Higher Sugawara operators for the quantum affine algebras of type A

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Abstract

We give explicit formulas for the elements of the center of the completed quantum affine algebra in type A at the critical level which are associated with the fundamental representations. We calculate the images of these elements under a Harish-Chandra-type homomorphism. These images coincide with those in the free field realization of the quantum affine algebra and reproduce generators of the q-deformed classical $\mathcal{W}$-algebra of Frenkel and Reshetikhin.

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1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $\mathrm{U}_q(\widehat{\mathfrak{g}})$ denote the quantum affine algebra associated with $\mathfrak{g}$. Due to the work of Reshetikhin and Semenov-Tian-Shansky [27], to every finite-dimensional representation V of $\mathrm{U}_q(\widehat{\mathfrak{g}})$ one can associate a formal Laurent series $\ell_V(z)$ whose coefficients belong to the center $\mathrm{Z}_q(\widehat{\mathfrak{g}})$ of the completion $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{g}})_{\mathrm{cri}}$ of the quantum affine algebra at the critical level. The map $V \mapsto \ell_V(z)$ was further studied by Ding and Etingof [9] who showed that if the coefficients of $\ell_V(z)$ are regarded as operators on highest weight modules at the critical level, then it possesses properties of a homomorphism from the Grothendieck ring $\operatorname{Rep} \mathrm{U}_q(\widehat{\mathfrak{g}})$ to formal series in z. Furthermore, the coefficients of $\ell_V(z)$ were shown to generate all singular vectors in Verma modules [9]. This relied upon connections of the series $\ell_V(z)$ with transfer matrices; see also work of Frenkel and Reshetikhin [18, Sec. 8] for its relationship with the q-characters of finite-dimensional representations of $\mathrm{U}_q(\widehat{\mathfrak{g}})$.

By a conjecture of Frenkel and Reshetikhin [17], [18], the center $\mathrm{Z}_q(\widehat{\mathfrak{g}})$ is isomorphic to the q-deformed classical $\mathcal{W}$-algebra, as a Poisson algebra. More precisely, Conjecture 1 of [17] applies to the completion of the central subalgebra of $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{g}})_{\mathrm{cri}}$ generated by the coefficients of the series $\ell_{V_1}(z), \dots, \ell_{V_n}(z)$ associated with all fundamental representations V_i of $\mathrm{U}_q(\widehat{\mathfrak{g}})$. Its proof was sketched in [17] for $\mathfrak{g} = \mathfrak{sl}_N$. The isomorphism is provided by the free field (or Wakimoto) realization of the quantum affine algebra due to Awata, Odake and Shiraishi [1] which extended an earlier work [19] on the vertex representations from level 1 to an arbitrary level.

The results of [17] generalize the Feigin–Frenkel theorem which establishes a Poisson algebra isomorphism between the center $\mathfrak{z}(\widehat{\mathfrak{g}})$ of the affine vertex algebra $V(\mathfrak{g})$ at the critical level and the classical $\mathcal{W}$-algebra associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$; see [15] for a detailed exposition. The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials which can be identified with a commutative subalgebra of the universal enveloping algebra $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$. As discovered by Feigin, Frenkel and Reshetikhin [13], the higher degree Hamiltonians in the Gaudin model can be obtained from generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$; see also [14] and [28]. Explicit constructions of generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ were given in [7], [8] and [24] for types A, B, C and D; see also [25] for their images in the classical $\mathcal{W}$-algebras and [26] for super-analogues of these constructions with $\mathfrak{g} = \mathfrak{gl}_{m|n}$.

Our goal in this paper is to give similar explicit formulas for higher degree Sugawara operators for $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$; i.e., for elements of the center of $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ (Theorem 3.2). The formulas express the operators in terms of the RLL-presentation $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$; see [20] and [27]. We use a version of the Poincaré–Birkhoff–Witt theorem for this presentation to introduce an analogue of the Harish-Chandra homomorphism and calculate the images of the central elements under the homomorphism.

Then we apply the Ding–Frenkel isomorphism [10] between the RLL and Drinfeld presentations to calculate the images of the Sugawara operators in the q-deformed classical $\mathcal{W}$-algebra by using the approach of [17] based on the free field realization [1]. Our generators correspond to the fundamental representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ and essentially coincide with those of [17] up to an appropriate identification of the parameters. The construction involves a fusion formula for the q-deformed antisymmetrizer expressing it in terms of the trigonometric R-matrices.

As an application of Theorem 3.2, we produce explicit invariants of the q-analogue $V_q(\mathfrak{gl}_n)$ of the vacuum module over the quantum affine algebra. The invariants are obtained by the action of the higher degree Sugawara operators on the vacuum vector; cf. [18].

2 Quantum affine algebra

We use the RLL presentation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ introduced in [27]; see also [20]. We regard q as a nonzero complex number which is not a root of unity. Introduce the two-parameter

R-matrix $R(u,v) \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$ by

$$R(u,v) = (u-v)\sum_{i\neq j} e_{ii}\otimes e_{jj} + (q^{-1}u - qv)\sum_{i} e_{ii}\otimes e_{ii} \\ + (q^{-1}-q)\,u\sum_{i>j} e_{ij}\otimes e_{ji} + (q^{-1}-q)\,v\sum_{i<j} e_{ij}\otimes e_{ji}, \tag{2.1}$$

where $e_{ij} \in \operatorname{End}\mathbb{C}^n$ are the standard matrix units. We will also need the one-parameter R-matrices

$$\overline{R}(x) = \frac{R(x,1)}{q^{-1}x - q} = \sum_{i} e_{ii}\otimes e_{ii} + \frac{1-x}{q-q^{-1}x}\sum_{i\neq j} e_{ii}\otimes e_{jj} \\ + \frac{(q-q^{-1})\,x}{q-q^{-1}x}\sum_{i>j} e_{ij}\otimes e_{ji} + \frac{q-q^{-1}}{q-q^{-1}x}\sum_{i<j} e_{ij}\otimes e_{ji}, \tag{2.2}$$

and

$$R(x) = f(x)\,\overline{R}(x), \tag{2.3}$$

where

$$f(x) = 1 + \sum_{k=1}^{\infty} f_k x^k, \qquad f_k = f_k(q),$$

is a formal power series in x whose coefficients f_k are rational functions in q uniquely determined by the relation

$$f(xq^{2n}) = f(x)\,\frac{(1-xq^2)\,(1-xq^{2n-2})}{(1-x)\,(1-xq^{2n})}. \tag{2.4}$$

They can be found by the recurrence

$$f_k = -\frac{(1-q^2)(1-q^{2n-2})}{1-q^{2n}}\sum_{i=1}^{k}\frac{1-q^{2ni}}{1-q^{2nk}}\,f_{k-i}, \qquad k \geqslant 1,$$

with $f_0 = 1$. Equivalently, $f(x)$ can be given by

$$f(x) = \frac{(x;q^{2n})_\infty\,(xq^{2n};q^{2n})_\infty}{(xq^2;q^{2n})_\infty\,(xq^{2n-2};q^{2n})_\infty}, \qquad (a;b)_\infty := \prod_{r=0}^{\infty}(1-ab^r),$$

where the coefficients of the powers x^k are power series in q converging to f_k for $|q| < 1$.

The *quantum affine algebra* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ is generated by elements

$$l^{+}_{ij}[-r], \qquad l^{-}_{ij}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant n, \qquad r = 0,1,\dots,$$

and the invertible central element q^c, subject to the defining relations

$$l_{ji}^{+}[0] = l_{ij}^{-}[0] = 0 \qquad \text{for} \qquad 1 \leqslant i < j \leqslant n, \tag{2.5}$$

$$l_{ii}^{+}[0]\, l_{ii}^{-}[0] = l_{ii}^{-}[0]\, l_{ii}^{+}[0] = 1 \qquad \text{for} \qquad i = 1, \ldots, n, \tag{2.6}$$

and

$$R(u/v)L_1^{\pm}(u)L_2^{\pm}(v) = L_2^{\pm}(v)L_1^{\pm}(u)R(u/v), \tag{2.7}$$

$$R(uq^{-c}/v)L_1^{+}(u)L_2^{-}(v) = L_2^{-}(v)L_1^{+}(u)R(uq^{c}/v). \tag{2.8}$$

In the last two relations we consider the matrices $L^{\pm}(u) = \big[l_{ij}^{\pm}(u)\big]$, whose entries are formal power series in u and u^{-1},

$$l_{ij}^{+}(u) = \sum_{r=0}^{\infty} l_{ij}^{+}[-r]\, u^{r}, \qquad l_{ij}^{-}(u) = \sum_{r=0}^{\infty} l_{ij}^{-}[r]\, u^{-r}. \tag{2.9}$$

Here and below we regard the matrices as elements

$$L^{\pm}(u) = \sum_{i,j=1}^{n} e_{ij} \otimes l_{ij}^{\pm}(u) \in \operatorname{End}\mathbb{C}^n \otimes \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)[[u^{\pm 1}]]$$

and use a subscript to indicate a copy of the matrix in the multiple tensor product algebra

$$\underbrace{\operatorname{End}\mathbb{C}^n \otimes \ldots \otimes \operatorname{End}\mathbb{C}^n}_{k} \otimes \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)[[u^{\pm 1}]] \tag{2.10}$$

so that

$$L_a^{\pm}(u) = \sum_{i,j=1}^{n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(k-a)} \otimes l_{ij}^{\pm}(u). \tag{2.11}$$

In particular, we take $k = 2$ for the defining relations (2.7) and (2.8).

This notation for elements of algebras of the form (2.10) will be extended as follows. For an element

$$C = \sum_{i,j,r,s=1}^{n} c_{ijrs}\, e_{ij} \otimes e_{rs} \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n,$$

and any two indices $a, b \in \{1, \ldots, k\}$ such that $a \neq b$, we denote by C_{ab} the element of the algebra $(\operatorname{End}\mathbb{C}^n)^{\otimes k}$ with $k \geqslant 2$ given by

$$C_{ab} = \sum_{i,j,r,s=1}^{n} c_{ijrs}\, (e_{ij})_a (e_{rs})_b, \qquad (e_{ij})_a = 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(k-a)}. \tag{2.12}$$

We regard the matrix transposition as the linear map

$$t : \operatorname{End}\mathbb{C}^n \to \operatorname{End}\mathbb{C}^n, \qquad e_{ij} \mapsto e_{ji}.$$

For any $a \in \{1, \dots, k\}$ we will denote by t_a the corresponding partial transposition on the algebra (2.10) which acts as t on the a-th copy of $\operatorname{End}\mathbb{C}^n$ and as the identity map on all the other tensor factors.

The R-matrix (2.3) satisfies the following *crossing symmetry relations* [20]:

$$\big(R_{12}(x)^{-1}\big)^{t_2} D_2 R_{12}(xq^{2n})^{t_2} = D_2 \qquad \text{and} \qquad R_{12}(xq^{2n})^{t_1} D_1 \big(R_{12}(x)^{-1}\big)^{t_1} = D_1, \quad (2.13)$$

where D denotes the diagonal $n \times n$ matrix

$$D = \operatorname{diag}\big[q^{n-1}, q^{n-3}, \dots, q^{-n+1}\big] \qquad (2.14)$$

with the meaning of subscripts as in (2.12).

3 Main theorem

Denote by $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ the quantum affine algebra *at the critical level* $c = -n$, which is the quotient of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ by the relation $q^c = q^{-n}$. Its completion $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ is defined as the inverse limit

$$\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}} = \lim_{\longleftarrow} \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}/J_p, \qquad p > 0, \qquad (3.1)$$

where J_p denotes the left ideal of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ generated by all elements $l^-_{ij}[r]$ with $r \geqslant p$. Elements of the center $\mathrm{Z}_q(\widehat{\mathfrak{gl}}_n)$ of $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ are known as *Sugawara operators*.

