BV FUNCTIONS, CACCIOPPOLI SETS AND DIVERGENCE THEOREM OVER WIENER SPACES*

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Abstract. Using finite dimensional approximation, we give a version of the definition of BV functions on abstract Wiener space introduced in Fukushima-Hino [17]. Then, we study Caccioppoli sets in the classical Wiener space and pinned Wiener space, and provide concrete examples of Caccioppoli sets, such as the balls and the level sets of solutions to SDEs. Moreover, without assuming the ray Hamza conditions in [16], we prove the infinite dimensional divergence theorem in any Caccioppoli set for any bounded continuous and $\mathbb{H}$-Lipschitz continuous vector field in the classical Wiener space. In particular, the isoperimetric inequality holds true for Caccioppoli sets.

1. Introduction

Let U be an open and bounded subset of the Euclidean space $\mathbb{R}^d$, with C^1-boundary. The classical divergence theorem states that for any $\varphi \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$

$$\int_U \mathrm{div}\varphi(x)\mathrm{d}x = \int_{\partial U} \varphi(x)\cdot \mathbf{n}_U(x)\mathcal{H}^{d-1}(\mathrm{d}x), \tag{1}$$

where $\mathbf{n}_U(x)$ is the unit exterior normal to U at x, and $\mathcal{H}^{d-1}$ is the Hausdorff measure on ∂U. This fundamental formula holds also for any Borel set U with finite perimeter; in this case ∂U is replaced by the measure theoretic boundary(cf. [13, p.209, Theorem 1] and [14]).

In [2], Airault-Malliavin established an infinite-dimensional version of the formula (1) in the classical Wiener space. Therein, they used the finite dimensional approximation, and considered the level sets of functionals that are non-degenerate and smooth in the sense of Malliavin calculus. We remark that infinite dimensional versions of the divergence theorem (1) have been also investigated in other contexts, see for example [26, 18, 15].

Fukushima [16] and Fukushima-Hino [17] studied BV functions over the abstract Wiener space $(\mathbb{X},\mathbb{H},\mu)$. Let $L(\log L)^{1/2}$ be the Orlicz space over $\mathbb{X}$. According to [17], a function $f\in L(\log L)^{1/2}$ is of bounded variation if

$$V(f) := \sup \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x)\cdot f(x)\mu(\mathrm{d}x) < +\infty, \tag{2}$$

where the supremum is taken over all $\mathbb{X}^*$-valued smooth cylindrical vector fields $\mathbf{f}$ with $\|\mathbf{f}(x)\|_{\mathbb{H}} \leqslant 1$ for each $x\in\mathbb{X}$. Here, $\mathbb{X}^*$ is the dual space of $\mathbb{X}$. The set of all functions

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$f : \mathbb{X} \to \mathbb{R}$ of bounded variation is denoted by $BV(\mathbb{X})$. In particular, they also showed that

$$\mathbb{D}^{1,1} \subsetneq BV(\mathbb{X}),$$

where $\mathbb{D}^{1,1}$ is the first order Mallavin-Sobolev space on $\mathbb{X}$.

Basing on the theory of their established BV functions, Fukushima and Hino extended the infinite dimensional like-formula (1) to the Caccioppoli set Γ (i.e. 1_Γ is a BV function). That is, for a Caccioppoli set $\Gamma \subset \mathbb{X}$, if 1_Γ satisfies the *ray Hamza condition*, Fukushima [16, Theorem 4.2] proved that for any smooth cylindrical vector field $\mathbf{f}$

$$\int_\Gamma \mathrm{div}\mathbf{f}(x)\mu(\mathrm{d}x) = \int_{\partial\Gamma} \langle \mathbf{f}(x), \mathbf{n}_\Gamma(x)\rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}x), \tag{3}$$

where $\|\partial\Gamma\|$ denotes the corresponding surface measure of bounded variation function 1_Γ. The proof of this formula in [16, 17] depends on the theory of Dirichlet form.

We may put forward the following three questions that are of some importance for the analysis of stochastic partial differential equations and the related deterministic Dirichlet considered in a bounded set of a Banach space, see for example [9].

(I) In the definition of $BV(\mathbb{X})$, can one start from a broader space rather than the Orlicz space $L(\log L)^{1/2}$?
(II) Can we give some concrete Caccioppoli sets such as the balls in $\mathbb{X}$?
(III) Is the ray Hamza condition satisfied by 1_Γ necessary for (3), and can the range of vector fields be enlarged to be such that the formula (3) holds?

For the first question, we note recent work [3], where there no additional integrability condition (except that of L^1) is imposed but the space $W^{1,1}$ is defined a somewhat non-standard way. In this paper we shall show that it is enough to require $f \in L^1_w(\mathbb{X}, \mu)$ in the definition of (2), where $L^1_w(\mathbb{X}, \mu)$ is the integrable functions space with weight $\|x\|_{\mathbb{X}} + 1$. By Fernique theorem, it is clear that $L(\log L)^{1/2} \subset L^1_w(\mathbb{X}, \mu)$. Using finite dimensional approximation, we finally prove that for $f \in L^1_w(\mathbb{X}, \mu)$, if $V(f)$ is finite, then $f \in L(\log L)^{1/2}$. This will be given in Section 2. Thus, in a priori position, we can work in a bigger space.

The second question was recently an object of intense study, see for example [4, 6, 10, 25]. We shall give an affirmative answer in the case of classical Wiener space. It should be noted that in the finite dimensional case, the indicator function of a Lipschitz domain is a BV function(cf. [13]). Naturally, we would ask if the indicator functions of a "smooth domain" in Wiener space such as the ball, is also a BV function so that the infinite dimensional divergence theorem holds. In [16] and [17], the authors used the coarea formula to assert the existence of Caccioppoli sets in $\mathbb{X}$. Moreover, in the pinned Wiener space, Zambotti [27] and Hariya [19] gave some concrete Caccioppoli sets. In their papers, the surface measures are explicitly given by some analytic method. In Section 3, we shall first give a criterion for Caccioppoli sets. Then, we prove that the balls in the classical Wiener space and the level sets of solutions to SDEs with constant diffusion coefficients are Caccioppoli sets. It is worthy to say that in [1], the authors proved that the indicator function of the ball in the classical Wiener space (resp. does not) belongs to $\mathbb{D}^{p,\alpha}$ provided $p\alpha < 1$ (resp. $p\alpha > 1$). However, it is not known in the critical case of $p\alpha = 1$.

Little is known about the third question, except some special cases, see for example [24, 27, 19]. In Section 4 we will approximate a general vector field by smooth cylindrical vector fields and will prove that without the ray Hamza condition assumption on 1_Γ, the formula (3) still holds true for any bounded continuous and $\mathbb{H}$-Lipschitz continuous vector fields on the classical Wiener space. The proof depends on the structure of the classical Wiener space. It is not known whether this is also true for an abstract Wiener spaces. As a simple conclusion, using the result of Bobkov [8](see also [5]), we obtain the isoperimetric inequality for Caccioppoli sets.

In Section 5, we also study the integration by parts formula in the pinned Wiener space, and give a more general integration by parts formula than Hariya [19]. The price to pay is that we can not explicitly give the surface measure.

2. BV Functions

Let $(\mathbb{X},\mathbb{H},\mu)$ be an abstract Wiener space. Namely, $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$ is a separable Banach space, $(\mathbb{H},\|\cdot\|_{\mathbb{H}})$ is a separable Hilbert space densely and continuously embedded in $\mathbb{X}$, and μ is the Gaussian measure over $\mathbb{X}$. If we identify the dual space $\mathbb{H}^*$ with itself, then $\mathbb{X}^*$ may be viewed as a dense linear subspace of $\mathbb{H}$ so that $\ell(x)=\langle \ell, x\rangle_{\mathbb{H}}$ whenever $\ell\in\mathbb{E}^*$ and $x\in\mathbb{H}$, where $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ denotes the inner product in $\mathbb{H}$.

For $p\geqslant 1$, let $L^p(\mathbb{X},\mu)$ be the usual L^p-space over $(\mathbb{X},\mu)$, the norm is denoted by $\|\cdot\|_p$. Let $L^1_w(\mathbb{X},\mu)$ be the following weighted L^1-space:

$$L^1_w(\mathbb{X},\mu):=\left\{f\in L^1(\mathbb{X},\mu):\|f\|_{1,w}<+\infty\right\},$$

where

$$\|f\|_{1,w}:=\int_{\mathbb{X}}|f(x)|\cdot(\|x\|_{\mathbb{X}}+1)\mu(\mathrm{d}x).$$

Define

$$\Phi_1(s):=\int_0^s\sqrt{\log(1+r)}\mathrm{d}r,\quad s\geqslant 0,$$

and

$$\Phi_2(s):=\int_0^s(e^{r^2}-1)\mathrm{d}r,\quad s\geqslant 0.$$

Then Φ_1 and Φ_2 are a pair of complementary Young functions. The corresponding Orlicz spaces $L^{\Phi_i}(\mathbb{X},\mu), i=1,2$ are defined by the following norms

$$\|f\|_{\Phi_i}:=\inf\left\{\alpha>0:\int_{\mathbb{X}}\Phi_i\Big(\frac{|f(x)|}{\alpha}\Big)\mu(\mathrm{d}x)\leqslant 1\right\},\quad i=1,2.$$

We also write $L^{\Phi_1}(\mathbb{X},\mu)$ as $L(\log L)^{1/2}$.

