# Quantum Polynomial Functors


Jiuzu Hong*

*Mathematics Department,*
*Yale University*

Oded Yacobi†

*School of Mathematics and Statistics,*
*University of Sydney*


April 6, 2015


**Abstract**

We construct a category of quantum polynomial functors which deforms Friedlander and Suslin's category of strict polynomial functors. The aim of this paper is to develop the basic structural properties of this category. We construct quantum Schur and Weyl functors and show that quantum divided powers form projective generators for the category of quantum polynomial functors of degree $d$. Using this result we prove that the category of quantum polynomial functors is braided, and give a new and streamlined proof of quantum $(GL(m), GL(n))$ duality, along with other results in quantum invariant theory.


## 1 Introduction

The category $\mathcal{P}$ of strict polynomial functors was introduced by Friedlander and Suslin in their study of the cohomology of finite group schemes [FS]. In this work, we define a new category $\mathcal{P}_q$ of quantum polynomial functors, which deforms Friedlander and Suslin's category. The aim of this paper is to develop the basic properties of this category, analogous to those of $\mathcal{P}$, and show that this category provides the framework for a functorial approach to the representation theory of the quantum general linear group. We apply these ideas to give a new approach to the invariant theory of quantum general linear groups.

We now describe the contents of the paper in more detail. In Section 2 we set up the basics of quantum linear algebra which we use throughout. Given two Hecke pairs $(V, R_V)$ and $(W, R_W)$ we associate a "rectangular Schur space" $S(V, W; d)$. The main result of this section is Proposition 2.4 which states that this space is isomorphic to the space of maps $\mathrm{Hom}_{\mathcal{H}_d}(V^{\otimes d}, W^{\otimes d})$, where $\mathcal{H}_d$ is the Iwahori-Hecke algebra of type A. We also recall the algebra of quantum $m \times n$ matrices $\mathcal{O}_q(M_{m,n})$.


*jiuzu.hong@yale.edu

†oded.yacobi@sydney.edu.au

In Section 3 we define the category $\mathcal{P}_q^d$ of quantum polynomial functors of degree $d$ over $\Bbbk$, where $q \in \Bbbk^\times$ and $\Bbbk$ is a field. Objects in $\mathcal{P}_q^d$ are functors $\Gamma_q^d\mathcal{V} \to \mathcal{V}$, where $\mathcal{V}$ is the category of finite dimensional vector spaces over $\Bbbk$, and $\Gamma_q^d\mathcal{V}$ is the category with objects natural numbers and morphisms given by

$$\mathrm{Hom}_{\Gamma^d\mathcal{V}}(m,n) := \mathrm{Hom}_{\mathcal{H}_d}((\Bbbk^m)^{\otimes d}, (\Bbbk^n)^{\otimes d}).$$

In Proposition 3.3, we show that there is another equivalent definition of quantum polynomial functors using quantum matrices. We use this to define a Frobenius twist functor $(-)^{(1)} : \mathcal{P}^d \to \mathcal{P}_q^{d\ell}$, where $q$ is an $\ell$-th root of unity and $\ell > 1$ is odd.

In Section 4 we prove our main theorem (Theorem 4.7) which describes projective generators of $\mathcal{P}_q^d$. This uses a finite generation property for quantum polynomial functors, which we prove in Proposition 4.5.

In Section 5 we prove that $\mathcal{P}_q$ is a braided category. We make crucial use of Theorem 4.7 to show that the R-matrices of the quantum general linear group are suitably functorial, and thereby define a braiding on $\mathcal{P}_q$. We also show that the braiding behaves well under the duality functor (Proposition 5.6). In Section 6 we introduce quantum Schur/Weyl functors in $\mathcal{P}_q$ and we show that they are dual to each other in Theorem 6.5. We also use them to describe the simple objects in $\mathcal{P}_q$.

Finally in Section 7 we specialize to the case when $q$ is generic and study the invariant theory of $GL_q(n)$. We use Theorem 4.7 to give an easy proof of the duality between $GL_q(m)$ and $GL_q(n)$. We also formulate and derive the equivalence of this duality to the quantum first fundamental theorem and Jimbo-Schur-Weyl duality. We remark that quantum $(GL(m), GL(n))$-duality is due to Zhang [Zh] and Phúng [Ph]. (Zhang also derives Jimbo-Schur-Weyl duality from $(GL_q(m), GL_q(n))$-duality.) The quantum FFT that we prove first appears in [GLR] with a much more complicated proof. (Other versions of the quantum FFT appear in [Ph] and [LZZ].) We remark also that our approach to quantum invariant theory applies to the other settings where a theory of strict polynomial functors has been constructed (cf. Remark 7.4).

**Acknowledgement.** We thank Joseph Bernstein for his encouragement and important discussions during the initial stages of this project, and also Roger Howe and Andrew Mathas for helpful conversations. We also thank Antoine Touzé for introducing us to polynomial functors. O.Y. is supported by an Australian Research Council Discovery Early Career Research Award.

# 2 Quantum linear algebra

## 2.1 Hecke pairs

We fix a field $\Bbbk$ and an element $q \in \Bbbk^\times$. Let $\mathcal{V}$ denote that category of finite dimensional vector spaces over $\Bbbk$. Let $\mathcal{H}_d$ be the Iwahori-Hecke algebra of type A: it is the $\Bbbk$-algebra generated by $T_1, T_2, ..., T_{d-1}$ subject to the relations:

$$\begin{aligned} T_iT_j &= T_jT_i \qquad \text{if } |i-j| > 1, \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \\ (T_i - q)(T_i + q^{-1}) &= 0. \end{aligned} \tag{2.1.1}$$

For $V \in \mathcal{V}$ a **Hecke operator** is a linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ such that

(i) $R$ satisfies the Yang-Baxter equation, i.e. the following equation holds in $\mathrm{End}(V^{\otimes 3})$:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

where $R_{12} = R \otimes 1_V$ and $R_{23} = 1_V \otimes R$.

(ii) $R$ satisfies the Hecke relation $(R - q)(R + q^{-1}) = 0$.

We call the tuple $(V, R)$ a **Hecke pair**. To a Hecke pair $(V, R)$ we associate the right module $\rho_{d,V} : \mathcal{H}_d \to \mathrm{End}(V^{\otimes d})$ via the formula

$$T_i \mapsto 1_{V^{\otimes i}} \otimes R \otimes 1_{V^{\otimes d-i-1}}.$$

Often we suppress $R$ in the notation and refer to a vector space $V$ as a "Hecke pair". In this case, the R-matrix is implicit and when necessary is denoted $R_V$.

Now consider two Hecke pairs $V, W$ and a $\Bbbk$-algebra $C$ with multiplication $m : C \otimes C \to C$. A **$q$-linear operator over** $C$ is a $\Bbbk$-linear operator $P : V \to W \otimes C$ such that the following diagram commutes:

$$\begin{tikzcd} V^{\otimes 2} \arrow[r, "P^{(2)}"] \arrow[d, "R_V"] & W^{\otimes 2} \otimes_{\Bbbk} C \arrow[d, "R_W"] \\ V^{\otimes 2} \arrow[r, "P^{(2)}"] & W^{\otimes 2} \otimes_{\Bbbk} C \end{tikzcd} \tag{2.1.2}$$

Here $P^{(2)}$ is the composition:

$$V^{\otimes 2} \xrightarrow{P \otimes P} W \otimes C \otimes W \otimes C \xrightarrow{\text{flip}} W^{\otimes 2} \otimes C \otimes C \xrightarrow{1 \otimes m} W^{\otimes 2} \otimes C.$$

Let $T(V, W)$ be the tensor algebra of $\mathrm{Hom}(V, W)$, which is graded

$$T(V, W) = \bigoplus_{d \geq 0} T(V, W)_d,$$

where

$$T(V, W)_d := \mathrm{Hom}(V, W)^{\otimes d} \simeq \mathrm{Hom}(V^{\otimes d}, W^{\otimes d}).$$

Let $I(V, W)$ be the two sided ideal generated by

$$R(V, W) := \{X \circ R_V - R_W \circ X \mid X \in \mathrm{Hom}(V^{\otimes 2}, W^{\otimes 2})\}.$$

The ideal $I(V, W)$ is homogeneous

$$I(V, W) = \bigoplus_{d \geq 0} I(V, W)_d,$$

where $I(V, W)_d$ is spanned by

$$\mathrm{Hom}(V, W)^{\otimes i-1} \otimes R(V, W) \otimes \mathrm{Hom}(V, W)^{\otimes d-i-1}$$

for $i = 1, 2, ..., d - 1$. We define

$$A(V, W) := T(V, W)/I(V, W).$$

The algebra $A(V, W)$ is called the **quantum Hom-space algebra** from $W$ to $V$ (cf. [Ph, §3] and [HH, §3]). It has a natural grading

$$A(V, W) = \bigoplus_{d \geq 0} A(V, W)_d,$$

where

$$A(V, W)_d = T(V, W)_d / I(V, W)_d. \tag{2.1.3}$$

Note that $A(V,W)_1 = \mathrm{Hom}(V,W)$ and hence the canonical map $V \to W \otimes \mathrm{Hom}(W,V)$ induces a linear operator

$$\delta_{V,W} : V \to W \otimes A(W,V).$$

By construction $\delta_{V,W}$ is a $q$-linear operator.

The following lemma shows that the quantum Hom-space is characterized by a universal property.

**Lemma 2.1.** *Let $V, W$ be two Hecke pairs and let $Q : V \to W \otimes C$ be a $q$-linear operator over a $\Bbbk$-algebra $C$. Then there exists a unique morphism of algebras $\tilde{Q} : A(W,V) \to C$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\delta_{V,W}"] \arrow[dr, "Q"'] & W \otimes A(W,V) \arrow[d, "1_W \otimes \tilde{Q}"] \\ & W \otimes C \end{tikzcd} \tag{2.1.4}$$

*Proof.* By construction of $A(W,V)$, a $q$-linear operator $Q : V \to W \otimes C$ is in fact equivalent to a homomorphism of algebras $\tilde{Q} : A(W,V) \to C$, and it is then easy to show diagram (2.1.4) commutes. □

Given three Hecke pairs $V, W, U$ and $q$-linear operators $P : V \to W \otimes C$ and $Q : W \to U \otimes D$ we denote by $Q \circ P : V \to U \otimes D \otimes C$ the composition of $P$ and $Q$. The following lemma is easy to check.

**Lemma 2.2.** *The composition $Q \circ P$ is a $q$-linear operator over $D \otimes C$.*

Applying Lemma 2.1 to the $q$-linear operator $\delta_{W,V} \circ \delta_{U,W}$ we obtain a morphism of algebras

$$\Delta_{V,W,U} : A(V,U) \to A(V,W) \otimes A(W,U).$$

It preserves degree, i.e. for each $d \geq 0$, we have

$$\Delta_{V,W,U} : A(V,U)_d \to A(V,W)_d \otimes A(W,U)_d.$$

Let $S(V,W;d) := (A(W,V)_d)^*$. From $\Delta_{V,W,U}$, we obtain a bilinear map

$$m_{U,W,V} : S(W,V;d) \times S(U,W;d) \to S(U,V;d).$$

For any $a \in S(W,V;d)$ and $b \in S(U,W;d)$, we denote by $b \circ a$ the element $m_{U,W,V}(a,b) \in S(U,V;d)$. It is given by the following composition

$$\begin{tikzcd} A(V,U)_d \arrow[r, "\Delta_{V,W,U}"] \arrow[dr, "b \circ a"] & A(V,W)_d \otimes A(W,U)_d \arrow[d, "a \otimes b"] \\ & \Bbbk \end{tikzcd} \tag{2.1.5}$$

**Lemma 2.3.** *For $V, W \in \mathcal{V}$ and $P_1, ..., P_n \in \mathrm{Hom}(V,W)$ we have*

$$\left( \bigcap_i \ker(P_i) \right)^* \cong \frac{V^*}{\sum_i \mathrm{im}(P_i^*)}.$$

*Proof.* Define $f : V \to W^{\oplus n}$ by $f(v) = (P_1(v), ..., P_n(v))$ and consider the short exact sequence $0 \to \mathrm{im}(f^*) \to V^* \to \ker(f)^* \to 0$. □

The following proposition generalizes [PW, Theorem 11.3.1].