Consider the q-permutation operator $P^q \in \operatorname{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \cong \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$ defined by

$$P^q = \sum_i e_{ii} \otimes e_{ii} + q \sum_{i>j} e_{ij} \otimes e_{ji} + q^{-1} \sum_{i<j} e_{ij} \otimes e_{ji}. \qquad (3.2)$$

The symmetric group $\mathfrak{S}_k$ acts on the space $(\mathbb{C}^n)^{\otimes k}$ by $s_i \mapsto P^q_{s_i} := P^q_{i,i+1}$ for $i = 1, \dots, k-1$, where s_i denotes the transposition $(i, i+1)$. If $\sigma = s_{i_1} \cdots s_{i_l}$ is a reduced decomposition of an element $\sigma \in \mathfrak{S}_k$ we set $P^q_\sigma = P^q_{s_{i_1}} \cdots P^q_{s_{i_l}}$. We denote by $A^{(k)}$ the image of the normalized antisymmetrizer associated with the q-permutations:

$$A^{(k)} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}\sigma \cdot P^q_\sigma \qquad (3.3)$$

so that $\big(A^{(k)}\big)^2 = A^{(k)}$.

For each $k = 1, \dots, n$ introduce the Laurent series $\ell_k(z)$ in z by

$$\ell_k(z) = \operatorname{tr}_{1,\dots,k} A^{(k)} L^+_1(z) \dots L^+_k(zq^{-2k+2}) L^-_k(zq^{-n-2k+2})^{-1} \dots L^-_1(zq^{-n})^{-1} D_1 \dots D_k, \quad (3.4)$$

where D is the diagonal matrix (2.14) and the trace is taken over all k copies of $\operatorname{End}\mathbb{C}^n$ in (2.10). All coefficients of the series $\ell_k(z)$ are elements of the algebra $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$. We will also need an equivalent formula for $\ell_k(z)$ which is obtained by using the following well-known particular case of the fusion procedure for the R-matrix (2.1); see [5].

Lemma 3.1. *Set $v_a = zq^{-2a+2}$ for $a = 1, \dots, k$. We have*

$$\prod_{1\leqslant a<b\leqslant k} R_{ab}(v_a, v_b) = k!\, z^{k(k-1)/2} \prod_{0\leqslant a<b\leqslant k-1} (q^{-2a} - q^{-2b})\, A^{(k)},$$

where the product is taken in the lexicographical order on the pairs (a, b). □

Applying (2.7) and Lemma 3.1 we obtain the relations

$$A^{(k)} L_1^+(v_1) \dots L_k^+(v_k) = L_k^+(v_k) \dots L_1^+(v_1)\, A^{(k)}, \tag{3.5}$$

$$A^{(k)} L_k^-(v_k q^{-n})^{-1} \dots L_1^-(v_1 q^{-n})^{-1} = L_1^-(v_1 q^{-n})^{-1} \dots L_k^-(v_k q^{-n})^{-1} A^{(k)}. \tag{3.6}$$

We also have

$$R(u, v) D_1 D_2 = D_2 D_1 R(u, v)$$

and $D_1 D_2 = D_2 D_1$ so that

$$A^{(k)} D_1 \dots D_k = D_1 \dots D_k\, A^{(k)}. \tag{3.7}$$

Thus, $\ell_k(z)$ defined in (3.4) can also be given by the formula

$$\ell_k(z) = \mathrm{tr}_{1,\dots,k}\, L_k^+(v_k) \dots L_1^+(v_1)\, L_1^-(v_1 q^{-n})^{-1} \dots L_k^-(v_k q^{-n})^{-1} D_1 \dots D_k\, A^{(k)}. \tag{3.8}$$

The following is our main result which provides explicit formulas for higher Sugawara operators.

Theorem 3.2. *The coefficients of $\ell_k(z)$ belong to the center of the completed quantum affine algebra at the critical level $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ for all $k = 1, \dots, n$.*

Proof. Introduce an extra copy of the endomorphism algebra $\mathrm{End}\,\mathbb{C}^n$ in (2.10) and label it by 0 to work with the algebra

$$\mathrm{End}\,\mathbb{C}^n \otimes \big(\mathrm{End}\,\mathbb{C}^n\big)^{\otimes k} \otimes \widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}. \tag{3.9}$$

It will be sufficient to verify what $\ell_k(z)$ commutes with $L_0^{\pm}(u)$. By using (2.7) we get

$$\begin{aligned} &L_0^+(u) L_k^+(v_k) \dots L_1^+(v_1) \\ &\qquad = R_{0k}(u/v_k)^{-1} \dots R_{01}(u/v_1)^{-1} R_{01}(u/v_1) \dots R_{0k}(u/v_k) L_0^+(u) L_k^+(v_k) \dots L_1^+(v_1) \\ &\qquad = R_{0k}(u/v_k)^{-1} \dots R_{01}(u/v_1)^{-1} L_k^+(v_k) \dots L_1^+(v_1) L_0^+(u) R_{01}(u/v_1) \dots R_{0k}(u/v_k). \end{aligned}$$

Relation (2.8) implies

$$L_0^+(u) R_{0a}(u/v_a) L_a^-(v_a q^{-n})^{-1} = L_a^-(v_a q^{-n})^{-1} R_{0a}(u q^{2n}/v_a) L_0^+(u), \qquad a = 1, \dots, k,$$

and so

$$L_0^+(u)R_{01}(u/v_1)\dots R_{0k}(u/v_k)L_1^-(v_1q^{-n})^{-1}\dots L_k^-(v_kq^{-n})^{-1}$$
$$= L_1^-(v_1q^{-n})^{-1}\dots L_k^-(v_kq^{-n})^{-1}R_{01}(uq^{2n}/v_1)\dots R_{0k}(uq^{2n}/v_k)L_0^+(u).$$

Thus, to conclude that $L_0^+(u)\,\ell_k(z)=\ell_k(z)\,L_0^+(u)$ we need to show that the trace

$$\operatorname{tr}_{1,\dots,k} R_{0k}(u/v_k)^{-1}\dots R_{01}(u/v_1)^{-1}L_k^+(v_k)\dots L_1^+(v_1)$$
$$\times L_1^-(v_1q^{-n})^{-1}\dots L_k^-(v_kq^{-n})^{-1}R_{01}(uq^{2n}/v_1)\dots R_{0k}(uq^{2n}/v_k)D_1\dots D_k\,A^{(k)} \quad (3.10)$$

equals $\ell_k(z)$. The R-matrix $R(u,v)$ satisfies the Yang–Baxter equation

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w)=R_{23}(v,w)R_{13}(u,w)R_{12}(u,v)$$

which implies

$$R_{12}(u,v)\,R_{13}(u/w)\,R_{23}(v/w)=R_{23}(v/w)\,R_{13}(u/w)\,R_{12}(u,v).$$

Therefore, Lemma 3.1 gives

$$R_{01}(uq^{2n}/v_1)\dots R_{0k}(uq^{2n}/v_k)\,A^{(k)}=A^{(k)}R_{0k}(uq^{2n}/v_k)\dots R_{01}(uq^{2n}/v_1), \quad (3.11)$$
$$R_{0k}(u/v_k)^{-1}\dots R_{01}(u/v_1)^{-1}A^{(k)}=A^{(k)}R_{01}(u/v_1)^{-1}\dots R_{0k}(u/v_k)^{-1}. \quad (3.12)$$

Applying (3.5), (3.6), (3.7) and (3.11) to the expression under the trace in (3.10), we will bring it to the form

$$\operatorname{tr}_{1,\dots,k} R_{0k}(u/v_k)^{-1}\dots R_{01}(u/v_1)^{-1}A^{(k)}L_1^+(v_1)\dots L_k^+(v_k)$$
$$\times L_k^-(v_kq^{-n})^{-1}\dots L_1^-(v_1q^{-n})^{-1}R_{0k}(uq^{2n}/v_k)\dots R_{01}(uq^{2n}/v_1)D_1\dots D_k. \quad (3.13)$$

Now write $A^{(k)}=(A^{(k)})^2$ and move one copy of $A^{(k)}$ to the left by using (3.12), and move the other copy back to its right-most position. We get

$$\operatorname{tr}_{1,\dots,k} A^{(k)}R_{01}(u/v_1)^{-1}\dots R_{0k}(u/v_k)^{-1}L_k^+(v_k)\dots L_1^+(v_1)$$
$$\times L_1^-(v_1q^{-n})^{-1}\dots L_k^-(v_kq^{-n})^{-1}R_{01}(uq^{2n}/v_1)\dots R_{0k}(uq^{2n}/v_k)D_1\dots D_k\,A^{(k)}. \quad (3.14)$$

Use the cyclic property of trace to move the left copy of $A^{(k)}$ to the right-most position and replace $(A^{(k)})^2$ with $A^{(k)}$. As a result of these transformations, the order of the first k factors in (3.10) will be reversed, while the rest of the expression remains unchanged. Therefore, we can also write it in the form (3.13) with the order of the first k factors reversed; that is, of the form $\operatorname{tr}_{1,\dots,k} XY$ with

$$X=R_{01}(u/v_1)^{-1}\dots R_{0k}(u/v_k)^{-1}A^{(k)}L_1^+(v_1)\dots L_k^+(v_k)L_k^-(v_kq^{-n})^{-1}\dots L_1^-(v_1q^{-n})^{-1}$$

and

$$Y = R_{0k}(uq^{2n}/v_k)\dots R_{01}(uq^{2n}/v_1)D_1\dots D_k.$$

Now use the property

$$\mathrm{tr}_{1,\dots,k}\, XY = \mathrm{tr}_{1,\dots,k}\, X^{t_1\dots t_k}Y^{t_1\dots t_k}.$$

We have

$$X^{t_1\dots t_k} = L^{t_1\dots t_k}\big(R_{01}(u/v_1)^{-1}\big)^{t_1}\dots\big(R_{0k}(u/v_k)^{-1}\big)^{t_k},$$

where we have set

$$L = A^{(k)}L_1^+(v_1)\dots L_k^+(v_k)L_k^-(v_kq^{-n})^{-1}\dots L_1^-(v_1q^{-n})^{-1}.$$

Furthermore,

$$Y^{t_1\dots t_k} = D_1\dots D_kR_{0k}(uq^{2n}/v_k)^{t_k}\dots R_{01}(uq^{2n}/v_1)^{t_1}.$$

Hence, the first crossing symmetry relation in (2.13) gives

$$\begin{aligned}\mathrm{tr}_{1,\dots,k}\, X^{t_1\dots t_k}Y^{t_1\dots t_k} &= \mathrm{tr}_{1,\dots,k}\, L^{t_1\dots t_k}D_1\dots D_k\\ &= \mathrm{tr}_{1,\dots,k}\, A^{(k)}L_1^+(v_1)\dots L_k^+(v_k)L_k^-(v_kq^{-n})^{-1}\dots L_1^-(v_1q^{-n})^{-1}D_1\dots D_k\end{aligned}$$

which coincides with $\ell_k(z)$ as defined in (3.4) thus completing the proof for $L_0^+(u)$.