In what follows, we shall fix an orthogonal basis $\mathscr{E}:=\{\ell_k,k\in\mathbb{N}\}\subset\mathbb{X}^*$ of $\mathbb{H}$. Let $\mathscr{F}C_b^\infty$ be the set of all smooth cylindrical functions with the following form:

$$f(x):=F(\ell_{i_1}(x),\cdots,\ell_{i_m}(x)),\ \ell_{i_j}\in\mathscr{E},\ F\in C_b^\infty(\mathbb{R}^m),$$

and $\mathscr{F}C_b^\infty(\mathbb{X}^*)$ be the set of all cylindrical $\mathbb{X}^*$-valued vector fields with the following form:

$$\mathbf{f}(x)=\sum_{j=1}^l f_j(x)\ell_{l_j},\quad \ell_{l_j}\in\mathscr{E},\ f_j\in\mathscr{F}C_b^\infty.$$

It is well known that for any $p\geqslant 1$, $\mathscr{F}C_b^\infty$ is dense in $L^p(\mathbb{X},\mu)$ and $\mathscr{F}C_b^\infty(\mathbb{X}^*)$ is dense in $L^p(\mathbb{X},\mu;\mathbb{H})$.

For $f\in\mathscr{F}C_b^\infty$, the Malliavin derivative of f is defined by(cf. [22])

$$Df(x):=\sum_{j=1}^m(\partial_jF)(\ell_{i_1}(x),\cdots,\ell_{i_m}(x))\ell_{i_j},\tag{4}$$

and for $\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*)$, the divergence of $\mathbf{f}$ is defined by

$$\mathrm{div}\mathbf{f}(x) := \sum_{j=1}^{l} \Big[f_j(x) \cdot \ell_{l_j}(x) - D_{\ell_{l_j}} f_j(x) \Big]. \tag{5}$$

where $D_{\ell_{l_j}} f_j(x) = \langle \ell_{l_j}, Df_j(x) \rangle_{\mathbb{H}}$. It is well known that

$$\int_{\mathbb{X}} \langle Df(x), \mathbf{f}(x) \rangle_{\mathbb{H}} \mu(\mathrm{d}x) = \int_{\mathbb{X}} f(x) \cdot \mathrm{div}\mathbf{f}(x) \mu(\mathrm{d}x),$$

which means that div is the dual operator of D.

The following lemma is direct from the definitions.

Lemma 2.1. *For any $\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*)$, we have $\mathrm{div}\mathbf{f} \in \mathscr{F}C_b^\infty$ and for some $C > 0$*

$$|\mathrm{div}\mathbf{f}(x)| \leqslant C(\|x\|_{\mathbb{X}} + 1), \quad \forall x \in \mathbb{X}. \tag{6}$$

For $p \geqslant 1$, the Malliavin Sobolev space $\mathbb{D}^{p,1}$ is defined as the completion of $\mathscr{F}C_b^\infty$ with respect to the norm:

$$\|f\|_{p,1} := \|f\|_p + \|Df\|_p.$$

Then

Proposition 2.2. $\mathbb{D}^{1,1} \subset L(\log L)^{1/2} \subset L^1_w(\mathbb{X}, \mu)$, *i.e., for some $C_1, C_2 > 0$*

$$\|f\|_{1,w} \leqslant C_1 \|f\|_{\Phi_1} \leqslant C_2 \|f\|_{1,1}.$$

Proof. The second inequality was proved in Proposition 3.2 [17]. Put $g(x) := \|x\|_{\mathbb{X}} + 1$. Then, by Fernique's theorem(cf. [22]), one has

$$\|g\|_{\Phi_2} < +\infty.$$

Thus, by the generalized Hölder inequality

$$\|f\|_{1,w} = \int_{\mathbb{X}} |f(x)| \cdot (\|x\|_{\mathbb{X}} + 1) \mu(\mathrm{d}x) \leqslant \|f\|_{\Phi_1} \cdot \|g\|_{\Phi_2} \leqslant C \|f\|_{\Phi_1}.$$

The result follows. □

For $f \in L^1_w(\mathbb{X}, \mu)$, define

$$V(f) := \sup_{\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*); \|\mathbf{f}(x)\|_{\mathbb{H}} \leqslant 1} \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x) \cdot f(x) \mu(\mathrm{d}x). \tag{7}$$

By (6), one knows that the above integral is well defined. Thus, we can introduce the following space of bounded variation functions.

Definition 2.3. *A function $f \in L^1_w(\mathbb{X}, \mu)$ is called bounded variation if $V(f) < +\infty$. All such functions are denoted by $BV(\mathbb{X})$.*

We have

Theorem 2.4. *The map $f \mapsto V(f)$ is lower semi-continuous in $L^1_w(\mathbb{X}, \mu)$, and $BV(\mathbb{X})$ is a Banach space under the norm:*

$$\|f\|_{BV} := \|f\|_{1,w} + V(f).$$

Proof. Let $f_n \to f$ in $L^1_w(\mathbb{X},\mu)$. For any $\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*)$ with $\|\mathbf{f}(x)\|_{\mathbb{H}} \leqslant 1$, and all $x \in \mathbb{X}$, we have

$$\int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x) \cdot f(x)\mu(\mathrm{d}x) = \lim_{n\to\infty} \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x) \cdot f_n(x)\mu(\mathrm{d}x) \leqslant \varliminf_{n\to\infty} V(f_n).$$

So,

$$V(f) \leqslant \varliminf_{n\to\infty} V(f_n). \tag{8}$$

It is clear that $BV(\mathbb{X})$ is a linear normed space under $\|\cdot\|_{BV}$. We now prove the completeness of $BV(\mathbb{X})$ with respect to $\|\cdot\|_{BV}$. Let f_n be a Cauchy sequence under $\|\cdot\|_{BV}$. Since $L^1_w(\mathbb{X},\mu)$ is complete, there is an $f \in L^1_w(\mathbb{X},\mu)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{1,w} = 0.$$

By (8), we know $f \in BV(\mathbb{X})$ and

$$\varlimsup_{n\to\infty} V(f - f_n) \leqslant \varlimsup_{n\to\infty} \varliminf_{k\to\infty} V(f_n - f_k) = 0.$$

The proof is complete. $\square$

Let $\{T_t\}_{t\geqslant 0}$ be the OU semigroup defined by the Mehler formula

$$T_t f(x) := \int_{\mathbb{X}} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\mu(\mathrm{d}y).$$

Then

Lemma 2.5. *T_t is a bounded operator from $L^1_w(\mathbb{X},\mu)$ to $L^1_w(\mathbb{X},\mu)$. More precisely, for some universal constant $C > 0$ and any $t > 0$, $f \in L^1_w(\mathbb{X},\mu)$*

$$\|T_t f\|_{1,w} \leqslant (e^{-t} + C\sqrt{1-e^{-2t}}) \cdot \|f\|_{1,w}. \tag{9}$$

Moreover,

$$\lim_{t\downarrow 0} \|T_t f - f\|_{1,w} = 0. \tag{10}$$

Proof. Notice that $\mu \otimes \mu$ is invariant under the rotation:

$$\begin{pmatrix} e^{-t}, & \sqrt{1-e^{-2t}} \\ -\sqrt{1-e^{-2t}}, & e^{-t} \end{pmatrix}.$$

Thus, we have for $f \in L^1_w(\mathbb{X},\mu)$

$$\begin{aligned} \|T_t f\|_{1,w} &= \int_{\mathbb{X}} |T_t f(x)| \cdot (\|x\|_{\mathbb{X}} + 1)\mu(\mathrm{d}x) \leqslant \\ &\leqslant \int_{\mathbb{X}}\int_{\mathbb{X}} |f(e^{-t}x + \sqrt{1-e^{-2t}}y)| \cdot (\|x\|_{\mathbb{X}} + 1)\mu(\mathrm{d}y)\mu(\mathrm{d}x) \\ &= \int_{\mathbb{X}}\int_{\mathbb{X}} |f(x)| \cdot (\|e^{-t}x - \sqrt{1-e^{-2t}}y\|_{\mathbb{X}} + 1)\mu(\mathrm{d}y)\mu(\mathrm{d}x) \\ &\leqslant e^{-t}\|f\|_{1,w} + \sqrt{1-e^{-2t}} \int_{\mathbb{X}} \|y\|_{\mathbb{X}}\mu(\mathrm{d}y) \cdot \|f\|_1, \end{aligned}$$

which gives (9).

