the following conditions hold.*

(a) *The continuous global section P preserves composability, and induces a continuous 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}_d$ satisfying*
$$P(\alpha)P(\beta)P(\alpha\beta)^{-1} = i\big(r(\alpha),\, \sigma(\alpha,\beta)\big),$$
for all $(\alpha,\beta) \in G^{(2)}$.

(b) *For all $(\alpha,\beta) \in G^{(2)}$, we have*
$$P(\alpha)P(\beta) = \sigma(\alpha,\beta)\cdot P(\alpha\beta) \quad \text{and} \quad P(\alpha)^{-1} = \overline{\sigma(\alpha,\alpha^{-1})}\cdot P(\alpha^{-1}).$$

(c) *Let $(G \times_\sigma \mathbb{T}_d, i_\sigma, q_\sigma)$ be the twist from Example 4.5. The map $\phi_P\colon G\times_\sigma \mathbb{T}_d \to \Sigma$ defined by $\phi_P(\alpha,z) := z\cdot P(\alpha)$ gives an isomorphism of the twists $G\times_\sigma\mathbb{T}_d$ and Σ.*

Proof. For (a), fix $(\alpha,\beta)\in G^{(2)}$. Since $q\circ P = \mathrm{id}_G$ and q is a groupoid homomorphism that restricts to a homeomorphism of unit spaces, we have
$$q(s(P(\alpha))) = s(q(P(\alpha))) = s(\alpha) = r(\beta) = r(q(P(\beta))) = q(r(P(\beta))),$$
and hence $(P(\alpha), P(\beta)) \in \Sigma^{(2)}$. We have
$$q\big(P(\alpha)P(\beta)P(\alpha\beta)^{-1}\big) = q(P(\alpha))\, q(P(\beta))\, q(P(\alpha\beta))^{-1} = r(\alpha) = q\big(P(r(\alpha))\big),$$
and so Lemma 4.7 implies that there is a unique value $\sigma(\alpha,\beta)\in\mathbb{T}_d$ such that
$$P(\alpha)P(\beta)P(\alpha\beta)^{-1} = \sigma(\alpha,\beta)\cdot P(r(\alpha)) = i\big(r(\alpha),\,\sigma(\alpha,\beta)\big). \tag{4.1}$$
Therefore, $\sigma(\alpha,\beta) = (\pi_2\circ i^{-1})\big(P(\alpha)P(\beta)P(\alpha\beta)^{-1}\big)$, where π_2 is the projection of $G^{(0)}\times\mathbb{T}_d$ onto the second coordinate. Noting that i is an open map by Lemma 4.3, we deduce that σ is continuous because it is a composition of continuous functions.

To check that σ satisfies the 2-cocycle identity, we fix $(\alpha,\beta,\gamma)\in G^{(3)}$ and show that
$$\sigma(\beta,\gamma) = \sigma(\alpha,\beta)\,\sigma(\alpha\beta,\gamma)\,\overline{\sigma(\alpha,\beta\gamma)}.$$
Since the image of i is central in Σ, we have
$$i\big(r(\alpha),\,\sigma(\beta,\gamma)\big)\,P(\alpha) = P(\alpha)\,i\big(s(\alpha),\,\sigma(\beta,\gamma)\big) = P(\alpha)\,i\big(r(\beta),\,\sigma(\beta,\gamma)\big). \tag{4.2}$$
Using Equation (4.2) for the first equality below and Equation (4.1) for the second and fourth equalities, we obtain
$$\begin{aligned}
i\big(r(\alpha),\,\sigma(\beta,\gamma)\big) &= P(\alpha)\,i\big(r(\beta),\,\sigma(\beta,\gamma)\big)\,P(\alpha)^{-1}\\
&= P(\alpha)P(\beta)P(\gamma)P(\beta\gamma)^{-1}P(\alpha)^{-1}\\
&= \big(P(\alpha)P(\beta)P(\alpha\beta)^{-1}\big)\big(P(\alpha\beta)P(\gamma)P(\alpha\beta\gamma)^{-1}\big)\big(P(\alpha\beta\gamma)P(\beta\gamma)^{-1}P(\alpha)^{-1}\big)\\
&= i\big(r(\alpha),\,\sigma(\alpha,\beta)\big)\,i\big(r(\alpha\beta),\,\sigma(\alpha\beta,\gamma)\big)\,i\big(r(\alpha),\,\sigma(\alpha,\beta\gamma)\big)^{-1}\\
&= i\big(r(\alpha),\,\sigma(\alpha,\beta)\,\sigma(\alpha\beta,\gamma)\,\overline{\sigma(\alpha,\beta\gamma)}\big).
\end{aligned}$$
Thus, by the injectivity of i, we deduce that σ satisfies the 2-cocycle identity.

To see that σ is normalised, first note that for all α in G,
$$q\big(i\big(r(\alpha),\,\sigma(r(\alpha),\alpha)\big)\big) = q\big(i\big(r(\alpha),\,\sigma(\alpha,s(\alpha))\big)\big) = q\big(i(r(\alpha),1)\big) = r(\alpha), \tag{4.3}$$

and $i(r(\alpha), 1) \in \Sigma^{(0)}$. Moreover, by Equation (4.1), we have

$$i\big(r(\alpha),\, \sigma(r(\alpha), \alpha)\big) = P(r(\alpha))P(\alpha)P(r(\alpha)\alpha)^{-1} = P(r(\alpha)) \in \Sigma^{(0)},$$

and, since $P(s(\alpha)) \in \Sigma^{(0)}$,

$$i\big(r(\alpha),\, \sigma(\alpha, s(\alpha))\big) = P(\alpha)P(s(\alpha))P(\alpha s(\alpha))^{-1} = P(\alpha)P(\alpha)^{-1} = r(P(\alpha)) \in \Sigma^{(0)}.$$

Since q restricts to a homeomorphism of unit spaces and i is injective, we deduce from Equation (4.3) that

$$\sigma(r(\alpha), \alpha) = \sigma(\alpha, s(\alpha)) = 1,$$

for all $\alpha \in G$.

For (b), fix $(\alpha, \beta) \in G^{(2)}$. Then Equation (4.1) implies that

$$P(\alpha)P(\beta) = i\big(r(\alpha\beta),\, \sigma(\alpha, \beta)\big)\, P(\alpha\beta) = \sigma(\alpha, \beta) \cdot P(\alpha\beta),$$

and also that

$$P(\alpha)P(\alpha^{-1})P(\alpha\alpha^{-1})^{-1} = i\big(r(\alpha),\, \sigma(\alpha, \alpha^{-1})\big).$$

Since $P(\alpha\alpha^{-1})^{-1} = P(r(\alpha)) \in \Sigma^{(0)}$, we deduce that

$$P(\alpha)^{-1} = P(\alpha^{-1})\, i\big(r(\alpha),\, \sigma(\alpha, \alpha^{-1})\big)^{-1} = P(\alpha^{-1}) \cdot \overline{\sigma(\alpha, \alpha^{-1})} = \overline{\sigma(\alpha, \alpha^{-1})} \cdot P(\alpha^{-1}).$$

For (c), define $\phi_P \colon G \times_\sigma \mathbb{T}_d \to \Sigma$ by $\phi_P(\alpha, z) \coloneqq z \cdot P(\alpha) = i(r(\alpha), z)\, P(\alpha)$. Then ϕ_P is continuous, because it is the pointwise product of the continuous maps $i \circ (r \times \mathrm{id})$ and $P \circ \pi_1$ from $G \times_\sigma \mathbb{T}_d$ to Σ, where π_1 is the projection of $G \times_\sigma \mathbb{T}_d$ onto the first coordinate. To see that ϕ_P is injective, suppose that $(\alpha, z), (\beta, w) \in G^{(2)}$ satisfy $\phi_P(\alpha, z) = \phi_P(\beta, w)$. Then

$$\alpha = q(i(r(\alpha), z))\, q(P(\alpha)) = q(\phi_P(\alpha, z)) = q(\phi_P(\beta, w)) = q(i(r(\beta), w))\, q(P(\beta)) = \beta.$$

Therefore,

$$i(r(\alpha), z) = \phi_P(\alpha, z)\, P(\alpha)^{-1} = \phi_P(\beta, w)\, P(\alpha)^{-1} = i(r(\beta), w)\, P(\beta)\, P(\alpha)^{-1} = i(r(\alpha), w),$$

and since i is injective, we have $z = w$. Thus ϕ_P is injective. To see that ϕ_P is surjective, fix $\epsilon \in \Sigma$. Then $q(\epsilon) = q\big(P(q(\epsilon))\big)$, and so by Lemma 4.7, there exists a unique $z_\epsilon \in \mathbb{T}_d$ such that

$$\phi_P\big(P(q(\epsilon)), z_\epsilon\big) = z_\epsilon \cdot P(q(\epsilon)) = i(r(\epsilon), z_\epsilon)\, P(q(\epsilon)) = \epsilon.$$

Thus ϕ_P is surjective, and we have $z_\epsilon = \pi_2\big(i^{-1}(\epsilon\, P(q(\epsilon))^{-1})\big)$, where π_2 is the projection of $G^{(0)} \times \mathbb{T}_d$ onto $\mathbb{T}_d$. Since $\phi_P^{-1}(\epsilon) = \big(P(q(\epsilon)), z_\epsilon\big)$ and Lemma 4.3 implies that i^{-1} is continuous, we deduce that ϕ_P^{-1} is continuous, because it is a composition of continuous maps. Hence ϕ_P is a homeomorphism.