The argument showing that $L_0^-(u)\,\ell_k(z) = \ell_k(z)\,L_0^-(u)$ is quite similar, so we only briefly outline it. Using (2.8) we get

$$\begin{aligned}L_0^-(u)L_k^+(v_k)\dots L_1^+(v_1) &= R_{k0}(v_kq^n/u)\dots R_{10}(v_1q^n/u)\\ &\times L_k^+(v_k)\dots L_1^+(v_1)L_0^-(u)R_{10}(v_1q^{-n}/u)^{-1}\dots R_{k0}(v_kq^{-n}/u)^{-1}.\end{aligned}$$

Next, due to (2.7), we have

$$\begin{aligned}&L_0^-(u)R_{10}(v_1q^{-n}/u)^{-1}\dots R_{k0}(v_kq^{-n}/u)^{-1}L_1^-(v_1q^{-n})^{-1}\dots L_k^-(v_kq^{-n})^{-1}\\ &\qquad= L_1^-(v_1q^{-n})^{-1}\dots L_k^-(v_kq^{-n})^{-1}R_{10}(v_1q^{-n}/u)^{-1}\dots R_{k0}(v_kq^{-n}/u)^{-1}L_0^-(u).\end{aligned}$$

The argument is completed by verifying that the trace

$$\begin{aligned}&\mathrm{tr}_{1,\dots,k}\, R_{k0}(v_kq^n/u)\dots R_{10}(v_1q^n/u)L_k^+(v_k)\dots L_1^+(v_1)L_1^-(v_1q^{-n})^{-1}\dots L_k^-(v_kq^{-n})^{-1}\\ &\qquad\times R_{10}(v_1q^{-n}/u)^{-1}\dots R_{k0}(v_kq^{-n}/u)^{-1}D_1\dots D_k\,A^{(k)}\end{aligned}$$

coincides with $\ell_k(z)$. This is done in the same way as for the expression (3.10) with the use of Lemma 3.1 and the second crossing symmetry relation in (2.13). □

As a corollary of Theorem 3.2, we obtain an explicit description of invariants of the vacuum module over the quantum affine algebra; cf. [7], [8]. By definition, the *vacuum module at the critical level* $V_q(\mathfrak{gl}_n)$ is the quotient of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ by the left ideal generated by all elements $l^-_{ij}[r]$ with $r > 0$ and by the elements $l^-_{ij}[0] - \delta_{ij}$ with $i \geqslant j$. The module $V_q(\mathfrak{gl}_n)$ is generated by the vector $\mathbf{1}$ (the image of $1 \in \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ in the quotient) such that

$$L^-(u)\,\mathbf{1} = I\,\mathbf{1},$$

where I denotes the identity matrix. As a vector space, $V_q(\mathfrak{gl}_n)$ can be identified with the subalgebra $\mathrm{Y}_q(\mathfrak{gl}_n)$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ generated by the coefficients of all series $l^+_{ij}(u)$ subject to the additional relations $l^+_{ii}[0] = 1$. This relies on the Poincaré–Birkhoff–Witt theorem for the quantum affine algebra; see, e.g., Sec. 5 below. The subspace of invariants of $V_q(\mathfrak{gl}_n)$ is defined by

$$\mathfrak{z}_q(\widehat{\mathfrak{gl}}_n) = \{v \in V_q(\mathfrak{gl}_n) \mid L^-(u)v = I\,v\};$$

cf. [15, Sec. 3.3] and [18, Sec. 8]. One can regard $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_n)$ as a subspace of $\mathrm{Y}_q(\mathfrak{gl}_n)$. Moreover, this subspace is closed under the multiplication in the quantum affine algebra. Therefore, $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_n)$ can be identified with a subalgebra of $\mathrm{Y}_q(\mathfrak{gl}_n)$. For $k = 1, \dots, n$ introduce the series $\bar{\ell}_k(z)$ with coefficients in $\mathrm{Y}_q(\mathfrak{gl}_n)$ by

$$\bar{\ell}_k(z) = \mathrm{tr}_{1,\dots,k}\, A^{(k)}\, L^+_1(z) \dots L^+_k(zq^{-2k+2}) D_1 \dots D_k. \tag{3.15}$$

Corollary 3.3. *All coefficients of the series* $\bar{\ell}_k(z)\mathbf{1}$ *with* $k = 1, \dots, n$ *belong to the algebra of invariants* $\mathfrak{z}_q(\widehat{\mathfrak{gl}}_n)$. *Moreover, the coefficients of all series* $\bar{\ell}_k(z)$ *pairwise commute.*

Proof. By Theorem 3.2, we have $L^-(u)\,\ell_k(z) = \ell_k(z)\,L^-(u)$. Apply both sides to the vector $\mathbf{1} \in V_q(\mathfrak{gl}_n)$ and observe that $\ell_k(z)\mathbf{1} = \bar{\ell}_k(z)\mathbf{1}$. This proves the first part of the corollary. The second part follows by the application of both sides of the identity $\ell_k(z)\ell_m(w) = \ell_m(w)\ell_k(z)$ to the vector $\mathbf{1}$. For the left hand side we get

$$\ell_k(z)\ell_m(w)\,\mathbf{1} = \ell_k(z)\bar{\ell}_m(w)\,\mathbf{1} = \bar{\ell}_m(w)\,\ell_k(z)\,\mathbf{1} = \bar{\ell}_m(w)\,\bar{\ell}_k(z)\,\mathbf{1}.$$

The same calculation for the right hand side gives $\bar{\ell}_k(z)\bar{\ell}_m(w) = \bar{\ell}_m(w)\bar{\ell}_k(z)$. □

The second part of the corollary is well known; the series $\bar{\ell}_k(z)$ essentially coincides with the *transfer matrix* associated with the k-th fundamental representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$; see e.g. [18]. The Harish-Chandra image of $\bar{\ell}_k(z)$ coincides with the *q-character* of this representation; see also Theorem 6.2 below which recovers the calculation of the image in a more general context.

Remark 3.4. The form of the series $\ell_k(z)$ and $\bar{\ell}_k(z)$ indicates a possible interpretation of their properties from the viewpoint of the quantum vertex algebra theory of [12]. □

4 Quantum minor formulas for $\ell_k(z)$ and $\bar{\ell}_k(z)$

By calculating the trace in (3.4), we can get two quantum minor-type expressions for $\ell_k(z)$ in terms of the entries of the matrices $L^+(z) = \big[l^+_{ij}(z)\big]$ and $\widetilde{L}(z) := L^-(z)^{-1}D = \big[\widetilde{l}_{ij}(z)\big]$. We will denote by $l(\sigma)$ the length of a reduced decomposition of a permutation $\sigma \in \mathfrak{S}_k$. The length $l(\sigma)$ coincides with the number of inversions in the sequence $\big(\sigma(1), \dots, \sigma(k)\big)$.

Proposition 4.1. *For $k = 1, \dots, n$ we have*

$$\ell_k(z) = \sum_{j_1,\dots,j_k} \sum_{i_1<\dots<i_k} \sum_{\sigma\in\mathfrak{S}_k} (-q)^{-l(\sigma)}\, l^+_{i_{\sigma(1)}j_1}(z)\dots l^+_{i_{\sigma(k)}j_k}(zq^{-2k+2}) \times \widetilde{l}_{j_k i_k}(zq^{-n-2k+2})\dots \widetilde{l}_{j_1 i_1}(zq^{-n}) \quad (4.1)$$

and

$$\ell_k(z) = \sum_{j_1,\dots,j_k} \sum_{i_1<\dots<i_k} \sum_{\sigma\in\mathfrak{S}_k} (-q)^{l(\sigma)}\, l^+_{i_{\sigma(k)}j_k}(z)\dots l^+_{i_{\sigma(1)}j_1}(zq^{-2k+2}) \times \widetilde{l}_{j_1 i_1}(zq^{-n-2k+2})\dots \widetilde{l}_{j_k i_k}(zq^{-n}). \quad (4.2)$$

Proof. Using (3.4) we interpret

$$A^{(k)} L^+_1(v_1)\dots L^+_k(v_k)\widetilde{L}_k(v_k q^{-n})\dots \widetilde{L}_1(v_1 q^{-n})$$

with $v_a = zq^{-2a+2}$ as an operator in the vector space $(\mathbb{C}^n)^{\otimes k}$. Since it is divisible on the right by $A^{(k)}$, the trace of the operator can be found as $k!$ times the sum of the diagonal matrix elements corresponding to basis vectors of the form $e_{i_1}\otimes\dots\otimes e_{i_k}$ with $i_1 < \dots < i_k$. Now (4.1) follows with the use of (3.3) and the action of the q-permutations on the basis vectors of this form: for any $\sigma \in \mathfrak{S}_k$

$$P^q_\sigma\big(e_{i_1}\otimes\dots\otimes e_{i_k}\big) = q^{l(\sigma)}\, e_{i_{\sigma^{-1}(1)}}\otimes\dots\otimes e_{i_{\sigma^{-1}(k)}}$$

and hence

$$P^q_\sigma\big(e_{i_{\sigma(1)}}\otimes\dots\otimes e_{i_{\sigma(k)}}\big) = q^{-l(\sigma)}\, e_{i_1}\otimes\dots\otimes e_{i_k}.$$

The proof of (4.2) is quite similar; use the basis vectors $e_{i_k}\otimes\dots\otimes e_{i_1}$ with the same condition $i_1 < \dots < i_k$ on the indices. □

Remark 4.2. Two more formulas for $\ell_k(z)$ analogous to (4.1) and (4.2) can be obtained by using (3.8) instead of (3.4). □

The series $\ell_n(z)$ can be factorized into a product of two quantum determinants. To derive the factorization formula, recall a construction of the quantum minors of the matrices $L^\pm(z)$. Lemma 3.1 implies the relations

$$A^{(k)} L^\pm_1(z)\dots L^\pm_k(q^{-2k+2}z) = L^\pm_k(q^{-2k+2}z)\dots L^\pm_1(z)A^{(k)}. \quad (4.3)$$

The *quantum minors* $L^{\pm}(z)^{a_1\cdots a_k}_{b_1\cdots b_k}$ are the coefficients in the expansion of the either side of (4.3) along the basis of matrix units:

$$\sum_{a_i,b_i} e_{a_1b_1}\otimes\ldots\otimes e_{a_kb_k}\otimes L^{\pm}(z)^{a_1\ldots a_k}_{b_1\ldots b_k}. \tag{4.4}$$

The following formulas are immediate from the definition. If $a_1<\cdots<a_k$ then

$$L^{\pm}(z)^{a_1\cdots a_k}_{b_1\cdots b_k}=\sum_{\sigma\in\mathfrak{S}_k}(-q)^{-l(\sigma)}\, l^{\pm}_{a_{\sigma(1)}b_1}(z)\cdots l^{\pm}_{a_{\sigma(k)}b_k}(q^{-2k+2}z) \tag{4.5}$$

and for any $\tau\in\mathfrak{S}_k$ we have

$$L^{\pm}(z)^{a_{\tau(1)}\cdots a_{\tau(k)}}_{b_1\ldots b_k}=(-q)^{l(\tau)}L^{\pm}(z)^{a_1\ldots a_k}_{b_1\ldots b_k}. \tag{4.6}$$

If $b_1<\cdots<b_k$ (and the a_i are arbitrary) then

$$L^{\pm}(z)^{a_1\ldots a_k}_{b_1\ldots b_k}=\sum_{\sigma\in\mathfrak{S}_k}(-q)^{l(\sigma)}\, l^{\pm}_{a_kb_{\sigma(k)}}(q^{-2k+2}z)\ldots l^{\pm}_{a_1b_{\sigma(1)}}(z), \tag{4.7}$$

and for any $\tau\in\mathfrak{S}_k$ we have

$$L^{\pm}(z)^{a_1\ldots a_k}_{b_{\tau(1)}\cdots b_{\tau(k)}}=(-q)^{-l(\tau)}L^{\pm}(z)^{a_1\ldots a_k}_{b_1\ldots b_k}. \tag{4.8}$$

Moreover, the quantum minor is zero if two top or two bottom indices are equal.

The following lemma is well-known. We give a proof for completeness.[1]

Lemma 4.3. *The coefficients of the quantum determinants*

$$\operatorname{qdet} L^{\pm}(z)=L^{\pm}(z)^{1\cdots n}_{1\cdots n} \tag{4.9}$$

belong to the center of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ *at the critical level.*

Proof. Introduce the product

$$R(v_0,v_1,\ldots,v_n)=\prod_{0\leqslant a<b\leqslant n}R_{ab}(v_a/v_b),$$

where the v_a are variables and the product is taken in the lexicographical order on the pairs (a,b). The defining relations (2.7) and (2.8) imply

$$R(uq^n,v_1,\ldots,v_n)L^+_0(u)L^-_1(v_1)\ldots L^-_n(v_n)=L^-_n(v_n)\ldots L^-_1(v_1)L^+_0(u)R(uq^{-n},v_1,\ldots,v_n).$$

[1]The critical level assumption $k=-2$ is omitted in the corresponding statement in [17, Lemma 2] in the case $n=2$. It was only used there under this assumption.

