TOPOLOGY OF EIGENSPACE POSETS FOR UNITARY REFLECTION GROUPS

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ABSTRACT. The eigenspace theory of unitary reflection groups, initiated by Springer and Lehrer, suggests that the following object is worthy of study: the poset of eigenspaces of elements of a unitary reflection group, for a fixed eigenvalue, ordered by the reverse of inclusion. We investigate topological properties of this poset. The new results extend the well-known work of Orlik and Solomon on the lattice of intersections of hyperplanes.

1. INTRODUCTION

Let V be a complex vector space of finite dimension, and $G \subseteq GL(V)$ a unitary reflection group in V. Denote by $\mathcal{A}(G)$ the set of reflecting hyperplanes of all reflections in G, and $\mathcal{M}_{\mathcal{A}(G)}$ the hyperplane complement – that is, the smooth manifold which remains when all the reflecting hyperplanes are removed from V. There is an extensive literature studying the topology of $\mathcal{M}_{\mathcal{A}(G)}$ ([1], [6], [26], [27], [20], [3]).

In particular, Orlik and Solomon [26, Corollary 5.7] showed that $H^*(\mathcal{M}_{\mathcal{A}(G)}, \mathbb{C})$ is determined (as a graded representation of G) by the poset $\mathcal{L}(\mathcal{A}(G))$ of intersections of the hyperplanes in $\mathcal{A}(G)$.

It is well-known that the poset $\mathcal{L}(\mathcal{A}(G))$ is a geometric lattice. Hence it is Cohen-Macaulay, and its reduced homology vanishes except in top dimension. The poset $\mathcal{L}(\mathcal{A}(G))$ is also known to coincide with the poset of fixed point subspaces (or 1-eigenspaces) of elements of G (see Theorem 2.10). Springer and Lehrer ([35], [22], [23]) developed a general theory of eigenspaces for unitary reflection groups. The purpose of this paper is to study topological properties of generalisations of $\mathcal{L}(\mathcal{A}(G))$ for arbitrary eigenvalues.

Namely, let ζ be a complex root of unity, and g be an element of G. Define $V(g, \zeta) \subseteq V$ to be the ζ-eigenspace of g. That is, $V(g, \zeta) := \{v \in V \mid gv = \zeta v\}$. Let $\mathcal{S}_\zeta^V(G)$ be the set $\{V(g, \zeta) \mid g \in G\}$, partially ordered by the reverse of inclusion. More generally, if $\gamma \in N_{GL(V)}(G)$ (the normaliser of G in $GL(V)$) and γG is a reflection coset, we may define $\mathcal{S}_\zeta^V(\gamma G)$ to be the set $\{V(x, \zeta) \mid x \in \gamma G\}$, partially ordered by the reverse of inclusion. This is a linear analogue of the poset of p-subgroups of a

group G first studied by Quillen [31]. The study of this poset was first suggested by Lehrer (see [19, Appendix C, p.270]).

The main result of this paper is the following:

Theorem 1.1. *Let G be a unitary reflection group acting on V. Then $\mathcal{S}_\zeta^V(G)$ is Cohen-Macaulay over $\mathbb{Z}$.*

The proof of this result depends on the following theorem, whose proof appears in [16] and [17]. The imprimitive reflection groups $G(r,p,n)$ are defined in Section 2.5.

Theorem 1.2. *Let G be any imprimitive unitary reflection group $G(r,p,n)$, and γ any element of $N_{GL(V)}(G)$. The poset $\mathcal{S}_\zeta^V(\gamma G)$ is Cohen-Macaulay over $\mathbb{Z}$.*

The proof of Theorem 1.2 is combinatiorial in flavour, relying on the establishment of an isomorphism between the posets $\mathcal{S}_\zeta^V(\gamma G)$ and certain subposets of Dowling lattices. The ideas used are very different from the invariant theoretic methods used in this paper, and for this reason the two papers have been separated.

The strategy for proving Theorem 1.1 is to reduce the statement to a computation of the reduced homology of a small number of posets. All except two of these computations are somewhat routine and were performed using MAGMA (see [5], [24]) and the GAP package Simplicial Homology (see [13], [12] for GAP and [10], [30] for the Simplicial Homology package). The other two posets, associated with the complexification of the real reflection group E_8, are very large. Additional techniques in computational algebraic topology were needed to compute their reduced homology. A second paper [18] describing these technques and their applications, written jointly with Mateusz Juda from the Jagiellonian University in Krakow, will appear shortly.

A third forthcoming paper [15] will consider an application of the main result in this paper to the representation theory of unitary reflection groups.

2. Preliminaries

2.1. **Notation.** Throughout this paper, G will denote a group, and V a complex vector space of dimension n. If H is a subgroup of G, write $N_G(H)$ for the normaliser of H in G. If $G \subseteq GL(V)$ and $x \in N_{GL(V)}(G)$, denote the centraliser of x in G by $C_G(x)$. If $g \in GL(V)$, write $\operatorname{Fix} g = \{v \in V \mid gv = v\}$ for the fixed-point subspace of g. If A and B are two sets, the set of elements in A but not B is denoted $A \backslash B$. If $x \in \operatorname{End}(V)$, let $V(x, \zeta)$ be the ζ-eigenspace of x acting on V.

2.2. **Partially ordered sets.** Let P denote a poset. In this paper, all posets will be assumed to be finite. If $a, b \in P$, b *covers* a if $a < c \leqslant b$ implies $c = b$. A *lower order ideal* of P is a subset $I \subseteq P$ such that if $x \in I$ and $y \leqslant x$ then $y \in I$. An *upper order ideal* of P is a subset I such that if $x \in I$ and $y \geqslant x$ then $y \in I$. If $x \in P$, denote by $P_{\leqslant x}$ the subposet of P consisting of elements less than or equal to x. Similarly define $P_{<x}, P_{\geqslant x}, P_{>x}$. Define the closed interval $[x, y] := \{z \in P \mid x \leqslant z \leqslant y\}$. Similarly, define the open interval $(x, y) := \{z \in P \mid x < z < y\}$. If P has a unique minimal element, denote this element $\hat{0}$. An *atom* $x \in P$ is an element which covers $\hat{0}$. Similarly, if P has a unique maximal element, denote this element $\hat{1}$. A *coatom* $x \in P$ is an element which is covered by $\hat{1}$.

A *chain* is a poset in which any two elements are comparable. Define the *length* of a chain C to be $|C| - 1$, and write this number as $l(C)$. The length of a poset P is defined to be:

$$l(P) := \max\{l(C) \mid C \text{ a chain in } P\}.$$

A poset P is *ranked* (or *pure*) of rank n if all maximal chains in P have the same length n. The *rank function* $r : P \longrightarrow \mathbb{N}$ of a ranked poset P is defined inductively by $r(x) = 0$ if x is a minimal element of P, and $r(y) = r(x) + 1$ if y covers x.