To see that ϕ_P is also a homomorphism, fix $(\alpha, \beta) \in G^{(2)}$ and $z, w \in \mathbb{T}_d$. Then, using part (b) for the third equality, we have

$$\begin{aligned}
\phi_P(\alpha, z)\, \phi_P(\beta, w) &= (z \cdot P(\alpha))(w \cdot P(\beta)) \\
&= (zw) \cdot (P(\alpha)P(\beta)) \\
&= (zw) \cdot \big(\sigma(\alpha, \beta) \cdot P(\alpha\beta)\big) \\
&= \big(\sigma(\alpha, \beta) zw\big) \cdot P(\alpha\beta) \\
&= \phi_P\big(\alpha\beta,\, \sigma(\alpha, \beta) zw\big) \\
&= \phi_P\big((\alpha, z)(\beta, w)\big).
\end{aligned}$$

Hence ϕ_P is a groupoid isomorphism.

Recall from Example 4.5 that $i_\sigma\colon G^{(0)} \times \mathbb{T}_d \to G \times_\sigma \mathbb{T}_d$ is the inclusion map and $q_\sigma\colon G \times_\sigma \mathbb{T}_d \to G$ is the projection onto the first coordinate. Fix $\alpha \in G$ and $w \in \mathbb{T}_d$. Then

$$(\phi_P \circ i_\sigma)(r(\alpha), w) = \phi_P(r(\alpha), w) = i(r(\alpha), w)\, P(r(\alpha)) = i(r(\alpha), w)\, i(r(\alpha), 1) = i(r(\alpha), w),$$

and

$$(q \circ \phi_P)(\alpha, w) = q\big(i(r(\alpha), w)\, P(\alpha)\big) = r(\alpha)\alpha = \alpha = q_\sigma(\alpha, w).$$

Therefore, Σ and $G \times_\sigma \mathbb{T}_d$ are isomorphic as twists over G. □

As one might expect, all twists constructed from locally constant 2-cocycles (as in Example 4.5) are topologically trivial, as we now prove.

Lemma 4.9. *Let G be a Hausdorff étale groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. The twist $(G \times_\sigma \mathbb{T}_d, i, q)$ described in Example 4.5 is topologically trivial, and the map $S\colon \gamma \mapsto (\gamma, 1)$ is a continuous global section from G to $G \times_\sigma \mathbb{T}_d$ that induces σ.*

Proof. It is clear that S is a continuous global section, and so $G \times_\sigma \mathbb{T}_d$ is topologically trivial. By Proposition 4.8, S induces a 2-cocycle $\omega\colon G^{(2)} \to \mathbb{T}_d$ satisfying

$$S(\alpha)S(\beta)S(\alpha\beta)^{-1} = i\big(r(\alpha),\, \omega(\alpha, \beta)\big) = (r(\alpha),\, \omega(\alpha, \beta)),$$

for all $(\alpha, \beta) \in G^{(2)}$. To see that S induces σ, fix $(\alpha, \beta) \in G^{(2)}$. Then

$$\begin{aligned}(r(\alpha),\, \omega(\alpha, \beta)) &= S(\alpha)S(\beta)S(\alpha\beta)^{-1}\\ &= (\alpha, 1)(\beta, 1)(\alpha\beta, 1)^{-1}\\ &= (\alpha\beta,\, \sigma(\alpha, \beta))\,\big((\alpha\beta)^{-1},\, \overline{\sigma(\alpha\beta, (\alpha\beta)^{-1})}\big)\\ &= \big(r(\alpha\beta),\, \sigma(\alpha\beta, (\alpha\beta)^{-1})\, \sigma(\alpha, \beta)\, \overline{\sigma(\alpha\beta, (\alpha\beta)^{-1})}\big)\\ &= (r(\alpha),\, \sigma(\alpha, \beta)).\end{aligned}$$

Therefore, $\sigma = \omega$, and so S induces σ. □

Together, Proposition 4.8 and Lemma 4.9 give us a one-to-one correspondence between discrete twists over a Hausdorff étale groupoid G which admit a continuous global section and twists over G arising from locally constant 2-cocycles on G.

As we shall see in Theorem 4.10, it turns out that all twists over a second-countable, ample, Hausdorff groupoid G admit a continuous global section. We are grateful to Elizabeth Gillaspy for alerting us to this folklore fact, citing conversations with Alex Kumjian. Because we know of no proofs in the literature, we give a detailed proof here in the discrete setting.

Theorem 4.10. *Let G be a second-countable, ample, Hausdorff groupoid, and (Σ, i, q) be a discrete twist over G. Then Σ is topologically trivial.*

In order to prove Theorem 4.10, we need the following lemma.

Lemma 4.11. *Let G be a second-countable, ample, Hausdorff groupoid, and suppose that $\mathcal{U}$ is an open cover of G. Then $\mathcal{U}$ has a countable refinement $\{B_j\}_{j=1}^\infty$ of mutually disjoint compact open bisections that form a cover of G.*

Proof. Let $\mathcal{U}$ be an open cover of G. By possibly passing to a refinement, we may assume that $\mathcal{U}$ consists of compact open bisections. Since G is second-countable, it is Lindelöf, and so we may assume that $\mathcal{U} = \{D_j\}_{j=1}^\infty$, where each D_j is a compact open bisection of G. Define $B_1 := D_1$, and for each $n \ge 2$, define $B_n := D_n \setminus \bigcup_{i=1}^{n-1} B_i$. Then each B_j is a compact open bisection contained in D_j, and $\{B_j\}_{i=j}^\infty$ forms a disjoint cover of G. □

Proof of Theorem 4.10. Since Σ is a twist over the ample groupoid G, for each $\alpha \in G$, there exists a compact open bisection $D_\alpha \subseteq G$ and a continuous local section $P_\alpha \colon D_\alpha \to \Sigma$ such that the map $\phi_\alpha \colon D_\alpha \times \mathbb{T}_d \to q^{-1}(D_\alpha)$ given by $\phi_\alpha(\beta, z) := i(r(\beta), z)\, P_\alpha(\beta) = z \cdot P_\alpha(\beta)$ is a homeomorphism. By Lemma 4.11, $\{D_\alpha\}_{\alpha \in G}$ has a countable refinement $\{B_j\}_{j=1}^\infty$ consisting of mutually disjoint compact open bisections that form a cover of G. For each $j \ge 1$, choose $\alpha_j \in G$ such that $B_j \subseteq D_{\alpha_j}$, and define $P_j := P_{\alpha_j}|_{B_j}$. For each $\beta \in G$, there is a unique $j_\beta \ge 1$ such that $\beta \in B_{j_\beta}$, and hence the map $P \colon G \to \Sigma$ given by $P(\beta) := P_{j_\beta}(\beta)$ is well-defined. Since $q(P(\beta)) = q(P_{j_\beta}(\beta)) = \beta = \mathrm{id}_G(\beta)$ for all $\beta \in G$, and $P_j(G^{(0)} \cap B_j) \subseteq \Sigma^{(0)}$ for each $j \ge 1$, P is a global section. We claim that P is continuous. Let U be an open subset of Σ. Then $P^{-1}(U) = \cup_{j=1}^\infty P_j^{-1}(U) = \cup_{j=1}^\infty \big(P_{\alpha_j}^{-1}(U) \cap B_j\big)$. Since each P_{α_j} is continuous and each B_j is open, $P^{-1}(U)$ is open in G. Hence P is a continuous global section, and Σ is topologically trivial. □

4.2. **Twists and 2-cocycles.** In this section we restrict our attention to twists arising from locally constant 2-cocycles, and we investigate the relationships between such twists. In particular, we prove the following theorem.

Theorem 4.12. *Let G be a Hausdorff étale groupoid, and $\sigma, \tau \colon G^{(2)} \to \mathbb{T}_d$ be continuous 2-cocycles. The following are equivalent:*

(1) $G \times_\sigma \mathbb{T}_d \cong G \times_\tau \mathbb{T}_d$;
(2) *σ is cohomologous to τ; and*
(3) *σ is induced by a continuous global section $P \colon G \to G \times_\tau \mathbb{T}_d$.*

We will split the proof of this theorem up into three lemmas. This proof has notable overlap with [12, Section 4], particularly the equivalence of (2) and (3). However, the two formulations are sufficiently different to warrant independent treatment here.

The following lemma expands on an argument given in [26, Remark 5.1.6] showing that the cohomology class of a continuous 2-cocycle $\sigma \colon G^{(2)} \to \mathbb{T}_d$ can always be recovered from the twist $G \times_\sigma \mathbb{T}_d$.

Lemma 4.13. *Let G be a Hausdorff étale groupoid and $\tau \colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. Suppose that $P \colon G \to G \times_\tau \mathbb{T}_d$ is a continuous global section, and $\sigma \colon G^{(2)} \to \mathbb{T}_d$ is the induced continuous 2-cocycle satisfying*

$$i\big(r(\alpha),\, \sigma(\alpha, \beta)\big) = P(\alpha)P(\beta)P(\alpha\beta)^{-1}$$

for all $(\alpha, \beta) \in G^{(2)}$, as in Proposition 4.8. Then σ is cohomologous to τ.