Use (2.1)–(2.3) to write this relation in terms of the R-matrix $R(u,v)$. By cancelling common factors we get

$$\widetilde{R}(uq^n, v_1, \dots, v_n) L_0^+(u) L_1^-(v_1) \dots L_n^-(v_n) = L_n^-(v_n) \dots L_1^-(v_1) L_0^+(u) \widetilde{R}(uq^{-n}, v_1, \dots, v_n),$$

where

$$\widetilde{R}(v_0, v_1, \dots, v_n) = \prod_{a=1}^n \frac{f(v_0/v_a)}{q^{-1}v_0 - qv_a} \prod_{0 \leqslant a < b \leqslant n} R_{ab}(v_a, v_b).$$

Now specialize the variables by setting $v_a = zq^{-2a+2}$ for $a = 1, \dots, n$ and replace the product of R-matrices $R_{ab}(v_a, v_b)$ over the set of pairs $1 \leqslant a < b \leqslant n$ using Lemma 3.1. Since

$$A^{(n)} L_1^{\pm}(v_1) \dots L_n^{\pm}(v_n) = L_n^{\pm}(v_n) \dots L_1^{\pm}(v_1) A^{(n)} = A^{(n)} \operatorname{qdet} L^{\pm}(z), \tag{4.10}$$

we get

$$\prod_{a=1}^n \frac{f(uq^n/v_a)}{uq^{n-1} - qv_a} \prod_{a=1,\dots,n}^{\longrightarrow} R_{0a}(uq^n, v_a) A^{(n)} L_0^+(u) \operatorname{qdet} L^-(z)$$
$$= \prod_{a=1}^n \frac{f(uq^{-n}/v_a)}{uq^{-n-1} - qv_a} \operatorname{qdet} L^-(z) L_0^+(u) A^{(n)} \prod_{a=1,\dots,n}^{\longleftarrow} R_{0a}(uq^{-n}, v_a).$$

Observe that

$$\prod_{a=1,\dots,n}^{\longrightarrow} R_{0a}(v_0, v_a) A^{(n)} = A^{(n)} \prod_{a=1,\dots,n}^{\longleftarrow} R_{0a}(v_0, v_a) = A^{(n)} (q^{-1}v_0 - qv_1) \prod_{a=2}^n (v_0 - v_a).$$

Indeed, by the first equality, it suffices to verify the second equality on the basis vectors of the form $e_i \otimes e_i \otimes e_1 \otimes \dots \otimes e_{i-1} \otimes e_{i-1} \otimes \dots \otimes e_n$ for $i = 1, \dots, n$ which is straightforward. Thus, we can conclude that $L_0^+(u)$ commutes with $\operatorname{qdet} L^-(z)$ due to the identity

$$\prod_{a=1}^n \frac{f(uq^n/v_a)}{f(uq^{-n}/v_a)} = \prod_{a=2}^n \frac{(uq^{n-1} - qv_a)(uq^{-n} - v_a)}{(uq^{-n-1} - qv_a)(uq^n - v_a)}$$

which follows from (2.4). The relation $L_0^-(u) \operatorname{qdet} L^+(z) = \operatorname{qdet} L^+(z) L_0^-(u)$ is verified by a similar argument with the use of the unitarity property of the R-matrix (2.2):

$$\overline{R}(x^{-1}) = \overline{R}_{21}(x)^{-1}.$$

The proof of the remaining two relations $L_0^{\pm}(u) \operatorname{qdet} L^{\pm}(z) = \operatorname{qdet} L^{\pm}(z) L_0^{\pm}(u)$ is simpler as it relies only on the defining relations (2.7). □

Remark 4.4. Both quantum determinants (4.9) are also known to be central in the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ at the zero level $c = 0$; the algebra is defined as in Sec. 2 with the factor $f(x)$ in (2.2) omitted. □

Corollary 4.5. *We have*

$$\ell_n(z) = \mathrm{qdet}\, L^+(z)\, \big(\mathrm{qdet}\, L^-(zq^{-n})\big)^{-1}.$$

Proof. Relation (4.10) implies

$$A^{(n)} L_n^-(v_n)^{-1} \dots L_1^-(v_1)^{-1} = \big(\mathrm{qdet}\, L^-(z)\big)^{-1} A^{(n)}.$$

Replacing here z by zq^{-n} and using $\mathrm{qdet}\, D = 1$ we get the desired formula from (3.4). □

We will now give a formulation of Corollary 3.3 by combining the series $\bar{\ell}_k(z)$ defined in (3.15) into a single determinant by some analogy with [7] and [8]. Introduce the extension $\mathrm{Y}_q^{\mathrm{ext}}(\mathfrak{gl}_n)$ of the algebra $\mathrm{Y}_q(\mathfrak{gl}_n)$ by adjoining pairwise commuting elements $\pi_1, \dots, \pi_n$ subject to the additional relations

$$\pi_i\, l_{km}^+(u) = \begin{cases} l_{km}^+(u)\, \pi_i & \text{if} \quad i \leqslant k, \\ q^2\, l_{km}^+(u)\, \pi_i & \text{if} \quad i > k. \end{cases} \tag{4.11}$$

Combine the elements π_i into the diagonal matrix $\Pi = \mathrm{diag}\, [\pi_1, \dots, \pi_n]$. We have a vector space isomorphism $\mathrm{Y}_q^{\mathrm{ext}}(\mathfrak{gl}_n)/J \cong \mathrm{Y}_q(\mathfrak{gl}_n)$, where J is the left ideal of $\mathrm{Y}_q^{\mathrm{ext}}(\mathfrak{gl}_n)$ generated by the elements $\pi_i - 1$ with $i = 1, \dots, n$. We will identify $\mathrm{Y}_q(\mathfrak{gl}_n)$ with the quotient via this isomorphism.

We will point out a connection with q-analogues of Manin matrices (also known as *right quantum matrices*); see, e.g., [6] for a detailed account of their properties. An $n \times n$ matrix M with entries in an associative algebra $\mathcal{A}$ is called *q-Manin* if it satisfies the relation

$$A^{(2)} M_1 M_2 = A^{(2)} M_1 M_2 A^{(2)}$$

in the algebra $\mathrm{End}\, \mathbb{C}^n \otimes \mathrm{End}\, \mathbb{C}^n \otimes \mathcal{A}$ with the meaning of the subscripts as in (2.11).

Introduce the operator δ which interacts with power series in z by the rule $\delta\, g(z) = g(zq^{-2})\, \delta$. Adjoining this element to the algebra $\mathrm{Y}_q(\mathfrak{gl}_n)$ we find that both $L^+(z)\delta$ and $L^+(z)\, D\, \delta$ are q-Manin matrices. Define the *q-determinant* of a square matrix M by

$$\mathrm{det}_q\, M = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)}\, M_{\sigma(1)1} \cdots M_{\sigma(n)n}.$$

In the following proposition we regard the q-determinant of the matrix $\Pi + L^+(z)\, D\, \delta$ as a polynomial in δ with coefficients in $\mathrm{Y}_q^{\mathrm{ext}}(\mathfrak{gl}_n)$.

Proposition 4.6. *We have the relation modulo the left ideal J:*

$$\mathrm{det}_q\big(\Pi + L^+(z)\, D\, \delta\big) = 1 + \sum_{k=1}^{n} \bar{\ell}_k(z)\, \delta^k.$$

Proof. Calculating the q-determinant we will write it as the sum of monomials of the form

$$(-q)^{-l(\sigma)} M_{\sigma(1)\,1} \cdots M_{\sigma(i_1-1)\,i_1-1}\, \pi_{i_1} M_{\sigma(i_1+1)\,i_1+1} \\ \times \cdots M_{\sigma(i_2-1)\,i_2-1}\, \pi_{i_2} M_{\sigma(i_2+1)\,i_2+1} \cdots \pi_{i_{n-k}} M_{\sigma(i_{n-k}+1)\,i_{n-k}+1} \cdots M_{\sigma(n)n},$$

where $M = L^+(z)D\delta$ and $\sigma(i_a) = i_a$ for $a = 1, \dots, n-k$. Now use relations (4.11) to move all the elements π_{i_a} so they will appear to the right from all factors M_{km} in the monomial. As a result, by moving each element π_{i_a} we get the factor q^{2r_a}, where r_a is the number of elements of the set $\{\sigma(i_a+1), \dots, \sigma(n)\}$ which are less than i_a. That is, r_a is the number of inversions formed by the index i_a with the indices of the set $\{\sigma(i_a+1), \dots, \sigma(n)\}$. Therefore, after moving all the elements π_{i_a}, the monomial will get the factor $(-q)^{-l(\widehat{\sigma})}$, where $\widehat{\sigma}$ is the sequence of elements obtained from $(\sigma(1), \dots, \sigma(n))$ by removing $i_1, \dots, i_{n-k}$, and $l(\widehat{\sigma})$ is the number of inversions in that sequence. This demonstrates that the coefficient of the product $\pi_{i_1} \dots \pi_{i_{n-k}}$ coincides with the q-determinant of the principal submatrix of M obtained by deleting rows and columns enumerated by $i_1, \dots, i_{n-k}$. This q-determinant equals $\big[L^+(z)D\big]^{a_1 \cdots a_k}_{a_1 \cdots a_k} \delta^k$ where $\{a_1, \dots, a_k\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_{n-k}\}$. Since

$$\bar{\ell}_k(z) = \sum_{1 \leqslant a_1 < \cdots < a_k \leqslant n} \big[L^+(z)D\big]^{a_1 \cdots a_k}_{a_1 \cdots a_k},$$

the required relation follows by taking the quotient over the left ideal J. □

The next lemma will be used in Sec. 6 below.

Lemma 4.7. *The entries of the inverse matrix $L^-(z)^{-1}$ are found by the formula*

$$\big[L^-(z)^{-1}\big]_{ij} = (-q)^{j-i} \big(\mathrm{qdet}\, L^-(zq^{2n-2})\big)^{-1} L^-(zq^{2n-2})^{1 \dots \widehat{j} \dots n}_{1 \dots \widehat{i} \dots n},$$

where the hats indicate indices to be skipped.

Proof. By (4.10) we have

$$A^{(n)} L^-_1(v_1) \dots L^-_{n-1}(v_{n-1}) = A^{(n)} \mathrm{qdet}\, L^-(z) L^-_n(v_n)^{-1},$$

where $v_a = zq^{-2a+2}$, as before. The desired formula follows by the application of both sides to the basis vector $e_1 \otimes \dots \otimes \widehat{e_i} \otimes \dots \otimes e_n \otimes e_j$ and the replacement $z \mapsto zq^{2n-2}$. □

5 Poincaré–Birkhoff–Witt theorem

To define analogues of the Harish-Chandra homomorphism, we will need a version of the Poincaré–Birkhoff–Witt theorem for the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. Introduce a total ordering $\prec$ on the set of generators as follows. First, each generator $l^+_{ij}[r]$ precedes each generator $l^-_{km}[s]$. Furthermore, $l^+_{ij}[r] \prec l^+_{km}[s]$ if and only if the triple $(j-i,i,r)$ precedes $(m-k,k,s)$ in the lexicographical order. Finally, we set $l^-_{ij}[r] \prec l^-_{km}[s]$ if and only if the triple $(i-j,i,r)$ precedes $(k-m,k,s)$ in the lexicographical order. Note that by the defining relations (2.7),

$$\big[l^\pm_{ij}[r], l^\pm_{ij}[s]\big] = 0 \tag{5.1}$$

for all r and s. Hence, the ordering $\prec$ induces a well-defined total ordering on the series (2.9) such that $l^+_{ij}(u) \prec l^-_{km}(u)$ and

$$l^+_{n1}(u) \prec l^+_{n-1\,1}(u) \prec l^+_{n\,2}(u) \prec \dots \prec l^+_{11}(u) \prec \dots \prec l^+_{nn}(u) \prec l^+_{12}(u) \prec \dots \prec l^+_{1n}(u),$$
$$l^-_{1n}(u) \prec l^-_{1\,n-1}(u) \prec l^-_{2\,n}(u) \prec \dots \prec l^-_{11}(u) \prec \dots \prec l^-_{nn}(u) \prec l^-_{21}(u) \prec \dots \prec l^-_{n1}(u).$$

Consider the ordered monomials in the generators $l^\pm_{ij}[r]$ multiplied by integer powers of the central element $\gamma = q^c$ (the zero elements $l^+_{ij}[0]$ for $i > j$ and $l^-_{ij}[0]$ for $i < j$ are excluded). Relations (2.7) and (2.8) imply that

$$l^+_{ii}[0]\, l^\pm_{km}(u) = q^{-\delta_{ik}+\delta_{im}}\, l^\pm_{km}(u)\, l^+_{ii}[0] \qquad \text{and} \qquad l^-_{ii}[0]\, l^\pm_{km}(u) = q^{\delta_{ik}-\delta_{im}}\, l^\pm_{km}(u)\, l^-_{ii}[0]. \tag{5.2}$$

Hence, using (2.6) we may suppose that for each $i = 1,\dots,n$ each monomial only contains either a nonnegative power of $l^+_{ii}[0]$ or a positive power of $l^-_{ii}[0]$. Under these assumptions we have the following version of the Poincaré–Birkhoff–Witt theorem.