**Proposition 2.4.** *Let $V, W$ be Hecke pairs. Then there exists a natural isomorphism*

$$S(V,W;d) \simeq \mathrm{Hom}_{\mathcal{H}_d}(V^{\otimes d}, W^{\otimes d}).$$

*Proof.* For $i = 1, ..., d-1$ define operators $\dot{T}_i$ on $\mathrm{Hom}(V^{\otimes d}, W^{\otimes d})$ by

$$\dot{T}_i(X) := X \circ \rho_{d,V}(T_i) - \rho_{d,W}(T_i) \circ X,$$

for $X \in \mathrm{Hom}(V^{\otimes d}, W^{\otimes d})$. Recall that $\rho_{d,V}$ (resp. $\rho_{d,W}$) denotes the right action of $\mathcal{H}_d$ on $V^{\otimes d}$ (resp. $W^{\otimes d}$). Note that

$$\mathrm{Hom}_{\mathcal{H}_d}(V^{\otimes d}, W^{\otimes d}) = \bigcap_{i=1}^{d-1} \ker(\dot{T}_i).$$

Similarly we define operators $\ddot{T}_i$ on $\mathrm{Hom}(W^{\otimes d}, V^{\otimes d})$ by

$$\ddot{T}_i(Y) := Y \circ \rho_{d,W}(T_i) - \rho_{d,V}(T_i) \circ Y,$$

for $Y \in \mathrm{Hom}(W^{\otimes d}, V^{\otimes d})$. Note that from (2.1.3) we have

$$A(W,V)_d \cong \frac{\mathrm{Hom}(W^{\otimes d}, V^{\otimes d})}{\sum_{i=1}^{d-1} \mathrm{im}(\ddot{\mathrm{T}}_\mathrm{i})}.$$

Therefore by Lemma 2.3 we have $A(W,V)_d \cong \left(\bigcap \ker(\ddot{T}_i^*)\right)^*$, where $\ddot{T}_i^*$ denotes the dual operator on $\mathrm{Hom}(W^{\otimes d}, V^{\otimes d})^*$. Now consider the non-degenerate pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}(V^{\otimes d}, W^{\otimes d}) \times \mathrm{Hom}(W^{\otimes d}, V^{\otimes d}) \to \mathbb{k},$$

given by $\langle X, Y \rangle := \mathrm{trace(Y \circ X)}$. For each $i$ we have

$$\langle \dot{T}_i(X), Y \rangle = -\langle X, \ddot{T}_i(Y) \rangle.$$

Therefore, under the identification $\mathrm{Hom}(V^{\otimes d}, W^{\otimes d}) \cong \mathrm{Hom}(W^{\otimes d}, V^{\otimes d})^*$ induced by this pairing, $\bigcap \ker(\dot{T}_i)$ is identified with $\bigcap \ker(\ddot{T}_i^*)$. Hence

$$A(W,V)_d \cong \mathrm{Hom}_{\mathcal{H}_d}(V^{\otimes d}, W^{\otimes d})^*,$$

proving the result.

$\square$

The following lemma is routine to check.

**Lemma 2.5.** *Given three Hecke pairs $V, W, U$, then the following diagram commutes:*

$$\begin{tikzcd} S(W,V;d) \times S(U,W;d) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{H}_d}(W^{\otimes d}, V^{\otimes d}) \times \mathrm{Hom}_{\mathcal{H}_d}(U^{\otimes d}, W^{\otimes d}) \arrow[d] \\ S(U,V;d) \arrow[r] & \mathrm{Hom}_{\mathcal{H}_d}(U^{\otimes d}, V^{\otimes d}) \end{tikzcd} \quad (2.1.6)$$

## 2.2 Quantum matrices

Let $V_n$ be the vector spaces $\Bbbk^n$ with standard basis $e_1, e_2, \cdots, e_n$. Let $R_n : V_n \otimes V_n \to V_n \otimes V_n$ be a linear operator defined as follows:

$$R_n(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{if } i < j \\ qe_i \otimes e_j & \text{if } i = j \\ (q - q^{-1})e_i \otimes e_j + e_j \otimes e_i & \text{if } i > j \end{cases}, \tag{2.2.7}$$

where $q \in \Bbbk$. The following is well-known and easy to check (see e.g. Lemma 4.8 in [T]).

**Lemma 2.6.** *For any $n$, $R_n : V_n^{\otimes 2} \to V_n^{\otimes 2}$ is a Hecke operator.*

Let $\rho_{d,n} : \mathcal{H}_d \to \mathrm{End}(V_n^{\otimes d})$ denote the corresponding right $\mathcal{H}_d$-module. Let

$$\mathcal{O}_q(M_{m,n}) = A(V_m, V_n)$$

denote the quantum Hom-space algebra from $(V_n, R_n)$ to $(V_m, R_m)$. This is the algebra of **quantum** $m \times n$ **matrices.** Let $\{x_{ji}\}$ be the standard basis of $\mathrm{Hom}(V_m, V_n)$ mapping $e_k \mapsto \delta_{ik} e_j$.

The following lemma is easy.

**Lemma 2.7.** *The algebra $\mathcal{O}_q(M_{m,n})$ is generated by $x_{ji}$, $1 \leq j \leq m$, $1 \leq i \leq n$, subject to the following relations: where $i > j$ and $k > \ell$:*

$$\begin{aligned} x_{ik}x_{i\ell} &= qx_{i\ell}x_{ik} \\ x_{ik}x_{jk} &= qx_{jk}x_{ik} \\ x_{i\ell}x_{jk} &= x_{jk}x_{i\ell} \\ x_{ik}x_{j\ell} - x_{j\ell}x_{ik} &= (q - q^{-1})x_{i\ell}x_{jk}. \end{aligned}$$

Set $\Delta_{\ell,m,n} : \mathcal{O}_q(M_{\ell,n}) \to \mathcal{O}_q(M_{\ell,m}) \otimes \mathcal{O}_q(M_{m,n})$. On generators $\Delta_{\ell,m,n}$ is given by

$$x_{ij} \mapsto \sum_{k=1}^{m} x_{ik} \otimes x_{kj}.$$

Usually $\ell, m, n$ are clear from context and we omit them from the notation.

Note that the algebra $\mathcal{O}_q(M_n) := \mathcal{O}_q(M_{n,n})$ is a bialgebra with counit $\epsilon : \mathcal{O}_q(M_n) \to \Bbbk$ given by $\epsilon(x_{ij}) = \delta_{ij}$. In fact, $\mathcal{O}_q(M_n)$ is the well-known quantum algebra of functions on $n \times n$ matrices (cf. [T, §4]).

Our definition of quantum $m \times n$ matrices is a direct generalization of $\mathcal{O}_q(M_n)$. In particular the ring $\mathcal{O}_q(M_{m,n})$ is a deformation of the ring of functions on the space $m \times n$ matrices over $\Bbbk$. Indeed by the above lemma we have $\mathcal{O}(M_1(m,n)) \cong \mathcal{O}(\mathrm{Hom}(\Bbbk^n, \Bbbk^m))$, the algebra of functions on $\mathrm{Hom}(\Bbbk^n, \Bbbk^m)$.

Let $\mathcal{O}_q^d(M_{m,n})$ denote the subspace of $\mathcal{O}_q(M_{m,n})$ spanned by monomials of degree $d$. Set $S_q(m,n;d) = (\mathcal{O}_q^d(M_{n,m}))^*$. Let $m_{\ell,m,n} : S_q(m,\ell;d) \otimes S_q(n,m;d) \to S_q(n,\ell;d)$ be the corresponding bilinear maps. The map $m_{n,n,n}$ induces an algebra structure on $S_q(n,d) := S_q(n,n;d)$ called the $q$**-Schur algebra** (cf. [T, §11]).

# 3 Main definitions

## 3.1 Classical polynomial functors

We recall the category of strict polynomial functors.

Let $\mathfrak{S}_d$ denote the symmetric group on $d$ letters. For any $V \in \mathcal{V}$ the symmetric group $\mathfrak{S}_d$ acts on the tensor product $V^{\otimes d}$ by permuting factors. For $V \in \mathcal{V}$ the $d$-th divided power of $V$ is defined as the invariants $\Gamma^d(V) = (\otimes^d V)^{\mathfrak{S}_d}$. Let $\Gamma^d\mathcal{V}$ denote the category consisting of objects $V \in \mathcal{V}$ and morphisms

$$\mathrm{Hom}_{\Gamma^d}(V, W) = \Gamma^d(\mathrm{Hom}(V, W)).$$

The diagonal inclusion $\mathfrak{S}_d \subset \mathfrak{S}_d \times \mathfrak{S}_d$ induces a morphism

$$\Gamma^d(U) \otimes \Gamma^d(V) \to \Gamma^d(U \otimes V).$$

Composition in $\Gamma^d\mathcal{V}$ is then defined as

$$\begin{tikzcd}
\Gamma^d(\mathrm{Hom}(V,U)) \otimes \Gamma^d(\mathrm{Hom}(W,V)) \arrow[r] & \Gamma^d(\mathrm{Hom}(V,U) \otimes \mathrm{Hom}(W,V)) \arrow[d] \\
 & \Gamma^d(\mathrm{Hom}(W,U)).
\end{tikzcd}$$

Let $\mathcal{P}^d$ be the category consisting of $\Bbbk$-linear functors $\Gamma^d\mathcal{V} \to \mathcal{V}$. Morphisms $\mathcal{P}^d$ are natural transformations of functors. $\mathcal{P}^d$ is the category of **polynomial functors of degree** $d$

We remark that this is not the definition of $\mathcal{P}^d$ which originally appears in e.g. Friedlander and Suslin's work [FS]. In their presentation polynomial functors have both source and target the category $\mathcal{V}$, and it is required that maps between Hom-spaces are polynomial. In the presentation we use, the polynomial condition is encoded in the category $\Gamma^d\mathcal{V}$. For details see [Kr, Ku] and references therein.

### 3.2 Definition of quantum polynomial functors

Note that in the above setup, $\Gamma^d(\mathrm{Hom}(V, W)) \cong \mathrm{Hom}_{\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d})$. This observation motivates our definition of quantum polynomial functors.

For any $d \geq 0$, we define a category $\Gamma^d_q\mathcal{V}$: it consists of objects $0, 1, 2, ...$ and the morphisms are defined as

$$\mathrm{Hom}_{\Gamma^d_q}(m, n) := \mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}).$$

A **quantum polynomial functor of degree** $d$ is defined to be a linear functor

$$M : \Gamma^d_q\mathcal{V} \to \mathcal{V}.$$

We denote by $\mathcal{P}^d_q$ the category of quantum polynomial functors of degree $d$. Morphisms are natural transformations of functors. Since $\mathcal{V}$ is abelian $\mathcal{P}^d_q$ is also an abelian category. Let $\mathcal{P}_q$ be the category of all quantum polynomial functors,

$$\mathcal{P}_q := \bigoplus_d \mathcal{P}^d_q.$$

Given $M \in \mathcal{P}_q$ we denote the map on hom-spaces by $M_{m,n} : \mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}) \to \mathrm{Hom}(M(m), M(n))$.

**Remark 3.1.** When $q = 1$ our construction recovers the classical category $\mathcal{P}^d$. Indeed the natural functor $\Gamma^d_1\mathcal{V} \to \Gamma^d\mathcal{V}$ defined by $n \mapsto \Bbbk^n$ is an equivalence of categories, and induces an equivalence $\mathcal{P}^d_1 \cong \mathcal{P}^d$.