Proof. To see that σ is cohomologous to τ, we will find a continuous function $b \colon G \to \mathbb{T}_d$ such that $b(x) = 1$ for all $x \in G^{(0)}$, and

$$\sigma(\alpha, \beta) = \tau(\alpha, \beta)\, b(\alpha)\, b(\beta)\, \overline{b(\alpha\beta)}$$

for all $(\alpha, \beta) \in G^{(2)}$. For each $\gamma \in G$, let $b(\gamma)$ be the unique element of $\mathbb{T}_d$ such that $P(\gamma) = (\gamma, b(\gamma))$. Since $P(G^{(0)}) \subseteq G^{(0)} \times \{1\}$, we have $b(x) = 1$ for all $x \in G^{(0)}$. Since $b = \pi_2 \circ P$, where π_2 is the projection of $G \times_\tau \mathbb{T}_d$ onto the second coordinate, b is continuous. For all $(\alpha, \beta) \in G^{(2)}$, we have

$$\begin{aligned} i\big(r(\alpha),\, \sigma(\alpha, \beta)\big) &= P(\alpha)P(\beta)P(\alpha\beta)^{-1} \\ &= (\alpha, b(\alpha))\, (\beta, b(\beta))\, (\alpha\beta, b(\alpha\beta))^{-1} \\ &= \big(\alpha\beta,\, \tau(\alpha, \beta)\, b(\alpha)\, b(\beta)\big)\big((\alpha\beta)^{-1},\, \overline{\tau(\alpha\beta, (\alpha\beta)^{-1})}\, \overline{b(\alpha\beta)}\big) \\ &= \big(r(\alpha\beta),\, \tau(\alpha\beta, (\alpha\beta)^{-1})\, \tau(\alpha, \beta)\, b(\alpha)\, b(\beta)\, \overline{\tau(\alpha\beta, (\alpha\beta)^{-1})}\, \overline{b(\alpha\beta)}\big) \\ &= \big(r(\alpha),\, \tau(\alpha, \beta)\, b(\alpha)\, b(\beta)\, \overline{b(\alpha\beta)}\big). \end{aligned}$$

Thus, noting that $i\colon G^{(0)} \times \mathbb{T}_d \to G \times_\sigma \mathbb{T}_d$ is the inclusion map, we deduce that
$$\sigma(\alpha, \beta) = \tau(\alpha, \beta)\, b(\alpha)\, b(\beta)\, \overline{b(\alpha\beta)},$$
as required □

We now show that cohomologous locally constant 2-cocycles give rise to isomorphic twists.

Lemma 4.14. *Let G be a Hausdorff étale groupoid and $\sigma, \tau\colon G^{(2)} \to \mathbb{T}_d$ be continuous 2-cocycles. If σ is cohomologous to τ, then the discrete twists $G \times_\sigma \mathbb{T}_d$ and $G \times_\tau \mathbb{T}_d$ are isomorphic.*

Proof. If σ is cohomologous to τ, then there is a continuous function $b\colon G \to \mathbb{T}_d$ such that $b(x) = 1$ for all $x \in G^{(0)}$, and
$$b(\alpha\beta)\, \sigma(\alpha, \beta) = \tau(\alpha, \beta)\, b(\alpha)\, b(\beta) \tag{4.4}$$
for all $(\alpha, \beta) \in G^{(2)}$. Define $\phi\colon G \times_\sigma \mathbb{T}_d \to G \times_\tau \mathbb{T}_d$ by $\phi(\alpha, z) \coloneqq (\alpha, b(\alpha)z)$. Then ϕ is bijective, with inverse given by $\phi^{-1}(\alpha, z) \coloneqq (\alpha, \overline{b(\alpha)}z)$. Since $\phi(\alpha, z) = (r(\alpha), b(\alpha))(\alpha, z)$, ϕ is continuous, because it is the pointwise product of the continuous map $(r \times b) \circ \pi_1$ and the identity map, where π_1 is the projection of $G \times_\sigma \mathbb{T}_d$ onto the first coordinate. A similar argument shows that ϕ^{-1} is continuous, and thus ϕ is a homeomorphism.

To see that ϕ is a homomorphism, fix $(\alpha, \beta) \in G^{(2)}$ and $z, w \in \mathbb{T}_d$. Using Equation (4.4) for the third equality, we obtain
$$\begin{aligned}
\phi((\alpha, z)(\beta, w)) &= \phi(\alpha\beta,\, \sigma(\alpha, \beta)\, zw) \\
&= (\alpha\beta,\, b(\alpha\beta)\, \sigma(\alpha, \beta)\, zw) \\
&= (\alpha\beta,\, \tau(\alpha, \beta)\, b(\alpha)\, b(\beta)\, zw) \\
&= (\alpha, b(\alpha)z)\, (\beta, b(\beta)w) \\
&= \phi(\alpha, z)\, \phi(\beta, w),
\end{aligned}$$
as required.

Let $i_\sigma\colon G^{(0)} \times \mathbb{T}_d \to G \times_\sigma \mathbb{T}_d$ and $i_\tau\colon G^{(0)} \times \mathbb{T}_d \to G \times_\tau \mathbb{T}_d$ be the inclusion maps, and $q_\sigma\colon G \times_\sigma \mathbb{T}_d \to G$ and $q_\tau\colon G \times_\tau \mathbb{T}_d \to G$ be the projections onto the first coordinate. Since $b(x) = 1$ for all $x \in G^{(0)}$, we have
$$\phi(i_\sigma(x, z)) = (x, b(x)z) = (x, z) = i_\tau(x, z),$$
and
$$q_\tau(\phi(\alpha, z)) = q_\tau(\alpha, b(\alpha)z) = \alpha = q_\sigma(\alpha),$$
for all $x \in G^{(0)}$, $\alpha \in G$, and $z \in \mathbb{T}_d$. Therefore, ϕ is an isomorphism of the twists $G \times_\sigma \mathbb{T}_d$ and $G \times_\tau \mathbb{T}_d$. □

Finally, we show that if σ and τ are locally constant 2-cocycles on G giving rise to isomorphic twists $G \times_\sigma \mathbb{T}_d$ and $G \times_\tau \mathbb{T}_d$, then $G \times_\tau \mathbb{T}_d$ admits a continuous global section that induces σ.

Lemma 4.15. *Let G be a Hausdorff étale groupoid and $\sigma, \tau\colon G^{(2)} \to \mathbb{T}_d$ be continuous 2-cocycles. If $(G \times_\sigma \mathbb{T}_d, i_\sigma, q_\sigma)$ and $(G \times_\tau \mathbb{T}_d, i_\tau, q_\tau)$ are isomorphic as twists, then σ is induced by a continuous global section $P\colon G \to G \times_\tau \mathbb{T}_d$.*

Proof. Suppose that $\phi\colon G \times_\sigma \mathbb{T}_d \to G \times_\tau \mathbb{T}_d$ is an isomorphism of twists. By Lemma 4.9, the map $S\colon \gamma \to (\gamma, 1)$ is a continuous global section from G to $G \times_\sigma \mathbb{T}_d$ that induces σ, in the sense that
$$S(\alpha)S(\beta)S(\alpha\beta)^{-1} = i_\sigma\big(r(\alpha),\, \sigma(\alpha, \beta)\big), \tag{4.5}$$
for all $(\alpha, \beta) \in G^{(2)}$.

Define $P \coloneqq \phi \circ S\colon G \to G \times_\tau \mathbb{T}_d$. We claim that P is a continuous global section. Since S is a continuous global section and ϕ is a groupoid isomorphism, P is continuous and $P(G^{(0)}) \subseteq G^{(0)} \times \{1\}$. Recall from Example 4.5 that $q_\sigma\colon G \times_\sigma \mathbb{T}_d \to G$ and $q_\tau\colon G \times_\tau \mathbb{T}_d \to G$ are the projections onto the first coordinate. Since ϕ is an isomorphism of twists, we have

$$q_\tau \circ P = q_\tau \circ (\phi \circ S) = (q_\tau \circ \phi) \circ S = q_\sigma \circ S = \mathrm{id}_G .$$

Hence P is a continuous global section.

We now show that P induces σ. By Proposition 4.8(a), P induces a continuous 2-cocycle $\omega\colon G^{(2)} \to \mathbb{T}_d$ satisfying

$$P(\alpha)P(\beta)P(\alpha\beta)^{-1} = i_\tau\big(r(\alpha),\, \omega(\alpha,\beta)\big), \tag{4.6}$$

for all $(\alpha,\beta) \in G^{(2)}$. Together, Equations (4.6) and (4.5) imply that

$$\begin{aligned} i_\tau\big(r(\alpha),\, \omega(\alpha,\beta)\big) &= P(\alpha)P(\beta)P(\alpha\beta)^{-1} \\ &= \phi\big(S(\alpha)S(\beta)S(\alpha\beta)^{-1}\big) \\ &= \phi\big(i_\sigma\big(r(\alpha),\, \sigma(\alpha,\beta)\big)\big) \\ &= i_\tau\big(r(\alpha),\, \sigma(\alpha,\beta)\big), \end{aligned}$$

for all $(\alpha,\beta) \in G^{(2)}$. Since i_σ and i_τ are both injective, we deduce that $\sigma = \omega$, and hence σ is induced by P. □

We now combine these three lemmas to prove our main theorem for this section.

Proof of Theorem 4.12. Lemma 4.15 gives (1) $\Longrightarrow$ (3), Lemma 4.13 gives (3) $\Longrightarrow$ (2), and Lemma 4.14 gives (2) $\Longrightarrow$ (1). □

We conclude this section with a corollary of Theorem 4.12.