Proposition 5.1. *The ordered monomials in the generators form a basis of the quantum affine algebra* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

Proof. First, we prove the claim for the quantum affine algebra $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$ which is defined in the same way as $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$; the only difference is the use of the R-matrix (2.2) instead of (2.3). Thus, we only replace (2.8) with the relation

$$\overline{R}(u\gamma^{-1}/v)\overline{L}{}^+_1(u)\overline{L}{}^-_2(v) = \overline{L}{}^-_2(v)\overline{L}{}^+_1(u)\overline{R}(u\gamma/v) \tag{5.3}$$

and leave all other defining relations unchanged. Here we use the bar symbol over the respective objects associated with the algebra $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$. We begin by showing that the ordered monomials in the generators of $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$ span the algebra. Given a monomial in the generators we will use the induction on its length to show that it equals a linear combination of ordered monomials. Writing (5.3) in terms of the entries of the matrices $\overline{L}{}^\pm(u)$ we get

$$\begin{aligned}
(\gamma\, q^{-\delta_{jm}} u - q^{\delta_{jm}} v)\, \bar{l}^-_{km}(v)\, \bar{l}^+_{ij}(u) + (q^{-1}-q)\,(u\,\gamma\,\delta_{m>j} + v\,\delta_{m<j})\, \bar{l}^-_{kj}(v)\, \bar{l}^+_{im}(u)\\
= \Big[(\gamma^{-1} q^{-\delta_{ik}} u - q^{\delta_{ik}} v)\, \bar{l}^+_{ij}(u)\, \bar{l}^-_{km}(v)\\
+ (q^{-1}-q)\,(u\,\gamma^{-1}\,\delta_{i>k} + v\,\delta_{i<k})\, \bar{l}^+_{kj}(u)\, \bar{l}^-_{im}(v)\Big]\, \frac{\gamma\, q^{-2}u - v}{\gamma^{-1} q^{-2} u - v},
\end{aligned} \tag{5.4}$$

where $\delta_{i<j}$ or $\delta_{i>j}$ equals 1 if the subscript inequality is satisfied and 0 otherwise. If $m=j$ then the relation allows us to write $\bar{l}^{-}_{kj}[s]\,\bar{l}^{+}_{ij}[r]$ as a linear combination of ordered products of generators. If $m\neq j$ then we swap m and j in the relation to get a system of two equations for $\bar{l}^{-}_{km}(v)\,\bar{l}^{+}_{ij}(u)$ and $\bar{l}^{-}_{kj}(v)\,\bar{l}^{+}_{im}(u)$. By solving the system we will be able to write $\bar{l}^{-}_{km}[s]\,\bar{l}^{+}_{ij}[r]$ as a linear combination of ordered products of generators.

Similarly, (2.7) gives the relation

$$
\begin{aligned}
&(q^{-\delta_{ik}}u-q^{\delta_{ik}}v)\,\bar{l}^{\pm}_{ij}(u)\,\bar{l}^{\pm}_{km}(v)+(q^{-1}-q)\,(u\,\delta_{i>k}+v\,\delta_{i<k})\,\bar{l}^{\pm}_{kj}(u)\,\bar{l}^{\pm}_{im}(v)\\
&=(q^{-\delta_{jm}}u-q^{\delta_{jm}}v)\,\bar{l}^{\pm}_{km}(v)\,\bar{l}^{\pm}_{ij}(u)+(q^{-1}-q)\,(u\,\delta_{m>j}+v\,\delta_{m<j})\,\bar{l}^{\pm}_{kj}(v)\,\bar{l}^{\pm}_{im}(u) \qquad (5.5)
\end{aligned}
$$

which implies that $\bar{l}^{\pm}_{ij}[r]\bar{l}^{\pm}_{km}[s]$ is a linear combination of ordered products of generators; see [21, Corollary 2.13] for a detailed argument.[2]

As a next step, we will show that the ordered monomials are linearly independent in $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$. Consider the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_0$ at the level zero which is the quotient of $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$ by the relation $\gamma=1$. We have the natural epimorphism

$$
\psi:\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)\to \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_0,\qquad \overline{L}^{\pm}(u)\mapsto L^{\pm}(u),\qquad \gamma\mapsto 1.
$$

Suppose that a Laurent polynomial in γ, whose coefficients are nontrivial linear combinations of ordered monomials in the generators $\bar{l}^{\pm}_{ij}[r]$, is zero in $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$. Multiplying by a power of γ if necessary, we get a polynomial in γ equal to zero. Choose such a polynomial $P=x_k\gamma^k+\dots+x_0$ of the minimal possible degree $k\geqslant 0$. Since $P=0$ in $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$ we have

$$
0=\psi(P)=\psi(x_k)+\dots+\psi(x_0)=\psi(x_k+\dots+x_0).
$$

The sum $x=x_k+\dots+x_0$ is a linear combination of ordered monomials in the generators $\bar{l}^{\pm}_{ij}[r]$. By the definition of ψ, the image $\psi(x)$ is the corresponding linear combination of ordered monomials in the generators $l^{\pm}_{ij}[r]$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_0$. On the other hand, by the arguments of [21, Sec. 2.3] applied to this particular ordering, the corresponding version of the Poincaré–Birkhoff–Witt theorem holds for $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_0$, so that the ordered monomials are linearly independent. Hence, $\psi(x)=0$ implies $x=0$. If $k=0$ then this is a contradiction. If $k\geqslant 1$ we can write

$$
P=x_k\gamma^k+\dots+x_0=(\gamma-1)(y_{k-1}\gamma^{k-1}+\dots+y_0),
$$

where the y_i are again linear combinations of ordered monomials in the generators $\bar{l}^{\pm}_{ij}[r]$. By the results of [10], the algebra $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$ can be defined by the Drinfeld generators. Due to the well-known relationship between the quantum affine algebras associated with $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ (see, e.g., [16, Sec. 2.6]), the Poincaré–Birkhoff–Witt theorem for the algebra

[2]The generator matrices $T(u)$ and $\overline{T}(u)$ of [21] correspond to $\overline{L}^{-}(u)$ and $\overline{L}^{+}(u)$, respectively.

$\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$ in its Drinfeld presentation [2, 3] implies that the relation $P = 0$ is possible only if $y_{k-1}\gamma^{k-1} + \dots + y_0 = 0$. This contradicts the minimality of the degree k thus completing the proof for $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$.

Finally, we extend the argument to the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ defined with the R-matrix (2.3) so that (2.8) with $\gamma = q^c$ should be used instead of (5.3). This affects only relation (5.4) (for the generators $l^{\pm}_{ij}[r]$ instead of $\bar{l}^{\pm}_{ij}[r]$) which will now get an extra factor $f(u\gamma^{-1}/v)/f(u\gamma/v)$ on the right hand side. However, this does not bring any change into the first part of the argument showing that the ordered monomials in the generators span the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

To prove the linear independence of the ordered monomials, we follow [10, Sec. V] and introduce a homomorphism

$$\phi : \mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \to \mathcal{H}_q(n) \otimes_{\mathbb{C}[\gamma,\gamma^{-1}]} \overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n), \tag{5.6}$$

where $\mathcal{H}_q(n)$ is the Heisenberg algebra with generators γ and $h[r]$, $r \in \mathbb{Z}$, $r \neq 0$. The defining relations of $\mathcal{H}_q(n)$ have the form

$$\big[h[r], h[s]\big] = \delta_{r,-s}\, \alpha[r], \qquad r \geqslant 1,$$

and γ is central and invertible; all other pairs of the generators commute. The elements $\alpha[r]$ are defined by the expansion

$$\exp \sum_{r=1}^{\infty} \alpha[r]\, x^r = \frac{f(x\gamma)}{f(x\gamma^{-1})}.$$

So we have the identity

$$f(u\gamma^{-1}/v)\, \exp\Big(\sum_{r=1}^{\infty} h[r]\, u^r\Big)\, \exp\Big(\sum_{s=1}^{\infty} h[-s]\, v^{-s}\Big) \\ = f(u\gamma/v)\, \exp\Big(\sum_{s=1}^{\infty} h[-s]\, v^{-s}\Big)\, \exp\Big(\sum_{r=1}^{\infty} h[r]\, u^r\Big).$$

Clearly, the monomials of the form $h[r_1] \dots h[r_k]$ with $k \geqslant 0$ and $r_1 \geqslant \dots \geqslant r_k$ (with $r_i \neq 0$) form a basis of the $\mathbb{C}[\gamma, \gamma^{-1}]$-module $\mathcal{H}_q(n)$. The homomorphism (5.6) is now defined by $\phi : \gamma \mapsto \gamma$ and

$$\phi : L^+(u) \mapsto \exp\Big(\sum_{r=1}^{\infty} h[r]\, u^r\Big)\, \overline{L}^{\,+}(u), \quad L^-(u) \mapsto \exp\Big(\sum_{r=1}^{\infty} h[-r]\, u^{-r}\Big)\, \overline{L}^{\,-}(u). \tag{5.7}$$

Suppose there is a linear combination of the ordered monomials in the generators of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ equal to zero. Consider its image under the homomorphism ϕ. Using the basis $\{h[r_1] \dots h[r_k]\}$ of $\mathcal{H}_q(n)$ and the Poincaré–Birkhoff–Witt basis for the algebra $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$, we conclude that all coefficients of the linear combination must be zero. $\square$

We will also need a version of the Poincaré–Birkhoff–Witt theorem for a different ordering of the generators. As we pointed out above, a total ordering can be defined on the generating series (2.9) due to (5.1). We set $l^+_{ij}(u) \prec l^-_{km}(u)$ as before, but the remaining conditions are swapped between $l^+_{ij}(u)$ and $l^-_{ij}(u)$:

$$l^+_{1n}(u) \prec l^+_{1\, n-1}(u) \prec l^+_{2\, n}(u) \prec \cdots \prec l^+_{11}(u) \prec \cdots \prec l^+_{nn}(u) \prec l^+_{21}(u) \prec \cdots \prec l^+_{n1}(u),$$

$$l^-_{n1}(u) \prec l^-_{n-1\, 1}(u) \prec l^-_{n\, 2}(u) \prec \cdots \prec l^-_{11}(u) \prec \cdots \prec l^-_{nn}(u) \prec l^-_{12}(u) \prec \cdots \prec l^-_{1n}(u).$$

Under the same assumptions on the monomials as for Proposition 5.1, the following holds.

Proposition 5.2. *The ordered monomials in the generators form a basis of the quantum affine algebra* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

Proof. The argument is the same as for Proposition 5.1 with some obvious minor changes taking into the account the ordering conditions. □

6 Harish-Chandra homomorphisms

Consider the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ at the critical level, $\gamma = q^{-n}$. By Proposition 5.1, any element $x \in \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ can be written as a unique linear combination of ordered monomials in the generators $l^{\pm}_{ij}[r]$. Denote by U^0 the subspace of the algebra spanned by those monomials which do not contain any generators $l^{\pm}_{ij}[r]$ with $i \neq j$. Let x_0 denote the component of the linear combination representing the element x, which belongs to U^0. The mapping $\theta : x \mapsto x_0$ defines the projection $\theta : \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}} \to \mathrm{U}^0$. Extending it by continuity we get the projection $\theta : \widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}} \to \widetilde{\mathrm{U}}^0$ to the corresponding completed vector space $\widetilde{\mathrm{U}}^0$.

Introduce the algebra $\Pi_q(n)$ as the quotient of the algebra of polynomials in independent variables $l^+_i[-r]$, $l^-_i[r]$ with $i = 1, \dots, n$ and $r = 0, 1, \dots$ by the relations $l^+_i[0]\, l^-_i[0] = 1$ for all i. The mapping $\eta : \mathrm{U}^0 \to \Pi_q(n)$ which takes each ordered monomial in the generators $l^{\pm}_{ii}[\mp r]$ to the corresponding monomial in the variables $l^{\pm}_i[\mp r]$ by the rule $l^{\pm}_{ii}[\mp r] \mapsto l^{\pm}_i[\mp r]$ extends to an isomorphism of vector spaces. Define the completion $\widetilde{\Pi}_q(n)$ of the algebra $\Pi_q(n)$ as the inverse limit

$$\widetilde{\Pi}_q(n) = \lim_{\longleftarrow} \Pi_q(n)/I_p, \qquad p > 0,$$

where I_p denotes the ideal of $\Pi_q(n)$ generated by all elements $l^-_i[r]$ with $r \geqslant p$; cf. (3.1). The isomorphism η extends to an isomorphism of the respective completed vector spaces $\eta : \widetilde{\mathrm{U}}^0 \to \widetilde{\Pi}_q(n)$. Thus we get a linear map

$$\chi : \widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}} \to \widetilde{\Pi}_q(n) \tag{6.1}$$

defined as the composition $\chi = \eta \circ \theta$. The next proposition provides an analogue of the Harish-Chandra homomorphism for the quantum affine algebra.