$\mathcal{P}_q$ has a monoidal structure. For any $M \in \mathcal{P}_q^d$ and $N \in \mathcal{P}_q^e$ define the tensor product $M \otimes N \in \mathcal{P}_q^{d+e}$ as follows: for any $n$, $(M \otimes N)(n) := M(n) \otimes N(n)$ and for any $m, n$, the map on morphisms is given by the composition

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}_{d+e}}(V_m^{\otimes d+e}, V_n^{\otimes d+e}) \arrow[r] & \mathrm{Hom}_{\mathcal{H}_d \otimes \mathcal{H}_e}(V_m^{\otimes d} \otimes V_m^{\otimes e}, V_n^{\otimes d} \otimes V_n^{\otimes e}) \arrow[d] \\
& \mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}) \otimes \mathrm{Hom}_{\mathcal{H}_e}(V_m^{\otimes e}, V_n^{\otimes e}) \arrow[d, "M_{m,n} \otimes N_{m,n}"] \\
& \mathrm{Hom}(M(m), M(n)) \otimes \mathrm{Hom}(N(m), N(n)) \arrow[d] \\
& \mathrm{Hom}(M(m) \otimes N(m), M(n) \otimes N(n))
\end{tikzcd}.$$

A duality is defined on $\mathcal{P}_q$ as follows. We first identify $V_m \cong V_m^*$ via the standard basis $e_i$, i.e. if $e_1^*, ..., e_m^*$ denotes the dual basis of $V_m^*$ then $V_m \to V_m^*$ is given by $e_i \mapsto e_i^*$. This induces an identification

$$\sigma : \mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}) \to \mathrm{Hom}_{\mathcal{H}_d}(V_n^{\otimes d}, V_m^{\otimes d}).$$

For $M \in \mathcal{P}_q^d$ we define $M^\sharp \in \mathcal{P}_q^d$ by:

(i) $M^\sharp(n) := M(n)^*$,

(ii) $M^\sharp_{m,n} : \mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}) \to \mathrm{Hom}(M^\sharp(m), M^\sharp(n))$ is given by the composition

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}) \arrow[r, "\sigma"] & \mathrm{Hom}_{\mathcal{H}_d}(V_n^{\otimes d}, V_m^{\otimes d}) \arrow[r, "M_{n,m}"] & \mathrm{Hom}(M(n), M(m)) \arrow[d, "\cong"] \\
& & \mathrm{Hom}(M(m)^*, M(n)^*)
\end{tikzcd}$$

Given a morphism $f : M \to N$ in $\mathcal{P}_q$, we define $f^\sharp : N^\sharp \to M^\sharp$ by $f^\sharp(n) = f(n)^*$. It is straightforward to check that $f^\sharp$ is a morphism of polynomial functors. Therefore duality defines a contravariant functor $\sharp : \mathcal{P}_q \to \mathcal{P}_q$. The following lemma is routine to check.

**Lemma 3.2.** *Given any two quantum polynomial functors $M, N$, then we have a canonical isomorphism*

$$(M \otimes N)^\sharp \simeq M^\sharp \otimes N^\sharp.$$

## 3.3 Examples

Here are some examples of quantum polynomial functors.

1. The identity functor $I \in \mathcal{P}_q^1$ is given by $I(n) = V_n$. On morphisms it is the identity map.
2. We denote by $\bigotimes^d$ the $d$-th tensor product functor. It is given by $n \mapsto V_n^{\otimes d}$ and on morphisms by the natural inclusion

$$\mathrm{Hom}_{\mathcal{H}_d}(V_n^{\otimes d}, V_m^{\otimes d}) \to \mathrm{Hom}(V_n^{\otimes d}, V_m^{\otimes d}).$$

   Notice that $\bigotimes^d = I^{\otimes d}$. It is also easy to see that the right action of Hecke algebra $\mathcal{H}_d$ on $V_n^{\otimes d}$ gives rise to endomrophisms of $\otimes^d$ as quantum polynomial functors, i.e. for any $w \in \mathfrak{S}_d$, $T_w : \otimes^d \to \otimes^d$ is a morphism.

3. Given a right $\mathcal{H}_d$-module $\rho$ define $M_\rho \in \mathcal{P}_q^d$ by:

$$M_\rho : n \mapsto \mathrm{Hom}_{\mathcal{H}_d}(\rho, V_n^{\otimes d}).$$

Similarly for a left $\mathcal{H}_d$-module $\tau$ define $N_\tau$ by:

$$N_\rho : n \mapsto V_n^{\otimes d} \otimes_{\mathcal{H}_d} \tau.$$

Let $\chi_+$ be the character of $\mathcal{H}_d$ given by $\chi_+(T_i) = q$, and let $\chi_-$ be the character given by $\chi_-(T_i) = -q^{-1}$. Define the $d$-th $q$-**divided power** by $\Gamma_q^d := M_{\chi_+}$, the $d$-**th q-symmetric power** by $S_q^d := N_{\chi+}$, and the $d$-**th** $q$-**exterior power** by $\bigwedge_q^d := N_{\chi_-}$.

An important role will also be played by the functors $\Gamma_q^{d,m} := M_{V_m^{\otimes d}}$. Under this functor $n \mapsto \mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d})$.

## 3.4 An equivalent characterization of quantum polynomial functors

Given a quantum polynomial functor $M$ of degree $d$ we get a vector space $M(n)$ for any $n \geq 0$ and for any $m, n$, by Proposition 2.4, we naturally get a map:

$$M_{m,n} : S_q(m, n; d) \to \mathrm{Hom}(M(m), M(n)).$$

This gives rise to maps

$$M'_{m,n} : S_q(m, n; d) \otimes M(m) \to M(n),$$

and

$$M''_{m,n} : M(m) \to M(n) \otimes \mathcal{O}_q^d(M_{n,m}).$$

The following proposition gives an equivalent characterization of quantum polynomial functors in terms of the quantum matrix algebra.

**Proposition 3.3.** *A quantum polynomial functor $M$ of degree $d$ is equivalent to the following data:*

1. *for each positive integer a vector space $M(n) \in \mathcal{V}$;*
2. *given any two nonnegative integers $m, n$ a linear map*

$$M''_{m,n} : M(m) \to M(n) \otimes \mathcal{O}_q^d(M_{n,m})$$

*such that, for any $\ell, m, n$, the following diagrams commute*

$$\begin{tikzcd} M(\ell) \arrow[r, "M''_{\ell,n}"] \arrow[d, "M''_{\ell,m}"] & M(n) \otimes \mathcal{O}_q^d(M_{n,\ell}) \arrow[d, "1\otimes\Delta_{n,m,\ell}"] \\ M(m) \otimes \mathcal{O}_q^d(M_{m,\ell}) \arrow[r, "1\otimes M''_{m,n}"] & M(n) \otimes \mathcal{O}_q^d(M_{n,m}) \otimes \mathcal{O}_q^d(M_{m,\ell}) \end{tikzcd} \tag{3.4.8}$$

*and for any $n$,*

$$\begin{tikzcd} M(n) \arrow[r, "M''_{n,n}"] \arrow[d, "id"] & M(n) \otimes \mathcal{O}_q^d(M_n) \arrow[dl, "1\otimes\epsilon"] \\ M(n) \otimes \Bbbk & \end{tikzcd} \tag{3.4.9}$$

*Here $\epsilon : \mathcal{O}_q(M_n) \to \Bbbk$ is the co-unit map.*

*Proof.* It follows from Proposition 2.4 and Lemma 2.5. $\square$

### 3.5 Frobenius twist

In this subsection assume that $q$ is a primitive $\ell^{\text{th}}$ root of unity, where $\ell > 1$ is an odd integer. Define an algebra homomorphism

$$\mathsf{Fr} : \mathcal{O}(M_{m,n}) \to \mathcal{O}_q(M_{m,n}) \text{ by } x_{ij} \mapsto x_{ij}^{\ell}.$$

By Lemma 7.2.2 in [PW] we have that $\Delta_{m,r,n} : \mathcal{O}_q(M_{m,n}) \to \mathcal{O}_q(M_{m,r}) \otimes \mathcal{O}_q(M_{r,n})$ satisfies

$$\Delta(x_{ij}^{\ell}) = \sum_{k=1}^{r} x_{ik}^{\ell} \otimes x_{kj}^{\ell}.$$

Therefore the following diagrams commutes:

$$\begin{tikzcd}
\mathcal{O}(M_{m,n}) \arrow[r, "\mathsf{Fr}"] \arrow[d, "\Delta"] & \mathcal{O}_q(M_{m,n}) \arrow[d, "\Delta"] \\
\mathcal{O}(M_{m,r}) \otimes \mathcal{O}(M_{r,n}) \arrow[r, "\mathsf{Fr}\otimes\mathsf{Fr}"] & \mathcal{O}_q(M_{m,r}) \otimes \mathcal{O}_q(M_{r,n})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{O}(M_n) \arrow[r, "\mathsf{Fr}"] \arrow[d, "\epsilon"] & \mathcal{O}_q(M_n) \arrow[dl, "\epsilon"] \\
\mathbb{k} &
\end{tikzcd}$$

Using this, we can define the **Frobenius twist** $(-)^{(1)} : \mathcal{P}^d \to \mathcal{P}_q^{\ell d}$. Indeed given a strict polynomial functor $M \in \mathcal{P}^d$, as in the quantum case we get maps

$$M''_{m,n} : M(\mathbb{k}^m) \to M(\mathbb{k}^n) \otimes \mathcal{O}(M_{n,m}).$$

Then set $M^{(1)}(n) = M(\mathbb{k}^n)$, and on morphisms define

$$(M^{(1)})''_{m,n} : M(\mathbb{k}^m) \xrightarrow{M''_{m,n}} M(\mathbb{k}^n) \otimes \mathcal{O}(M_{n,m}) \xrightarrow{1\otimes\mathsf{Fr}} M(\mathbb{k}^n) \otimes \mathcal{O}_q(M_{n,m}).$$

By Proposition 3.3 and the commutativity of the above two diagrams, $M^{(1)}$ is a quantum polynomial functor. The definition of $(-)^{(1)} : \mathcal{P}^d \to \mathcal{P}_q^{\ell d}$ on natural transformations is straightforward.

## 4 Finite generation and the representability theorem

**Definition 4.1.** The quantum polynomial functors $M \in \mathcal{P}_q^d$ is $m$**-generated** if for any $n$ the map

$$M'_{m,n} : S_q(m,n;d) \otimes M(m) \to M(n)$$

is surjective. $M$ is **finitely generated** if it is $m$-generated for some $m$.

Let $\underline{i} = \{i_1, ..., i_r\}$ be a set of positive integers. Define a homomorphism

$$\phi_{\underline{i}} : \mathcal{O}_q(M_{n,m}) \to \mathbb{k}$$

by $x_{k\ell} \mapsto 1$ if $k = \ell$ and $k \in \underline{i}$, and otherwise $x_{k\ell} \mapsto 0$. By restriction we get a linear map

$$\phi_{\underline{i}}^d : \mathcal{O}_q^d(M_{n,m}) \to \mathbb{k}.$$

In other words, $\phi_{\underline{i}}^d \in S_q(m, n, d)$. For $M \in \mathcal{P}_q^d$ set

$$M_{m,n,\underline{i}} = M_{m,n}(\phi_{\underline{i}}^d) \in \mathrm{Hom}(M(m), M(n)).$$

**Lemma 4.2.** *Let $V$ be a vector space over $\mathbb{k}$. We fix vectors $v_1, v_2, \cdots, v_n \in V$. For any homogeneous polynomial of degree $d$, if $d < n$ then*

$$f(v_1 + v_2 + \cdots + v_n) = \sum_{\underline{i} \subset \{1,2,\cdots,n\}, |\underline{i}| \leq d} (-1)^{n-|\underline{i}|} f(\sum_{k \in \underline{i}} v_k),$$

*where $|\underline{i}|$ is the cardinality of the set $\underline{i} \subset \{1, 2, \cdots, n\}$.*

*Proof.* This is [FS, Lemma 2.8]. □

**Lemma 4.3.** *If $m > d$ then $\phi_{\{1,\ldots,m\}}^d \in S_q(m, d)$ is an integral linear combination of $\phi_{\underline{i}}^d$ where $|\underline{i}| \leq d$.*

*Proof.* There is a homomorphism of algebras $\delta : \mathcal{O}_q(M_m) \to \mathbb{k}[t_1, t_2, \cdots, t_m]$ given by $x_{kk} \mapsto t_k$; $x_{k\ell} \mapsto 0$ if $k \neq \ell$. Note that for any $\underline{i}$, $\phi_{\underline{i}}$ factors through the homomorphism $\tilde{\phi}_{\underline{i}} : \mathbb{k}[t_1, t_2, \cdots, t_m] \to \mathbb{k}$, where

$$\tilde{\phi}_{\underline{i}}(t_k) = \begin{cases} 1 & \text{if } k \in \underline{i} \\ 0 & \text{otherwise} \end{cases},$$

i.e. we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{O}_q(M_m) \arrow[r, "\delta"] \arrow[dr, "\phi_{\underline{i}}"] & \mathbb{k}[t_1, \cdots, t_m] \arrow[d, "\tilde{\phi}_{\underline{i}}"] \\ & \mathbb{k} \end{tikzcd} . \tag{4.0.10}$$

Let $\tilde{\phi}_{\underline{i}}^d$ be the restriction of $\tilde{\phi}_{\underline{i}}$ to $\mathbb{k}^d[t_1, t_2, \cdots, t_m]$. Observe that for any polynomial $f \in \mathbb{k}^d[t_1, t_2, \cdots, t_m]$, we have

$$\tilde{\phi}_{\underline{i}}^d(f) = f(\sum_{i \in \underline{i}} e_i),$$

where $e_i$ is the $i$-th basis in $\mathbb{k}^m$. Therefore the lemma follows from Lemma 4.2.