The limit (10) is clearly true for $f \in \mathscr{F}C_b^\infty$. For general $f \in L^1_w(\mathbb{X},\mu)$, it follows by using (9) and a simple approximation. $\square$

It was proved in [17, Proposition 3.5] that T_t maps $L(\log L)^{1/2}$ into $\mathbb{D}^{1,1}$. It is not known whether T_t maps $L^1_w(\mathbb{X},\mu)$ into $\mathbb{D}^{1,1}$. Thus, we can not simply use $T_t f$ as a mollifier to approximate the function $f \in BV(\mathbb{X})$. So, we need to introduce a finite dimensional approximation operator(cf. [22]).

For $n \in \mathbb{N}$, let $\mathbb{H}_n := \overline{\text{span}\{\ell_1, \ell_2, \cdots, \ell_n\}}$, and Π_n the orthogonal projection from $\mathbb{H}$ to $\mathbb{H}_n$, $\mathbb{H}_n^\perp$ the orthogonal complement of $\mathbb{H}_n$ in $\mathbb{H}$. It is clear that Π_n can be extended to $\mathbb{X}$ by defining

$$\Pi_n(x) := \ell_1(x)\cdot\ell_1 + \cdots + \ell_n(x)\cdot\ell_n.$$

Then $\mu_n := \mu\circ\Pi_n^{-1}$ is the finite dimensional Gaussian measure on $(\mathbb{H}_n, \mathscr{B}(\mathbb{H}_n))$. We have the following decomposition

$$(\mathbb{X},\mathbb{H},\mu) := (\mathbb{H}_n,\mathbb{H}_n,\mu_n) \oplus (\mathbb{Y}_n, \mathbb{H}_n^\perp, \mu_n^\perp),$$

where $(\mathbb{H}_n,\mathbb{H}_n,\mu_n)$ is the finite dimensional Gaussian space, $(\mathbb{Y}_n, \mathbb{H}_n^\perp, \mu_n^\perp)$ is still an abstract Wiener space and

$$\mathbb{X} = \mathbb{H}_n \oplus \mathbb{Y}_n.$$

In particular, we have the following disintegrated formula(cf. [21, 22]):

$$\int_{\mathbb{X}} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{H}_n}\int_{\mathbb{Y}_n} f(\ell+y)\mu_n^\perp(\mathrm{d}y)\mu_n(\mathrm{d}\ell),$$

and we shall write

$$\mathbb{H}_n \ni \ell \mapsto \int_{\mathbb{Y}_n} f(\ell+y)\mu_n^\perp(\mathrm{d}y) = P_n f(\ell).$$

We remark that $(P_n f)\circ\Pi_n$ actually equals to $\mathbb{E}(f|\mathscr{B}_n(\mathbb{X}))$, where $\mathscr{B}_n(\mathbb{X}) := \sigma\{\Pi_n^{-1}(B) : B \in \mathscr{B}(\mathbb{H}_n)\}$. In the following, if there are no dangers of confusions, we shall not distinguish $P_n f$ with $(P_n f)\circ\Pi_n$.

We have

Lemma 2.6. *For any $f \in L^1(\mathbb{X},\mu)$, it holds that*

$$\|P_n f\|_1 \leqslant \|f\|_1 \tag{11}$$

and

$$\lim_{n\to\infty}\|P_n f - f\|_1 = 0. \tag{12}$$

Proof. Since $\mathscr{B}_n(\mathbb{X}) \uparrow \mathscr{B}(\mathbb{X})$, $\{\mathbb{E}(f|\mathscr{B}_n(\mathbb{X})), n \in \mathbb{N}\}$ is a martingale. The result follows from the martingale convergence theorem. $\square$

Define the following approximation operator

$$A_n f := P_n T_{1/n} f.$$

The following lemma is easy to verify.

Lemma 2.7. *For any $f \in \mathscr{F}C_b^\infty$ and $\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*)$, we have*

$$P_n f, T_{1/n} f \in \mathscr{F}C_b^\infty, \quad P_n\Pi_n\mathbf{f} = \Pi_n P_n\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*),$$

and

$$P_n\Pi_n(Df) = D(P_n f), \quad P_n(\mathrm{div}\mathbf{f}) = \mathrm{div}(P_n\Pi_n\mathbf{f}).$$

We now prove the following mollifying property of A_n.

Proposition 2.8. *The operator $DA_n : \mathscr{F}C_b^\infty \to L^1(\mathbb{X},\mu;\mathbb{H})$ can be extended to a bounded linear operator from $L^1_w(\mathbb{X},\mu)$ to $L^1(\mathbb{X},\mu;\mathbb{H})$. In particular, A_n is a bounded linear operator from $L^1_w(\mathbb{X},\mu)$ to $\mathbb{D}^{1,1}$.*

Proof. Let $f \in \mathscr{F}C_b^\infty$. Noting that $A_n f$ is in fact a function on $\mathbb{H}_n$, we have by Lemma 2.7

$$\|DA_n f(x)\|_{\mathbb{H}} = \left(\sum_{i=1}^n |D_{\ell_i} A_n f(x)|^2\right)^{1/2} \leqslant \sum_{i=1}^n |P_n D_{\ell_i} T_{1/n} f(x)|.$$

On the other hand, a direct calculation in finite dimensional Gaussian space leads to(cf. [22, 23])

$$D_{\ell_i} T_{1/n} f(x) = \frac{e^{-1/n}}{\sqrt{1-e^{-2/n}}} \int_{\mathbb{X}} f(e^{-1/n}x + \sqrt{1-e^{-2/n}}y)\ell_i(y)\mu(\mathrm{d}y).$$

Hence,

$$\begin{aligned}\int_{\mathbb{X}} \|DA_n f(x)\|_{\mathbb{H}}\mu(\mathrm{d}x) \;\leqslant\; & \int_{\mathbb{X}}\int_{\mathbb{X}} |f(e^{-1/n}x + \sqrt{1-e^{-2/n}}y)| \cdot \|y\|_{\mathbb{X}}\mu(\mathrm{d}y)\mu(\mathrm{d}x) \\ & \times \frac{e^{-1/n}}{\sqrt{1-e^{-2/n}}} \sum_{i=1}^n \|\ell_i\|_{\mathbb{X}^*},\end{aligned}$$

and by (9) we get

$$\int_{\mathbb{X}} \|DA_n f(x)\|_{\mathbb{H}}\mu(\mathrm{d}x) \leqslant C_n \|f\|_{1,w}.$$

The proof is complete. □

Basing on this proposition, we can prove the following characterization for $BV(\mathbb{X})$.

Theorem 2.9. *If $f \in BV(\mathbb{X})$, then $f \in L(\log L)^{1/2}$ and there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{D}^{1,1}$ such that $f_n \to f$ in $L^1(\mathbb{X},\mu)$ and*

$$\lim_{n\to\infty} \|Df_n\|_1 = V(f). \tag{13}$$

Conversely, for $f \in L^1(\mathbb{X},\mu)$, if there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{D}^{1,1}$ such that $f_n \to f$ in $L^1(\mathbb{X},\mu)$ and $\sup_{n\in\mathbb{N}} \|Df_n\|_1 < +\infty$. Then $f \in BV(\mathbb{X})$ and

$$V(f) \leqslant \varliminf_{n\to\infty} \|Df_n\|_1. \tag{14}$$

In particular,

$f \in BV(\mathbb{X})$ if and only if $f \in L(\log L)^{1/2}$ and there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{D}^{1,1}$ such that $f_n \to f$ in $L^1(\mathbb{X},\mu)$ and $\sup_{n\in\mathbb{N}} \|Df_n\|_1 < +\infty$.

Proof. Put

$$f_n := A_n f.$$

It is clear by (10) (12) and Proposition 2.8 that

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0 \quad \text{and} \quad f_n \in \mathbb{D}^{1,1}. \tag{15}$$

Noticing that $A_n f$ is indeed a function defined on the finite dimensional Gaussian probability space $(\mathbb{H}_n, \mu_n)$, we know

$$V(f_n) = \|Df_n\|_1. \tag{16}$$

For any $\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*)$, we have by Lemma 2.7

$$\begin{aligned}\int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x) \cdot f_n(x)\mu(\mathrm{d}x) &= \int_{\mathbb{X}} P_n \mathrm{div}\mathbf{f}(x) \cdot T_{1/n} f(x)\mu(\mathrm{d}x) \\ &= \int_{\mathbb{X}} \mathrm{div} P_n \Pi_n \mathbf{f}(x) \cdot T_{1/n} f(x)\mu(\mathrm{d}x)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{X}} T_{1/n}\mathrm{div} P_n \Pi_n \mathbf{f}(x)\cdot f(x)\mu(\mathrm{d}x) \\
&= e^{-1/n}\int_{\mathbb{X}} \mathrm{div} T_{1/n} P_n \Pi_n \mathbf{f}(x)\cdot f(x)\mu(\mathrm{d}x),
\end{aligned}
$$

which implies that

$$V(f_n) \leqslant e^{-1/n} V(f). \tag{17}$$

So,

$$\sup_{n\in\mathbb{N}} \|Df_n\|_1 = \sup_{n\in\mathbb{N}} V(f_n) \leqslant V(f). \tag{18}$$

In particular, this also produces that $f \in L(\log L)^{1/2}$ by Proposition 2.2 and (15).