Proposition 6.1. *The restriction of the map* (6.1) *to the center* $\mathrm{Z}_q(\widehat{\mathfrak{gl}}_n)$ *of the algebra* $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ *is a homomorphism of commutative algebras*

$$\chi : \mathrm{Z}_q(\widehat{\mathfrak{gl}}_n) \to \widetilde{\Pi}_q(n). \tag{6.2}$$

Proof. For $x, y \in \mathrm{Z}_q(\widehat{\mathfrak{gl}}_n)$ set $x_0 = \chi(x)$ and $y_0 = \chi(y)$. Write y as a (possibly infinite) linear combination of ordered monomials in the generators $l_{ij}^{\pm}[r]$. Suppose that

$$m = \prod_a l^{+}_{i_a j_a}[r_a] \prod_b l^{-}_{i_b j_b}[r_b]$$

is an ordered monomial which occurs in the linear combination. Note its property

$$\sum_a (i_a - j_a) + \sum_b (i_b - j_b) = 0 \tag{6.3}$$

implied by (5.2). Suppose that $m \in \ker\chi$. Since x is in the center, we have

$$xm = \prod_a l^{+}_{i_a j_a}[r_a]\, x \prod_b l^{-}_{i_b j_b}[r_b].$$

To write xm as a linear combination of ordered monomials we will only need to use the defining relations (2.7) which are also given in (5.5) where the series $\bar{l}^{\pm}_{ij}(u)$ should be replaced with $l^{\pm}_{ij}(u)$, respectively. Since the relations (5.5) are homogeneous with respect to the weight parameter $i - j + k - m$, we derive that $xm \in \ker\chi$. Hence a nonzero contribution to the image $\chi(xy)$ can only come from $\chi(xy_0)$, that is, from expressions of the form

$$\prod_a l^{+}_{i_a i_a}[r_a]\, x \prod_b l^{-}_{i_b i_b}[r_b].$$

If p is an ordered monomial which occurs in the linear combination representing x and $\chi(p) = 0$, then applying property (6.3) to the monomial p we conclude that

$$\chi : \prod_a l^{+}_{i_a i_a}[r_a]\, p \prod_b l^{-}_{i_b i_b}[r_b] \mapsto 0.$$

Finally, observe that by the defining relations (5.5), any two generators $l^{-}_{ii}[r]$ and $l^{-}_{jj}[s]$ (resp., $l^{+}_{ii}[r]$ and $l^{+}_{jj}[s]$) can be permuted modulo $\ker\chi$ within any monomial of the form

$$\prod_a l^{+}_{i_a i_a}[r_a] \prod_b l^{-}_{i_b i_b}[r_b].$$

This proves that $\chi(xy) = x_0 y_0$. □

Now we are in a position to calculate the Harish-Chandra images of the higher Sugawara operators provided by Theorem 3.2. Combine the generators of the algebra $\Pi_q(n)$ into the series

$$l_i^+(z) = \sum_{r=0}^{\infty} l_i^+[-r]\, z^r, \qquad l_i^-(z) = \sum_{r=0}^{\infty} l_i^-[r]\, z^{-r}$$

and for $i = 1, \dots, n$ set

$$\lambda_i(z) = q^{n-2i+1}\, \frac{l_i^+(z)\, l_1^-(zq^{-n+2}) \dots l_{i-1}^-(zq^{-n+2i-2})}{l_1^-(zq^{-n}) \dots l_i^-(zq^{-n+2i-2})}.$$

This is a Laurent series in z whose coefficients are elements of the completed algebra $\widetilde{\Pi}_q(n)$.

Theorem 6.2. *For each* $k = 1, \dots, n$ *the image of the series* $\ell_k(z)$ *under the Harish-Chandra homomorphism* (6.2) *is found by*

$$\chi : \ell_k(z) \mapsto \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} \lambda_{i_1}(z) \lambda_{i_2}(zq^{-2}) \dots \lambda_{i_k}(zq^{-2k+2}).$$

Proof. We will use formula (4.1) for $\ell_k(z)$. Apply Lemma 4.7 to express the series $\widetilde{l}_{ji}(z) = q^{n-2i+1} \left[L^-(z)^{-1}\right]_{ji}$ in terms of quantum minors. By (4.8) we have

$$L^-(z)^{1\dots\widehat{i}\dots n}_{1\dots\widehat{j}\dots n} = (-q)^{l(\omega)}\, L^-(z)^{1\dots\widehat{i}\dots n}_{n\dots\widehat{j}\dots 1},$$

where $\omega \in \mathfrak{S}_{n-1}$ reverses the order of the lower indices. Expanding this quantum minor by (4.5) we find that a nonzero contribution to the image $\chi\big(\ell_k(z)\big)$ can only come from the summands in (4.1) with $i_{\sigma(1)} \leqslant j_1 \leqslant i_1$. These conditions imply that $\sigma(1) = 1$ and $j_1 = i_1$. By the defining relations in $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$, the same observation gives $\sigma(2) = 2$ and $j_2 = i_2$, etc., so that a nonzero contribution comes only from the terms with $\sigma = 1$ and $j_a = i_a$ for all $a = 1, \dots, k$. Applying Lemma 4.7 and formulas (4.5) and (4.8) again we find that the contributions of the quantum minors are found by

$$\mathrm{qdet}\, L^-(z) \mapsto l_1^-(zq^{-2n+2}) \dots l_n^-(z)$$

and

$$L^-(z)^{1\dots\widehat{i}\dots n}_{1\dots\widehat{i}\dots n} \mapsto l_1^-(zq^{-2n+4}) \dots l_{i-1}^-(zq^{-2n+2i})\, l_{i+1}^-(zq^{-2n+2i+2}) \dots l_n^-(z).$$

This completes the calculation of the Harish-Chandra image of $\ell_k(z)$. □

Consider the restriction of the map (6.1) to the subalgebra $\mathrm{Y}_q(\mathfrak{gl}_n)$ of $\widetilde{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)_{\text{cri}}$. As in Sec. 3, we impose the conditions $l_{ii}^+[0] = 1$ for all i so that

$$\chi : \mathrm{Y}_q(\mathfrak{gl}_n) \to \Pi_q^+(n), \tag{6.4}$$

where $\Pi_q^+(n)$ is the subalgebra of $\Pi_q(n)$ generated by the variables $l_i^+[-r]$ with $i = 1, \dots, n$ and $r = 0, 1, \dots$ subject to the relations $l_i^+[0] = 1$ for all i. Recall the q-determinant calculated in Proposition 4.6. The following corollary essentially reproduces the q-*deformed Miura transformation* of [17].

Corollary 6.3. *We have*

$$\chi : \det_q\big(\Pi + L^+(z)\, D\delta\big) \mapsto \big(1 + \bar{\lambda}_1(z)\delta\big) \dots \big(1 + \bar{\lambda}_n(z)\delta\big),$$

where $\bar{\lambda}_i(z) = q^{n-2i+1}\, l_i^+(z)$.

Proof. This is immediate from Proposition 4.6 and Theorem 6.2. □

In the remainder of this section we outline an alternative construction of the Harish-Chandra homomorphism for the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$. The starting point is the version of the Poincaré–Birkhoff–Witt theorem for a different ordering on the generators as provided by Proposition 5.2. The arguments are essentially the same, with only minor changes in notation. As above, we define the projection $\theta' : \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}} \to \mathrm{U}^0$ in the same way. Proposition 6.1 holds in the same form but for the different Harish-Chandra homomorphism

$$\chi' : \mathrm{Z}_q(\widehat{\mathfrak{gl}}_n) \to \widetilde{\Pi}_q(n) \tag{6.5}$$

defined as the restriction of the composition $\chi' = \eta \circ \theta'$. For $i = 1, \dots, n$ set

$$\lambda'_i(z) = q^{n-2i+1}\, \frac{l_i^+(z)\, l_{i+1}^-(zq^{n-2i}) \dots l_n^-(zq^{-n+2})}{l_i^-(zq^{n-2i}) \dots l_n^-(zq^{-n})}.$$

This is a Laurent series in z whose coefficients are elements of the completed algebra $\widetilde{\Pi}_q(n)$.

Theorem 6.4. *For each* $k = 1, \dots, n$ *the image of the series* $\ell_k(z)$ *under the Harish-Chandra homomorphism* (6.5) *is found by*

$$\chi' : \ell_k(z) \mapsto \sum_{n \geqslant i_1 > \dots > i_k \geqslant 1} \lambda'_{i_1}(z)\lambda'_{i_2}(zq^{-2}) \dots \lambda'_{i_k}(zq^{-2k+2}).$$

Proof. The starting point is formula (4.2) and the argument is quite similar to the proof of Theorem 6.2. □

7 Eigenvalues in Wakimoto modules

Our goal in this section is to relate the image of the series $\ell_k(z)$ under the Harish-Chandra homomorphism provided by Theorem 6.2 with their eigenvalues in the q-deformed Wakimoto modules constructed by Awata, Odake and Shiraishi [1]. Equivalently, due to the work of Frenkel and Reshetikhin [17], these eigenvalues can be interpreted as elements of the q-deformed classical $\mathcal{W}$-algebra $\mathcal{W}_q(\mathfrak{gl}_n)$. They were associated in [17] to the series $\ell_{V_k}(z)$ corresponding to fundamental representations V_k of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$.

To establish the relationship, we consider the Wakimoto modules at the critical level over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. The coefficients of the series $\ell_k(z)$ act as multiplications by scalars in the irreducible modules. We will show that these scalars can be found from Theorem 6.2 by an appropriate identification of the parameters of the Wakimoto modules with elements of $\Pi_q(n)$.

The free field realization of [1] is given in terms of Drinfeld's "new realization" [11] of the quantum affine algebra. Following [17], we will use the Ding–Frenkel isomorphism [10] to get the formulas for the action of the generators of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ in the Wakimoto modules in terms of the RLL presentation. Introduce the series $e^{\pm}_{ij}(u)$, $f^{\pm}_{ij}(u)$ and $k^{\pm}_i(u)$ which are uniquely determined by the Gauss decompositions of the respective matrices $L^{\pm}(u)$:

$$L^{\pm}(u) =$$
$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ e^{\pm}_{21}(u) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ e^{\pm}_{n1}(u) & \dots & e^{\pm}_{n\,n-1}(u) & 1 \end{bmatrix} \begin{bmatrix} k^{\pm}_1(u) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & k^{\pm}_n(u) \end{bmatrix} \begin{bmatrix} 1 & f^{\pm}_{12}(u) & \dots & f^{\pm}_{1n}(u) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f^{\pm}_{n-1\,n}(u) \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

We will need the following quantum minor expressions for these series. Their Yangian counterparts go back to [11] and detailed arguments were given in [4]; see also [23, Sec. 1.11]. The quantum affine algebra case is quite similar so we only sketch the main steps of the proof.

Lemma 7.1. *We have*

$$k^{\pm}_i(u) = L^{\pm}(q^{2i-2}u)^{1\cdots i}_{1\cdots i}\Big[L^{\pm}(q^{2i-2}u)^{1\cdots i-1}_{1\cdots i-1}\Big]^{-1} \tag{7.1}$$

for $i = 1, \dots, n$ *and*

$$\begin{aligned} e^{\pm}_{ji}(u) &= L^{\pm}(q^{2i-2}u)^{1\cdots i-1\,j}_{1\cdots i}\Big[L^{\pm}(q^{2i-2}u)^{1\cdots i}_{1\cdots i}\Big]^{-1} \\ &= q^{-1}\Big[L^{\pm}(q^{2i}u)^{1\cdots i}_{1\cdots i}\Big]^{-1} L^{\pm}(q^{2i}u)^{1\cdots i-1\,j}_{1\cdots i}, \end{aligned} \tag{7.2}$$

$$f^{\pm}_{ij}(u) = \Big[L^{\pm}(q^{2i-2}u)^{1\cdots i}_{1\cdots i}\Big]^{-1} L^{\pm}(q^{2i-2}u)^{1\cdots i}_{1\cdots i-1\,j} = q\,L^{\pm}(q^{2i}u)^{1\cdots i}_{1\cdots i-1\,j}\Big[L^{\pm}(q^{2i}u)^{1\cdots i}_{1\cdots i}\Big]^{-1}$$

for $1 \leqslant i < j \leqslant n$.