□

**Lemma 4.4.** *Let $\underline{i}, \underline{j}$ be sets of positive integers and consider $\phi_{\underline{i}}^d \in S_q(\ell, m, d)$ and $\phi_{\underline{j}}^d \in S_q(m, n, d)$. Furthermore consider $\phi_{\underline{i} \cap \underline{j}}^d \in S_q(\ell, n, d)$. Then we have*

$$\phi_{\underline{j}}^d \circ \phi_{\underline{i}}^d = \phi_{\underline{i} \cap \underline{j}}^d.$$

*Therefore $M_{m,n,\underline{j}} \circ M_{\ell,m,\underline{i}} = M_{\ell,n,\underline{i} \cap \underline{j}}$.*

*Proof.* It suffices to show that $(\phi_{\underline{j}} \otimes \phi_{\underline{i}}) \circ \Delta_{n,m,\ell} = \phi_{\underline{i} \cap \underline{j}}$, and for this it suffices to show that both sides of the equation agree on $x_{ab} \in \mathcal{O}_q(M_{n,\ell})$:

$$(\phi_{\underline{j}} \otimes \phi_{\underline{i}})(\Delta_{n,m,\ell}(x_{ab})) = (\phi_{\underline{j}} \otimes \phi_{\underline{i}})(\sum_{p=1}^{m} x_{ap} \otimes x_{pb}) \tag{4.0.11}$$

$$= \sum_{p=1}^{m} \phi_{\underline{j}}(x_{ap})\phi_{\underline{i}}(x_{pb}) \tag{4.0.12}$$

Since $\phi_{\underline{j}}(x_{ap})\phi_{\underline{i}}(x_{pb}) = 1$ if and only if $a = b = p$ and $a \in \underline{i} \cap \underline{j}$ we have that

$$\sum_{p=1}^{m} \phi_{\underline{j}}(x_{ap})\phi_{\underline{i}}(x_{pb}) = \phi_{\underline{i} \cap \underline{j}}(x_{ab}).$$

The second statement of the lemma follows immediately. $\square$

**Proposition 4.5.** *$M \in \mathcal{P}_q^d$ is $m$-generated for any $m \geq d$.*

*Proof.* We need to show that $M'_{m,n} : S_q(m,n,d) \otimes M(m) \to M(n)$ given by $\phi \otimes v \mapsto M_{m,n}(\phi)(v)$ is surjective for any $n$.

Suppose $m \geq n$ and choose $\underline{i} = \{1, ..., n\}$. By Lemma 4.4 $M_{n,n,\underline{i}} = M_{m,n,\underline{i}} \circ M_{n,m,\underline{i}}$. Now note that $\phi_{\underline{i}}^d \in S_q(n,d)$ is the unit element, and hence $M_{n,n,\underline{i}} = 1_{M(n)}$. Therefore $M_{m,n,\underline{i}}$ is surjective which implies that $M'_{m,n}$ is as well.

Now suppose $m < n$. By Lemma 4.3 the identity operator $1_{M(n)}$ is an integral linear combination of $M_{m,m,\underline{i}}$, where $|\underline{i}| \leq d$. Therefore we have, by Lemma 4.4,

$$\begin{aligned} 1_{M(n)} &= \sum_{|\underline{i}| \leq d} a_{\underline{i}} M_{n,n,\underline{i}} \\ &= \sum_{|\underline{i}| \leq d} a_{\underline{i}} M_{m,n,\underline{i}} \circ M_{n,m,\underline{i}}, \end{aligned}$$

where $a_{\underline{i}} \in \mathbb{Z}$ and only finitely many are nonzero. Given $v \in M(n)$ let $v_{\underline{i}} = M_{n,m,\underline{i}}(v)$. Then we have that $v = \sum_{|\underline{i}| \leq d} a_{\underline{i}} M_{m,n,\underline{i}}(v_{\underline{i}})$, i.e.

$$M'_{m,n} \left( \sum_{|\underline{i}| \leq d} a_{\underline{i}} \phi_{\underline{i}}^d \otimes v_{\underline{i}} \right) = v$$

proving that $M'_{m,n}$ is surjective. $\square$

**Proposition 4.6.** *For any $n \geq 0$, the divided power $\Gamma_q^{d,n}$ represents the evaluation functor $\mathcal{P}_q^d \to \mathcal{V}$ given by $M \mapsto M(n)$, i.e. there exists a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{P}_q^d}(\Gamma_q^{d,n}, M) \simeq M(n).$$

*Hence $\Gamma_q^{d,n}$ is a projective object in $\mathcal{P}_q^d$.*

*Proof.* We first show that given $M \in \mathcal{P}_q^d$ there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{P}_q^d}(\Gamma_q^{d,n}, M) \cong M(n)$$

for any $n$. Consider the map $\phi : M(n) \to \mathrm{Hom}_{\mathcal{P}_q^d}(\Gamma_q^{d,n}, M)$ given by $w \mapsto \phi_w$, where $\phi_w : \Gamma_q^{d,n} \to M$ is the natural transformation

$$\phi_w(-) = \mathsf{ev}_w \circ M_{n,-}.$$

In other words, $\phi_w(m) : \Gamma_q^{d,n}(m) \to M(m)$ is the map

$$x \in \mathrm{Hom}_{\mathcal{H}_d}(V_n^{\otimes d}, V_m^{\otimes d}) \mapsto M_{n,m}(x)(w) \in M(m).$$

Conversely, consider the map $\psi : \mathrm{Hom}_{\mathcal{P}_q^d}(\Gamma_q^{d,n}, M) \to M(n)$ defined as follows:

$$\begin{aligned} f \in \mathrm{Hom}_{\mathcal{P}_q^d}(\Gamma_q^{d,n}, M) &\rightsquigarrow f(n) : \mathrm{End}_{\mathcal{H}_d}(V_n^{\otimes d}) \to M(n) \\ &\rightsquigarrow f(n)(1_n) \in M(n) \end{aligned}$$

where $1_n \in \mathrm{End}_{\mathcal{H}_d}(V_n^{\otimes d})$ is the identity operator.

Unpackaging these definitions it is easy to see that $\phi$ is inverse to $\psi$, proving that $\Gamma_q^{d,n}$ represents the evaluation functor. It follows that $\Gamma_q^{d,n}$ is projective since the evaluation functor $ev_n : \mathcal{P}_q^d \to \mathcal{V}$, $M \mapsto M(n)$ is exact. □

For an algebra $A$ we let $\mathsf{Mod}(A)$ denote the category of finitely generated left $A$-modules.

**Theorem 4.7.** *If $n \geq d$ then $\Gamma_q^{d,n}$ is a projective generator of $\mathcal{P}_q^d$. Hence the evaluation functor $\mathcal{P}_q^d \to \mathsf{Mod}(S_q(n,d))$ is an equivalence of categories.*

*Proof.* By Proposition 4.6 we have that $\Gamma_q^{d,n}$ is projective. To see that it's a generator when $n \geq d$ it suffices to show that $M'_{n,-} : \Gamma_q^{d,n} \otimes M(n) \to M$ is surjective. This follows immediately from Proposition 4.5, which gives us that for every $m$ the map $M'_{n,m}$ is surjective. Hence the equivalence follows.

□

We now state a seriues of corollaries of Theorem 4.7. The first is well-known (cf. [BDK, p.26]) and it is an immediate consequence.

**Corollary 4.8.** *Let $d \geq 0$ be an integer. For any two integers $m, n \geq d$ the $q$-Schur algebras $S_q(n,d)$ and $S_q(m,d)$ are Morita equivalence.*

To state another corollary, we first note that the functor $\Gamma_q^{d,n}$ has a natural decomposition

$$\Gamma_q^{d,n} \cong \bigoplus_{d_1+\cdots+d_n=d} \Gamma_q^{d_1} \otimes \cdots \otimes \Gamma_q^{d_n}. \tag{4.0.13}$$

Indeed, by Frobenius reciprocity we have

$$\begin{aligned} \Gamma_q^{d_1}(m) \otimes \cdots \otimes \Gamma_q^{d_n}(m) &\cong \mathrm{Hom}_{\mathcal{H}_{d_1}\otimes\cdots\otimes\mathcal{H}_{d_n}}(\chi_+ \otimes \cdots \otimes \chi_+, V_m^{\otimes d}) \\ &\cong \mathrm{Hom}_{\mathcal{H}_d}(\mathrm{Ind}_{\mathcal{H}_{d_1}\otimes\cdots\otimes\mathcal{H}_{d_n}}^{\mathcal{H}_d}(\chi_+ \otimes \cdots \otimes \chi_+), V_m^{\otimes d}) \end{aligned}$$

and so (4.0.13) follows from the isomorphism which is due to Dipper-James (cf. [T, Proposition 11.5])

$$V_n^{\otimes d} \cong \bigoplus_{d_1+\cdots+d_n=d} \mathrm{Ind}_{\mathcal{H}_{d_1}\otimes\cdots\otimes\mathcal{H}_{d_n}}^{\mathcal{H}_d}(\chi_+ \otimes \cdots \otimes \chi_+). \tag{4.0.14}$$

By Proposition 2.4, (4.0.14) induces a partition of the unit of $S_q(n,d)$ into orthogonal idempotents: $1 = \sum 1_{\vec{d}}$, where the sum ranges over all $\vec{d} = (d_1, ..., d_n)$ such that

$d_1 + \cdots + d_n = d$. For $M \in \mathsf{Mod}(S_q(n,d))$ there is a corresponding decomposition into weight spaces

$$M = \bigoplus M_{\vec{d}},$$

where $M_{\vec{d}} = 1_{\vec{d}} M$.