Denote the *product* of two posets P and Q by $P \times Q$, and the *join* by $P * Q$ (see [36, §3.2] for definitions).

The poset P is a *join semilattice* if $x \vee y$ exists for all $x, y \in P$ (which implies that P has a unique maximum element $\hat{1}$). The poset P is a *meet semilattice* if $x \wedge y$ exists for all $x, y \in P$ (which implies that P has a unique minimum element $\hat{0}$). If P is both a join semilattice and a meet semilattice, then P is called a *lattice.*

A lattice L is said to be *atomic* if every element of L is the join of atoms (by convention, the unique minimal element $\hat{0}$ is the join of the empty set of atoms). A lattice L is *semimodular* if it is ranked, and if the associated rank function r of L has the property that $r(x \vee y) + r(x \wedge y) \leqslant r(x) + r(y)$ for all $x, y \in L$. A lattice is *geometric* if it is both atomic and semimodular.

2.3. **Poset topology.** For the basics of poset topology, see [39].

The *order complex* of a poset P, denoted $\Delta(P)$, is the abstract simplicial complex whose vertices are the elements of P and whose simplices are the chains of P.

The simplicial homology of P over a commutative ring $\mathbb{A}$, denoted $H(P, \mathbb{A})$, is the simplicial homology of the order complex $\Delta(P)$. Where the ring $\mathbb{A}$ is clear, we may denote this simply by $H(P)$. Similarly, reduced homology can be defined for P, and is denoted $\tilde{H}(P, \mathbb{A})$. The nth homology and reduced homology modules are denoted $H_n(P, \mathbb{A})$ and $\tilde{H}_n(P, \mathbb{A})$ respectively.

A poset is said to be *connected* (resp. *contractible*) if the associated simplicial complex is connected (resp. contractible).

2.4. **Cohen-Macaulay posets.** The property of being Cohen-Macaulay has its origins in commutative algebra, in the context of Cohen-Macaulay rings. For more details, see [39, Lecture 4].

Let $C = (x_0 < \ldots < x_i)$ be a chain of P. Define the *link* of C in P by

$$\mathrm{lk}_P(C) = \{z \in P \mid z \notin C \text{ and } \{z\} \cup C \text{ is a chain of } P\}.$$

Clearly, $\mathrm{lk}_P(C) = (< x_0) * (x_0, x_1) * \ldots * (x_{n-1}, x_n) * (> x_n)$ when $C \neq \emptyset$, while $\mathrm{lk}_P(C) = P$ when $C = \emptyset$.

Then P is said to be *Cohen-Macaulay* over the ring $\mathbb{A}$ if for every chain C in P, including $C = \emptyset$,

$$\widetilde{H}_i(\mathrm{lk}_P(C), \mathbb{A}) = 0 \quad \text{for } i \neq l(\mathrm{lk}_P(C)).$$

Unless stated otherwise, 'Cohen-Macaulay' will mean 'Cohen-Macaulay over $\mathbb{A}$'.

Lemma 2.1. *(Garst [14]) The poset P is Cohen-Macaulay over the ring $\mathbb{A}$ if and only if $\widetilde{H}_i(P, \mathbb{A}) = 0$ for $i \neq l(P)$, and $\mathrm{lk}_P(x)$ is Cohen-Macaulay for all $x \in P$.*

It is known (see [33, Lemma 1.16], [39, Exercise 4.1.3]) that a Cohen-Macaulay poset is ranked. The following proposition is also well-known (see [33, Proposition 1.14] or [17, Proposition 2.4.7] for a proof):

Proposition 2.2. *[33, Proposition 1.14] Let $P_1, \ldots, P_n$ be posets. Then $P_1 * \cdots * P_n$ is Cohen-Macaulay over the field $\mathbb{F}$ if and only if $P_1, \ldots, P_n$ are Cohen-Macaulay over $\mathbb{F}$.*

In particular,

Corollary 2.3. *Let $P^a = P * \{a\}$, where $\{a\}$ is the poset with one element. Then P is Cohen-Macaulay over $\mathbb{F}$ if and only if P^a is Cohen-Macaulay over $\mathbb{F}$. Similarly let $P_a = \{a\} * P$. Then P is Cohen-Macaulay over $\mathbb{F}$ if and only if P_a is Cohen-Macaulay over $\mathbb{F}$.*

2.5. **Unitary reflection groups and the Shephard-Todd classification.** Let V be a vector space over $\mathbb{C}$ of dimension n.

Definition 2.4. An element $g \in GL(V)$ is a *reflection* if the order of g is finite and if $\dim(\mathrm{Fix}\, g) = n - 1$. If g is a reflection, the subspace $\mathrm{Fix}\, g$ is a hyperplane, called the reflecting hyperplane of g.

Definition 2.5. A *unitary reflection group* is a finite subgroup of $GL(V)$, which is generated by reflections.

Any such group is a group of unitary reflections with respect to some hermitian form, which explains the term unitary.

If G is a unitary reflection group, the G-module V is called the *natural* (or *reflection*) representation of G. If V is irreducible as a G-module, we say that G is an *irreducible unitary reflection group*. The irreducible unitary reflection groups were first classified by Shephard and Todd [34].

There is an infinite family $G(r, p, n)$ of unitary reflection groups indexed by triples of positive integers (r, p, n) such that p is a divisor of r. The group $G(r, p, n)$ is defined to be the subgroup of $GL_n(\mathbb{C})$ consisting of all $n \times n$ monomial matrices whose non-zero entries are complex rth roots of unity such that the product of the non-zero entries is an $\frac{r}{p}$th root of unity.

In addition, there are 34 *exceptional* irreducible unitary reflection groups. This includes (complexifications of) the six exceptional irreducible real reflection groups (of types H_3, H_4, F_4, E_6, E_7 and E_8).

2.6. **Invariant theory for unitary reflection groups.** Let S denote the algebra of polynomial functions on V, which may be identified with the symmetric algebra of the dual space, $S(V^*)$. Then $GL(V)$ acts on S, and if $G \subseteq GL(V)$ is a finite group, we denote by S^G the subalgebra of all G-invariant functions. Both S and S^G have a natural grading by polynomial degree.

It is well-known that if G is a unitary reflection group then S^G is free, and in fact this property characterises unitary reflection groups among finite subgroups of $GL(V)$ (see [35, Theorem 2.4]).

Let F denote the ideal of S generated by the elements of S^G with nonzero constant term. The quotient S/F is called the *coinvariant algebra* of G and is denoted S_G.

The coinvariant algebra S_G inherits a grading from S. It is known (see [19, Corollary 3.29 and Proposition 3.32] that $\dim S_G = |G|$ and that the representation of G on S_G is the regular representation.

Let t be an indeterminate, and M any finite-dimensional G-module. Define the *fake degree* of M to be the polynomial

$$f_M(t) := \sum_i \langle (S_G)_i, M \rangle t^i = \sum_{j=1}^r t^{q_j(M)}$$

where $\langle , \rangle$ denotes the usual intertwining number for complex representations of G. The integers $q_1(M) \leqslant q_2(M) \leqslant \ldots \leqslant q_r(M)$ are called the M-exponents of G. Since S_G affords the regular representation of G, $r = \dim M$.