Corollary 4.16. *Let G be a Hausdorff étale groupoid and Σ be a topologically trivial discrete twist over G. Suppose that $\sigma_1, \sigma_2\colon G^{(2)} \to \mathbb{T}_d$ are continuous 2-cocycles induced by two different continuous global sections $P_1, P_2\colon \Sigma \to G$, as in Proposition 4.8(a). Then σ_1 is cohomologous to σ_2.*

Proof. By Proposition 4.8(c), we have $G \times_{\sigma_1} \mathbb{T}_d \cong \Sigma \cong G \times_{\sigma_2} \mathbb{T}_d$, and hence Theorem 4.12 implies that σ_1 is cohomologous to σ_2. □

4.3. **Twisted Steinberg algebras arising from discrete twists.** In this section we give a construction of a twisted Steinberg algebra $A(G;\Sigma)$ coming from a topologically trivial discrete twist Σ over an ample Hausdorff groupoid G. We prove that if two such twists are isomorphic, then they give rise to isomorphic twisted Steinberg algebras. We also prove that if $\Sigma \cong G \times_\sigma \mathbb{T}_d$ for some continuous 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}_d$, then the twisted Steinberg algebras $A(G;\Sigma)$ and $A(G,\sigma)$ are $*$-isomorphic.

Definition 4.17. Let G be an ample Hausdorff groupoid and (Σ, i, q) be a topologically trivial discrete twist over G. We say that $f \in C(\Sigma, \mathbb{C}_d)$ is $\mathbb{T}_d$-*equivariant* if $f(z \cdot \epsilon) = z\, f(\epsilon)$ for all $z \in \mathbb{T}_d$ and $\epsilon \in \Sigma$, and we define

$$A(G;\Sigma) \coloneqq \{f \in C(\Sigma, \mathbb{C}_d) \,:\, f \text{ is } \mathbb{T}_d\text{-equivariant and } \overline{q(\mathrm{supp}(f))} \text{ is compact}\}.$$

We first show that $A(G;\Sigma)$ is a vector space under the pointwise operations inherited from $C(\Sigma, \mathbb{C}_d)$.

Lemma 4.18. *Let G be an ample Hausdorff groupoid and (Σ, i, q) be a topologically trivial discrete twist over G. Then $A(G;\Sigma)$ is a linear subspace of $C(\Sigma, \mathbb{C}_d)$.*

Proof. Fix $f, g \in A(G;\Sigma)$ and $\lambda \in \mathbb{C}\backslash\{0\}$. Then $\lambda f + g$ is continuous and $\mathbb{T}_d$-equivariant. Since $q\big(\operatorname{supp}(\lambda f + g)\big)$ is contained in the compact set $\overline{q(\operatorname{supp}(f))} \cup \overline{q(\operatorname{supp}(g))}$, we deduce that $q\big(\operatorname{supp}(\lambda f + g)\big)$ has compact closure. Hence $\lambda f + g \in A(G;\Sigma)$. □

Since we are assuming that the twist Σ is topologically trivial, it necessarily admits a continuous global section $P\colon G \to \Sigma$. We now show that Definition 4.17 can be rephrased in terms of any such P.

Lemma 4.19. *Let G be an ample Hausdorff groupoid and (Σ, i, q) be a topologically trivial discrete twist over G. Let $P\colon G \to \Sigma$ be any continuous global section. Then*

$$A(G;\Sigma) = \{f \in C(\Sigma, \mathbb{C}_d) \,:\, f \text{ is } \mathbb{T}_d\text{-equivariant and } f \circ P \in C_c(G, \mathbb{C}_d)\}.$$

Proof. Fix $f \in C(\Sigma, \mathbb{C}_d)$. Then $f \circ P$ is continuous. It suffices to show that $q(\operatorname{supp}(f)) = \operatorname{supp}(f \circ P)$, because then $\overline{q(\operatorname{supp}(f))}$ is compact if and only if $f \circ P \in C_c(G, \mathbb{C}_d)$. By Proposition 4.8(c), we know that $\Sigma = \{z \cdot P(\alpha) : (\alpha, z) \in G \times \mathbb{T}_d\}$. Therefore, we have

$$\begin{aligned} q(\operatorname{supp}(f)) &= \{q(\epsilon) \,:\, \epsilon \in \Sigma,\ f(\epsilon) \neq 0\} \\ &= \{q(z \cdot P(\alpha)) \,:\, (\alpha, z) \in G \times \mathbb{T}_d,\ f(z \cdot P(\alpha)) \neq 0\} \\ &= \{\alpha \,:\, (\alpha, z) \in G \times \mathbb{T}_d,\ z\, f(P(\alpha)) \neq 0\} \\ &= \{\alpha \in G \,:\, (f \circ P)(\alpha) \neq 0\} \\ &= \operatorname{supp}(f \circ P), \end{aligned}$$

as required. □

Remarks 4.20.

(1) It is crucial here that we are dealing with discrete twists. Suppose that σ is a 2-cocycle on an ample Hausdorff groupoid G that is continuous with respect to the standard topology on $\mathbb{T}$, and consider the classical twist $G \times_\sigma \mathbb{T}$ over G. Suppose that $f \in C(G \times_\sigma \mathbb{T})$ is a $\mathbb{T}$-equivariant function that is locally constant. Then, for any $\alpha \in G$, there is an open subset V of G containing α and an open subset W of $\mathbb{T}$ containing 1 such that f is constant on $V \times W$. Since W is open in the standard topology on $\mathbb{T}$, we have $W \neq \{1\}$. For each $z \in W\backslash\{1\}$, we have

$$f(\alpha, 1) = f(\alpha, z) = f(z \cdot (\alpha, 1)) = z\, f(\alpha, 1),$$

and hence $f|_{G \times \{1\}} \equiv 0$. But this implies that $f(\beta, w) = 0$ for all $(\beta, w) \in G \times_\sigma \mathbb{T}$, because f is $\mathbb{T}$-equivariant. In other words, if singleton sets are not open in $\mathbb{T}$, then the only locally constant $\mathbb{T}$-equivariant function on $G \times_\sigma \mathbb{T}$ is the zero function.

(2) Suppose that G is an ample Hausdorff groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ is a continuous 2-cocycle. Since $\mathbb{T}_d$ has the discrete topology, nonzero functions in $A(G; G \times_\sigma \mathbb{T}_d)$ are not compactly supported. To see this, fix $f \in A(G; G \times_\sigma \mathbb{T}_d)$ such that $f(\alpha, w) \neq 0$ for some $(\alpha, w) \in G \times_\sigma \mathbb{T}_d$. Then, for all $z \in \mathbb{T}_d$, we have

$$f(\alpha, z) = f(\alpha, z\,\overline{w}\,w) = f\big((z\,\overline{w}) \cdot (\alpha, w)\big) = z\,\overline{w}\, f(\alpha, w) \neq 0.$$

Thus $\{\alpha\} \times \mathbb{T}_d$ is a closed subset of $\operatorname{supp}(f)$ which is not compact, and hence f is not compactly supported.

Note that Definition 4.17 differs from the C*-algebraic analogue defined in [26, Definition 5.1.7 and Theorem 5.1.11], which is the completion of the subalgebra of continuous compactly supported $\mathbb{T}$-equivariant functions on a (non-discrete) twist over G with respect to a C*-norm.

Proposition 4.21. *Let G be an ample Hausdorff groupoid and (Σ, i, q) be a topologically trivial discrete twist over G. There is a multiplication on $A(G;\Sigma)$ given by*

$$(f *_\Sigma g)(\epsilon) := \sum_{\substack{(\delta,\eta)\in\Sigma^{(2)},\\ \delta\eta=\epsilon}} f(\delta)\, g(\eta) = \sum_{\zeta\in\Sigma^{s(\epsilon)}} f(\epsilon\zeta)\, g(\zeta^{-1}), \tag{4.7}$$

and an involution given by

$$f^*(\epsilon) := \overline{f(\epsilon^{-1})}.$$

Under these operations, along with pointwise addition and scalar multiplication, $A(G;\Sigma)$ is a $*$-algebra.

We call $A(G;\Sigma)$ the twisted Steinberg algebra *associated to the pair (G,Σ).*

Proof. By Lemma 4.18, $A(G;\Sigma)$ is a vector space. To see that $A(G;\Sigma)$ is a $*$-algebra, we will just show that it is closed under the involution and convolution, as it is routine to check that the multiplication and involution satisfy all of the other necessary properties.

We first prove that $A(G;\Sigma)$ is closed under the involution. Fix $f \in A(G;\Sigma) \subseteq C(\Sigma,\mathbb{C}_d)$. Then f^* is a composition of continuous maps, so $f^* \in C(\Sigma,\mathbb{C}_d)$. For all $z \in \mathbb{T}_d$ and $\epsilon \in \Sigma$, we have

$$f^*(z\cdot\epsilon) = \overline{f((z\cdot\epsilon)^{-1})} = \overline{f(\overline{z}\cdot(\epsilon^{-1}))} = \overline{\overline{z}\, f(\epsilon^{-1})} = z\, f^*(\epsilon),$$

and so f^* is $\mathbb{T}_d$-equivariant. Since $\operatorname{supp}(f^*) = (\operatorname{supp}(f))^{-1}$ and q is a continuous homomorphism, we have $q(\operatorname{supp}(f^*)) \subseteq \big(\overline{q(\operatorname{supp}(f))}\big)^{-1}$, and hence $\overline{q(\operatorname{supp}(f^*))}$ is compact because it is a closed subset of a compact set. Thus $f^* \in A(G;\Sigma)$.