On the other hand, we have for n sufficiently large

$$
\begin{aligned}
\int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x)\cdot f_n(x)\mu(\mathrm{d}x) &= \int_{\mathbb{X}} P_n \mathrm{div}\mathbf{f}(x)\cdot T_{1/n} f(x)\mu(\mathrm{d}x) \\
&= \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x)\cdot T_{1/n} f(x)\mu(\mathrm{d}x).
\end{aligned}
$$

So, by (10) we have

$$\lim_{n\to\infty} \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x)\cdot f_n(x)\mu(\mathrm{d}x) = \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x)\cdot f(x)\mu(\mathrm{d}x),$$

which then gives

$$V(f) \leqslant \varliminf_{n\to\infty} V(f_n). \tag{19}$$

The limit (13) thus follows by combining (16) (17) and (19).

We now look at the converse part. For $m, n \in \mathbb{N}$, set

$$
\begin{aligned}
f_n^m(x) &:= (-m) \vee (f_n(x) \wedge m), \\
f^m(x) &:= (-m) \vee (f(x) \wedge m).
\end{aligned}
$$

Then $f_n^m \in \mathbb{D}^{1,1}$, and $\|Df_n^m\|_1 \leqslant \|Df_n\|_1$,

$$\lim_{n\to\infty} \|f_n^m - f^m\|_1 = 0, \quad \lim_{m\to\infty} \|f^m - f\|_1 = 0.$$

Thus, for any $\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{X}^*)$ with $\|\mathbf{f}(x)\|_{\mathbb{H}} \leqslant 1$ for all $x \in \mathbb{X}$, we have by the dominated convergence theorem

$$
\begin{aligned}
\int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x)\cdot f^m(x)\mu(\mathrm{d}x) &= \lim_{n\to\infty} \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x)\cdot f_n^m(x)\mu(\mathrm{d}x) \\
&= \lim_{n\to\infty} \int_{\mathbb{X}} \langle \mathbf{f}(x), Df_n^m(x)\rangle_{\mathbb{H}}\mu(\mathrm{d}x) \\
&\leqslant \varliminf_{n\to\infty} \int_{\mathbb{X}} \|Df_n^m(x)\|_{\mathbb{H}}\mu(\mathrm{d}x) \\
&\leqslant \varliminf_{n\to\infty} \|Df_n\|_1.
\end{aligned}
$$

This implies by Theorem 2.4 that

$$V(f) \leqslant \varliminf_{m\to\infty} V(f^m) \leqslant \varliminf_{n\to\infty} \|Df_n\|_1 < +\infty.$$

The proof is complete. □

Remark 2.10. *From this theorem, one sees that our definition is equivalent to the one in* [17]*. In particular, for $f \in BV(\mathbb{X})$ we have*

$$V(f) = \sup \int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x) \cdot f(x)\mu(\mathrm{d}x), \tag{20}$$

where the supremum runs over all the following vector fields $\mathbf{f}$ with $\|\mathbf{f}(x)\|_{\mathbb{H}} \leqslant 1$:

$$\mathbf{f}(x) = \sum_{j=1}^{n} F_j(h_{n_1}(x), \cdots, h_{n_{m_j}}(x))\ell_j, \quad h_{n_j}, \ell_j \in \mathbb{X}^*, \ F_j \in C_b^\infty(\mathbb{R}^{m_j}). \tag{21}$$

All such vector fields will be denoted by $\tilde{\mathscr{F}}C_b^\infty(\mathbb{X}^*)$. Indeed, (20) follows from the above characterization.

Using this characterization, we have the following useful proposition.

Proposition 2.11. *Let $f \in BV(\mathbb{X}) \cap L^\infty(\mathbb{X}, \mu)$ and $g \in \mathbb{D}^{1,1} \cap L^\infty(\mathbb{X}, \mu)$. Then $fg \in BV(\mathbb{X}) \cap L^\infty(\mathbb{X}, \mu)$, and*

$$V(fg) \leqslant V(f) \cdot \|g\|_\infty + \|f\|_\infty \cdot \|Dg\|_1.$$

Proof. Let $f_n := A_n f$. Then

$$\begin{aligned} \|D(f_n g)\|_1 &\leqslant \|Df_n\|_1 \cdot \|g\|_\infty + \|f_n\|_\infty \cdot \|Dg\|_1 \\ &\leqslant V(f) \cdot \|g\|_\infty + \|f\|_\infty \cdot \|Dg\|_1. \end{aligned}$$

and

$$\lim_{n\to\infty} \|f_n g - fg\|_1 \leqslant \|g\|_\infty \lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

The result follows by Theorem 2.9. □

We also have the following isoperimetric inequality for BV functions(cf. [5, 8]).

Theorem 2.12. *Let $f \in BV(\mathbb{X})$ with $0 \leqslant f \leqslant 1$. Then*

$$\mathscr{U}\Big(\int_{\mathbb{X}} f\mathrm{d}\mu\Big) - \int_{\mathbb{X}} \mathscr{U}(f)\mathrm{d}\mu \leqslant V(f),$$

where $\mathscr{U}(s) := \Phi' \circ \Phi^{-1}(s)$ and $\Phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-r^2/2}\mathrm{d}s$.

Proof. Set $f_n := A_n 1_\Gamma$. Then $f_n \in \mathbb{D}^{1,1}$ and $0 \leqslant f_n \leqslant 1$. Noting that f_n is in fact a function defined on $\mathbb{H}_n$, we thus have by using [8, Corollary 2] and mollifying technique

$$\mathscr{U}\Big(\int_{\mathbb{X}} f_n\mathrm{d}\mu\Big) - \int_{\mathbb{X}} \mathscr{U}(f_n)\mathrm{d}\mu \leqslant \int_{\mathbb{X}} \|Df_n(x)\|_{\mathbb{H}}\mu(\mathrm{d}x).$$

Taking the limits $n \to \infty$ yields the desired inequality by Theorem 2.9. □

The following result was proved in [17, Theorem 3.9].

Theorem 2.13. *For each $f \in BV(\mathbb{X})$, there exists a positive finite measure ν(also written as $\|Df\|$) on $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$ and an $\mathbb{H}$-valued Borel function $\mathbf{n}_f$ with $\|\mathbf{n}_f\|_{\mathbb{H}} = 1$ $\nu-$ a.e. such that for all $\mathbf{f} \in \tilde{\mathscr{F}}C_b^\infty(\mathbb{X}^*)$(see Remark 2.10)*

$$\int_{\mathbb{X}} \mathrm{div}\mathbf{f}(x) \cdot f(x)\mu(\mathrm{d}x) = \int_{\mathbb{X}} \langle \mathbf{f}(x), \mathbf{n}_f(x)\rangle_{\mathbb{H}}\nu(\mathrm{d}x), \tag{22}$$

and

$$V(f) = \|Df\|(\mathbb{X}).$$

Moreover, ν and $\mathbf{n}_f$ are uniquely determined; namely, if ν' and $\mathbf{n}'_f$ are another pair of satisfying (22), then $\nu = \nu'$ and $\mathbf{n}_f = \mathbf{n}'_f$ $\nu-$ a.e..

The following co-area formula can be proved by using the same method as in [13, p.185 Theorem 1](see [16, 17]).

Theorem 2.14. *Let $f \in BV(\mathbb{X})$. Then*

$$V(f) = \int_{-\infty}^{\infty} V(1_{\{f>t\}}) \mathrm{d}t.$$

In particular, for a.e. t, $1_{\{f>t\}} \in BV(\mathbb{X})$.

3. Caccioppoli Sets

Definition 3.1. *A Borel set $\Gamma \subset \mathbb{X}$ is called a Caccioppoli set if $1_\Gamma \in BV(\mathbb{X})$. The total of all the Caccioppoli sets is denoted by $\mathscr{C}(\mathbb{X})$.*

The following proposition is obvious.

Proposition 3.2. *Let $\Gamma_1, \Gamma_2 \in \mathscr{C}(\mathbb{X})$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then, $\Gamma_1^c, \Gamma_1 \cup \Gamma_2 \in \mathscr{C}(\mathbb{X})$.*

For a Borel set $\Gamma \subset \mathbb{X}$, the lower Minkowski content of the boundary of Γ is defined by

$$\mu_s(\partial\Gamma) := \varliminf_{\epsilon \to 0} \frac{\mu(\Gamma_\epsilon) - \mu(\Gamma)}{\epsilon}, \tag{23}$$

where $\Gamma_\epsilon := \{x \in \mathbb{X} : \mathrm{dis}(x, \Gamma) < \epsilon\}$.

We have the following simple proposition for a Borel set to be a Caccioppoli set.