Proof. The arguments for the matrices $L^+(u)$ and $L^-(u)$ are the same so we will work with $L^+(u)$ and use the notation $\mathrm{Y}_q(\mathfrak{gl}_n)$ for the subalgebra of the quantum affine algebra generated by the coefficients of all series $l^+_{ij}(u)$. We will also use the algebra $\mathrm{Y}_{q^{-1}}(\mathfrak{gl}_n)$ and denote its generator matrix by $\widetilde{L}^+(u) = [\,\widetilde{l}^{\,+}_{ij}(u)]$. Due to the property

$$R_{21}(v,u) = -R(u,v)\big|_{q\mapsto q^{-1}}$$

of the R-matrix (2.1), the mapping

$$\omega_n : L^+(u) \mapsto \widetilde{L}^+(u)^{-1}$$

defines a homomorphism $\omega_n : \mathrm{Y}_q(\mathfrak{gl}_n) \to \mathrm{Y}_{q^{-1}}(\mathfrak{gl}_n)$. For any $m \geqslant 0$ consider another homomorphism

$$\jmath_m : \mathrm{Y}_{q^{-1}}(\mathfrak{gl}_n) \to \mathrm{Y}_{q^{-1}}(\mathfrak{gl}_{m+n}) \tag{7.3}$$

which takes the coefficients of the series $\widetilde{l}^{\,+}_{ij}(u)$ to the respective coefficients of the series $\widetilde{l}^{\,+}_{m+i\, m+j}(u)$. Consider the composition $\phi_m = \omega_{m+n}^{-1} \circ \jmath_m \circ \omega_n$ which is an algebra homomorphism

$$\phi_m : \mathrm{Y}_q(\mathfrak{gl}_n) \to \mathrm{Y}_q(\mathfrak{gl}_{m+n}).$$

By [22, Lemma 3.7][3]

$$\phi_m : l^+_{ij}(u) \mapsto \Big[L^+(q^{2m}u)^{1\cdots\, m}_{1\cdots\, m}\Big]^{-1}\, L^+(q^{2m}u)^{1\cdots\, m\, m+i}_{1\cdots\, m\, m+j}. \tag{7.4}$$

Now apply [23, Lemmas 1.11.2 and 1.11.5] to the algebra $\mathrm{Y}_q(\mathfrak{gl}_n)$ to get

$$\begin{aligned} k^+_i(u) &= \phi_{i-1}\big(l^+_{11}(u)\big),\\ e^+_{ji}(u) &= \phi_{i-1}\big(l^+_{j-i+1\,1}(u)\, l^+_{11}(u)^{-1}\big),\\ f^+_{ij}(u) &= \phi_{i-1}\big(l^+_{11}(u)^{-1}\, l^+_{1\,j-i+1}(u)\big). \end{aligned}$$

Together with (7.4) this proves the formula for $k^+_i(u)$ and the first expressions for $e^+_{ji}(u)$ and $f^+_{ij}(u)$. To prove the second expressions, use the relations

$$l^+_{j-i+1\,1}(u)\, l^+_{11}(u)^{-1} = q^{-1}\, l^+_{11}(q^2u)^{-1}\, l^+_{j-i+1\,1}(q^2u)$$

and

$$l^+_{11}(u)^{-1}\, l^+_{1\,j-i+1}(u) = q\, l^+_{1\,j-i+1}(q^2u)\, l^+_{11}(q^2u)^{-1}$$

in $\mathrm{Y}_q(\mathfrak{gl}_n)$ implied by (2.7) (or (5.5)) and apply (7.4) again. □

Corollary 7.2. *We have the expansions for the quantum determinants*

$$\operatorname{qdet} L^\pm(z) = k^\pm_1(z)\, k^\pm_2(zq^{-2}) \dots k^\pm_n(zq^{-2n+2}).$$

Proof. This is immediate from Lemma 7.1. □

We adopt the notation $\psi^i_\pm(z)$ and $E^{\pm,i}(z)$ of [1] for the series of Drinfeld generators of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$ and identify them with the respective elements of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ by using the isomorphism of [10] in the normalization of [16]. The formulas are given in [16] only for the zero level

[3]The generator matrices $T(u)$ and $\overline{T}(u)$ of [22] correspond to $L^-(u)$ and $L^+(u)$, respectively.

case, but they can be easily modified to include the central element q^c as in [10]. For $i = 1, \dots, n-1$ we have

$$\psi^i_\pm(z) = k^\mp_{i+1}(zq^{-i})\, k^\mp_i(zq^{-i})^{-1}, \tag{7.5}$$

$$E^{+,i}(z) = \frac{e^+_{i+1\,i}(zq^{\frac{c}{2}-i}) - e^-_{i+1\,i}(zq^{-\frac{c}{2}-i})}{(q-q^{-1})\,z}, \tag{7.6}$$

$$E^{-,i}(z) = \frac{f^+_{i\,i+1}(zq^{-\frac{c}{2}-i}) - f^-_{i\,i+1}(zq^{\frac{c}{2}-i})}{(q-q^{-1})\,z}.$$

In fact, these relations were established in [10] for the respective elements of the algebra $\overline{\mathrm{U}}_q(\widehat{\mathfrak{gl}}_n)$ which we defined in Sec. 5. However, the map (5.7) connecting the two algebras only results in the multiplication of the series $k^+_i(z)$ by a scalar series with coefficients in the Heisenberg algebra $\mathcal{H}_q(n)$ and in the multiplication of the series $k^-_i(z)$ by a different scalar series. The series $e^\pm_{ij}(u)$ and $f^\pm_{ij}(u)$ are not affected and hence the above relations apply to the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ in the same form.

The q-deformed Wakimoto modules over $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$ are realized in the boson Fock space by an explicit action of the series $\psi^i_\pm(z)$ and $E^{\pm,i}(z)$ described in [1]. We recall this construction assuming that the level is critical. That is, we take $k = -n$ in the notation of [1] and for all $1 \leqslant i < j \leqslant n$ consider *free boson fields*

$$b^{ij}(z) = -\sum_{r\neq 0} \frac{b^{ij}_r}{[r]}\, z^{-r} + \frac{q-q^{-1}}{2\log q} b^{ij}_0 \log z + Q^{ij}_b,$$

$$c^{ij}(z) = -\sum_{r\neq 0} \frac{c^{ij}_r}{[r]}\, z^{-r} + \frac{q-q^{-1}}{2\log q} c^{ij}_0 \log z + Q^{ij}_c,$$

and

$$b^{ij}_\pm(z) = \pm(q-q^{-1})\Big(\frac{b^{ij}_0}{2} + \sum_{\pm r>0} b^{ij}_r\, z^{-r}\Big),$$

$$c^{ij}_\pm(z) = \pm(q-q^{-1})\Big(\frac{c^{ij}_0}{2} + \sum_{\pm r>0} c^{ij}_r\, z^{-r}\Big),$$

where

$$[r] = \frac{q^r - q^{-r}}{q-q^{-1}}$$

and the coefficients satisfy the relations

$$[b^{ij}_r, b^{kl}_s] = -\frac{[r]^2}{r}\,\delta_{ik}\,\delta_{jl}\,\delta_{r,-s}, \qquad [b^{ij}_0, Q^{kl}_b] = -\frac{2\log q}{q-q^{-1}}\,\delta_{ik}\,\delta_{jl}, \tag{7.7}$$

$$[c^{ij}_r, c^{kl}_s] = \frac{[r]^2}{r}\,\delta_{ik}\,\delta_{jl}\,\delta_{r,-s}, \qquad [c^{ij}_0, Q^{kl}_c] = \frac{2\log q}{q-q^{-1}}\,\delta_{ik}\,\delta_{jl}, \tag{7.8}$$

all other pairs of coefficients commute. The quantum Heisenberg algebra $\mathcal{A}_q(n)$ is generated by the elements $e^{\pm Q_b^{ij}}$, $e^{\pm Q_c^{ij}}$, $e^{\pm(q-q^{-1})b_0^{ij}/2}$, $e^{\pm(q-q^{-1})c_0^{ij}/2}$ and b_r^{ij}, c_r^{ij} with $r \neq 0$. The defining relations are those implied by the above commutations relations for the coefficients of the free boson fields. The Fock representation $F_q(n)$ of $\mathcal{A}_q(n)$ is generated by the vacuum vector $|0\rangle$ such that

$$b_r^{ij}|0\rangle = c_r^{ij}|0\rangle = 0 \qquad \text{for all } i < j \text{ and } r \geqslant 0$$

so that, in particular,

$$e^{\pm(q-q^{-1})b_0^{ij}/2}|0\rangle = e^{\pm(q-q^{-1})c_0^{ij}/2}|0\rangle = |0\rangle.$$

The generators a_r^i with $i = 1, \dots, n-1$ and $r \in \mathbb{Z}$ used in the free field realization of [1] pairwise commute at the critical level. Therefore, they may be regarded as numerical parameters of the q-deformed Wakimoto modules and we must have $a_r^i = 0$ for $r \geqslant 0$. For the use in the formulas below we set

$$a_{\pm}^i(z) = \pm(q-q^{-1})\Big(\frac{1}{2}\,a_0^i + \sum_{\pm r>0} a_r^i\, z^{-r}\Big),$$

as in [1], and we have $a_+^i(z) = 0$ for all i. The (q-deformed) Wakimoto module over $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$ at the critical level is defined by the action of the Drinfeld generators in the space $F_q(n)$. For the action of the coefficients of the series $\psi_{\pm}^i(z)$ we have

$$\psi_{\pm}^i(zq^{\mp n/2}) = \exp\Big(\sum_{j=1}^{i}\big(b_{\pm}^{j\,i+1}(zq^{\pm(j-n-1)}) - b_{\pm}^{j\,i}(zq^{\pm(j-n)})\big) \\ + a_{\pm}^i(z) + \sum_{j=i+1}^{n}\big(b_{\pm}^{ij}(zq^{\pm(j-n)}) - b_{\pm}^{i+1\,j}(zq^{\pm(j-n-1)})\big)\Big), \tag{7.9}$$

where $b_{\pm}^{ii}(z) = c_{\pm}^{ii}(z) = 0$. The coefficients of $E^{+,i}(z)$ act by

$$E^{+,i}(z) = \frac{1}{(q-q^{-1})\,z}\sum_{j=1}^{i} :\exp\big((b+c)^{j\,i}(zq^{j-1})\big) \\ \times\Big(\exp\big(b_-^{j\,i+1}(zq^{j-1}) - (b+c)^{j\,i+1}(zq^{j-2})\big) - \exp\big(b_+^{j\,i+1}(zq^{j-1}) - (b+c)^{j\,i+1}(zq^{j})\big)\Big) \\ \times\exp\Big(\sum_{l=1}^{j-1}\big(b_+^{l\,i+1}(zq^{l-1}) - b_+^{l\,i}(zq^{l})\big)\Big):, \tag{7.10}$$

where we have used the notation $(b+c)^{ij}(z) = b^{ij}(z)+c^{ij}(z)$ and set $b^{ii}(z) = c^{ii}(z) = 0$. The colons indicate normal ordering so that the coefficients b_r^{ij} with $r < 0$ or $\exp Q_b^{ij}$ should be placed to the left of the coefficients b_r^{ij} with $r \geqslant 0$. The same rule applies to the coefficients of $c^{ij}(z)$. We will not reproduce the formulas for the action of $E^{-,i}(z)$ as they are given by longer expressions and will not be used; see [1, (3.7)]. Our notation is the same as in [1] for b^{ij}_r and c^{ij}_r, whereas $Q^{ij}_b = \hat{q}^{ij}_b$ and $Q^{ij}_c = \hat{q}^{ij}_c$.