We now prove that $A(G;\Sigma)$ is closed under the convolution. To see this, first note that since Σ is topologically trivial, it admits a continuous global section $P\colon G \to \Sigma$. Moreover, by Proposition 4.8, P induces a continuous 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}_d$ such that the map $\phi_P\colon G\times_\sigma \mathbb{T}_d \to \Sigma$ given by $\phi_P(\alpha,z) := z\cdot P(\alpha)$ is an isomorphism of twists. Fix $f,g \in A(G;\Sigma)$, and define $f_P := \overline{f\circ P}$ and $g_P := \overline{g\circ P}$. By Lemma 4.19, f_P and g_P are elements of $C_c(G,\mathbb{C}_d)$, which is equal (as a vector space) to $A(G,\sigma)$, by Lemma 3.1. We will express $f *_\Sigma g$ in terms of $f_P * g_P$, which we know is in $A(G,\sigma)$ by Proposition 3.2. Fix $(\alpha,z) \in G\times_\sigma \mathbb{T}_d$. Then

$$\begin{aligned}
\Sigma^{s(z\cdot P(\alpha))} &= \{\epsilon\in\Sigma : r(\epsilon) = s(z\cdot P(\alpha))\}\\
&= \{w\cdot P(\beta) : (\beta,w)\in G\times\mathbb{T}_d,\ r(w\cdot P(\beta)) = s(z\cdot P(\alpha))\}\\
&= \{w\cdot P(\beta) : (\beta,w)\in G^{s(\alpha)}\times\mathbb{T}_d\}.
\end{aligned}$$

Using $\mathbb{T}_d$-equivariance and Proposition 4.8(b) for the fourth equality below, we obtain

$$\begin{aligned}
(f *_\Sigma g)(z\cdot P(\alpha)) &= \sum_{\zeta\in\Sigma^{s(z\cdot P(\alpha))}} f\big((z\cdot P(\alpha))\,\zeta\big)\, g(\zeta^{-1})\\
&= \sum_{(\beta,w)\in G^{s(\alpha)}\times\mathbb{T}_d} f\big((z\cdot P(\alpha))(w\cdot P(\beta))\big)\, g\big((w\cdot P(\beta))^{-1}\big)\\
&= \sum_{(\beta,w)\in G^{s(\alpha)}\times\mathbb{T}_d} f\big((zw)\cdot(P(\alpha)P(\beta))\big)\, g\big(\overline{w}\cdot P(\beta)^{-1}\big)\\
&= \sum_{(\beta,w)\in G^{s(\alpha)}\times\mathbb{T}_d} z\, w\, f\big(\sigma(\alpha,\beta)\cdot P(\alpha\beta)\big)\, \overline{w}\, g\big(\overline{\sigma(\beta,\beta^{-1})}\cdot P(\beta^{-1})\big)\\
&= \sum_{\beta\in G^{s(\alpha)}} z\,\sigma(\alpha,\beta)\, f\big(P(\alpha\beta)\big)\, \overline{\sigma(\beta,\beta^{-1})}\, g\big(P(\beta^{-1})\big)\\
&= z \sum_{\beta\in G^{s(\alpha)}} \sigma(\alpha,\beta)\, \overline{\sigma(\beta,\beta^{-1})}\, \overline{f_P(\alpha\beta)}\, \overline{g_P(\beta^{-1})}.
\end{aligned} \tag{4.8}$$

We also have

$$\overline{(f_P * g_P)(\alpha)} = \sum_{\beta \in G^{s(\alpha)}} \overline{\sigma(\alpha\beta, \beta^{-1})}\, \overline{f_P(\alpha\beta)}\, \overline{g_P(\beta^{-1})}. \tag{4.9}$$

Since σ is normalised and satisfies the 2-cocycle identity, we have

$$\sigma(\alpha, \beta)\, \sigma(\alpha\beta, \beta^{-1}) = \sigma(\alpha, \beta\beta^{-1})\, \sigma(\beta, \beta^{-1}) = \sigma(\beta, \beta^{-1}),$$

and hence

$$\sigma(\alpha, \beta)\, \overline{\sigma(\beta, \beta^{-1})} = \overline{\sigma(\alpha\beta, \beta^{-1})}, \tag{4.10}$$

for each $\beta \in G^{s(\alpha)}$. Together, Equations (4.8), (4.9), and (4.10) imply that

$$(f *_\Sigma g)(\phi_P(\alpha, z)) = (f *_\Sigma g)(z \cdot P(\alpha)) = z\, \overline{(f_P * g_P)(\alpha)}. \tag{4.11}$$

Define $\psi_P^{f,g} \colon G \times_\sigma \mathbb{T}_d \to \mathbb{C}_d$ by $\psi_P^{f,g}(\alpha, z) \coloneqq z\, \overline{(f_P * g_P)(\alpha)}$. Since $f_P, g_P \in A(G, \sigma)$, we have $f_P * g_P \in A(G, \sigma) \subseteq C(G, \mathbb{C}_d)$. Thus $\psi_P^{f,g}$ is continuous. Since ϕ_P is a homeomorphism and $f *_\Sigma g = \psi_P^{f,g} \circ \phi_P^{-1}$, we deduce that $f *_\Sigma g \in C(\Sigma, \mathbb{C}_d)$. Taking $z = 1$ in Equation (4.11) shows that $(f *_\Sigma g) \circ P = \overline{f_P * g_P} \in C_c(G, \mathbb{C}_d)$, and Lemma 4.19 implies that this is equivalent to showing that $q(\operatorname{supp}(f *_\Sigma g))$ is compact. Finally, to see that $f *_\Sigma g$ is $\mathbb{T}_d$-equivariant, fix $z \in \mathbb{T}_d$ and $\epsilon \in \Sigma$. Then $\epsilon = w \cdot P(\beta)$ for a unique pair $(\beta, w) \in G \times_\sigma \mathbb{T}_d$. Thus, Equation (4.11) implies that $(f *_\Sigma g)(\epsilon) = w\, \overline{(f_P * g_P)(\beta)}$, and hence

$$(f *_\Sigma g)(z \cdot \epsilon) = (f *_\Sigma g)\big((zw) \cdot P(\beta)\big) = z\, w\, \overline{(f_P * g_P)(\beta)} = z\, (f *_\Sigma g)(\epsilon).$$

Therefore, $f *_\Sigma g \in A(G; \Sigma)$, and so $A(G; \Sigma)$ is a $*$-algebra. □

We now show that isomorphic twists give rise to isomorphic twisted Steinberg algebras.

Proposition 4.22. *Let G be an ample Hausdorff groupoid, and (Σ_1, i_1, q_1) and (Σ_2, i_2, q_2) be topologically trivial discrete twists over G. If $\phi\colon \Sigma_1 \to \Sigma_2$ is an isomorphism of twists, then the map $\Phi\colon f \mapsto f \circ \phi$ is a $*$-isomorphism from $A(G; \Sigma_2)$ to $A(G; \Sigma_1)$.*

Proof. We first show that $f \circ \phi \in A(G; \Sigma_1)$ for each $f \in A(G; \Sigma_2)$. Let $P_1\colon G \to \Sigma_1$ be a continuous global section, and define $P_2 \coloneqq \phi \circ P_1\colon G \to \Sigma_2$. Then P_2 is continuous, $P_2(G^{(0)}) \subseteq \phi\big(\Sigma_1^{(0)}\big) = \Sigma_2^{(0)}$, and

$$q_2 \circ P_2 = q_2 \circ (\phi \circ P_1) = (q_2 \circ \phi) \circ P_1 = q_1 \circ P_1 = \mathrm{id}_G .$$

Hence P_2 is a continuous global section. Fix $f \in A(G; \Sigma_2) \subseteq C(\Sigma_2, \mathbb{C}_d)$. Since ϕ is continuous, $f \circ \phi \in C(\Sigma_1, \mathbb{C}_d)$. By Lemma 4.6, ϕ respects the action of $\mathbb{T}_d$, and hence the $\mathbb{T}_d$-equivariance of f implies that $f \circ \phi$ is $\mathbb{T}_d$-equivariant. Moreover, Lemma 4.19 implies that $f \circ \phi \circ P_1 = f \circ P_2 \in C_c(G, \mathbb{C}_d)$, and thus $f \circ \phi \in A(G; \Sigma_1)$.

Therefore, there is a map $\Phi\colon A(G; \Sigma_2) \to A(G; \Sigma_1)$ given by $\Phi(f) \coloneqq f \circ \phi$. Routine calculations show that Φ is a $*$-homomorphism. Furthermore, Φ is bijective with inverse given by $\Phi^{-1}(g) \coloneqq g \circ \phi^{-1}$, and hence Φ is a $*$-isomorphism. □

By Proposition 4.8, we know that for every topologically trivial twist Σ over an ample Hausdorff groupoid G, there is a continuous 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}_d$ such that $\Sigma \cong G \times_\sigma \mathbb{T}_d$. Hence $A(G; \Sigma)$ is $*$-isomorphic to $A(G;\, G \times_\sigma \mathbb{T}_d)$, by Proposition 4.22. We now prove that $A(G; \Sigma)$ is also $*$-isomorphic to $A(G, \sigma)$.