Proposition 3.3. *Let $\Gamma \in \mathscr{B}(\mathbb{X})$. If $\mu_s(\partial\Gamma) < +\infty$, then $\Gamma \in \mathscr{C}(\mathbb{X})$.*

Proof. Noting that $x \mapsto \mathrm{dis}(x, \Gamma)$ is a Lipschitz continuous function on $\mathbb{X}$ with Lipschitz constant 1, we have by [12]

$$\mathrm{ess}\sup_{x \in \mathbb{X}} \|D[\mathrm{dis}(\cdot, \Gamma)](x)\|_{\mathbb{H}} \leqslant c_{\mathbb{H} \hookrightarrow \mathbb{X}},$$

where $c_{\mathbb{H} \hookrightarrow \mathbb{X}}$ is the embedding constant of $\mathbb{H} \hookrightarrow \mathbb{X}$.

Let $\chi_\epsilon(s)$ be defined by

$$\chi_\epsilon(s) := \begin{cases} 1, & s \in [0, \epsilon/2], \\ 1 - [(2s - \epsilon) \wedge \epsilon]/\epsilon, & s \in [\epsilon/2, \infty). \end{cases}$$

Then

$$\|D\chi_\epsilon(\mathrm{dis}(\cdot, \Gamma))\|_1 \leqslant \frac{2c_{\mathbb{H} \hookrightarrow \mathbb{X}}}{\epsilon}[\mu(\Gamma_\epsilon) - \mu(\Gamma_{\epsilon/2})] \leqslant \frac{2c_{\mathbb{H} \hookrightarrow \mathbb{X}}}{\epsilon}[\mu(\Gamma_\epsilon) - \mu(\Gamma)],$$

and

$$\|\chi_\epsilon(\mathrm{dis}(\cdot, \Gamma)) - 1_\Gamma\|_1 \leqslant \mu(\Gamma_\epsilon \setminus \Gamma) = \mu(\Gamma_\epsilon) - \mu(\Gamma).$$

The result now follows by (14) (23) and $\mu_s(\partial\Gamma) < +\infty$. □

From this proposition, it is immediate that the balls in the classical Wiener space are Caccioppoli sets. However, it seems difficult to verify that $\mu_s(\partial\Gamma) < +\infty$ for a general set Γ. We now give a more useful criterion for the level set of a functional to be a Caccioppoli set.

Theorem 3.4. *Let $f := (f_1, \cdots, f_d) \in \mathbb{D}^{2,1}(\mathbb{R}^d)$ be a d-dimensional real valued random variable on $\mathbb{X}$. Let U be a Borel subset of $\mathbb{R}^d$ with compact and Lipschitz boundary ∂U. Assume that the law of f has a bounded density $\rho_{U_{\epsilon_0}}$ on some neighbourhood U_{ϵ_0} of ∂U with respect to the Lebesgue measure, and*

$$C_f := \mathrm{ess} \sup_{x \in f^{-1}(U_{\epsilon_0})} \|Df(x)\|_{\mathbb{H}} < +\infty, \tag{24}$$

where for $\epsilon > 0$

$$U_\epsilon := \{z \in \mathbb{R}^d : \mathrm{dis}(z, \partial U) < \epsilon\}.$$

Then, the set $\Gamma := \{x : f(x) \in U\}$ *belongs to* $\mathscr{C}(\mathbb{X})$.

Proof. First of all, since ∂U is Lipschitz, noticing the following Minkowski content formula(cf. [14, Theorem 3.2.39])

$$\lim_{\epsilon \downarrow 0} \frac{\mathrm{Vol}(U_\epsilon)}{\epsilon} = \mathcal{H}^{d-1}(\partial U),$$

we have for $\epsilon < \epsilon_0$

$$\mathrm{Vol}(U_\epsilon) \leqslant C_{\partial U} \cdot \epsilon, \tag{25}$$

where the constant $C_{\partial U}$ is independent of ϵ.

Define

$$\chi_\epsilon(z) := \begin{cases} 1, & \mathrm{dis}(z, U^c) \in [\epsilon, \infty), \\ \mathrm{dis}(z, U^c)/\epsilon, & \mathrm{dis}(z, U^c) \in [0, \epsilon]. \end{cases}$$

Then $z \mapsto \chi_\epsilon(z)$ is a Lipschitz function with Lipschitz constant $\frac{1}{\epsilon}$, and by (25)

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \|\chi_\epsilon(f) - 1_\Gamma\|_1 &\leqslant \lim_{\epsilon \downarrow 0} \|1_{U_\epsilon}(f)\|_1 = \lim_{\epsilon \downarrow 0} \int_{U_\epsilon} \rho_{U_{\epsilon_0}}(z)\mathrm{d}z \\ &\leqslant \sup_{z \in U_{\epsilon_0}} \rho_{U_{\epsilon_0}}(z) \cdot \lim_{\epsilon \downarrow 0} \mathrm{Vol}(U_\epsilon) = 0. \end{aligned}$$

On the other hand, by the local property of the operator D(cf. [23, Proposition 1.3.7]), we know

$$D(\chi_\epsilon(f))(x) = 1_{U_\epsilon}(f(x)) \cdot \sum_{j=1}^d (\partial_j \chi_\epsilon)(f(x)) \cdot Df^j(x).$$

Hence, by (24) and (25) we have for $\epsilon < \epsilon_0$

$$\begin{aligned} \|D(\chi_\epsilon(f))\|_1 &\leqslant \sum_{j=1}^d \int_{\mathbb{X}} 1_{U_\epsilon}(f(x)) \cdot |(\partial_j \chi_\epsilon)(f(x))| \cdot \|Df^j(x)\|_{\mathbb{H}} \mu(\mathrm{d}x) \\ &\leqslant \operatorname*{ess\,sup}_{x \in \mathbb{X}} (1_{U_{\epsilon_0}}(f(x)) \cdot \|Df(x)\|_{\mathbb{H}}) \cdot \frac{1}{\epsilon} \mu\Big\{x : f(x) \in U_\epsilon\Big\} \\ &= C_f \cdot \frac{1}{\epsilon} \int_{U_\epsilon} \rho_{U_{\epsilon_0}}(z)\mathrm{d}z \\ &\leqslant C_f \cdot \sup_{z \in U_{\epsilon_0}} \rho_{U_{\epsilon_0}}(z) \cdot C_{\partial U}, \end{aligned}$$

and by (14)

$$V(1_\Gamma) \leqslant \varliminf_{\epsilon \downarrow 0} \|D(\chi_\epsilon(f))\|_1 < +\infty.$$

The proof is thus finished. □

Remark 3.5. *It was proved in* [12] *that (24) is equivalent to the local* μ*-a.e.* $\mathbb{H}$*-Lipschitz continuity. Moreover, in place of the assumption that the law density of* f *is uniformly bounded on a neighbourhood of* ∂U*, we can only require the following estimation:*

$$\mu\Big\{x : f(x) \in U_\epsilon\Big\} \leqslant C\epsilon.$$

In what follows, we shall study the sets in the classical Wiener space $(\mathbb{W}, \mathbb{H}, \mu)$. Namely, $\mathbb{W}$ is the space of all $\mathbb{R}^d$-valued continuous functions on $[0,1]$ starting from 0 at 0 with the norm

$$\|w\|_{\mathbb{W}} := \left(\sum_{i=1}^d \|w_i\|_\infty^2\right)^{1/2}, \quad \|w_i\|_\infty := \sup_{s\in[0,1]} |w_i(s)|,$$

where $w = (w_1, \cdots, w_d)$ denotes a generic element in $\mathbb{W}$; $\mathbb{H} \subset \mathbb{X}$ is the Cameron-Martin space, in which the elements have absolutely continuous and square integrable derivatives, and endowed with the norm,

$$\|\ell\|_{\mathbb{H}} := \left(\int_0^1 |\dot{\ell}(s)|^2 \mathrm{d}s\right)^{1/2};$$

and μ is the Wiener measure.

Let $0 < r < \frac{1}{2}$, $q > \frac{1}{2r} \vee \frac{1}{1-2r}$. We also consider the subspace $\mathbb{W}_{q,r}$ of $\mathbb{W}$ with the following Sobolev pseudo-norms(cf.[2]):

$$\|w\|_{q,r} := \left(\sum_{i=1}^d \|w_i\|_{q,r}^2\right)^{1/2} < \infty,$$

where

$$\|w_i\|_{q,r} := \left(\int_0^1 \int_0^1 \frac{|w(t) - w(s)|^{2q}}{|t-s|^{1+2qr}} \mathrm{d}t\mathrm{d}s\right)^{\frac{1}{2q}}.$$

Using Theorem 3.4, we have

Theorem 3.6. *Let $U \in \mathscr{B}(\mathbb{R}^d)$ have compact and Lipschitz boundary. Then we have*

$$\{w \in \mathbb{W} : (\|w_1\|_\infty, \cdots, \|w_d\|_\infty) \in U\} \in \mathscr{C}(\mathbb{W}),$$

and for $0 < r < \frac{1}{2}$, $q > \frac{1}{2r} \vee \frac{1}{1-2r}$,

$$\{w \in \mathbb{W}_{q,r} : (\|w_1\|_{q,r}, \cdots, \|w_d\|_{q,r}) \in U\} \in \mathscr{C}(\mathbb{W}_{q,r}).$$

In particular, the balls in $\mathbb{W}$ and $\mathbb{W}_{q,r}$ are Caccioppoli sets.