In the case $M = V$ set $f_V(t) = \sum_j t^{m_j}$. The integers $m_i = q_i(V)$ are called the *exponents* of G. It is known (see [19, Corollary 10.23]) that $m_i = d_i - 1$, where $d_1, \ldots, d_r$ are the degrees of the basic invariants of G.

When $M = V^*$, the corresponding exponents $m_i^* := q_i(V^*)$ are called the *coexponents* of G. Define the *codegrees* d_i^* of G by $d_i^* := m_i^* - 1$.

Let $\mathcal{H}$ be the space of G-harmonic polynomials – that is, those polynomials which are annihilated by all G-invariant polynomial differential operators on S with no constant term (see [19, §§9.5-6] for more details). Clearly $\mathcal{H}$ is graded by degree. It is known ([19, Corollary 9.37]) that $\mathcal{H}$ is a G-stable complement of F in S, so that $\mathcal{H}$ may be identified with S_G. If $\mathcal{N} = N_{GL(V)}(G)$ then $\mathcal{H}$ is $\mathcal{N}$-stable (as is S^G), and there is an isomorphism $S = S^G \otimes \mathcal{H}$ of $\mathcal{N}$-modules (see [19, Proposition 12.2]).

If M is any finite-dimensional G-module, define a grading on the G-module $S \otimes M$ by defining the degree j component of $S \otimes M$ to be $S_j \otimes M$. The module $(\mathcal{H} \otimes M^*)^G$ has a homogeneous linear basis $u_1, \ldots, u_r$, where $r = \dim M$, and the degree of u_i is $q_i(M)$. Such a basis is also an S^G-basis of $(S \otimes M^*)^G$ (see [19, Proposition 10.3]). The elements $u_1, \ldots, u_n$ are called *covariants* of G.

2.7. Parabolic subgroups.

Definition 2.6. Suppose $G \subseteq GL(V)$ is a unitary reflection group on V, and that U is a subset of V. Denote by G_U the subgroup

$$\{g \in G \mid \mathrm{Fix}(g) \supseteq U\}.$$

These groups are called *parabolic subgroups* of G. This generalises the notion of a parabolic subgroup for a real reflection group.

The following is known as Steinberg's fixed point theorem ([38, Theorem 1.5]). For a more recent proof, see [21], and [19, §9.7].

Theorem 2.7. *Let G be a finite reflection group. If $v \in V$, the stabiliser $G_v = \{g \in G \mid gv = v\}$ is the reflection group generated by the reflections which fix v.*

Corollary 2.8. *[19, Corollary 9.51] Let U be any subset of V. The parabolic subgroup G_U of G which fixes U pointwise is the reflection group generated by the reflections in G whose reflecting hyperplanes contain U.*

2.8. **Hyperplane arrangements and posets of eigenspaces.** For background on hyperplane arrangements, see [29]. For a discussion of hyperplane arrangements in the context of unitary reflection groups, see [28]. In the following section, let V be a complex vector space of dimension n.

A *hyperplane* of V is a subspace H of codimension 1 in V - that is, a linear hyperplane. A *hyperplane arrangement* $\mathcal{A}$ in V is a finite set of hyperplanes in V. A *subarrangement* $\mathcal{A}'$ of $\mathcal{A}$ is a subset of $\mathcal{A}$. A hyperplane arrangement is *essential* if $\cap_{H \in \mathcal{A}} H = \{0\}$.

Definition 2.9. Let G be a unitary reflection group, and let $\mathcal{A} = \mathcal{A}(G)$ be the set of reflecting hyperplanes of G – i.e. the set of hyperplanes of V which are the fixed point subspaces of elements in G. Define $\mathcal{L}(\mathcal{A}(G))$ to be the poset of intersections of the hyperplanes in $\mathcal{A}$. By convention, $V \in \mathcal{L}(\mathcal{A}(G))$ – it is the empty intersection of elements of $\mathcal{L}(\mathcal{A}(G))$.

It is well known that $\mathcal{L}(\mathcal{A}(G))$ is a geometric lattice (see [29, Lemma 2.3]), and hence (see [11, Theorem 4.1]) that it has vanishing reduced homology except in top dimension.

Theorem 2.10. *[29, Theorem 6.27] Let $G \subseteq GL(V)$ be a unitary reflection group. Then:*

(i) *If $g \in G$, then $\mathrm{Fix}(g) \in \mathcal{L}$.*
(ii) *If $X \in \mathcal{L}(\mathcal{A}(G))$, then there exists $g \in G$ with $\mathrm{Fix}(g) = X$.*

Thus the set of subspaces $\{\mathrm{Fix}\, g \mid g \in G\}$ coincides with $\mathcal{L}(\mathcal{A}(G))$.

2.9. **Reflection cosets.**

Definition 2.11. Suppose G is a unitary reflection group in V. Let γ be a linear transformation of finite order in V, such that $\gamma G = G\gamma$. The coset γG is called a *reflection coset*.

Write $\widetilde{G} = \langle G, \gamma \rangle$ for the group generated by G and γ. Note that $\widetilde{G}$ is finite.

Let M be any finite-dimensional G-module. Recall from §2.6 that the module $(\mathcal{H} \otimes M^*)^G$ has a homogeneous linear basis $u_1, \ldots, u_r$, where $r = \dim M$, and the degree of u_i is $q_i(M)$. In fact (see [19, Proposition 12.2(ii)]) such a basis may be chosen such that for each i, $\gamma u_i = \varepsilon_i(M) u_i$ for some complex root of unity $\varepsilon_i(M) \in \mathbb{C}^\times$. Furthermore, the (multi)set of pairs $(\deg u_i, \varepsilon_i(M))$ depends only on the coset γG and the module M, and not on the particular choice of γ [19, Proposition 12.2(iii)]. The multiset $\{\varepsilon_i(M)\}$ is called the *set of M-factors of γG*. In the special case $M = V$, the V-factors $\varepsilon_i(V)$ will be denoted ε_i and called *factors*. Analogously, in the special case $M = V^*$, the V^*-factors $\varepsilon_i(V^*)$ will be denoted ε_i^* and called *cofactors*.

Irreducible reflection cosets were discussed by Cohen in [9] and classified in [7], where they are referred to as *reflection data*. Also see [19, Table D.5, p.278].

Let γG be a reflection coset in V. For $\zeta \in \mathbb{C}^\times$ a root of unity, define

$$A(\zeta) = \{i \mid \varepsilon_i \zeta^{d_i} = 1\},$$

where d_i is the degree of the basic invariant corresponding to ε_i. Let $a(\zeta) = |A(\zeta)|$.