Theorem 4.23. *Let G be an ample Hausdorff groupoid and Σ be a topologically trivial discrete twist over G. Let $P\colon G \to \Sigma$ be a continuous global section and let $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be the continuous 2-cocycle induced by P, as in Proposition 4.8(a). The map $\psi\colon f \mapsto \overline{f \circ P}$ is a $*$-isomorphism from $A(G; \Sigma)$ to $A(G, \sigma)$.*

Proof. By Lemma 3.1(a), $A(G,\sigma)$ and $C_c(G,\mathbb{C}_d)$ agree as sets, and hence Lemma 4.19 implies that

$$A(G;\Sigma) = \{f \in C(\Sigma,\mathbb{C}_d) \,:\, f \text{ is } \mathbb{T}_d\text{-equivariant and } f \circ P \in A(G,\sigma)\}. \tag{4.12}$$

Thus there is a map $\psi\colon A(G;\Sigma) \to A(G,\sigma)$ given by $\psi(f) \coloneqq \overline{f \circ P}$.

To see that ψ is injective, suppose that $\psi(f) = \psi(g)$ for some $f,g \in A(G;\Sigma)$. Fix $(\alpha,z) \in G \times_\sigma \mathbb{T}_d$. Since f and g are $\mathbb{T}_d$-equivariant, we have

$$f(z \cdot P(\alpha)) = z\, f(P(\alpha)) = z\,\overline{\psi(f)(\alpha)} = z\,\overline{\psi(g)(\alpha)} = z\, g(P(\alpha)) = g(z \cdot P(\alpha)). \tag{4.13}$$

By Proposition 4.8(c), we have $\Sigma = \{z \cdot P(\alpha) : (\alpha,z) \in G \times_\sigma \mathbb{T}_d\}$, and so Equation (4.13) implies that $f = g$, and thus ψ is injective.

To see that ψ is surjective, fix $h \in A(G,\sigma)$, and recall from Proposition 4.8(c) that the map $\phi_P\colon G \times_\sigma \mathbb{T}_d \to \Sigma$ given by $\phi_P(\alpha,z) \coloneqq z \cdot P(\alpha)$ is an isomorphism of twists. Define $f\colon \Sigma \to \mathbb{C}_d$ by $f(z \cdot P(\alpha)) \coloneqq z\,\overline{h(\alpha)}$, and $\widetilde{f}\colon G \times_\sigma \mathbb{T}_d \to \mathbb{C}_d$ by $\widetilde{f}(\alpha,z) \coloneqq z\,\overline{h(\alpha)}$. Since $h \in C(G,\mathbb{C}_d)$, we have $\widetilde{f} \in C(G \times_\sigma \mathbb{T}_d,\, \mathbb{C}_d)$, and hence $f = \widetilde{f} \circ \phi_P^{-1} \in C(\Sigma,\mathbb{C}_d)$ because ϕ_P^{-1} is continuous. For all $\alpha \in G$ and $z,w \in \mathbb{T}_d$, we have

$$f\big(z \cdot (w \cdot P(\alpha))\big) = f\big((zw) \cdot P(\alpha)\big) = z\,w\,\overline{h(\alpha)} = z\, f\big(w \cdot P(\alpha)\big),$$

and so f is $\mathbb{T}_d$-equivariant. We also have $f \circ P = \overline{h} \in A(G,\sigma)$, and thus Equation (4.12) implies that $f \in A(G;\Sigma)$. Since $\psi(f) = \overline{f \circ P} = h$, ψ is surjective.

It is clear that ψ is linear. We claim that it is a $*$-homomorphism. Fix $f,g \in A(G;\Sigma)$ and $\alpha \in G$. By Proposition 4.8(b), we have $P(\alpha)^{-1} = \overline{\sigma(\alpha,\alpha^{-1})} \cdot P(\alpha^{-1})$, and hence

$$\psi(f^*)(\alpha) = \overline{f^*(P(\alpha))} = f\big(P(\alpha)^{-1}\big) = f\big(\overline{\sigma(\alpha,\alpha^{-1})} \cdot P(\alpha^{-1})\big). \tag{4.14}$$

We also have

$$\psi(f)^*(\alpha) = \overline{\sigma(\alpha,\alpha^{-1})}\,\overline{\psi(f)(\alpha^{-1})} = \overline{\sigma(\alpha,\alpha^{-1})}\, f\big(P(\alpha^{-1})\big) = f\big(\overline{\sigma(\alpha,\alpha^{-1})} \cdot P(\alpha^{-1})\big). \tag{4.15}$$

Together, Equations (4.14) and (4.15) imply that $\psi(f^*) = \psi(f)^*$. In the notation defined in the proof of Proposition 4.21, we have $\psi(f) = f_P$ and $\psi(g) = g_P$, and hence Equation (4.11) implies that for all $\alpha \in G$, we have

$$\psi(f *_\Sigma g)(\alpha) = \overline{(f *_\Sigma g)(P(\alpha))} = (\psi(f) * \psi(g))(\alpha).$$

So $\psi(f *_\Sigma g) = \psi(f) * \psi(g)$, and thus ψ is a $*$-isomorphism. ☐

Corollary 4.24. *Let G be an ample Hausdorff groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. There is a $*$-isomorphism $\psi\colon A(G;\, G \times_\sigma \mathbb{T}_d) \to A(G,\sigma)$ such that $\psi(f)(\gamma) = \overline{f(\gamma,1)}$ for all $\gamma \in G$.*

Proof. By Lemma 4.9, the map $S\colon \gamma \mapsto (\gamma,1)$ is a continuous global section from G to $G \times_\sigma \mathbb{T}_d$ that induces σ, so the result follows from Theorem 4.23. ☐

Remark 4.25. If G is an ample Hausdorff groupoid, then $G \times_\sigma \mathbb{T}_d$ is also an ample Hausdorff groupoid for any continuous 2-cocycle $\sigma\colon G^{(2)} \to \mathbb{T}_d$, and hence there is an associated (untwisted) Steinberg algebra $A(G \times_\sigma \mathbb{T}_d)$. As a set, $A(G \times_\sigma \mathbb{T}_d)$ is equal to

$$\{f \in C(G \times_\sigma \mathbb{T}_d,\, \mathbb{C}_d) \,:\, \operatorname{supp}(f) \text{ is compact}\},$$

and is dense in $C_r^*(G \times_\sigma \mathbb{T}_d)$, by [6, Proposition 4.2] and [28, Proposition 5.7]. Moreover, by Theorem 4.23, we have $A(G;\, G \times_\sigma \mathbb{T}_d) \cong A(G,\sigma)$, and we know from Proposition 3.2 that $A(G,\sigma)$ is dense in $C_r^*(G,\sigma)$. We saw in Remarks 4.20(2) that the only compactly supported function in $A(G;\, G \times_\sigma \mathbb{T}_d) \subseteq C(G \times_\sigma \mathbb{T}_d,\, \mathbb{C}_d)$ is the zero function, and hence

$$A(G;\, G \times_\sigma \mathbb{T}_d) \cap A(G \times_\sigma \mathbb{T}_d) \;=\; \{0\}.$$

However, this does not preclude $C^*_r(G,\sigma)$ from embedding into $C^*_r(G \times_\sigma \mathbb{T}_d)$. It would be interesting to know how these two C*-algebras are related.

5. Examples of twisted Steinberg algebras

In this section we discuss two important classes of examples of twisted Steinberg algebras: twisted group algebras and twisted Kumjian–Pask algebras.

5.1. **Twisted discrete group algebras.** Suppose that G is a topological group. (That is, G is a group endowed with a topology with respect to which multiplication and inversion are continuous.) Then G is an ample groupoid if and only if G has the discrete topology, in which case any $\mathbb{T}_d$-valued 2-cocycle on G is locally constant. One defines a twist over a discrete group G via a split extension by an abelian group A, as in [4, Chapter IV.3]. When $A = \mathbb{T}_d$, the twist gives rise to a $\mathbb{T}_d$-valued 2-cocycle on G, with which one can define a twisted group algebra over $\mathbb{C}_d$. The twisted convolution and involution defined in Proposition 3.2 generalise those of classical twisted group algebras over $\mathbb{C}_d$, and hence our twisted Steinberg algebras generalise these twisted (discrete) group algebras. Interesting questions still exist about this class of algebras, even for finite groups. See, for example, [20]. Moreover, twisted group C*-algebras (as studied in [21]) have featured prominently in the study of C*-algebras associated with groups and group actions; in particular, they have proved essential in establishing superrigidity results for certain nilpotent groups (see [8]).

5.2. **Twisted Kumjian–Pask algebras.** For each finitely-aligned higher-rank graph (or k-graph) Λ, there is both a C*-algebra $C^*(\Lambda)$ called the *Cuntz–Krieger algebra* (see [23]) and a dense subalgebra $\mathrm{KP}(\Lambda)$ called the *Kumjian–Pask algebra* (see [7]). Letting G_Λ denote the boundary-path groupoid defined in [30], we have

$$C^*(\Lambda) \cong C^*(G_\Lambda) \quad \text{and} \quad \mathrm{KP}(\Lambda) \cong A(G_\Lambda).$$

Twisted higher-rank graph C*-algebras were introduced and studied in a series of papers by Kumjian, Pask, and Sims [15, 16, 17, 18]. Twisted higher-rank graph C*-algebras provide a class of (somewhat) tractable examples that can be used to demonstrate more general C*-algebraic phenomena. See also [1, 11, 27]. We introduce twisted Kumjian–Pask algebras for row-finite higher-rank graphs with no sources using a twisted Steinberg algebra approach.