Proof. Without loss of generality, we assume $d = 1$. It is well known that for any $a > 0$(cf. [20, p. 30])

$$\mu\Big\{w : \sup_{s\in[0,1]} |w(s)| \leqslant a\Big\} = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} (-1)^n \int_{(2n-1)a}^{(2n+1)a} e^{-r^2/2} \mathrm{d}r.$$

That is, the random variable $\|\cdot\|_\infty$ has a smooth density.

Moreover, Airault and Malliavin [2] proved that $\|\cdot\|_{q,r}^{2q} \in \mathbb{D}^\infty = \cap_{p,\alpha} \mathbb{D}^{p,\alpha}$, and Fang [11] proved that $\|\cdot\|_{q,r}^{2q}$ is non-degenerate in the Malliavin calculus sense. Thus, $\|\cdot\|_{q,r}$ has an infinitely differentiable density.

On the other hand, $\|\cdot\|_\infty$ and $\|\cdot\|_{q,r}$ are Lipschitz continuous, the condition (24) holds by [12]. The result thus follows by Theorem 3.4. $\square$

We now consider the following SDE:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}w(t), \quad X_0 = x \in \mathbb{R}^d.$$

Then

Theorem 3.7. *Let $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a Lipschitz continuous function, U a Borel subset of $\mathbb{R}^d$ with compact and Lipschitz boundary. Then for any $t \in [0,1]$, $\{w : X_t(w) \in U\} \in \mathscr{C}(\mathbb{W})$.*

Proof. First of all, X_t has a continuous law density with respect to the Lebesgue measure because the generator $\mathcal{L} := \Delta + b^i(x)\partial_i$ is strictly elliptic. So, we only need to check (24). By Theorem 2.2.1 in [23], we know

$$D_r X_t = I + \int_r^t B_k(s) \cdot D_r X_s^k \mathrm{d}s,$$

where $B_k(s)$ is a uniformly bounded and adapted d-dimensional process.

Thus

$$\|DX_t\|_{\mathbb{H}} = \left(\int_0^1 |D_r X_t|^2 \mathrm{d}r \right)^{1/2} \leqslant 1 + C \int_0^t \|DX_s\|_{\mathbb{H}}^2 \mathrm{d}s.$$

By the Gronwall inequality, we get

$$\operatorname*{ess\,sup}_{w \in \mathbb{W}} \|DX_t(w)\|_H \leqslant C, \quad \forall t \in [0, 1].$$

By Theorem 3.4, we complete the proof. □

4. Divergence Theorem

In this section, we still work in the classical Wiener space. For any $n \in \mathbb{N}$, define a family of functions in $\mathbb{W}^*$ by

$$\dot{h}_{nk}(s) := 2^{n/2} 1_{(t_k^n, t_{k+1}^n]}(s), \quad k = 0, 1, \cdots, 2^n - 1,$$

where $t_k^n = k2^{-n}$ and the dot denotes the derivative with respect to s. It is easy to see that

$$\langle h_{nk}, h_{nm} \rangle_{\mathbb{H}} = \delta_{km}, \quad k, m = 0, 1, \cdots, 2^n - 1, \tag{26}$$

where δ_{km} is the Kronecker symbol.

For $n \in \mathbb{N}$ and $w \in \mathbb{W}$, define

$$\pi_n w(t) := \sum_{k=0}^{2^n-1} \Big[w(t_k^n) + w(t_{k+1}^n) - w(t_k^n)(2^n t - k) \Big] \cdot 1_{[t_k^n, t_{k+1}^n]}(t).$$

Then $\pi_n w = \sum_{k=0}^{2^n-1} h_{nk}(w) \cdot h_{nk}$ and

$$\lim_{n\to\infty} \|\pi_n w - w\|_{\mathbb{W}} \leqslant 2 \lim_{n\to\infty} \sup_{|t-s|\leqslant 2^{-n}} |w(t) - w(s)| = 0. \tag{27}$$

We now prove the following generalized Green formula:

Theorem 4.1. *Let $\Gamma \in \mathscr{C}(\mathbb{W})$. Then the support of $\|\partial\Gamma\| := \|D1_\Gamma\|$ is contained in the boundary $\partial\Gamma := \bar{\Gamma} - \Gamma^\circ$, where $\bar{\Gamma}$(resp. Γ°) denotes the closure(resp. interior) of Γ in $\mathbb{W}$. Moreover, for any bounded continuous and $\mathbb{H}$-Lipschitz continuous vector field $\mathbf{f}$ on $\mathbb{W}$,*

$$\int_\Gamma \mathrm{div}\mathbf{f}(w) \ \mu(\mathrm{d}w) = \int_{\partial\Gamma} \langle \mathbf{f}(w), \mathbf{n}_\Gamma(w) \rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w), \tag{28}$$

where $\mathbf{n}_\Gamma := \mathbf{n}_{1_\Gamma}$ and $\|\partial\Gamma\| := \|D1_\Gamma\|$ are from Theorem 2.13.

Proof. In the first three steps, we shall prove

$$\int_\Gamma \mathrm{div}\mathbf{f}(w) \ \mu(\mathrm{d}w) = \int_{\mathbb{W}} \langle \mathbf{f}(w), \mathbf{n}_\Gamma(w) \rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w). \tag{29}$$

In the last step, we prove (28).

(Step 1): In this step, we assume that $\mathbf{f}$ has the form:

$$\mathbf{f}(w) := F(h_{n0}(w), \cdots, h_{n(2^n-1)}(w))\ell,$$

where $\ell \in \mathbb{W}^*$ and $F : \mathbb{R}^{2^n} \to \mathbb{R}$ is a bounded and Lipschitz continuous function.

Let $\varphi \in C_c^\infty(\mathbb{R}^{2^n})$ be a positive smooth function with compact support and satisfy $\int_{\mathbb{R}^{2^n}} \varphi(x)\mathrm{d}x = 1$. Define for $\epsilon > 0$

$$F_\epsilon(x) := \epsilon^{2^n} \int_{\mathbb{R}^{2^n}} F(y)\varphi(\epsilon^{-1}(x-y))\mathrm{d}y,$$

and

$$\mathbf{f}_\epsilon(w) := F_\epsilon(h_{n0}(w), \cdots, h_{n(2^n-1)}(w))\ell, \tag{30}$$

Then $\mathbf{f}_\epsilon \in \mathscr{F}C_b^\infty(\mathbb{W}^*)$ and by (22)

$$\int_\Gamma \mathrm{div}\mathbf{f}_\epsilon(w)\mu(\mathrm{d}w) = \int_{\mathbb{W}} \langle \mathbf{f}_\epsilon(w), \mathbf{n}_\Gamma(w)\rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w). \tag{31}$$

Since for each $w \in \mathbb{W}$

$$\lim_{\epsilon\downarrow 0} \|\mathbf{f}_\epsilon(w) - \mathbf{f}(w)\|_{\mathbb{H}} = 0, \tag{32}$$

the right hand side of (31) converges to the right hand side of (29) by the dominated convergence theorem.

Note that

$$\begin{aligned} \mathrm{div}\mathbf{f}_\epsilon(w) &= F_\epsilon(h_{n0}(w), \cdots, h_{n(2^n-1)}(w)) \cdot \ell(w) \\ &\quad - D_\ell F_\epsilon(h_{n0}(w), \cdots, h_{n(2^n-1)}(w)). \end{aligned}$$

Since F is Lipschitz continuous, we know

$$\|DF_\epsilon(h_{n0}(w), \cdots, h_{n(2^n-1)}(w))\|_{\mathbb{H}} \leqslant \mathrm{Lip}(F_\epsilon) \leqslant \mathrm{Lip}(F).$$

So

$$\sup_{\epsilon\in(0,1]} \|\mathrm{div}\mathbf{f}_\epsilon\|_2 < +\infty.$$

Thus, thanks to (32)

$$\mathrm{div}\mathbf{f}_\epsilon \to \mathrm{div}\mathbf{f} \text{ weakly in } L^2(\mathbb{W}, \mu) \text{ possible a subsequence},$$

and hence, the left hand side of (31) converges to the left hand side of (29).