**Corollary 4.9.** *Let $M \in \mathcal{P}_q^d$, $n \geq 0$ and $d_1, ..., d_n \geq 0$ such that $d_1 + \cdots + d_n = d$. Then under the isomorphism $\mathrm{Hom}_{\mathcal{P}_q}(\Gamma_q^{d,n}, M) \cong M(n)$ we have*

$$\mathrm{Hom}_{\mathcal{P}_q}(\Gamma_q^{d_1} \otimes \cdots \otimes \Gamma_q^{d_n}, M) \cong M(n)_{d_1,...,d_n}.$$

*Proof.* There is a canonical element $\iota_{(d_1,...,d_n)} \in \Gamma_q^{d_1}(n) \otimes \cdots \otimes \Gamma_q^{d_n}(n)$ corresponding to the inclusion

$$\mathrm{Ind}_{\mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_n}}^{\mathcal{H}_d} (\chi_+ \otimes \cdots \otimes \chi_+) \hookrightarrow V_n^{\otimes d}$$

under (4.0.14). The map $\mathrm{Hom}_{\mathcal{P}_q}(\Gamma_q^{d_1} \otimes \cdots \otimes \Gamma_q^{d_n}, M) \to M(n)_{d_1,...,d_n}$ is given by $f \mapsto f(n)(\iota_{(d_1,...,d_n)})$. This map lands in the $(d_1, ..., d_n)$ weight space since $f$ is a natural transformation. More precisely, under our identifications we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}_d}(\mathrm{Ind}_{\mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_n}}^{\mathcal{H}_d} (\chi_+ \otimes \cdots \otimes \chi_+), V_n^{\otimes d}) \arrow[r, "f(n)"] \arrow[d, "1_{(d_1,...,d_n)}"] & M(n) \arrow[d, "1_{(d_1,...,d_n)}"] \\
\mathrm{Hom}_{\mathcal{H}_d}(\mathrm{Ind}_{\mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_n}}^{\mathcal{H}_d} (\chi_+ \otimes \cdots \otimes \chi_+), V_n^{\otimes d}) \arrow[r, "f(n)"] & M(n)
\end{tikzcd}$$

which implies that $f(n)(\iota_{(d_1,...,d_n)}) = 1_{(d_1,...,d_n)} f(n)(\iota_{(d_1,...,d_n)})$. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}_q}(\Gamma_q^{d_1} \otimes \cdots \otimes \Gamma_q^{d_n}, M) \arrow[r] \arrow[d] & M(n)_{d_1,...,d_n} \arrow[d] \\
\mathrm{Hom}_{\mathcal{P}_q}(\Gamma_q^{d,n}, M) \arrow[r] & M(n)
\end{tikzcd}$$

This diagram clearly commutes. Since both vertical maps are inclusions and the bottom map is an isomorphism by Theorem 4.7, the top map is an isomorphism. □

The final corollary recovers a basic result relating the Hecke algebra and the $q$-Schur algebra.

**Corollary 4.10.** *The map $\mathcal{H}_d \to \mathrm{Hom}_{\mathcal{P}_q}(\otimes^d, \otimes^d)$ is an isomorphism. Hence for any $n \geq d$, the map $\mathcal{H}_d \to \mathrm{Hom}_{S_q(n,d)}(V_n^{\otimes d}, V_n^{\otimes d})$ is an isomorphism.*

*Proof.* By Corollary 4.9, we have $\mathrm{Hom}_{\mathcal{P}_d}(\otimes^d, \otimes^d) \simeq (V_d^{\otimes d})_{1,...,1}$. The space $(V_d^{\otimes d})_{1,...,1}$ has of basis $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$, where $i_1, i_2, \cdots, i_d$ are all distinct. Under this isomorphism, the map $\mathcal{H}_d \to (V_d^{\otimes d})_{1,...,1}$ is given by $T_w \mapsto e_{w(1)} \otimes e_{w(2)} \otimes \cdots \otimes e_{w(d)}$, for any $w \in \mathfrak{S}_d$. It is easy to see that this is a bijection.

The second statement now follows from the first one using Theorem 4.7. □

# 5 Braiding on $\mathcal{P}_q$

In this section we will use Theorem 4.7 to define a braiding on the category of quantum polynomial functors, thus showing that $\mathcal{P}_q$ is a braided monoidal category.

Observe first that if $M \in \mathcal{P}_q^d$ then, by Proposition 3.3, the map $M''_{n,n}$ induces on $M(n)$ the structure of an $\mathcal{O}_q^d(M_n)$-comodule:

$$M''_{n,n} : M(n) \to M(n) \otimes \mathcal{O}_q^d(M_n).$$

We will use the Sweedler notation to denote this coaction:

$$v \in M(n) \mapsto \sum v_0 \otimes v_1 \in M(n) \otimes \mathcal{O}_q^d(M_n)$$

For a coalgebra $C$ we let $\mathsf{CoMod}(C)$ be the category of finitely generated right $C$-comodules. Now suppose we are given $V \in \mathsf{CoMod}(\mathcal{O}_q(m)_d)$ and $W \in \mathsf{CoMod}(\mathcal{O}_q(m)_e)$. Then $V \otimes W \in \mathsf{CoMod}(\mathcal{O}_q(m)_{d+e})$ and there is a well-known morphism induced from the R-matrix

$$R_{V,W} : V \otimes W \to W \otimes V,$$

which is an isomorphism of $\mathcal{O}_q^{d+e}(M_n)$-comodules. We recall the construction of $R_{V,W}$ following Takeuchi [T, §12].

Define $\sigma : \mathcal{O}_q^1(M_n) \times \mathcal{O}_q^1(M_n) \to \Bbbk$ by

$$\sigma(x_{ii}, x_{jj}) = \begin{cases} 1 & \text{if } i < j \\ q & \text{if } i = j \\ 1 & \text{if } i > j \end{cases}$$

and in addition $\sigma(x_{ij}, x_{ji}) = q - q^{-1}$ if $i < j$ and $\sigma(x_{ij}, x_{kl}) = 0$ otherwise.

We extend $\sigma$ to a braiding on $\mathcal{O}_q(M_n)$ [T, Proposition 12.9]. This means that it is an invertible bilinear form on $\mathcal{O}_q(M_n)$ such that for all $x, y, z \in \mathcal{O}_q(M_n)$:

$$\begin{aligned} \sigma(xy, z) &= \sum \sigma(x, z_1)\sigma(y, z_2) \\ \sigma(x, yz) &= \sum \sigma(x_1, z)\sigma(x_2, y) \\ \sigma(x_1, y_1)x_2y_2 &= \sum y_1x_1\sigma(x_2, y_2) \end{aligned}$$

Here we again we use the Sweedler notation for the coproduct $\Delta : \mathcal{O}_q(M_n) \to \mathcal{O}_q(M_n) \otimes \mathcal{O}_q(M_n)$ so $\Delta(x) = \sum x_1 \otimes x_2$. The R-matrix is given by

$$R_{V,W}(v \otimes w) = \sum \sigma(v_1, w_1)w_0 \otimes v_0.$$

Note that $R_{V_n,V_n} = R_n$, where $R_n$ is defined in Section 2.2.

**Lemma 5.1.** *Let $d, e \geq 0$. Then there exists $\kappa \in \mathcal{H}_{d+e}$ such that for all $m \geq 1$*

$$R_{V_m^d, V_m^e} = \rho_{d+e,m}(\kappa).$$

*In particular $\kappa = T_{w_{d,e}}$ where $w_{d,e} \in \mathfrak{S}_{d+e}$ is given by*

$$w(i) = \begin{cases} i + e & \text{if } 1 \leq i \leq d \\ i - d & \text{if } d < i \end{cases}$$

*Proof.* For $U \in \mathsf{CoMod}(\mathcal{O}_q(m)_d), V \in \mathsf{CoMod}(\mathcal{O}_q(m)_e)$ and $W \in \mathsf{CoMod}(\mathcal{O}_q(m)_f)$ the following two diagrams commute:

$$\begin{tikzcd} U \otimes V \otimes W \arrow[rr, "R_{U,V\otimes W}"] \arrow[dr, "R_{U,V}\otimes 1_W"'] & & V \otimes W \otimes U \\ & V \otimes U \otimes W \arrow[ur, "1_V \otimes R_{U,W}"'] & \end{tikzcd} \tag{5.0.15}$$

and

$$\begin{tikzcd} U \otimes V \otimes W \arrow[rr, "R_{U\otimes V,W}"] \arrow[dr, "1_U\otimes R_{V,W}"'] & & W \otimes U \otimes V \\ & U \otimes W \otimes V \arrow[ur, "R_{U,W} \otimes 1_V"'] & \end{tikzcd} \tag{5.0.16}$$

These are well-known properties of the R-matrix, and follow from the fact that $\sigma$ is a braiding.

We will use these diagrams to prove the lemma by induction on $d+e$. If $d+e=2$ then the statement is tautological. If $d+e>2$ then suppose first $e \geq 2$. By (5.0.15) and the inductive hypothesis we have:

$$\begin{aligned} R_{V_m^{\otimes d},V_m^{\otimes e}} = R_{V_m^{\otimes d},V_m^{\otimes e-1}\otimes V_m} &= (1_{V_m^{\otimes e-1}} \otimes R_{V_m^{\otimes d},V_m}) \circ (R_{V_m^{\otimes d},V_m^{\otimes e-1}} \otimes 1_{V_m}) \\ &= (1_{V_m^{\otimes e-1}} \otimes \rho_{d+1,m}(T_{w_{d,1}}) \circ (\rho_{d+e-1,m}(T_{w_{d,e-1}}) \otimes 1_{V_m}) \\ &= \rho_{d+e,m}(T_{w_1}) \circ \rho_{d+e,m}(T_{w_2}) \\ &= \rho_{d+e,m}(T_{w_1}T_{w_2}) \end{aligned}$$

where $w_1, w_2 \in \mathfrak{S}_{d+e}$ are given by

$$w_1(i) = \begin{cases} i & \text{if } 1 \leq i \leq e-1 \\ i+1 & \text{if } e \leq i \leq e+d-1 \\ e & \text{if } i = e+d \end{cases}$$

and

$$w_2(i) = \begin{cases} e-1+i & \text{if } 1 \leq i \leq d \\ i-d & \text{if } d+1 \leq i \leq e+d-1 \\ e+d & \text{if } i = e+d \end{cases}$$

Since $w_1 w_2 = w_{d,e}$ and $\ell(w_1)+\ell(w_2) = \ell(w_{d,e})$ (where $\ell$ is the usual length function), we have that $T_{w_1}T_{w_2} = T_{w_{d,e}}$ and the result follows.

In the case that $e<2$ then $d \geq 2$ and a similar induction applies, where one uses (5.0.16) instead of (5.0.15) . □

Now suppose $M \in \mathcal{P}_q^d$ and $N \in \mathcal{P}_q^e$. Define

$$R_{M,N} : M \otimes N \to N \otimes M$$

by $R_{M,N}(m) = R_{M(m),N(m)}$.

**Theorem 5.2.** *$R$ induces a braiding on the category $\mathcal{P}_q$. In other words, let $M \in \mathcal{P}_q^d$ and $N \in \mathcal{P}_q^e$. Then $R_{M,N} \in \mathrm{Hom}_{\mathcal{P}_q}(M \otimes N, N \otimes M)$ and moreover $R_{M,N}$ is an isomorphism.*

*Proof.* We only need to show that $R_{M,N} \in \mathrm{Hom}_{\mathcal{P}_q}(M \otimes N, N \otimes M)$; the fact that $R_{M,N}$ is an isomorphism then follows immediately.

We first prove $R_{M,N} \in \mathrm{Hom}_{\mathcal{P}_q}(M \otimes N, N \otimes M)$ in the case where $M = \bigotimes^d$ and $N = \bigotimes^e$. In that case we need to show that for any $x \in \mathrm{Hom}_{\mathcal{H}_{d+e}}(V_m^{\otimes d+e}, V_n^{\otimes d+e})$ the diagram

$$\begin{tikzcd} V_m^{\otimes d} \otimes V_m^{\otimes e} \arrow[r, "\bigotimes^{d+e}(x)"] \arrow[d, "R_{V_m^{\otimes d}, V_m^{\otimes e}}"'] & V_n^{\otimes d} \otimes V_n^{\otimes e} \arrow[d, "R_{V_n^{\otimes d}, V_n^{\otimes e}}"] \\ V_m^{\otimes e} \otimes V_m^{\otimes d} \arrow[r, "\bigotimes^{d+e}(x)"] & V_n^{\otimes e} \otimes V_n^{\otimes d} \end{tikzcd} \tag{5.0.17}$$

commutes. Cleary we have that $\bigotimes^{d+e}(x) \in \mathrm{Hom}_{\mathcal{H}_{d+e}}(V_m^{\otimes d+e}, V_n^{\otimes d+e})$, i.e. for all $\tau \in \mathcal{H}_{d+e}$

$$\bigotimes^{d+e}(x) \circ \rho_{d+e,m}(\tau) = \rho_{d+e,n}(\tau) \circ \bigotimes^{d+e}(x).$$

In particular this is true for $\tau = \kappa$, which, by Lemma 5.1, is precisely the commutativity of (5.0.17).