Let Λ be a row-finite higher-rank graph with no sources and c be a continuous $\mathbb{T}$-valued 2-cocycle on Λ, as defined in [17, Definition 3.5]. Then $C^*(\Lambda, c)$ is the C*-algebra generated by a universal Cuntz–Krieger (Λ, c)-family, as defined in [17, Definition 5.2]. In [17, Theorem 6.3(iii)], the authors describe how Λ and c give rise to a 2-cocycle $\sigma_c\colon G_\Lambda^{(2)} \to \mathbb{T}$ such that

$$C^*(\Lambda, c) \cong C^*(G_\Lambda, \sigma_c).$$

By the last two sentences of the proof of [17, Lemma 6.3], the 2-cocycle σ_c is normalised and locally constant. We define

$$\mathrm{KP}(\Lambda, c) := A(G_\Lambda, \sigma_c),$$

and call this the *(complex) twisted Kumjian–Pask algebra* associated to the pair (Λ, c). By Proposition 3.2, $\mathrm{KP}(\Lambda, c)$ is dense in $C^*(\Lambda, c)$.

In [17, Definition 5.2], Kumjian, Pask, and Sims construct $C^*(\Lambda, c)$ using a generators and relations model involving the same generating partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ as $C^*(\Lambda)$, but with the relation $t_\mu t_\nu = t_{\mu\nu}$ replaced by $t_\mu t_\nu = c(\mu, \nu)\, t_{\mu\nu}$. We expect that there is a similar construction of $\mathrm{KP}(\Lambda, c)$ using these generators and relations, but we do not pursue this here.

6. A Cuntz–Krieger uniqueness theorem and simplicity of twisted Steinberg algebras of effective groupoids

In this section we extend the Cuntz–Krieger uniqueness theorem and a part of the simplicity characterisation for Steinberg algebras from [3] to the twisted Steinberg algebra setting. Throughout this section, we will assume that G is an effective, ample, Hausdorff groupoid.

Theorem 6.1 (Cuntz–Krieger uniqueness theorem)**.** *Let G be an effective, ample, Hausdorff groupoid, and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. Suppose that Q is a ring and $\pi\colon A(G,\sigma) \to Q$ is a ring homomorphism. Then π is injective if and only if $\pi(1_V) \neq 0$ for every nonempty compact open subset V of $G^{(0)}$.*

Proof. It is clear that if π is injective, then $\pi(1_V) \neq 0$ for every nonempty compact open subset V of $G^{(0)}$. Suppose that π is not injective. Then there exists $f \in A(G,\sigma)$ such that $f \neq 0$ and $\pi(f) = 0$. We aim to find a nonempty compact open subset V of $G^{(0)}$ such that $\pi(1_V) = 0$. Since σ is locally constant, we can use Lemma 3.1(b) to write $f = \sum_{D\in F} a_D 1_D$, where F is a finite collection of disjoint nonempty compact open bisections of G such that $\sigma(\alpha^{-1},\alpha)$ is constant for all $\alpha \in D$, and $a_D \in \mathbb{C}\backslash\{0\}$, for each $D \in F$. Let $g \coloneqq 1_{D_0^{-1}} f$ for some $D_0 \in F$. Then $g \in \ker(\pi)$, because π is a homomorphism. Fix $\alpha \in D_0$, and define $c_{D_0} \coloneqq \sigma(\alpha^{-1},\alpha)\, a_{D_0} \neq 0$. Then

$$g(s(\alpha)) = g(\alpha^{-1}\alpha) = \sigma(\alpha^{-1},\alpha)\, 1_{D_0^{-1}}(\alpha^{-1})\, f(\alpha) = \sigma(\alpha^{-1},\alpha)\, a_{D_0} = c_{D_0} \neq 0.$$

Let $g_0 \coloneqq g|_{G^{(0)}}$, and define $H \coloneqq \operatorname{supp}(g - g_0) \subseteq G\backslash G^{(0)}$. The calculation above implies that $s(\alpha) \in \operatorname{supp}(g_0)$. Since G is ample and effective, [3, Lemma 3.1] implies that there is a nonempty compact open subset V of $\operatorname{supp}(g_0) \cap s(D_0)$ such that $VHV = \varnothing$. Therefore, since $\operatorname{supp}(1_V(g-g_0)1_V) \subseteq VHV$, we have $1_V(g-g_0)1_V = 0$, and hence

$$1_V\, g\, 1_V = 1_V\, g_0\, 1_V = c_{D_0}\, 1_V. \tag{6.1}$$

Thus, using that $\pi(g) = 0$, we deduce from Equation (6.1) that

$$\pi(1_V) = c_{D_0}^{-1}\, \pi(c_{D_0}\, 1_V) = c_{D_0}^{-1}\, \pi(1_V)\, \pi(g)\, \pi(1_V) = 0,$$

as required. □

Given a groupoid G, one calls a subset $U \subseteq G^{(0)}$ *invariant* if, for any $\gamma \in G$, we have

$$s(\gamma) \in U \iff r(\gamma) \in U.$$

One says that G is *minimal* if $G^{(0)}$ has no nontrivial open invariant subsets. Equivalently, G is minimal if and only if $\overline{s(r^{-1}(x))} = G^{(0)}$ for every $x \in G^{(0)}$.

Theorem 6.2. *Let G be an effective, ample, Hausdorff groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. Then G is minimal if and only if $A(G,\sigma)$ is simple.*

Proof. Suppose that G is minimal, and let I be a nonzero ideal of $A(G,\sigma)$. Then I is the kernel of some noninjective ring homomorphism of $A(G,\sigma)$, so Theorem 6.1 implies that there is a compact open subset $V \subseteq G^{(0)}$ such that $1_V \in I$. We claim that the ideal generated by 1_V is the whole of $A(G,\sigma)$. Since the twisted convolution product of characteristic functions on the unit space is the same as the untwisted convolution product, the proof follows directly from the arguments used in the proof of [3, Proposition 4.5].

For the converse, suppose that G is not minimal. Then there exists a nonempty open invariant subset $U \subsetneq G^{(0)}$. The set

$$G_U \coloneqq s^{-1}(U) = \{\gamma \in G : s(\gamma) \in U\} = \{\gamma \in G : r(\gamma) \in U\}$$

is an open subgroupoid of G, so we can view $I := A\big(G_U, \sigma|_{G_U^{(2)}}\big)$ as a proper subset of $A(G,\sigma)$. Since U is a nonempty open set and G is ample, we can find a nonempty compact open bisection B of U, and thus $I \neq \{0\}$, because $1_B \in I$. We claim that I is an ideal of G. Since the vector-space operations are defined pointwise, it is straightforward to show that I is a subspace. To see that I is an ideal, fix $f \in I$ and $g \in A(G,\sigma)$. Since U is invariant, we have

$$\operatorname{supp}(fg) \subseteq \operatorname{supp}(f)\operatorname{supp}(g) \subseteq G_U\, G \subseteq G_U,$$

and so $fg \in I$. Similarly, $gf \in I$, and thus I is an ideal. (Note that I is also a $*$-ideal.) □

Remark 6.3. By [3, Theorem 4.1], the untwisted Steinberg algebra $A(G)$ is simple if and only if G is minimal and effective. Note that Theorem 6.2 does not give necessary and sufficient conditions on G and σ for simplicity of twisted Steinberg algebras. This is a hard problem. We expect, as in the C*-setting of [17, Remark 8.3], that there exist simple twisted Steinberg algebras for which the groupoid G is not effective.

7. Gradings and a graded uniqueness theorem

In this section we describe the graded structure that twisted Steinberg algebras inherit from the underlying groupoid, and we prove a graded uniqueness theorem. The arguments are similar to those used in the untwisted setting (see [6]). Let Γ be a discrete group and suppose that $c\colon G \to \Gamma$ is a continuous groupoid homomorphism (or 1-*cocycle*). Then we call G a *graded groupoid*, and we define $G_\gamma := c^{-1}(\gamma)$ for each $\gamma \in \Gamma$. Since c is continuous and Γ is discrete, each G_γ is clopen. Since c is a homomorphism, we have

$$G_\gamma^{-1} = G_{\gamma^{-1}} \quad \text{and} \quad G_\gamma\, G_\delta \subseteq G_{\gamma\delta}$$

for all $\gamma, \delta \in \Gamma$. Note that all groupoids are graded with respect to the groupoid homomorphism into the trivial group.

Proposition 7.1. *Let G be an ample Hausdorff groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. Let Γ be a discrete group and $c\colon G \to \Gamma$ be a continuous groupoid homomorphism. For each $\gamma \in \Gamma$, define the set of* homogeneous elements *by*

$$A(G,\sigma)_\gamma := \{f \in A(G,\sigma) \,:\, \operatorname{supp}(f) \subseteq G_\gamma\}.$$

Then $A(G,\sigma)$ is a Γ-graded algebra.

Proof. It is clear that $A(G,\sigma)_\gamma$ is a $\mathbb{C}$-submodule of $A(G,\sigma)$, for each $\gamma \in \Gamma$. Since $A(G,\sigma)$ and $A(G)$ agree as vector spaces, [6, Lemma 3.5] implies that every $f \in A(G,\sigma)$ can be written as a linear combination of homogeneous elements. Thus, to see that

$$A(G,\sigma) = \bigoplus_{\gamma\in\Gamma} A(G,\sigma)_\gamma,$$

it suffices to show that any finite collection

$$\{f_i \in A(G,\sigma)_{\gamma_i} \,:\, 1 \le i \le n, \text{ and each } \gamma_i \text{ is distinct from the others}\}$$

is linearly independent. But this is clear, because $\operatorname{supp}(f_i) \cap \operatorname{supp}(f_j) = \varnothing$ when $i \neq j$. Finally, we have

$$A(G,\sigma)_\gamma\, A(G,\sigma)_\delta \subseteq A(G,\sigma)_{\gamma\delta},$$

because $\operatorname{supp}(fg) \subseteq \operatorname{supp}(f)\operatorname{supp}(g)$ and $G_\gamma\, G_\delta \subseteq G_{\gamma\delta}$. □

As in the untwisted case [6, Theorem 5.4], the graded uniqueness theorem follows from the Cuntz–Krieger uniqueness theorem. Note that if e is the identity of Γ, then G_e is a clopen subgroupoid of G, and so we can identify $A(G,\sigma)_e$ with $A(G_e,\sigma)$, just as we can identify $A(G_e)$ with $A(G)_e$.