The following are two key theorems of Springer-Lehrer theory:

Theorem 2.12. *[19, Theorem 12.19] Let γG be a reflection coset in V, and $\zeta \in \mathbb{C}^\times$ be a complex root of unity. Suppose E is maximal among ζ-eigenspaces corresponding to elements of γG. Then:*

(i) $\dim E = a(\zeta)$,

(ii) *any two maximal eigenspaces $V(x, \gamma)$ are conjugate under the action of G.*

Theorem 2.13. *[19, Theorem 12.20] Let γG be a reflection coset in V, and for $\zeta \in \mathbb{C}^\times$, let $E = V(x, \zeta)$ be a ζ-eigenspace, maximal among the ζ-eigenspaces of elements of γG. Let $N(E) = \{g \in G \mid gE = E\}$, and $C(E) = \{g \in G \mid gv = v \text{ for all } v \in E\}$. Then $N(E)/C(E)$ acts as a unitary reflection group on E, whose hyperplanes are the intersections with E of the hyperplanes of G.*

There is also a theory of regular elements for reflection cosets. As with reflection groups, the element $x \in \gamma G$ is ζ-regular if $V(x, \zeta)$ contains a G-regular vector. If $x \in \gamma G$ is ζ-regular for some ζ, then x is said to be regular. The following results will be important later:

Proposition 2.14. *[19, Proposition 12.21] Let γG be a reflection coset and let M be a $\widetilde{G}$-module of dimension r, where $\widetilde{G} = \langle G, \gamma \rangle$. If x is a regular element of γG, then the eigenvalues of x on M are $\{\varepsilon_i(M^*)\zeta^{q_i(M^*)} \mid i = 1, \ldots, r\}$.*

In particular,

Corollary 2.15. *Let $M = V$, with the notation as in Proposition 2.14. If x is a regular element of γG, then the eigenvalues of x acting on V are $\{\varepsilon_i^* \zeta^{m_i^*} \mid i = 1, \ldots r\}$, where the m_i^* are the coexponents of G, and the ε_i^* are the cofactors defined above.*

For a reflection group it is trivial that the element $1 \in G$ is regular. For a reflection coset the existence of a regular element is not obvious, and indeed if G acts reducibly on V, γ could be chosen so that there are no γG-regular vectors in V. See [19, Remark 12.24]. However, the following was proved in [4, Corollary 7.3], and independently in [25, Theorem 3.4]:

Theorem 2.16. *If γG is a reflection coset and G acts irreducibly on V, then there is a ζ-regular element in γG for some $\zeta \in \mathbb{C}^\times$.*

2.10. **Posets of eigenspaces.** The following definition contains the central object of study in this paper:

Definition 2.17. Let γG be a reflection coset in $V = \mathbb{C}^n$, and $\zeta \in \mathbb{C}^\times$ be a complex root of unity. Define $\mathcal{S}_\zeta^V(\gamma G)$ to be the set $\{V(x, \zeta) \mid x \in \gamma G\}$, partially ordered by the reverse of inclusion.

It will be seen (Corollary 3.3) that the poset $\mathcal{S}_\zeta^V(\gamma G)$ always has a unique maximal element $\hat{1}$, and that it may or may not have a unique minimal element $\hat{0}$ as well (for example, the full space V). Thus the associated simplicial complex is a cone, which is contractible. Hence its homology is uninteresting, and we have reason to study the following poset:

Definition 2.18. Define $\widetilde{\mathcal{S}}_\zeta^V(\gamma G)$ to be the subposet of $\mathcal{S}_\zeta^V(\gamma G)$ obtained by removing the unique maximal element, as well as the unique minimal element if it exists.

3. General structure theorems for $\mathcal{S}_V^\zeta(\gamma G)$

This section examines the structure of the posets $\mathcal{S}_\zeta^V(\gamma G)$ for arbitrary unitary reflection groups G and $\gamma \in N_{GL(V)}(G)$. With the idea of showing that the posets $\mathcal{S}_\zeta^V(\gamma G)$ are Cohen-Macaulay in mind, we consider closed intervals in $\mathcal{S}_\zeta^V(\gamma G)$. The key is the following theorem:

Theorem 3.1. *Let γG be a reflection coset in V. Let $E = V(\gamma g, \zeta) \in \mathcal{S}_\zeta^V(\gamma G)$. Then the following three posets are identical (as sets of subspaces of V):*

  (i) $\{V(\gamma h, \zeta) \in \mathcal{S}_\zeta^V(\gamma G) \mid h \in G, V(\gamma h, \zeta) \subseteq E\}$;
  (ii) $\{V(\gamma h, \zeta) \cap E \mid h \in G\}$;
  (iii) $\{E \cap \mathrm{Fix}_V(x) \mid x \in G\}$.

In addition, if E is maximal among the ζ-eigenspaces of elements of γG, then there is a fourth poset

  (iv) $\mathcal{S}_1^E(N(E)/C(E))$

equal to each of the above.

Before embarking on the proof of this theorem, we point out some important corollaries:

Corollary 3.2. *With notation as above, suppose $E_1, E_2 \in \mathcal{S}_\zeta^V(\gamma G)$. Then $E_1 \cap E_2 \in \mathcal{S}_\zeta^V(\gamma G)$. In particular, $\mathcal{S}_\zeta^V(\gamma G)$ is a join semilattice.*

Proof of Corollary 3.2. Assuming that the posets (i) and (ii) are equal, it follows that the intersection of any two ζ-eigenspaces is again a ζ-eigenspace. Hence any two elements of $\mathcal{S}_\zeta^V(\gamma G)$ have a least upper bound, and so $\mathcal{S}_\zeta^V(\gamma G)$ is a join semilattice. $\square$

Corollary 3.3. *With notation as above, the poset $\mathcal{S}_\zeta^V(\gamma G)$ always has a unique maximal element $\hat{1}$.*

Proof of Corollary 3.3. By repeated application of Corollary 3.2, $\bigcap_{E \in \mathcal{S}_\zeta^V(\gamma G)} E$ is an eigenspace, and by definition it is contained within every element $E \in \mathcal{S}_\zeta^V(\gamma G)$. $\square$

Remark 3.4. As a result of Corollary 3.3, the first poset mentioned in Theorem 3.1 can be written $[E, \hat{1}]$.

Corollary 3.5. *Suppose $E = V(\gamma g, \zeta) \in \mathcal{S}_\zeta^V(\gamma G)$. The interval $[E, \hat{1}]$ is a geometric lattice.*

Proof of Corollary 3.5. The eigenspace $E = V(\gamma g, \zeta)$ is contained in some maximal eigenspace E'. Hence by Theorem 3.1(iv), $[E, \hat{1}]$ is a principal upper order ideal of $\mathcal{S}_1^{E'}(N(E')/C(E'))$. The latter is known to be a geometric lattice by Proposition 2.10 and [29, Lemma 2.3]. Since closed intervals of geometric lattices are also geometric lattices (see [37, Lecture 3]), the result follows. $\square$

We now prove Theorem 3.1.