Now, by Theorem 4.7, any $M \in \mathcal{P}_q^d$ is a subquotient of some copies of $\bigotimes^d$. Therefore to prove the theorem in general it suffices to prove it for $M = M'/M''$ and $N = N'/N''$, where $M'' \subset M' \subset \bigotimes^d$ and $N'' \subset N' \subset \bigotimes^e$. In other words, we need to show that for $M$ and $N$ as in the previous sentence and any $x \in \mathrm{Hom}_{\mathcal{H}_{d+e}}(V_m^{\otimes d+e}, V_n^{\otimes d+e})$ the diagram

$$\begin{tikzcd} M(m) \otimes N(m) \arrow[r, "M\otimes N(x)"] \arrow[d, "R_{M(m),N(m)}"'] & M(n) \otimes N(n) \arrow[d, "R_{M(n),N(n)}"] \\ N(m) \otimes M(m) \arrow[r, "N\otimes M(x)"] & N(n) \otimes M(n) \end{tikzcd}$$

the diagram commutes. This is a consequence of the commutativity of (5.0.17) and the fact that the R-matrix is compatible with restriction. In other words, given $V \in \mathsf{CoMod}(\mathcal{O}_q(m)_d)$ and $W \in \mathsf{CoMod}(\mathcal{O}_q(m)_e)$ and sub-comodules $V' \subset V$ and $W' \subset W$ then $R_{V',W'} = R_{V,W}|_{V' \otimes W'}$. $\square$

Let $\Omega(n,d)$ be the set of tuples $I = (i_1, i_2, \cdots, i_d)$, where $1 \leq i_k \leq n$ for any $1 \leq k \leq d$. We call $I$ **increasing** if $i_1 \leq i_2 \leq \cdots \leq i_d$ and $I$ is **strictly increasing** if $i_1 < i_2 < \cdots < i_d$. We denote by $e_I$ the element $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} \in V_n^{\otimes d}$. We now introduce a pairing $(,)$ on $V_n^{\otimes d}$, for any $I, J \in \Omega(n,d)$,

$$(e_I, e_J) := \delta_{IJ},$$

where $\delta_{IJ}$ is the Kronecker symbol.

**Lemma 5.3.** *Given any $w \in \mathfrak{S}_d$, $I, J \in \Omega(n,d)$, we have*

$$(e_I \cdot T_w, e_J) = (e_I, e_J \cdot T_{w^{-1}}).$$

*Proof.* It can be reduced to the case $d = 2$. In this case, it suffices to check that for any $i, j, k, \ell$,

$$(R_n(e_i \otimes e_j), e_k \otimes e_\ell) = (e_i \otimes e_j, R_n(e_k \otimes e_\ell)).$$

This is a straightforward computation from the definition of the R-matrix $R_n$. $\square$

**Lemma 5.4.** *There exists a canonical isomorphism.* $(\bigotimes^d)^\sharp \simeq \bigotimes^d$.

*Proof.* It easily follow from the definition of duality functor $\sharp$. $\square$

By this lemma, we can identify $\bigotimes^d$ and $(\bigotimes^d)^\sharp$.

**Proposition 5.5.** *Given any $w \in \mathfrak{S}_d$, we have*

$$(T_w)^\sharp = T_{w^{-1}} : \bigotimes^d \to \bigotimes^d .$$

*Proof.* It follows from Lemma 5.3 and Lemma 5.4. □

The following proposition is about the compatibility between the duality functor $\sharp$ and the braiding $R$.

**Proposition 5.6.** *Given any two quantum polynomial functors $M, N \in \mathcal{P}_q$, we have*

$$(R_{M,N})^\sharp = R_{N^\sharp, M^\sharp}.$$

*Proof.* It suffices to check the following diagram commutes,

$$\begin{tikzcd} (N \otimes M)^\sharp \arrow[r, "(R_{M,N})^\sharp"] \arrow[d, "\simeq"] & (M \otimes N)^\sharp \arrow[d, "\simeq"] \\ N^\sharp \otimes M^\sharp \arrow[r, "R_{N^\sharp, M^\sharp}"] & M^\sharp \otimes N^\sharp \end{tikzcd} \tag{5.0.18}$$

where the horizontal maps are the canonical isomorphisms in Lemma 3.2. By the functoriality of $R$, as the argument in Theorem 5.2 we can reduce to the case $M = \bigotimes^d$ and $N = \bigotimes^e$. Under the identification $(\bigotimes^n)^\sharp \simeq \bigotimes^n$ for any $n$, it is enough for us to check $(R_{\otimes^d, \otimes^e})^\sharp = R_{\otimes^e, \otimes^d}$. By Theorem 5.2 and Proposition 5.5, we only need to show that $w_{d,e}^{-1} = w_{e,d}$, which is clearly true. □

# 6 Quantum Schur and Weyl Functors

In this section we assume $q^2 \neq -1$. In this section we define quantum Schur and Weyl functors. As in the setting of classical strict polynomial functors, these families of functors play a fundamental role, and we use them here to construct the simple objects in $\mathcal{P}_q$ (up to isomorphism). In several key calculations in this section we appeal to theorems in [HH].

## 6.1 Quantum symmetric and exterior powers

We call $I \in \Omega(n, d)$ **strict** if for any $1 \le k \neq \ell \le d$, $i_k \neq i_\ell$. Let $\Omega^{++}(n, d)$ be the set of strictly increasing tuples of integers in $\Omega(n, d)$. We denote by $x_{IJ}$ the monomials $x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_d j_d}$ in $\mathcal{O}_q(M_{nm})$ where $I = (i_1, i_2, \cdots, i_d) \in \Omega(n, d)$ and $J = (j_1, j_2, \cdots, j_d) \in \Omega(m, d)$.

Recall that we defined $\bigwedge_q^d(n) = V_n^{\otimes d} \otimes_{\mathcal{H}_d} \chi_-$. Note that $\bigwedge_q^d(n)$ is isomorphic to the $d^{\text{th}}$ graded component of

$$\textstyle\bigwedge_q^\bullet(n) := T(V_n)/I(R_n)$$

where $T(V_n)$ is the tensor algebra of $V_n$ and $I(R_n)$ is the two-sided ideal of $T(V_n)$, generated in degree two by $R_n(v \otimes w) + q^{-1} w \otimes v$, for $v, w \in V_n$.

As usual for exterior algebras, we use $\wedge$ to denote the product in the algebra $\bigwedge_q^\bullet(n)$. For any $I \in \Omega(n, d)$ we denote by $\bar{e}_I$ the image of $e_I$ in $\bigwedge_q^d(n)$:

$$\bar{e}_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}.$$

Moreover we have the following basic calculus of q-wedge products:

$$e_i \wedge e_j = \begin{cases} 0 & \text{if } i = j \\ -q^{-1} e_j \wedge e_i & \text{if } i > j \end{cases}$$

**Lemma 6.1.** *Let $I = (i_1, i_2, \cdots, i_d) \in \Omega(n, d)$.*

1. *If there exists $1 \leq k \neq \ell \leq d$ such that $i_k = i_\ell$ then $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} = 0$.*
2. *If $I$ is strictly increasing $\sigma \in \mathfrak{S}_d$, then*

$$e_{i_{\sigma(1)}} \wedge e_{i_{\sigma(2)}} \wedge \cdots \wedge e_{i_{\sigma(d)}} = (-q^{-1})^{\ell(\sigma)} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d},$$

*where $\ell(\sigma)$ is the length of $\sigma$.*

*Proof.* Both parts follow easily from the definition of the $q$ wedge products, cf. Equations (2.3),(2.4) in [HH]. □

A consequence of above lemma is that $\bigwedge_q^d(n)$ has a basis $e_{i_1} \wedge \cdots \wedge e_{i_d}$ for $1 \leq i_1 < \cdots < i_d \leq n$. The $q$-antisymmetrization map $\alpha_d(n) : \bigwedge_q^d(n) \to V_n^{\otimes d}$ is given by

$$e_{i_1} \wedge \cdots \wedge e_{i_d} \mapsto \sum_{w \in \mathfrak{S}_d} (-q^{-1})^{\ell(w)} e_{i_{w(1)}} \otimes \cdots \otimes e_{i_{w(d)}},$$

for $1 \leq i_1 < \cdots < i_d \leq n$.

We define the following elements of $\mathcal{H}_d$:

$$\begin{aligned} x_d &= \sum_{w \in \mathfrak{S}_d} q^{\ell(w)} T_w \\ y_d &= \sum_{w \in \mathfrak{S}_d} (-q^{-1})^{\ell(w)} T_w. \end{aligned}$$

In the current setting, it is convenient for us to denote the right action of $\mathcal{H}_d$ on $V_n^{\otimes d}$ by a dot.

**Lemma 6.2.** *Given any tuple $I = (i_1, i_2, \cdots, i_d) \in \Omega(n, d)$ we have*

$$\alpha_d(n)(e_{i_1} \wedge \cdots \wedge e_{i_d}) = e_I \cdot y_d.$$

*Proof.* Suppose first that $I$ is strict. Let $I_0$ be the strictly increasing tuple such that $I = I_0 \cdot \sigma$ for a unique permutation $\sigma \in \mathfrak{S}_d$. The following computation proves the lemma in this case:

$$\begin{aligned} \alpha_d(n)(\bar{e}_I) &= (-q^{-1})^{\ell(\sigma)} \alpha_q^d(\bar{e}_{I_0}) \\ &= (-q^{-1})^{\ell(\sigma)} \sum_{w \in \mathfrak{S}_d} (-q^{-1})^{\ell(w)} e_{I_0 \cdot w} \\ &= (-q^{-1})^{\ell(\sigma)} e_{I_0} \cdot y_d \\ &= e_{I_0} \cdot (T_\sigma \cdot y_d \\ &= e_I \cdot y_d \end{aligned} \tag{6.1.19}$$

where the first equality follows from Lemma 6.1 (2), the third and the last equalities holds because $I_0$ is strictly increasing and the fourth equality follows from the following fact:

$$T_\sigma \cdot y_d = (-q^{-1})^{\ell(\sigma)} y_d.$$

Now suppose that $I$ is not strict. Then by Lemma 6.1 (1) it is enough to show

$$e_I \cdot y_d = 0. \tag{6.1.20}$$

Let $I = (i_1, i_2, \cdots, i_d)$. Assume that $k$ is the maximal number such that $i_1, i_2, \cdots, i_k$ are all distinct but $i_{k+1}$ is equal to one of $i_1, i_2, \cdots, i_k$. Let $\sigma$ be the (unique) element in $\mathfrak{S}_k \subset \mathfrak{S}_d$, such that $(i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \cdots, i_{\sigma^{-1}(k)})$ are strictly increasing. Then $e_I = e_{I\cdot\sigma^{-1}} T_\sigma$ and

$$e_I \cdot y_d = e_{I\cdot\sigma^{-1}}(T_\sigma y_d) = (-q^{-1})^{\ell(\sigma)} e_{I\cdot\sigma^{-1}} \cdot y_d.$$

Hence to show the formula (6.1.20), we can always assume that $i_1 < i_2 < \cdots < i_k$ and $i_{k+1} = i_a$, where $1 \le a \le k$. Take the element $S = T_{a+1} \cdots T_{k-1} T_k \in \mathcal{H}_d$. Then $e_I = e_{I'} \cdot S$, where $I' = (i_1, i_2, \cdots, i_a, i_{k+1}, i_{a+1}, i_{a+2}, \cdots, i_k, i_{k+2}, i_{k+3}, \cdots, i_d)$ and then

$$e_I \cdot y_d = e_{I'}(S y_d) = (-q^{-1})^{k-a} e_{I'} \cdot y_d.$$

Note that $e_{I'} T_a = q e_{I'}$. On the other hand

$$e_{I'}(T_a \cdot y_d) = (-q^{-1})(e_{I'} \cdot y_d).$$

By the assumption that $q^2 \neq -1$, it forces $e_{I'} \cdot y_d = 0$, and hence $e_I \cdot y_d = 0$. □

Recall also that we define the quantum symmetric power

$$S_q^d(n) = V_n^{\otimes d} \otimes_{\mathcal{H}_d} \mathbb{C}_+,$$

and the quantum divided power functor

$$\Gamma_q^d(n) = \mathrm{Hom}_{\mathcal{H}_d}(\chi_+, V_n^{\otimes d}).$$

Let $p_d$ be the projection map $p_d : \bigotimes^d \to \bigwedge_q^d$ and let $q_d$ be the projection morphism $q_d : \bigotimes^d \to S_q^d$. Let $i_d : \Gamma_q^d \to \bigotimes^d$ be the natural inclusion map. It is clear that $p_d, q_d, i_d$ are morphisms of quantum polynomial functors.