Theorem 7.2 (Graded uniqueness theorem)**.** *Let G be an ample Hausdorff groupoid and $\sigma\colon G^{(2)} \to \mathbb{T}_d$ be a continuous 2-cocycle. Let Γ be a discrete group with identity e, and let $c\colon G \to \Gamma$ be a continuous groupoid homomorphism such that the subgroupoid G_e is effective. Suppose that Q is a Γ-graded ring and $\pi\colon A(G,\sigma) \to Q$ is a graded ring homomorphism. Then π is injective if and only if $\pi(1_K) \neq 0$ for every nonempty compact open subset K of $G^{(0)}$.*

Proof. It is clear that if π is injective, then $\pi(1_K) \neq 0$ for every nonempty compact open subset K of $G^{(0)}$. Suppose that π is not injective. We claim that there exists $f \in A(G_e,\sigma)$ such that $f \neq 0$ and $\pi(f) = 0$. To see this, fix $g \in \ker(\pi)$ such that $g \neq 0$. By the proof of Proposition 7.1, g can be expressed as a finite sum of homogeneous elements; that is, $g = \sum_{\gamma\in F} g_\gamma$, where F is a finite subset of Γ, and $g_\gamma \in A(G,\sigma)_\gamma$ for each $\gamma \in F$. Thus

$$\sum_{\gamma\in F} \pi(g_\gamma) = \pi\Big(\sum_{\gamma\in F} g_\gamma\Big) = \pi(g) = 0.$$

Since π is graded, we have $\pi(g_\gamma) \in Q_\gamma$ for each $\gamma \in \Gamma$. Thus each $\pi(g_\gamma) = 0$, because elements of different graded subspaces of Q are linearly independent. Since $g \neq 0$, we can choose $\gamma \in F$ such that $g_\gamma \neq 0$. Since g_γ is locally constant and G_γ is open, there exists a compact open bisection $B \subseteq G_\gamma$ such that $g_\gamma(B) = \{k\}$, for some $k \in \mathbb{C}_d\backslash\{0\}$. Define

$$f \coloneqq 1_{B^{-1}}\, g_\gamma \in A(G_e,\sigma) \cap \ker(\pi).$$

For all $\alpha \in B$, we have

$$f(s(\alpha)) = f(\alpha^{-1}\alpha) = \sigma(\alpha^{-1},\alpha)\, 1_{B^{-1}}(\alpha^{-1})\, g_\gamma(\alpha) = \sigma(\alpha,\alpha^{-1})\, k \neq 0,$$

and hence $f \neq 0$. Thus the restriction π_e of π to $A(G_e,\sigma)$ is not injective.

Since $G^{(0)} \subseteq G_e$ and we have assumed that the groupoid G_e is effective, we can apply Theorem 6.1 to the restricted homomorphism π_e to obtain a nonempty compact open subset $K \subseteq G^{(0)}$ such that $\pi(1_K) = 0$, as required. □

Acknowledgements. This research collaboration began as part of the project-oriented workshop "Women in Operator Algebras" (18w5168) in November 2018, which was funded and hosted by the Banff International Research Station. The attendance of the first-named author at this workshop was supported by an AustMS WIMSIG Cheryl E. Praeger Travel Award, and the attendance of the third-named author was supported by SFB 878 Groups, Geometry & Actions. The research was also funded by the Australian Research Council grant DP170101821, and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC 2044 – 390685587, Mathematics Münster – Dynamics – Geometry – Structure, and under SFB 878 Groups, Geometry & Actions.

References

[1] B. Armstrong and N. Brownlowe, *Product-system models for twisted* C*-*algebras of topological higher-rank graphs*, J. Math. Anal. Appl. **466** (2018), no. 2, 1443–1475.

[2] S. Barlak and X. Li, *Cartan subalgebras and the UCT problem*, Adv. Math. **316** (2017), 748–769.

[3] J.H. Brown, L.O. Clark, C. Farthing, and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), no. 2, 433–452.

[4] K.S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics, vol. 87, Springer, New York, 1982.

[5] L.O. Clark, R. Exel, and E. Pardo, *A generalized uniqueness theorem and the graded ideal structure of Steinberg algebras*, Forum Math. **30** (2018), no. 3, 533–552.

[6] L.O. Clark, C. Farthing, A. Sims, and M. Tomforde, *A groupoid generalization of Leavitt path algebras*, Semigroup Forum **89** (2014), no. 3, 501–517.
[7] L.O. Clark and Y.E.P. Pangalela, *Kumjian–Pask algebras of finitely aligned higher-rank graphs*, J. Algebra **482** (2017), 364–397.
[8] C. Eckhardt and S. Raum, *C*-superrigidity of 2-step nilpotent groups*, Adv. Math. **338** (2018), 175–195.
[9] R. Exel, *Inverse semigroups and combinatorial C*-algebras*, Bull. Braz. Math. Soc. (N.S.) **39** (2008), no. 2, 191–313.
[10] ———, *Reconstructing a totally disconnected groupoid from its ample semigroup*, Proc. Amer. Math. Soc. **138** (2010), no. 8, 2991–3001.
[11] E. Gillaspy, *K-theory and homotopies of 2-cocycles on higher-rank graphs*, Pacific J. Math. **278** (2015), no. 2, 407–426.
[12] A. Kumjian, *On C*-diagonals*, Canad. J. Math. **38** (1986), no. 4, 969–1008.
[13] A. Kumjian and D. Pask, *Higher rank graph C*-algebras*, New York J. Math. **6** (2000), 1–20.
[14] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz–Krieger algebras*, J. Funct. Anal. **144** (1997), no. 2, 505–541.
[15] A. Kumjian, D. Pask, and A. Sims, *Homology for higher-rank graphs and twisted C*-algebras*, J. Funct. Anal. **263** (2012), no. 6, 1539–1574.
[16] ———, *On the K-theory of twisted higher-rank-graph C*-algebras*, J. Math. Anal. Appl. **401** (2013), no. 1, 104–113.
[17] ———, *On twisted higher-rank graph C*-algebras*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 5177–5216.
[18] ———, *Simplicity of twisted C*-algebras of higher-rank graphs and crossed products by quasifree actions*, J. Noncommut. Geom. **10** (2016), no. 2, 515–549.
[19] X. Li, *Every classifiable simple C*-algebra has a Cartan subalgebra*, Invent. Math. (2019), 1–46, DOI 10.1007/s00222-019-00914-0.
[20] L. Margolis and O. Schnabel, *Twisted group ring isomorphism problem*, Q. J. Math. **69** (2018), no. 4, 1195–1219.
[21] J.A. Packer and I. Raeburn, *On the structure of twisted group C*-algebras*, Trans. Amer. Math. Soc. **334** (1992), no. 2, 685–718.
[22] A.L.T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Mathematics, vol. 170, Birkhäuser Boston, Inc., Boston, MA, 1999.
[23] I. Raeburn, A. Sims, and T. Yeend, *The C*-algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. **213** (2004), no. 1, 206–240.
[24] J. Renault, *A groupoid approach to C*-algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, New York, 1980.
[25] ———, *Cartan subalgebras in C*-algebras*, Irish Math. Soc. Bulletin **61** (2008), 29–63.
[26] A. Sims, *Hausdorff étale groupoids and their C*-algebras*, Operator algebras and dynamics: groupoids, crossed products and Rokhlin dimension, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, to appear, arXiv:1710.10897v2 [math.OA].
[27] A. Sims, B. Whitehead, and M.F. Whittaker, *Twisted C*-algebras associated to finitely aligned higher-rank graphs*, Documenta Math. **19** (2014), 831–866.
[28] B. Steinberg, *A groupoid approach to discrete inverse semigroup algebras*, Adv. Math. **223** (2010), no. 2, 689–727.
[29] J.L. Tu, *La conjecture de Baum-Connes pour les feuilletages moyennables*, *K*-Theory **17** (1999), no. 3, 215–264.
[30] T. Yeend, *Groupoid models for the C*-algebras of topological higher-rank graphs*, J. Operator Theory **57** (2007), no. 1, 95–120.

(B. Armstrong and J. Ramagge) School of Mathematics and Statistics, The University of Sydney, NSW 2006, AUSTRALIA
E-mail address: becky.armstrong, jacqui.ramagge@sydney.edu.au

(L.O. Clark) School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington 6140, NEW ZEALAND
E-mail address: lisa.clark@vuw.ac.nz

(K. Courtney) Mathematical Institute, WWU Münster, Einsteinstr. 62, 48149 Münster, GERMANY
E-mail address: kcourtne@uni-muenster.de

(Y.-F. Lin) Mathematical Sciences Research Centre, Queen's University Belfast, Belfast, BT7 1NN, UNITED KINGDOM
E-mail address: y.lin@qub.ac.uk

(K. McCormick) Department of Mathematics, University of Minnesota, Minneapolis, MN, UNITED STATES
E-mail address: kamccorm@umn.edu