Proof of Theorem 3.1. The equality of (iii) and (iv) follows easily from Theorem 2.13. Hence we concentrate on the equality of (i), (ii) and (iii).

(ii) = (iii). First we show that, for any element $h \in G$, $V(\gamma h, \zeta) \cap E = E \cap \text{Fix}_V(\gamma g h^{-1} \gamma^{-1})$. Note that $\gamma g h^{-1} \gamma^{-1} \in G$ since γ normalises G.

If $v \in V(\gamma h, \zeta) \cap E$ then $\gamma g v = \zeta v$ and $\gamma h v = \zeta v$. So $v = (\gamma g)(\gamma h)^{-1} v = (\gamma g h^{-1} \gamma^{-1}) v$. Hence $v \in \text{Fix}_V(\gamma g h^{-1} \gamma^{-1})$. Conversely, if $v \in E \cap \text{Fix}_V(\gamma g h^{-1} \gamma^{-1})$ then $\gamma g v = \zeta v$, and $\gamma g h^{-1} \gamma^{-1} v = v$. Hence $(\gamma h)^{-1} v = h^{-1} \gamma^{-1} v = (\gamma g)^{-1}(\gamma g h^{-1} \gamma^{-1}) v = (\gamma g)^{-1} v = \zeta^{-1} v$. Thus $(\gamma h) v = \zeta v$, and so $v \in V(\gamma h, \zeta)$. This proves that $V(\gamma h, \zeta) \cap E = E \cap \text{Fix}_V(\gamma g h^{-1} \gamma^{-1})$, and therefore that (ii) $\subseteq$ (iii). To prove the reverse inclusion, if x is any element of G, let $h = \gamma^{-1} x^{-1} \gamma g$. Substituting into the formula just proved, we have $V(\gamma h, \zeta) \cap E = E \cap \text{Fix}_V(x)$. Thus the reverse inclusion is true, and (ii) = (iii).

By definition, (i) $\subseteq$ (ii). Hence the theorem will be proved if we can show that (iii) $\subseteq$ (i). In other words, we must prove

Proposition 3.6. *Let V be a complex vector space, and γG a reflection coset in V. Suppose that $E = V(\gamma, \zeta) \in \mathcal{S}^V_\zeta(\gamma G)$, and $X \in \mathcal{L}(G)$. Then $E \cap X = V(\gamma h, \zeta)$ for some $h \in G$.*

Note that we can assume without loss of generality that $E = V(\gamma, \zeta)$ rather than $V(\gamma g, \zeta)$ by replacing γ with γg, since if $\gamma \in N_{GL(V)}(G)$ then $\gamma g \in N_{GL(V)}(G)$ also.

Consider the following three propositions:

Proposition 3.7. *Let V be a complex vector space, and γG any reflection coset in V. Suppose that $E = V(\gamma, \zeta) \in \mathcal{S}^V_\zeta(\gamma G)$ and $X \in \mathcal{L}(G)$ such that $\gamma X = X$. Then $E \cap X = V(\gamma h, \zeta)$ for some $h \in G$.*

Note that we are not assuming that E is maximal.

Proposition 3.8. *Let V be a complex vector space, and γG any reflection coset in V, where G acts essentially on V (i.e. $\cap_{H \in \mathcal{A}_G} H = \{0\}$). Then there exists $h \in G$ such that $V(\gamma h, \zeta) = \{0\}$.*

Proposition 3.9. *Let V be a complex vector space, and γG any reflection coset in V, where G acts essentially on V (i.e. $\cap_{H \in \mathcal{A}_G} H = \{0\}$). Then there exists $h \in G$ such that $V(\gamma h, 1) = \{0\}$.*

The key to proving Theorem 3.1 is the following reduction theorem:

Theorem 3.10. *We have the implications (3.9) $\Rightarrow$ (3.8) $\Rightarrow$ (3.7) $\Rightarrow$ (3.6).*

Proof of Theorem 3.10. (3.7) $\Rightarrow$ (3.6). Let E and X be as in Proposition 3.6. Define $X' := \bigcap_{i=0}^{\infty} \gamma^i X$. Note that this intersection is actually finite as γ has finite order. For each i, $\gamma^i X \in \mathcal{L}(G)$, since γ normalises G. Thus $X' \in \mathcal{L}(G)$, and so by Proposition 2.10(ii), $X' = \text{Fix}_V(y)$ for some $y \in G$.

Clearly $\gamma X' = X'$. Applying Proposition 3.7 to X', $E \cap X' = V(\gamma h, \zeta)$ for some $h \in G$.

Now $E \cap X' = E \cap (\cap_{i=0}^{\infty} \gamma^i X) = \cap_{i=0}^{\infty}(E \cap \gamma^i X)$. Also,

$$E \cap \gamma^i X = \gamma^{-i}(E \cap \gamma^i X) \quad \text{(since } \gamma \text{ acts by scalar multiplication on } E)$$

$$= \gamma^{-i} E \cap X$$

$$= E \cap X.$$

Hence $V(\gamma h, \zeta) = E \cap X' = \bigcap_{i=1}^{\infty}(E \cap \gamma^i X) = \bigcap_{i=1}^{\infty}(E \cap X) = E \cap X$, as required.

$(3.8) \Rightarrow (3.7)$. Let E and X be as in Proposition 3.7. Now G_X acts as a reflection group on V by Steinberg's fixed point theorem (Corollary 2.8), and it acts essentially on V/X. Since $\gamma X = X$ we have $\gamma \in N_{GL(V)}(G_X)$. Hence γG_X is a reflection coset in V/X.

Applying Proposition 3.8 to γG_X, there is some $h \in G_X$ such that $(V/X)(\gamma h, \zeta) = \{0\}$. In other words, $V(\gamma h, \zeta) \subseteq X$. Also if $v \in V(\gamma h, \zeta)$ and $h \in G_X$, then

$$\gamma v = \gamma(hv) = (\gamma h)v = \zeta v.$$

Hence $V(\gamma h, \zeta) \subseteq E$. Thus $V(\gamma h, \zeta) \subseteq E \cap X$. Conversely, $E \cap X \subseteq V(\gamma h, \zeta)$, so $E \cap X = V(\gamma h, \zeta)$.