**Proposition 6.3.** *The $q$-antisymmetrization $\alpha_d : \bigwedge_q^d \to \bigotimes^d$ is a morphism of quantum polynomial functors.*

*Proof.* We work with the characterization of quantum polynomial functors given by Proposition 3.3. We need to check that, for any $n, m$, the following diagram commutes:

$$\begin{tikzcd} \bigwedge_q^d(m) \arrow[r] \arrow[d, "\alpha_d(n)"] & \bigwedge_q^d(n) \otimes \mathcal{O}_q^d(M_{n,m}) \arrow[d, "\alpha_d(n)\otimes 1"] \\ V_m^{\otimes d} \arrow[r] & V_n^{\otimes d} \otimes \mathcal{O}_q^d(M_{n,m}) \end{tikzcd} \tag{6.1.21}$$

The quantum polynomial functor $\bigotimes^d$ gives rise to the bottom map, which for any $I \in \Omega(m, d)$, is given by

$$e_I \mapsto \sum_{J \in \Omega(n,d)} e_J \otimes x_{JI}.$$

It also induces the quantum polynomial functor structure on $\bigwedge^d$, and so for any $m, n$ and for any $I \in \Omega(n, d)$ the top map is given by

$$\bar{e}_I \mapsto \sum_{J \in \Omega(n,d)} \bar{e}_J \otimes x_{JI},$$

where $\bar{e}_I \in \bigwedge_q^d(m)$ and $\bar{e}_J \in \bigwedge_q^d(n)$.

We start with an element $\bar{e}_I \in \bigwedge_q^d(m)$, where $I$ is strictly increasing. In the diagram (6.1.21), if we go up-horizontal and then downward, then by Lemma 6.2, $\bar{e}_I$ is mapped to

$$\begin{aligned}\sum_{J\in\Omega(n,d)} e_J \cdot y_d \otimes x_{JI} &= \sum_{J\in\Omega(n,d)} \sum_{w\in\mathfrak{S}_d} (-q^{-1})^{\ell(w)} e_J \cdot T_w \otimes x_{JI} \\ &= \sum_{w\in\mathfrak{S}_d} (-q^{-1})^{\ell(w)} \Big( \sum_{J\in\Omega(n,d)} e_J \cdot T_w \otimes x_{JI} \Big), \\ &= \sum_{w\in\mathfrak{S}_d} (-q^{-1})^{\ell(w)} \Big( \sum_{J\in\Omega(n,d)} e_J \otimes x_{J(I\cdot w)} \Big)\end{aligned} \tag{6.1.22}$$

where the last equality holds since $T_w$ is an endomorphism of the quantum polynomial functor $\bigotimes^d$, and also $e_I \cdot T_w = e_{I\cdot w}$.

If we go downward and then down-horizontal, $\bar{e}_I$ is exactly mapped to

$$\sum_{w\in\mathfrak{S}_d} (-q^{-1})^{\ell(w)} \Big( \sum_{J\in\Omega(n,d)} e_J \otimes x_{J(I\cdot w)} \Big)$$

showing the commutativity of the diagram (6.1.21). $\square$

**Proposition 6.4.** *There exist canonical isomorphisms*

$$(\textstyle\bigwedge_q^d)^\sharp \simeq \bigwedge_q^d, \quad (S_q^d)^\sharp \simeq \Gamma_q^d.$$

*Under these identifications, we have the following equalities:*

$$(p_d)^\sharp = \alpha_d, \quad (q_d)^\sharp = i_d.$$

*Proof.* We first consider $\bigwedge_q^d$. Let $\{(\bar{e}_I)^*\}_{I\in\Omega(n,d)^{++}}$ be the dual basis of $\{\bar{e}_I\}_{I\in\Omega(n,d)^{++}}$ in $(\bigwedge_q^d(n))^*$, where $\bar{e}_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \in \bigwedge_q^d(n)$ for $I = (i_1, i_2, \cdots, i_n)$. It naturally gives a set of elements in $(V_n^{\otimes d})^*$ via the inclusion map $(\bigwedge_q^d(n))^* \to (V_n^{\otimes d})^*$. By Lemma 6.1, the element $(\bar{e}_I)^*$ can be identified with

$$e_I \cdot y_d = \sum_{w\in\mathfrak{S}_d} (-q^{-1})^{\ell(w)} e_{I\cdot w}.$$

It exactly coincides with the of image of $\bar{e}_I$ after the $q$-antisymmetrization map $\alpha_d(n)$. It implies that $(\bigwedge_q^d)^\sharp \simeq \bigwedge_q^d$ under the correspondences $(\bar{e}_I)^* \mapsto \bar{e}_I$, moreover $\alpha_d = (p_d)^\sharp$.

We now consider the projection map $q_d : \bigotimes^d \to S_q^d$. Note that the $q$-symmetric power $S_q^d(n)$ is the quotient

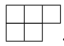
$$\frac{V_n^{\otimes d}}{\sum_{i=1}^{d-1} \mathrm{Im}(\mathrm{T_i} - \mathrm{q})},$$

and the $q$-divided power $\Gamma_q^d(n)$ is the subspace of $V_n^{\otimes d}$,

$$\bigcap_{i=1}^{d-1} \mathrm{Ker}(T_i - q).$$

By Lemma 5.3, the operator $T_i - q : V_n^{\otimes d} \to V_n^{\otimes d}$ is self-adjoint, with respect to the bilinear form $(,)$. Therefore by Lemma 2.3, we have $(S_q^d)^\sharp \simeq \Gamma_q^d$ and $(q_d)^\sharp = i_d$. $\square$

## 6.2 Definition and properties of Quantum Schur and Weyl functors

Let $\lambda = (\lambda_1, ..., \lambda_s)$ be a partition. By convention our partitions have no zero parts, so $\lambda_1 \geq \cdots \geq \lambda_s > 0$. The size of $\lambda$ is $|\lambda| := \lambda_1 + \cdots + \lambda_s$ and the length of $\lambda$ is $\ell(\lambda) := s$. We depict partitions using diagrams, e.g. $(3, 2) =$ . Let $\lambda'$ denote the conjugate partition.

The canonical tableau of shape $\lambda$ is the tableau with entries $1, ..., |\lambda|$ in sequence along the rows. For example

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | |

is the canonical tableau of shape $(3, 2)$. Let $\sigma_\lambda \in \mathfrak{S}_d$ be given by the column reading word of the canonical tableau. For instance, if $\lambda = (3, 2)$ then $\sigma_\lambda = 14253$ (in one-line notation). Define the following quantum polynomial functors of degree $d$:

$$\begin{aligned} \textstyle\bigwedge_q^\lambda &= \textstyle\bigwedge_q^{\lambda_1} \otimes \cdots \otimes \textstyle\bigwedge_q^{\lambda_s} \\ S_q^\lambda &= S_q^{\lambda_1} \otimes \cdots \otimes S_q^{\lambda_s} \\ \Gamma_q^\lambda &= \Gamma_q^{\lambda_1} \otimes \cdots \otimes \Gamma_q^{\lambda_s} \end{aligned}$$

and the following morphisms:

$$\begin{aligned} \alpha_\lambda &= \alpha_{\lambda_1} \otimes \alpha_{\lambda_2} \otimes \cdots \otimes \alpha_{\lambda_s} \\ i_\lambda &= i_{\lambda_1} \otimes i_{\lambda_2} \otimes \cdots \otimes i_{\lambda_s} \\ p_\lambda &= p_{\lambda_1} \otimes p_{\lambda_2} \otimes \cdots \otimes p_{\lambda_s} \\ q_\lambda &= q_{\lambda_1} \otimes q_{\lambda_2} \otimes \cdots \otimes q_{\lambda_s}. \end{aligned}$$

We define the **quantum Schur functor** $S_\lambda$ as the image of the composition of the following morphiphs

$$\textstyle\bigwedge_q^\lambda \xrightarrow{\alpha_\lambda} \bigotimes^d \xrightarrow{T_{\sigma_\lambda}} \bigotimes^d \xrightarrow{q_{\lambda'}} S_q^{\lambda'},$$

Define the **quantum Weyl functor** $W_\lambda$ as the image of the composition of the following morphisms:

$$\Gamma_q^\lambda \xrightarrow{i_\lambda} \bigotimes^d \xrightarrow{T_{\sigma_\lambda}} \bigotimes^d \xrightarrow{p_{\lambda'}} \textstyle\bigwedge_q^{\lambda'}.$$

**Theorem 6.5.** *For any partition $\lambda$, we have a canonical isomorphism*

$$W_{\lambda'} \simeq (S_\lambda)^\sharp.$$

*Proof.* We first note that $\sigma_{\lambda'} = (\sigma_\lambda)^{-1}$. Then the theorem follows from Proposition 5.5, 6.4. $\square$

Suppose that $\ell(\lambda) \leq n$. Then $S_\lambda(n)$ is the Schur module and $W_\lambda(n)$ is the Weyl module of $S_q(n, d)$ (cf. Definition 6.7, Theorem 6.19, and Definition 6.21 [HH]). Let $L_\lambda$ be the socle of the functor $S_{\lambda'}$. Recall that this is the maximal semisimple subfunctor of $S_{\lambda'}$.

**Proposition 6.6.** *The functors $L_\lambda$, where $\lambda$ ranges over all partitions of $d$, form a complete set of representatives for the isomorphism classes of irreducible objects in $\mathcal{P}_q^d$.*

*Proof.* By Theorem 4.7, $\mathcal{P}_q^d \cong \mathsf{Mod}(S_q(n,d))$ for any $n \geq d$. To prove the statement it suffices to show that $\{L_\lambda(n)\}$ form a complete set of representatives for irreducible $S_q(n,d)$-modules. This follows from Lemma 8.3 and Proposition 8.4 in [HH]. $\square$

# 7 Invariant theory of $GL_q(n)$

In this section, we assume $q$ is generic. Our aim is to show that the theory of quantum polynomial functors affords a streamlined derivation of the invariant theory of the quantum group $GL_q(n)$.

Following Howe's approach to classical invariant theory (cf.[Ho]), we first prove a quantum analog of $(GL(m), GL(n))$ duality. In the classical case the proof is based on a geometric argument that the matrix space is spherical. While this geometric argument fails in the quantum case, we show that $(GL_q(m), GL_q(n))$ duality is a direct consequence of the Theorem 4.7. We then show that, as in the classical case, quantum analogs of the first fundamental theorem and Schur-Weyl duality follow from $(GL_q(m), GL_q(n))$ duality.

Let $\mathcal{O}_q(GL(n))$ be the coordinate ring of the quantum group $GL_q(n)$. Recall that, by definition, this is the localization of $\mathcal{O}_q(M_n)$ by the quantum determinant,

$$det_q := \sum_{\sigma \mathfrak{S}_n} (-q^{-1})^{\ell(\sigma)} x_{1\sigma(1)} \cdots x_{n\sigma(n)}.$$

$\mathcal{O}_q(GL(n))$ is a Hopf algebra, and we denote its antipode by $\iota$. For details and precise definitions see e.g. Chapter 5 of [PW].

By definition an action of $GL_q(n)$ on $V$ is a right coaction of $\mathcal{O}_q(GL(n))$ on $V$. A representation of $GL_q(n)$ is a $GL_q(n)$-module. A module over the $q$-Schur algebra $S_q(n,d)$ is naturally a representation of $\mathcal{O}_q(GL(n))$. By analogy with the classical setting, any representation of $\mathcal{O}_q(GL(n))$ coming from $S_q(n,d)$ is a polynomial representation of degree $d$.