(Step 2): In this step, we assume that $\mathbf{f}$ has the following form:

$$\mathbf{f}(w) = f(w)\ell, \quad \ell \in \mathbb{W}^*,$$

where f is a bounded continuous and and $\mathbb{H}$-Lipschitz continuous function on $\mathbb{W}$, i.e.

$$|f(w+h) - f(w)| \leqslant \mathrm{Lip}_{\mathbb{H}}(f) \cdot \|h\|_{\mathbb{H}}, \quad \forall w \in \mathbb{W}, \ h \in \mathbb{H}. \tag{33}$$

Define

$$F_n(x_0, \cdots, x_{2^n-1}) := f\Big(\sum_{k=0}^{2^n-1} x_k h_{nk}\Big), \quad x_k \in \mathbb{R},$$

and the approximation of $\mathbf{f}$ by

$$\mathbf{f}_n(w) := F_n(h_{n0}(w), \cdots, h_{n(2^n-1)}(w))\ell.$$

It is clear that by (26) and (33)

$$|F_n(x_0, \cdots, x_{2^n-1}) - F_n(x_0', \cdots, x_{2^n-1}')| \leqslant \mathrm{Lip}_{\mathbb{H}}(\mathrm{f}) \left(\sum_{k=0}^{2^n-1} |x_k - x_k'|^2\right)^{1/2} \tag{34}$$

and

$$F_n(h_{n0}(w), \cdots, h_{n(2^n-1)}(w)) = f(\pi_n(w)), \quad \mathbf{f}_n(w) = \mathbf{f}(\pi_n(w)).$$

Thus, by Step 1 we have

$$\int_\Gamma \mathrm{div}\mathbf{f}_n(w)\ \mu(\mathrm{d}w) = \int_{\mathbb{W}} \langle \mathbf{f}_n(w), \mathbf{n}_\Gamma(w)\rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w). \tag{35}$$

Moreover, it is easy to see that

$$\sup_{w\in\mathbb{W}} \|\mathbf{f}_n(w)\|_{\mathbb{H}} \leqslant \sup_{w\in\mathbb{W}} \|\mathbf{f}(w)\|_{\mathbb{H}}, \tag{36}$$

and by (27), for each $w \in \mathbb{W}$

$$\lim_{n\to\infty} \|\mathbf{f}_n(w) - \mathbf{f}(w)\|_{\mathbb{H}} = 0, \tag{37}$$

Therefore, the right hand side of (35) converges to the right hand side of (29) as $n \to \infty$.

On the other hand, noting that

$$\mathrm{div}\mathbf{f}_n(w) = \sum_{k=0}^{2^n-1} (\partial_k F_n)(h_{n0}(w), \cdots, h_{n(2^n-1)}(w)) \langle \ell, h_{nk}\rangle_{\mathbb{H}} + f(\pi_n(w)) \cdot \ell(w),$$

we have by (34) and the Rademacher theorem(cf. [13])

$$|\mathrm{div}\mathbf{f}_n(w)| \leqslant \mathrm{Lip}_{\mathbb{H}}(\mathrm{f}) \cdot \|\ell\|_{\mathbb{H}} + |\ell(w)| \cdot \sup_{w\in\mathbb{W}} |f(w)|,$$

Therefore, it is the same reason as in Step 1 that the left hand side of (35) converges to the left hand side of (29).

(Step 3:) We now assume that $\mathbf{f}$ is a bounded continuous and $\mathbb{H}$-Lipschitz continuous vector field $\mathbf{f}$ on $\mathbb{W}$. For $n \in \mathbb{N}$, define

$$\mathbf{f}_n(w) = \sum_{k=1}^{n} \langle \mathbf{f}(w), \ell_k\rangle_{\mathbb{H}} \ell_k.$$

Then clearly, (36) and (37) hold. Moreover, by Krée-Meyer inequality(cf. [22]), we have by [12]

$$\|\mathrm{div}\mathbf{f}_n\|_2 \leqslant C(\|D\mathbf{f}_n\|_2 + \|\mathbf{f}_n\|_2) \leqslant C \cdot \mathrm{Lip}_{\mathbb{H}}(\mathbf{f}) + C\|\mathbf{f}\|_2,$$

where the constant C is independent of n.

Thus, as above, using the result obtained in Step 2 we prove (29).

(Step 4:) For any $\epsilon > 0$, let $\varphi_\epsilon \geqslant 0$ be a smooth function on $\mathbb{R}^+$ satisfying

$$\varphi_\epsilon(s) = 0, \quad s \in [0, \epsilon/2], \quad \varphi_\epsilon(s) = 1, \quad s \in [\epsilon, \infty).$$

Set

$$\chi_\epsilon(w) := \varphi_\epsilon(\mathrm{dis}(w, \bar{\Gamma}))$$

and

$$\Gamma_\epsilon := \{w \in \mathbb{W} : \mathrm{dis}(w, \bar{\Gamma}) < \epsilon\}.$$

Then for any $\mathbf{f} \in \mathscr{F}C_b^\infty(\mathbb{W}^*)$, $\mathbf{f} \cdot \chi_\epsilon$ is a bounded and $\mathbb{W}$-Lipschitz continuous vector field on $\mathbb{W}$, and

$$\mathrm{div}(\mathbf{f} \cdot \chi_\epsilon)(w) = 0, \quad \mu\text{-a.e. on } \Gamma_{\epsilon/2}.$$

Hence, by (29)

$$\int_{\mathbb{W}} \langle \mathbf{f} \cdot \chi_\epsilon, \mathbf{n}_\Gamma\rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w) = 0. \tag{38}$$

Set for $j = 1, 2, \cdots$

$$n_\Gamma^j(w) := \langle \mathbf{n}_\Gamma(w), \ell_j\rangle_{\mathbb{H}}.$$

As in Fukushima [16, p.240], for each n we can find $\{v_{j,m}, j = 1, \cdots, n, m \in \mathbb{N}\} \subset \mathscr{F}C_b^\infty$ such that

$$\lim_{m\to\infty} v_{j,m}(w) = n_\Gamma^j(w), \quad \|\partial\Gamma\| - a.e., \quad j = 1, \cdots, n.$$

Define

$$g_{j,m}(w) := \frac{v_{j,m}(w)}{\sqrt{\sum_{k=1}^n v_{j,m}^2(w) + 1/m}} \in \mathscr{F}C_b^\infty,$$

and

$$\mathbf{f}_{n,m}(w) := \sum_{j=1}^n g_{j,m}(w)\ell_j \in \mathscr{F}C_b^\infty(\mathbb{X}^*).$$

Then

$$\|\mathbf{f}_{n,m}(w)\|_{\mathbb{H}} \leqslant 1, \quad \forall w \in \mathbb{W},$$

and by (38)

$$\int_{\mathbb{W}} \langle \mathbf{f}_{n,m} \cdot \chi_\epsilon, \mathbf{n}_\Gamma \rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w) = 0.$$

Taking the limits $m \to \infty$ and $n \to \infty$ yields by the dominated convergence theorem that

$$\int_{\mathbb{W}} \chi_\epsilon(w) \|\partial\Gamma\|(\mathrm{d}w) = 0.$$

By the arbitrariness of $\epsilon > 0$, we obtain

$$\|\partial\Gamma\|(\mathbb{W} - \bar{\Gamma}) = 0. \tag{39}$$

In view of

$$\int_{\mathbb{W}} \mathrm{div}\mathbf{f} \ \mu(\mathrm{d}w) = 0,$$

we have

$$\int_{\mathbb{W}} \langle \mathbf{f}, \mathbf{n}_\Gamma \rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w) = -\int_{\Gamma^c} \mathrm{div}\mathbf{f} \ \mu(\mathrm{d}w).$$

Using the same argument as above, we also have

$$\|\partial\Gamma\|(\mathbb{W} - \bar{\Gamma}^c) = 0.$$

which together with (39) gives

$$\|\partial\Gamma\|(\mathbb{W} - \partial\Gamma) = 0.$$

The proof is complete. □

As a corollary of Theorems 4.1 and 3.6, we have

Corollary 4.2. *Let $a > 0$ and $B_a := \{w \in \mathbb{W} : \|w\|_{\mathbb{W}} < a\}$. Then for any $\epsilon > 0$ and any bounded continuous and $\mathbb{H}$-Lipschitz continuous vector field $\mathbf{f}$ on $\mathbb{W}$*

$$\int_{B_a} \mathrm{div}\mathbf{f}(w)\mu(\mathrm{d}w) = \int_{\partial B_a} \langle \mathbf{f}(w), \mathbf{n}_\Gamma(w) \rangle_{\mathbb{H}} \|\partial B_a\|(\mathrm{d}w).$$

As a corollary of Theorems 2.12 and 4.1, we have the following isoperimetric inequality.