$(3.9) \Rightarrow (3.8)$. If γG is a reflection coset in V, then so is $\zeta^{-1}\gamma G$. Applying Proposition 3.9 to $\zeta^{-1}\gamma G$, there exists $h \in G$ such that $V(\zeta^{-1}\gamma h, 1) = \{0\}$. But $V(\gamma h, \zeta) = V(\zeta^{-1}\gamma h, 1)$, as required. $\square$

We now focus on the proof of Proposition 3.9. The first task is to reduce Proposition 3.9 to the case where G is an irreducible reflection group. Consider the following two Propositions:

Proposition 3.11. *Proposition 3.9 holds when V is irreducible as a $\widetilde{G}$-module, where $\widetilde{G} = \langle G, \gamma \rangle$.*

Proposition 3.12. *Proposition 3.9 holds when V is irreducible as a G-module.*

Clearly $(3.9) \Rightarrow (3.11) \Rightarrow (3.12)$. The reverse implications are also true:

Theorem 3.13. *We have the implications $(3.12) \Rightarrow (3.11) \Rightarrow (3.9)$.*

Proof of Theorem 3.13. $(3.11) \Rightarrow (3.9)$. Suppose $V = \bigoplus_{i=1}^{k} V_i$ is a decomposition of V into irreducible $\widetilde{G}$-modules. Then $G = G_1 \times G_2 \times \ldots \times G_k$ and $\gamma = \gamma_1 \oplus \gamma_2 \oplus \ldots \ldots \oplus \gamma_k$, where G_i acts on V_i as a reflection group and trivially on V_j for $j \neq i$, and γ_i normalises G_i.

Note that $\mathcal{A}(G) = \cup_{i=1}^{k}\mathcal{A}(G_i)$.

Thus $\bigcap_{H \in \cup \mathcal{A}(G_j), j \neq i} H \supseteq V_i$ (and by hypothesis equals V_i).

It follows that $\bigcap_{H \in \mathcal{A}(G_i)} H = \bigoplus_{j \neq i} V_j$, and hence that $\bigcap_{H \in \mathcal{A}(G_i)}(H \cap V_i) = \{0\}$.

Hence (G_i, γ_i, V_i) satisfies the conditions of Proposition 3.11. So by assumption there exists elements $x_i \in G_i$ such that $V_i(\gamma_i x_i, 1) = \{0\}$.

Hence $V(\gamma x, 1) = \{0\}$, where $x = x_1 \oplus x_2 \oplus \ldots \oplus x_k$, and Proposition 3.9 follows.

$(3.12) \Rightarrow (3.11)$. Assume that V is irreducible as a $\widetilde{G}$-module. Then $G = G_1 \times G_2 \times \ldots \times G_k$, $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$, and all (G_i, V_i) are isomorphic, irreducible, and permuted cyclically by γ. Thus γ^k fixes all the V_i and in particular normalises G_i for each i. (See [3, Proposition 6.9] for further details.)

This means that $V_i = \gamma^{i-1}V_1$, $i = 1, 2, \ldots, k$ and $\gamma^k V_1 = V_1$. Thus we may write an arbitrary element $v \in V$ uniquely as $v = v_1 + \gamma v_2 + \ldots + \gamma^{k-1} v_k$ where $v_i \in V_1$ for all i.

Then

$$\gamma v = \gamma(v_1 \oplus \gamma v_2 \oplus \ldots \oplus \gamma^{k-1} v_k)$$
$$= \gamma^k v_k \oplus \gamma v_1 \oplus \ldots \oplus \gamma^{k-1} v_{k-1}.$$

Thus v is fixed by γ if and only if $v_1 = v_2 = \ldots = v_k$ and $\gamma^k v_1 = v_1$.

Now by Proposition 3.12 applied to (V_1, γ^k, G_1), there exists $y \in G_1$ such that $\mathrm{Fix}_{V_1}(\gamma^k y) = V_1(\gamma^k y, 1) = \{0\}$.

We may write $x \in G$ uniquely as $x = (x_1, {}^{\gamma} x_2, \ldots, {}^{\gamma^{k-1}} x_k)$, where $x_i \in G_1$ for all i, and the left superscript denotes conjugation.

Then

$$(\gamma x)v = \gamma(xv)$$
$$= \gamma(x_1 v_1 \oplus \gamma x_2 v_2 \oplus \ldots \oplus \gamma^{k-1} x_k v_k)$$
$$= \gamma^k x_k v_k \oplus \gamma x_1 v_1 \oplus \ldots \oplus \gamma^{k-1} x_{k-1} v_{k-1}.$$

This equals $v = v_1 \oplus \gamma v_2 \oplus \ldots \oplus \gamma^{k-1} v_k$ if and only if

$$\gamma^k x_k v_k = v_1, x_1 v_1 = v_2, x_2 v_2 = v_3, \ldots, x_{k-1} v_{k-1} = v_k.$$

That is,

$$v_k = x_{k-1} v_{k-1} = x_{k-1} x_{k-2} v_{k-2} = \ldots = x_{k-1} x_{k-2} \ldots x_2 v_2 = x_{k-1} x_{k-2} \ldots x_1 v_1.$$

Therefore $\gamma^k x_k \ldots x_1 v_1 = v_1$.

To summarise, we have proved that if $v \in \mathrm{Fix}_V(\gamma x) = V(\gamma x, 1)$, then $\gamma^k x_k x_{k-1} \ldots x_1 v_1 = v_1$.

Now take $x = (y, 1, 1, \ldots, 1)$. Then if $(\gamma x)v = v$, we must have $(\gamma^k y)v_1 = v_1$, whence by assumption $v_1 = 0$. But then $v_i = 0$ for all i, and so $v = 0$. Hence $V(\gamma x, 1) = \{0\}$, as required. $\qquad\square$

Therefore, in view of Theorems 3.10 and 3.13, Theorem 3.1 will follow from Proposition 3.12.

Before proving Proposition 3.12, we record a lemma which will be useful later:

Lemma 3.14. *With notation as above, $V(\gamma x, \zeta)$ and $V(\gamma^k x_k x_{k-1} \ldots x_1, \zeta^k)$ are isomorphic as vector spaces.*

Proof. As in the proof of Theorem 3.13,

$$(\gamma x)v = \gamma(xv)$$
$$= \gamma(x_1 v_1 \oplus \gamma x_2 v_2 \oplus \ldots \oplus \gamma^{k-1} x_k v_k)$$
$$= \gamma^k x_k v_k \oplus \gamma x_1 v_1 \oplus \ldots \oplus \gamma^{k-1} x_{k-1} v_{k-1}.$$

This equals $\zeta v = \zeta v_1 \oplus \zeta \gamma v_2 \oplus \ldots \oplus \zeta \gamma^{k-1} v_k$ if and only if

$$\gamma^k x_k v_k = \zeta v_1, x_1 v_1 = \zeta v_2, x_2 v_2 = \zeta v_3, \ldots, x_{k-1} v_{k-1} = \zeta v_k.$$

That is,

$$v_k = \zeta^{-1} x_{k-1} v_{k-1} = \zeta^{-2} x_{k-1} x_{k-2} v_{k-2} = \ldots = \zeta^{-(k-2)} x_{k-1} x_{k-2} \ldots x_2 v_2 = \zeta^{-(k-1)} x_{k-1} x_{k-2} \ldots x_1 v_1.$$

Therefore $\gamma^k x_k \ldots x_1 v_1 = \zeta^k v_1$. That is, $v_1 \in V(\gamma^k x_k x_{k-1} \ldots x_1, \zeta^k)$. On the other hand, the above equations make it clear that if $v = v_1 + \gamma v_2 + \ldots + \gamma^{k-1} v_k \in V(\gamma x, \zeta)$ then $v_2, \ldots, v_k$ are uniquely determined by v_1. Thus the map which sends v to v_1 is a vector space isomorphism. $\qquad\square$

We now prove Proposition 3.12.