By Theorem 6.6 $L_\lambda(n)$ is an irreducible representation $GL_q(n)$, and any irreducible representation of $GL_q(n)$ is isomorphic to $L_\lambda(n)$ for a unique $\lambda$ such that $\ell(\lambda) \leq n$.

The comultiplication $\Delta : \mathcal{O}_q(M_{\ell,n}) \to \mathcal{O}_q(M_{\ell,m}) \otimes \mathcal{O}_q(M_{m,n})$ induces actions of the quantum general linear group by left and right multiplication on quantum $m \times n$ matrices:

$$\mu'_L : \mathcal{O}_q(M_{m,n}) \to \mathcal{O}_q(GL(m)) \otimes \mathcal{O}_q(M_{m,n})$$
$$\mu_R : \mathcal{O}_q(M_{m,n}) \to \mathcal{O}_q(M_{m,n}) \otimes \mathcal{O}_q(GL(n))$$

These maps commute and preserve degree. We define $\mu_L := P \circ (\iota \otimes 1) \circ \mu'_L$, where $P$ is the flip map. Then using $(\mu_L \otimes 1) \circ \mu_R$, we regard $\mathcal{O}_q^d(M_{m,n})$ as a $GL_q(m) \times GL_q(n)$-module.

Given a representation $V$ of $GL_q(n)$ let $V^*$ be the contragredient represenation, i.e. twist the left coaction of $\mathcal{O}_q(GL(n))$ on the dual space $V^*$ by the antipode $\iota$.

**Theorem 7.1** ($(GL_q(m), GL_q(n))$ duality)**.** *As a $GL_q(m) \times GL_q(n)$-module we have a multiplicity-free decomposition:*

$$\mathcal{O}_q^d(M_{m,n}) \cong \bigoplus_\lambda L_\lambda(m)^* \otimes L_\lambda(n),$$

*where $\lambda$ runs over all partitions of $d$ such that $\ell(\lambda) \leq min(m,n)$.*

*Proof.* By Theorem 4.7 the category $\mathcal{P}_q^d$ is equivalent to the category $\mathsf{Mod}(S_q(n,d))$. Hence the category $\mathcal{P}_q^d$ is semi-simple, and the simple objects are, up to equivalence, the functors $L_\lambda$ where $\lambda$ ranges over partitions of $d$. (Since $q$ is generic $L_\lambda \cong W_\lambda \cong S_{\lambda'}$.) By Proposition 4.6 for any $m \geq 0$ there exists a natural isomorphism $\mathrm{Hom}_{\mathcal{P}_q}(\Gamma_q^{d,m}, L_\lambda) \simeq L_\lambda(m)$. Moreover, $L_\lambda(m) = 0$ if $m > \ell(\lambda)$. Hence we have the following decomposition

$$\begin{aligned}\Gamma_q^{d,m} &\cong \bigoplus_\lambda L_\lambda \otimes \mathrm{Hom}_{\mathcal{P}_q^d}(L_\lambda, \Gamma_q^{d,m}) \\ &\cong \bigoplus_\lambda L_\lambda \otimes \mathrm{Hom}_{\mathcal{P}_q^d}(\Gamma_q^{d,m}, L_\lambda)^* \\ &\cong \bigoplus_\lambda L_\lambda \otimes L_\lambda(m)^*,\end{aligned}$$

where the second isomorphism follows from the natural pairing

$$\mathrm{Hom}_{\mathcal{P}_q^d}(L_\lambda, \Gamma_q^{d,m}) \times \mathrm{Hom}_{\mathcal{P}_q^d}(\Gamma_q^{d,m}, L_\lambda) \to \mathrm{Hom}_{\mathcal{P}_q^d}(L_\lambda, L_\lambda) \simeq \Bbbk.$$

Evaluating both sides at $n$ yields

$$\mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}) \cong \bigoplus_\lambda L_\lambda(n) \otimes L_\lambda(m)^*. \tag{7.0.23}$$

This proves the theorem, since

$$\mathcal{O}_q^d(M_{m,n}) \cong S_q(n,m;d)^* \cong (\mathrm{Hom}_{\mathcal{H}_d}(V_n^{\otimes d}, V_m^{\otimes d}))^* \simeq \bigoplus_\lambda L_\lambda(m)^* \otimes L_\lambda(n),.$$

□

In analogy with the classical setting, $(GL_q(m), GL_q(n))$ duality is equivalent to quantum FFT and Jimbo-Schur-Weyl duality. We briefly mention these connections.

Given three numbers $\ell, m, n$ define a representation of $GL_q(m)$ on $\mathcal{O}_q(M_{n,m}) \otimes \mathcal{O}_q(M_{m,\ell})$ as follows:

$$\begin{tikzcd} \mathcal{O}_q(M_{n,m}) \otimes \mathcal{O}_q(M_{m,\ell}) \arrow[r, "\mu_R \otimes \mu_L"] & \mathcal{O}_q(M_{n,m}) \otimes \mathcal{O}_q(GL(m)) \otimes \mathcal{O}_q(M_{m,\ell}) \otimes \mathcal{O}_q(GL(m)) \arrow[d] \\ & \mathcal{O}_q(M_{n,m}) \otimes \mathcal{O}_q(M_{m,\ell}) \otimes \mathcal{O}_q(GL(m)) \end{tikzcd}$$

In the above diagram, the downward map is given by multiplication in $\mathcal{O}_q(GL(m))$.

**Theorem 7.2** (Quantum FFT)**.** *For any $\ell, m, n$ the image of the comultiplication*

$$\Delta : \mathcal{O}_q(M_{n,\ell}) \to \mathcal{O}_q(M_{n,m}) \otimes \mathcal{O}_q(M_{m,\ell})$$

*lies in the subspace of $GL_q(m)$-invariants, and, moreover, gives rise to a surjective map*

$$\mathcal{O}_q(M_{n,\ell}) \twoheadrightarrow (\mathcal{O}_q(M_{n,m}) \otimes \mathcal{O}_q(M_{m,\ell}))^{GL_q(m)}.$$

*Proof.* First we note that for any representation $V$ of $GL_q(m)$, by complete reducibility, we have $(V^*)^{GL_q(m)} \simeq (V^{GL_q(m)})^*$. Then taking duals, by Proposition 2.4, it suffices to show that the following map is injective:

$$(\mathrm{Hom}_{\mathcal{H}_d}(V_\ell^{\otimes d}, V_m^{\otimes d}) \otimes \mathrm{Hom}_{\mathcal{H}_d}(V_m^{\otimes d}, V_n^{\otimes d}))^{GL_q(m)} \to \mathrm{Hom}_{\mathcal{H}_d}(V_\ell^{\otimes d}, V_n^{\otimes d}), \tag{7.0.24}$$

where $GL_q(m)$ acts diagonally on the left hand side. This follows immediately from $(GL_q(m), GL_q(n))$ duality, since by Equation 7.0.23 the above map is precisely the inclusion

$$\bigoplus_{\ell(\lambda)\leq \ell,m,n} L_\lambda(\ell)^* \otimes L_\lambda(n) \to \bigoplus_{\ell(\lambda)\leq \ell,n} L_\lambda(\ell)^* \otimes L_\lambda(n).$$

□

Finally, consider tensor space $V_m^{\otimes d}$. As a representation of $GL_q(m)$ we have a decomposition

$$V_m^{\otimes d} \cong \bigoplus_{\lambda} L_\lambda(m) \otimes M_\lambda \tag{7.0.25}$$

where the $\lambda$ runs over all partitions of $d$, and $M_\lambda = \mathrm{Hom}_{GL_q(m)}(L_\lambda(m), V_m^{\otimes d})$. Notice that by the construciton of $L_\lambda$, we have that $L_\lambda(m) = 0$ if $\ell(\lambda) > m$. Hence the sum above is over all partitions $\lambda$ of $d$ such that $\ell(\lambda) \leq m$. Note also that $M_\lambda$ are naturally $\mathcal{H}_d$-modules.

**Theorem 7.3** (Jimbo-Schur-Weyl duality)**.** *Equation (7.0.25) is a multiplicity-free decomposition of $V_m^{\otimes d}$ as a $GL_q(m) \times \mathcal{H}_d$-module. In particular, the modules $M_\lambda$ are irreducible pairwise inequivalent $\mathcal{H}_d$-modules.*

*Proof.* We will deduce this result from the quantum FFT. Indeed, applying Theorem (7.2) to the case $n = m = \ell$, it follows that for any partition $\lambda$ of $d$ such that $\ell(\lambda) \leq m$, the following map is injective:

$$\bigoplus_{\mu} \mathrm{Hom}_{\mathcal{H}_d}(M_\lambda, M_\mu) \otimes \mathrm{Hom}_{\mathcal{H}_d}(M_\mu, M_\lambda) \to \mathrm{Hom}_{\mathcal{H}_d}(M_\lambda, M_\lambda),$$

where $\mu$ runs over all partition of $d$ with $\ell(\mu) \leq m$. This implies that $M_\lambda$ is irreducible as $\mathcal{H}_d$-module and for any $\lambda \neq \mu$, $M_\lambda$ and $M_\mu$ are non-isomorphic, proving the result. □

**Remark 7.4.**

1. One can easily show that Jimbo-Schur-Weyl duality implies $(GL_q(m), GL_q(n)$ duality using Proposition 2.4. This completes the chain of equivalences, and hence the three basic theorems of quantum invariant theory ($(GL_q(m), GL_q(n))$ duality, the quantum FFT, and Jimbo-Schur-Weyl duality) are all equivalent, as in the classical case done by Howe[Ho].
2. The approach taken here uses only basic facts about quantum polynomial functors which have analogs in other settings where a theory of strict polynomial functors, namely the classical and super cases [FS, Ax]. Therefore this approach can be used to give a new and uniform development for the classical, quantum and super invariant theories of the general linear group.

# References

[Ax] J. Axtell. Spin polynomial functors and representations of Schur superalgebras. Represent. Theory 17 (2013), 584-609.

[BDK] J. Brundan, R. Dipper and A. Kleshchev. Quantum linear groups and representations of $\mathrm{GL}_n(\mathbb{F}_q)$. Mem.

[FS] E. M. Friedlander and A. Suslin. Cohomology of finite group schemes over a field. Invent. Math. 127 (1997), no. 2, 209-270. Amer. Math. Soc. 149 (2001), no. 706, viii+112 pp.

[GLR] K. Goodearl, T. Lenagan and L. Rigal. The first fundamental theorem of coinvariant theory for the quantum general linear group. Publ. Res. Inst. Math. Sci. 36 (2000), no. 2, 269-296.

[Ho] R. Howe. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. The Schur lectures (1992) (Tel Aviv), 1-182, Israel Math. Conf. Proc., 8, Bar-Ilan Univ., Ramat Gan, 1995.

[HH] M. Hashimoto and T. Hayashi. Quantum multilinear algebra. Tohoku Math. J. (2) 44 (1992), no. 4, 471-521.

[Kr] H. Krause. Koszul, Ringel and Serre duality for strict polynomial functors. Compos. Math. 149 (2013), no. 6, 996-1018.

[Ku] N. J. Kuhn. Rational cohomology and cohomological stability in generic representation theory, Amer. J. Math. 120 (1998), 1317-1341.

[LZZ] G. I. Lehrer, H. Zhang and R. B. Zhang. A quantum analogue of the first fundamental theorem of classical invariant theory. Comm. Math. Phys. 301 (2011), no. 1, 131-174.

[Ph] H. H. Phúng. Realizations of quantum hom-spaces, invariant theory, and quantum determinantal ideals. J. Algebra 248 (2002), no. 1, 50-84.

[PW] B. Parshall and J. P. Wang. Quantum linear groups. Mem. Amer. Math. Soc. 89 (1991), no. 439, vi+157 pp.

[T] M. Takeuchi. A short course on quantum matrices, New Directions in Hopf Algebras, MSRI Publications, Vol. 43, 2002.

[Zh] R. B. Zhang. Howe duality and the quantum general linear group. (English summary) Proc. Amer. Math. Soc. 131 (2003), no. 9, 2681-2692 (electronic).