By Lemma 4.3, the coefficients of the quantum determinants are central in the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ at the critical level. Therefore, the irreducible Wakimoto modules can be extended to $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ by specifying the eigenvalues $K^{\pm}(z)$ of $\mathrm{qdet}\, L^{\pm}(z)$. By Corollary 7.2, this gives the conditions

$$k^{\pm}_1(z)\, k^{\pm}_2(zq^{-2}) \dots k^{\pm}_n(zq^{-2n+2}) \mapsto K^{\pm}(z),$$

where $K^{+}(z)$ and $K^{-}(z)$ are power series in z and z^{-1}, respectively. Hence, relations (7.5) and (7.9) allow us to define the action of the coefficients of all series $k^{\pm}_i(z)$ on the Fock space.

For any $X \in \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ we will write $\langle 0|X|0\rangle$ to denote the coefficient of $|0\rangle$ in the expansion of $X|0\rangle$ along the basis of the Fock space.[4] More generally, a relation of the form $\langle 0|X = d\,\langle 0|$ for a constant d will be understood in the sense that $\langle 0|XY|0\rangle = d\,\langle 0|Y|0\rangle$ for any element $Y \in \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$. Using this notation we can parameterize the corresponding modules over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ by the power series $\varkappa^{+}_i(z)$ and $\varkappa^{-}_i(z)$ in z and z^{-1}, respectively, such that

$$k^{-}_i(z)|0\rangle = \varkappa^{-}_i(z)|0\rangle \qquad \text{and} \qquad \langle 0|k^{+}_i(z) = \langle 0|\varkappa^{+}_i(z) \tag{7.11}$$

for all $i = 1, \dots, n$ satisfying the relations

$$\varkappa^{-}_{i+1}(z)\varkappa^{-}_i(z)^{-1} = \exp\big(a^i_+(zq^{\frac{n}{2}+i})\big) \qquad \text{and} \qquad \varkappa^{+}_{i+1}(z)\varkappa^{+}_i(z)^{-1} = \exp\big(a^i_-(zq^{-\frac{n}{2}+i})\big)$$

for $i = 1, \dots, n-1$. Since $a^i_+(z) = 0$, the series $\varkappa^{-}_i(z)$ is the same for each i and we denote it by $\varkappa^{-}(z)$.

The following theorem is essentially due to [17] subject to the identification of $\ell_k(z)$ with the series $\ell_{V_k}(z)$ corresponding to the fundamental representation V_k of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$, although the arguments were only outlined there. The eigenvalues of $\ell_{V_k}(z)$ were interpreted in [17] as generators of the q-deformed classical $\mathcal{W}$-algebras and the Poisson brackets between the generators were explicitly calculated.

Theorem 7.3. *Given an irreducible Wakimoto module over* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ *with the parameters* $\varkappa^{+}_i(z)$ *and* $\varkappa^{-}(z)$*, the eigenvalues of the coefficients of the series* $\ell_k(z)$ *in the module are found by*

$$\ell_k(z) \mapsto \sum_{1\leqslant i_1<\dots<i_k\leqslant n} \Lambda_{i_1}(z)\Lambda_{i_2}(zq^{-2})\dots\Lambda_{i_k}(zq^{-2k+2}), \qquad k = 1, \dots, n,$$

where

$$\Lambda_i(z) = q^{n-2i+1}\, \varkappa^{+}_i(z)\varkappa^{-}(zq^{-n})^{-1}, \qquad i = 1, \dots, n.$$

[4] An equivalent interpretation would involve an inner product on the Fock space which we will not introduce.

Proof. Any irreducible Wakimoto module coincides with the cyclic span over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)_{\mathrm{cri}}$ of the vacuum vector $|0\rangle$. Hence, the eigenvalues of the coefficients of the series $\ell_k(z)$ can be found by calculating the series $\langle 0|\ell_k(z)|0\rangle$.

We find from (7.10) that $E^{+,i}(z)|0\rangle$ is a power series in z for all i. Therefore, (7.6) implies $e^-_{i+1\,i}(z)|0\rangle = 0$ for $i = 1, \dots, n-1$. Note the following relations for the action of the generators $l^-_{ij}[0]$ on the vacuum vector:

$$l^-_{i+1\,i}[0]|0\rangle = 0 \qquad \text{and} \qquad l^-_{ii}[0]|0\rangle = \varkappa^-[0]|0\rangle, \tag{7.12}$$

where $\varkappa^-[0]$ denotes the constant term of the series $\varkappa^-(z)$. Indeed, using (7.1) and (7.11), we find that $L^-(z)^{1\cdots i}_{1\cdots i}|0\rangle$ is a scalar power series in z^{-1}. Expanding the quantum minor by (4.5) and taking the constant term we find that

$$\sum_{\sigma\in\mathfrak{S}_i}(-q)^{-l(\sigma)}\, l^-_{\sigma(1)\,1}[0]\cdots l^-_{\sigma(i)\,i}[0]|0\rangle \tag{7.13}$$

is a scalar multiple of the vacuum vector $|0\rangle$. However, $l^-_{ji}[0] = 0$ for $j < i$ by (2.5) so that the only nonzero term in (7.13) corresponds to the identity permutation σ. Therefore, taking the constant terms in (7.1) and (7.11) we derive the second relation in (7.12). Now use the relation $L^-(z)^{1\cdots i-1\, i+1}_{1\cdots i}|0\rangle = 0$ implied by (7.2). Exactly as above, we get

$$l^-_{11}[0]\cdots l^-_{i-1\,i-1}[0]\, l^-_{i+1\,i}[0]|0\rangle = 0 \tag{7.14}$$

which gives the first relation in (7.12) by (5.2). As a next step, we will derive the relations

$$e^-_{j\,i}(z)|0\rangle = 0 \qquad \text{for all} \quad j > i. \tag{7.15}$$

It follows from (2.7) that

$$\big[l^-_{ij}[0], l^-_{jm}(z)\big] = (q-q^{-1})\, l^-_{im}(z)\, l^-_{jj}[0] \qquad \text{for} \quad i > j > m \tag{7.16}$$

and

$$\big[l^-_{ij}[0], l^-_{km}(z)\big] = 0 \qquad \text{for} \quad i, j > k, m.$$

Hence, (7.2) gives

$$\big[l^-_{j+1\,j}[0], e^-_{j\,i}(z)\big] = (q-q^{-1})\, e^-_{j+1\,i}(z)\, l^-_{jj}[0]$$

and (7.15) follows by induction from (7.12).

Thus, we may conclude that

$$L^-(z)^{1\cdots i-1\, j}_{1\cdots i}|0\rangle = 0 \qquad \text{for all} \quad j > i.$$

Expanding the quantum minor with the use of (4.5) and (4.8) we get

$$\sum_{\sigma}(-q)^{l(\sigma)}\, l^-_{\sigma(j)\,i}(z)\, l^-_{\sigma(i-1)\,i-1}(q^{-2}z)\cdots l^-_{\sigma(1)\,1}(q^{-2i+2}z)|0\rangle = 0, \tag{7.17}$$

where the sum is taken over permutations σ of the set $\{1,\dots,i-1,j\}$. Together with the property that

$$L^-(z)_{1\cdots i}^{1\cdots i}|0\rangle = \sum_{\sigma\in\mathfrak{S}_i}(-q)^{l(\sigma)}\, l^-_{\sigma(i)\,i}(z)\cdots l^-_{\sigma(1)\,1}(q^{-2i+2}z)|0\rangle \tag{7.18}$$

is a scalar power series in z^{-1}, these relations imply that

$$l^-_{j\,i}(z)|0\rangle = 0 \qquad \text{for all} \ \ j>i \tag{7.19}$$

and $l^-_{i\,i}(z)|0\rangle$ is a scalar power series in z^{-1} by an obvious induction. Moreover, it follows from (7.1) and (7.11) that

$$l^-_{i\,i}(z)|0\rangle = \varkappa^-(z)|0\rangle \qquad \text{for } i=1,\dots,n. \tag{7.20}$$

Now we will apply similar arguments to derive that

$$\langle 0|l^+_{j\,i}(z) = 0 \quad \text{for all} \ \ j>i \qquad \text{and} \qquad \langle 0|l^+_{i\,i}(z) = \langle 0|\varkappa^+_i(z) \quad \text{for} \ \ i=1,\dots,n. \tag{7.21}$$

The first step is to observe that $\langle 0|zE^{+,i}(z)$ is a power series in z^{-1}. Indeed, this follows from (7.10) with the use of the relations

$$\langle 0|b^{ij}_r = \langle 0|c^{ij}_r = 0 \qquad \text{for all} \ \ r\geqslant 0.$$

One additional step is to use the commutation relations (7.7) and (7.8) which imply

$$\exp Q^{ij}_b\cdot z^{\frac{q-q^{-1}}{2\log q}\, b^{ij}_0} = z^{\frac{q-q^{-1}}{2\log q}\, b^{ij}_0}\cdot\exp Q^{ij}_b\cdot z$$

and

$$\exp Q^{ij}_c\cdot z^{\frac{q-q^{-1}}{2\log q}\, c^{ij}_0} = z^{\frac{q-q^{-1}}{2\log q}\, c^{ij}_0}\cdot\exp Q^{ij}_c\cdot z^{-1}.$$

Although extra powers of z occur as a result of swapping the coefficients, these powers arising from the coefficients of the series $b^{ij}(z)$ and $c^{ij}(z)$ cancel each other. Thus, using (7.6) and noting that the constant term of $e^+_{i+1\,i}(z)$ is zero, we come to the relation $\langle 0|e^+_{i+1\,i}(z)=0$ for $i=1,\dots,n-1$. The rest of the arguments is essentially the same with some obvious adjustments. In particular, to evaluate the constant term of the power series $\langle 0|L^+(z)_{1\cdots i}^{1\cdots i}$ we use the expansion

$$\langle 0|\sum_{\sigma\in\mathfrak{S}_i}(-q)^{l(\sigma)} l^+_{\sigma(i)\,i}[0]\cdots\, l^+_{\sigma(1)\,1}[0]$$

instead of (7.13) and note that $l^+_{j\,i}[0]=0$ for $j>i$. Together with the corresponding counterpart of (7.14) this implies

$$\langle 0|l^+_{i+1\,i}[0] = 0 \qquad \text{and} \qquad \langle 0|l^+_{i\,i}[0] = \langle 0|\varkappa^+_i[0],$$

where $\varkappa_i^+[0]$ denotes the constant term of the series $\varkappa_i^+(z)$. In the final part we use the relations

$$\langle 0|\sum_\sigma (-q)^{-l(\sigma)}\, l^+_{\sigma(1)\,1}(z)\dots l^+_{\sigma(i-1)\,i-1}(q^{-2i+4}z)\, l^+_{\sigma(j)\,i}(q^{-2i+2}z)=0$$

with the sum over permutations σ of the set $\{1,\dots,i-1,j\}$, and

$$\langle 0|L^+(z)^{1\cdots i}_{1\cdots i}=\langle 0|\sum_{\sigma\in\mathfrak{S}_i}(-q)^{-l(\sigma)} l^+_{\sigma(1)\,1}(z)\,\dots l^+_{\sigma(i)\,i}(q^{-2i+2}z)$$

instead of (7.17) and (7.18).

Relations (7.19), (7.20) and (7.21) allow us to conclude that the eigenvalue $\langle 0|\ell_k(z)|0\rangle$ coincides with the image of the series $\ell_k(z)$ under the Harish-Chandra homomorphism calculated in Theorem 6.2 for the specialization

$$l^+_i(z)=\varkappa^+_i(z) \qquad \text{and} \qquad l^-_i(z)=\varkappa^-(z)$$

for $i=1,\dots,n$. Clearly, then $\lambda_i(z)$ specializes to $\Lambda_i(z)$ and the proof is complete. □

Remark 7.4. The fact that the eigenvalues of the coefficients of $\ell_k(z)$ in the Wakimoto modules are consistent with the Harish-Chandra images provided by Theorem 6.2 relies on the properties (7.19), (7.20) and (7.21). It was essential for their derivation that the "zero mode matrices" $L^+[0]$ and $L^-[0]$ are upper and lower triangular, respectively. These properties do not hold for the presentation of the quantum affine algebra used in [17], where the triangularity of the zero mode matrices is opposite. □

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