Proof of Proposition 3.12. The proof starts with a further reduction:

Lemma 3.15. *If γ induces an inner automorphism of G, then Proposition 3.12 holds.*

Proof of Lemma 3.15. By hypothesis, there exists $h \in G$ such that γh is central in $\langle G, \gamma \rangle$ and hence is scalar. If $\gamma h \neq 1$ then $\mathrm{Fix}_V(\gamma h) = \{0\}$, while if $\gamma h = 1$, then $\gamma \in G$ and $\gamma G = G$. In the latter case, since $G \neq \{1\}$, there exists at least one reflecting hyperplane and so $\{0\} = \cap_\alpha H_a = \mathrm{Fix}_V(g)$ for some $g \in G$ (by Proposition 2.10). $\qquad\square$

In [19, Table D.5, p.278], there is a list of reflection cosets (G, γ_0). Any reflection coset (G, γ) satisfying the conditions of Proposition 3.12, in which γ does not induce an inner automorphism, has the form $(G, \gamma) = (G, \alpha\gamma_0)$, where (G, γ_0) appears in this table, and $\alpha \in \mathbb{C}^\times$. (See also [7, Proposition 3.13]). We check Proposition 3.12 for all such cosets case by case. That is, it is necessary to show that the reflection coset $(\alpha\gamma_0)G$ contains an element x such that $\mathrm{Fix}_V(x) = \{0\}$, where $\alpha \in \mathbb{C}^\times$.

The proof relies on the following lemma:

Lemma 3.16. *Suppose $\alpha\gamma_0 G$ is a reflection coset in V which satisfies the conditions of Proposition 3.12. If there exists $x \in \alpha\gamma_0 G$ which acts on V in such a way that none of its eigenvalues is equal to 1, then $\mathrm{Fix}_V(x) = \{0\}$.*

Proof of Lemma 3.16. This is immediate from the definition of an eigenvalue. $\qquad\square$

We now proceed to check that Lemma 3.16 holds for all reflection cosets satisfying the conditions of Proposition 3.12. Let ξ_m denote a fixed primitive m-th root of unity.

Case 1: $G = G(m, p, n), \gamma = \mathrm{diag}(\xi_{\frac{em}{p}}, 1, \ldots, 1)$, where $e \mid p$ and $\xi_{\frac{em}{p}}$ is a fixed primitive $\frac{em}{p}$-th root of unity. Note that [19, Table D.5, p.278] treats the cases $m \neq p$ and $m = p$ separately. For the purposes of this argument they can be treated together.

In this case $\alpha\gamma = \mathrm{diag}(\alpha\xi_{\frac{em}{p}}, \alpha, \ldots, \alpha)$. The eigenvalues of $\alpha\gamma$ acting on V are $\alpha\xi_{\frac{em}{p}}$ and α. Hence if $\alpha \neq 1, \xi_{\frac{em}{p}}^{-1}$ by Lemma 3.16 there is nothing more to prove.

If $\alpha = 1$ we may assume that $e \neq 1$ (and hence $p, m \neq 1$, as $e \mid p$), since if $e = 1$, $\alpha\gamma$ induces an inner automorphism. Thus $\xi_{\frac{em}{p}} \neq 1$.

If n is even define $y \in G$ by $y_{1,2} = y_{2,1} = 1$, $y_{i,i} = (\xi_m)^{(-1)^{i+1}}$ if $i \geqslant 3$, $y_{i,j} = 0$ otherwise. Then $y \in G$ and $x := \alpha\gamma y$ is defined by $x_{1,2} = \xi_{\frac{em}{p}}, x_{2,1} = 1, x_{i,i} = (\xi_m)^{(-1)^{i+1}}$ if $i \geqslant 3$, $x_{i,j} = 0$ otherwise. It is easy to check that $\mathrm{Fix}_V(x) = \{0\}$.

If n is odd define $y \in G$ by $y := \mathrm{diag}(1, \xi_m, \xi_m^{-1}, \ldots, \xi_m, \xi_m^{-1})$. Then $x := \alpha\gamma y$ is equal to $\mathrm{diag}(\xi_{\frac{em}{p}}, \xi_m, \xi_m^{-1}, \ldots, \xi_m, \xi_m^{-1})$ and $\mathrm{Fix}_V(x) = \{0\}$.

If on the other hand $\alpha = \xi_{\frac{em}{p}}^{-1}$ then $\alpha\gamma = \mathrm{diag}(1, \xi_{\frac{em}{p}}^{-1}, \ldots, \xi_{\frac{em}{p}}^{-1})$. Again we may assume that $e \neq 1$ as otherwise $\alpha\gamma$ induces an inner automorphism. Define $y \in G$ by $y_{1,2} = y_{2,1} = 1, y_{i,i} = 1$ if $i \geqslant 3$, $y_{i,j} = 0$ otherwise. Then $x \in \alpha\gamma G$ is defined by

$x_{1,2} = \xi_{\frac{em}{p}}^{-1}, x_{2,1} = 1, x_{i,i} = \xi_{\frac{em}{p}}^{-1}$ if $i \geqslant 3$, $x_{i,j} = 0$ otherwise. Once again, it is easy to check that $\mathrm{Fix}_V(x) = \{0\}$. This completes the proof of Lemma 3.16 for Case 1.

The proof of Proposition 3.12 in the remaining cases relies on Corollary 2.15, Theorem 2.16, and the following lemma:

Lemma 3.17. *If γG is a reflection coset, α, $\zeta \in \mathbb{C}^\times$, and $x \in \gamma G$ then:*
  (i) *x is ζ-regular for γG if and only if αx is $\alpha\zeta$-regular for $\alpha\gamma G$*
  (ii) *$\{\zeta \mid \zeta$ is regular for $\alpha\gamma G\} = \{\alpha\zeta \mid \zeta$ is regular for $\gamma G\}$.*

Proof of Lemma 3.17.   (i) The element $x \in \gamma G$ is ζ-regular

$$\Leftrightarrow V(x, \zeta) \text{ contains a regular vector } v$$
$$\Leftrightarrow V(\alpha x, \alpha\zeta) \text{ contains a regular vector } v$$
$$\Leftrightarrow \alpha x \in \alpha\gamma G \text{ is } \alpha\zeta\text{-regular for } \alpha\gamma G.$$

  (ii) The eigenvalue $\zeta \in \mathbb{C}$ is regular for γG

$$\Leftrightarrow V(x, \zeta) \text{ contains a regular vector } v \text{ for some } x \in \gamma G$$
$$\Leftrightarrow V(\alpha x, \alpha\zeta) \text{ contains a regular vector } v \text{ for some } \alpha x \in \alpha\gamma G$$
$$\Leftrightarrow \alpha\zeta \in \mathbb{C} \text{ is regular for } \alpha\gamma G.$$

$\square$

Corollary 3.18. *If $x \in \gamma G$ is ζ-regular, then $\alpha x \in \alpha\gamma G$ is $\alpha\zeta$-regular (in $\alpha\gamma G$), and the eigenvalues of αx acting on V are $\{\alpha\varepsilon_i^* \zeta^{m_i^*} \mid i = 1, \ldots, n\}$. (See §2.6 for a definition of the coexponents m_i^*, and §2.9 for a definition of the V^*-factors ε_i^*.)*

Proof of Corollary 3.18. This follows directly from Corollary 2.15 and Lemma 3.17.