Corollary 4.3. *For any $\Gamma \in \mathscr{C}(\mathbb{W})$, $\mathscr{U}(\mu(\Gamma)) \leqslant \|\partial\Gamma\|(\partial\Gamma)$.*

5. Application to pinned Wiener space

Fix an $a \in \mathbb{R}^d$, the pinned Wiener space is a closed linear subspace of $\mathbb{W}$ in which each path ends a at time 1,

$$\mathbb{W}_a := \{w \in \mathbb{W} : w(1) = a\},$$

and the pinned Wiener measure is defined by the regular conditional probability of μ with respect to $w(1)$:

$$\mu_a(\cdot) := \mu(\cdot \mid w(1) = a).$$

More precisely, for any bounded measurable function F on $\mathbb{R}^{dn}$

$$\begin{aligned} p_1(0,a) \cdot \int_{\mathbb{W}_a} & F(w(t_1), \cdots, w(t_n))\mu_a(w) \\ &= \int_{\mathbb{R}^{dn}} F(x_1, \cdots, x_n)\Pi_{i=0}^n p_{t_{i+1}-t_i}(x_{i+1}, x_i)\mathrm{d}x_1 \cdots \mathrm{d}x_n, \end{aligned} \tag{40}$$

where $x_0 = 0, x_{n+1} = a, t_{n+1} = 1$ and $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-|x-y|^2/(2t)}$. The corresponding Cameron-Martin space is given by $\mathbb{H}_0 := \mathbb{H} \cap \mathbb{W}_0$.

Notice that the pinned Wiener measure μ_a can also be regarded as the law of pinned Brownian motion $w(t)-t(w(1)-a)$ under μ. It is clear that $(\mathbb{W}_0, \mathbb{H}_0, \mu_0)$ forms an abstract Wiener space, and for $a \neq 0$, the translation $w \mapsto w - \cdot a$ establishes an isomorphism between $(\mathbb{W}_a, \mu_a)$ and $(\mathbb{W}_0, \mu_0)$. Therefore, it suffices to consider the pinned Wiener space $(\mathbb{W}_0, \mathbb{H}_0, \mu_0)$. In this case, for any $n \in \mathbb{N}$, let $\{\ell_{nk}, k = 1, \cdots, 2^n - 1\}$ be a linearly independent subset of $\mathbb{W}_0^*$ given by

$$\dot{\ell}_{nk}(s) := 1_{(t_{k-1}^n, t_k^n]}(s) - 1_{(t_k^n, t_{k+1}^n]}(s), \quad k = 1, \cdots, 2^n - 1,$$

where $t_k^n = k2^{-n}$. Let $\{h_{nk}, k = 1, \cdots, 2^n - 1\}$ be the Gram-Schmidt orthogonalization of $\{\ell_{nk}, k = 1, \cdots, 2^n - 1\}$. Then, for any $w \in \mathbb{W}_0$

$$\sum_{k=1}^{2^n-1} h_{nk}(w) \cdot h_{nk}(t) = \sum_{k=0}^{2^n-1} \Big[w(t_k^n) + \Delta w(t_k^n)(2^n t - k) \Big] \cdot 1_{[t_k^n, t_{k+1}^n]}(t) =: \pi_n w(t),$$

where $\Delta w(t_k^n) := w(t_{k+1}^n) - w(t_k^n)$. Indeed, it follows from $h_{nk}(w) = h_{nk}(\pi_n w)$ and $\pi_n w \in \tilde{\mathbb{H}}_n := \overline{\mathrm{span}\{h_{nk}, k = 1, \cdots, 2^n - 1\}}$. Thus, using the same method as in proving Theorem 4.1, we find that the conclusions of Theorem 4.1 still holds for the pinned Wiener measure.

On the other hand, for $d = 1$, it is well known that for any $r > a$(cf. [7, (4.12)])

$$\mu_a(w \in \mathbb{W}_a : \|w\|_\infty < r) = \sum_{n=-\infty}^{\infty} (-1)^n \exp\{-2nr(nr - a)\}.$$

Thus, combining Theorems 3.4 and 4.1 yields that

Theorem 5.1. *Let $U \in \mathscr{B}(\mathbb{R}^d)$ have compact and Lipschitz boundary. Define*

$$\Gamma := \{w \in \mathbb{W}_a : (\|w_1\|_\infty, \cdots, \|w_d\|_\infty) \in U\}.$$

Then $\Gamma \in \mathscr{C}(\mathbb{W}_a)$, and for any bounded continuous and $\mathbb{H}_0$-Lipschitz continuous vector field $\mathbf{f}$ on $\mathbb{W}_a$,

$$\int_\Gamma \mathrm{div}\mathbf{f}(w)\mu_a(\mathrm{d}w) = \int_{\partial\Gamma} \langle \mathbf{f}(w), \mathbf{n}_\Gamma(w)\rangle_{\mathbb{H}} \|\partial\Gamma\|(\mathrm{d}w),$$

where $\partial\Gamma = \{w \in \mathbb{W}_a : (\|w_1\|_\infty, \cdots, \|w_d\|_\infty) \in \partial U\}$.

References

[1] Airault, H.; Zhang, X.: Smoothness of indicator functions of some sets in Wiener spaces. J. Math. Pures Appl. (9) 79 (2000), no. 5, 515–523.

[2] Airault, H. and Malliavin, P.: Intégration géométrique sur l'espace de wiener. *Bull. Sci math.' 2 série*, 112(1998),3-52.

[3] Ambrosio, L., Da Prato G., Goldys B. and Pallara D.: Bounded variation with respect to a log-concave measure. *Comm. Partial Differential Equations* 37 (2012), 2272-2290

[4] Ambrosio L., Figalli A. and Runa E.: On sets of finite perimeter in Wiener spaces: reduced boundary and convergence to halfspaces. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) Mat. Appl. 24 (2013), 111-122

[5] Bakry, D. and Ledoux, M.: Lévy Gromov's isoperimetric inequality for an infinite dimensional generator. *Invent. Math.*, 123, 259-281(1996).

[6] Ambrosio L., Miranda M. Jr. and Pallara D.: Sets with finite perimeter in Wiener spaces, perimeter measure and boundary rectifiability. *Discrete Contin. Dyn. Syst.* 28 (2010), 591-606

[7] Beghin, L. and Orsingher, E.: On the maximum of the generalized Brownian Bridge. *Lithuanian Math. Journal*, Vol. 39, No. 2, 157-167(1999).

[8] Bobkov, S.G.: An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space. *Annals of Prob.*, Vol. 25, No.1 206-214(1997).

[9] Da Prato G. and Lunardi A.: On the Dirichlet semigroup for Ornstein-Uhlenbeck operators in subsets of Hilbert spaces. *J. Funct. Anal.* 259 (2010), 2642-2672

[10] Caselles V., Lunardi A., Miranda M. jr and Novaga M.: Perimeter of sublevel sets in infinite dimensional spaces. *Adv. Calc. Var.* 5 (2012), 59-76

[11] Fang, S.: Non-dégénérescence des pseudo-normes de sobolev sur l'espace de wiener. *Bull. Sc. math., 2 série*, 115 (1991), 223-234

[12] Enchev O. and Stroock D.W.: Rademacher's theorem for Wiener functionals. *Annals of Prob.*, Vol. 21, No.1 25-38(1993).

[13] Evans, L.; Gariepy, R.F.: Measure Theory and Fine Properties of Functions. Studyies in Advanced Mathematics, CRC Press, New York, London,1992.

[14] Federer H.: Geometric Measure Theory. Springer-Verlag, Berlin, 1969.

[15] Feyel D. and Pradelle A.: Hausdorff measures on the Wiener space, Potential Anal., 1(1992), 177-189.

[16] Fukushima, M.: *BV* functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space. J. Funct. Anal. 174 (2000), no. 1, 227–249.

[17] Fukushima, M.; Hino, M.: On the space of BV functions and a related stochastic calculus in infinite dimensions. J. Funct. Anal. 183 (2001), no. 1, 245–268.

[18] Goodman, V.: A divergence theorem for Hilbert space. Trans, Amer. Math. Soc., 164(1972), 411-426.

[19] Hariya, Y. Integration by parts formulae for Wiener measures restricted to subsets in $\mathbb{R}^d$. *J. Func. Anal.*, 239(2006) 594-610.

[20] Itô, K.; McKean, H.P.: Diffusion processes and their sample paths. Second printing, corrected. Die Grundlehren der mathematischen Wissenschaften, Band 125. Springer-Verlag, Berlin-New York, 1974. xv+321pp.

[21] Malliavin, P.: *Differential Analysis in Stochastic Analysis.* Cours MIT, 1978.

[22] Malliavin, P.: *Stochastic analysis.* Grundlehren der Mathematischen Wissenschaften, 313. Springer-Verlag, Berlin, 1997.

[23] Nualart, D.: The Malliavin calculus and related topics. Second edition. Probability and its Applications (New York). Springer-Verlag, Berlin, 2006. xiv+382 pp.

[24] Otobe Y.: A type of Gauss divergence formula on Wiener spaces. *Electron. Commun. Probab.* 14 (2009), 457-463

[25] Röckner M., Zhu Rong-Chan and Zhu Xiang-Chan: The stochastic reflection problem on an infinite dimensional convex set and BV functions in a Gelfand triple. *Ann. Probab.* 40 (2012), 1759-1794

[26] Skorohod, A.: Integration in Hilbert space. Ergeb. Math. Grenzgeb. 79, Springer(1974).

[27] Zambotti, L.: Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection. *Probab. Theory Relat. Fields*, 123, 579-600(2002).