$\square$

By Lemma 3.16 and Corollary 3.18, if we can find $\zeta \in \mathbb{C}$ which is regular and satisfies $\alpha\varepsilon_i^* \zeta^{m_i^*} \neq 1$ for all $i = 1, \ldots, r$), it will follow that $\mathrm{Fix}_V(\alpha x) = \{0\}$, where $x \in \gamma G$ is any ζ-regular element.

As the proofs for the remaining six cases are very similar, we present only the most difficult case here. The full proof is contained in [17].

Case 2: $G = G(2, 2, 4)$, $\gamma = \dfrac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

$m_i^* = 1, 3, 3, 5$; $\varepsilon_i^* = 1, \omega, \omega^2, 1$; ζ regular for $\gamma G \Leftrightarrow |\zeta| \in \{1, 2, 3, 6, 12\}$.

We must choose ζ such that $\alpha\zeta \neq 1$, $\alpha\omega\zeta^3 \neq 1$, $\alpha\omega^2\zeta^3 \neq 1$, and $\alpha\zeta^5 \neq 1$. Upon simplification, this becomes $\zeta \neq \alpha^{-1}$, $\zeta^3 \neq \alpha^{-1}\omega$, $\alpha^{-1}\omega^2$ and $\zeta^5 \neq \alpha^{-1}$.

There are 10 complex numbers ζ whose order $|\zeta| \in \{1, 2, 3, 6, 12\}$ (all 12th roots of unity, except i and $-i$, which have order 4). Of these, three satisfy $\zeta^3 = 1$, three satisfy $\zeta^3 = -1$, two satisfy $\zeta^3 = i$ and two satisfy $\zeta^3 = -i$. Hence there are at least four 12th roots of unity ζ (in fact 7, if we argue more carefully) which satisfy $\zeta^3 \neq \alpha^{-1}\omega, \alpha^{-1}\omega^2$. Since all 5th powers of 12th roots of unity are distinct (as $\gcd(5, 12) = 1$), at most one of these satisfies $\zeta^5 = \alpha^{-1}$, and also at most one satisfies $\zeta = \alpha^{-1}$. Hence we can choose such a ζ as required.

As mentioned above, the other cases are similar but easier. This completes the proof of Proposition 3.12 and Theorem 3.1. $\qquad\square$

$\qquad\square$

Theorem 3.1 describes the principal upper order ideals of the poset $\mathcal{S}_\zeta^V(\gamma G)$. The description of principal lower order ideals is much simpler. Recall (Corollary 2.8) that if U is any subset of $V = \mathbb{C}^n$, then G_U is the parabolic subgroup of G which fixes U pointwise.

Lemma 3.19. *Let γG be a reflection coset in $V = \mathbb{C}^n$. Let $E = V(\gamma, \zeta) \in \mathcal{S}_\zeta^V(\gamma G)$. Then $\widetilde{\mathcal{S}}_\zeta^V(\gamma G)_{\leqslant E} \cong \widetilde{\mathcal{S}}_\zeta^V(\gamma G_E)$*

Proof. Suppose $V(\gamma h, \zeta) \leqslant E$. Then γ and γh both act on E as multiplication by ζ. Then $h = \gamma^{-1}(\gamma h) \in G_E$.

Conversely, if $h \in G_E$ then certainly $V(\gamma h, \zeta)$ contains E. Note that G_E is itself a unitary reflection group by Steinberg's fixed point theorem (Corollary 2.8). $\qquad\square$

4. Homology of $\widetilde{\mathcal{S}}_\zeta^V(\gamma G)$

Recall (Definition 2.18) that $\widetilde{\mathcal{S}}_\zeta^V(\gamma G)$ is the poset obtained from $\mathcal{S}_\zeta^V(\gamma G)$ by removing the unique maximal element, as well as the unique minimal element if it exists. In this section we assume that G is an irreducible unitary reflection group.

The main theorem in this section is the following:

Theorem 4.1. *Suppose γG is a reflection coset in V, that G acts irreducibly on V, and that $\zeta \in \mathbb{C}^\times$ is a complex root of unity. Then $\widetilde{H}_i(\widetilde{\mathcal{S}}_\zeta^V(\gamma G), \mathbb{Z}) = 0$ for $i \neq l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G))$.*

Proof. Consider the following propositions:

Proposition 4.2. *Theorem 4.1 holds when $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G)) \geqslant 2$.*

Proposition 4.3. *Theorem 4.1 holds when $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G)) \geqslant 2$ and G is an exceptional irreducible unitary reflection group.*

Clearly $(4.1) \Rightarrow (4.2) \Rightarrow (4.3)$. The reverse implications are also true:

Theorem 4.4. *We have the implications $(4.3) \Rightarrow (4.2) \Rightarrow (4.1)$.*

Proof of Theorem 4.4. $(4.2) \Rightarrow (4.1)$. It is sufficient to show that Theorem 4.1 holds in the case $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G)) \leqslant 1$.

The poset $\widetilde{\mathcal{S}}_\zeta^V(\gamma G)$ can have nonzero homology in dimension at most $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G))$. Hence Theorem 4.1 is trivially true if $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G)) = 0$.

For the case $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G)) = 1$ we require the following proposition:

Proposition 4.5. *If $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G)) > 0$, then $\widetilde{\mathcal{S}}_\zeta^V(\gamma G)$ is connected.*

Proof of Proposition 4.5. Given two eigenspaces X_1, X_2, we must find a sequence of eigenspaces $X_1 = Y_1, Y_2, \ldots, Y_k = X_2$ such that for each i, either Y_i covers Y_{i+1} or Y_{i+1} covers Y_i. First note that it suffices to consider the case where $X_1 = E_1$, $X_2 = E_2$ are maximal eigenspaces, since every eigenspace is contained within a maximal eigenspace.

We now require a lemma:

Lemma 4.6. *Let $X = V(\gamma g, \zeta)$ be a ζ-eigenspace in $\widetilde{\mathcal{S}}^V_\zeta(\gamma G)$, and let r be a reflection in G such that $r \notin N(X)$. Then $\dim(rX \cap X) = \dim X - 1$.*

Proof of Lemma 4.6. It is clear that $rX \cap X \supseteq \mathrm{Fix}(r) \cap X$. Since r is a reflection, $\mathrm{Fix}(r)$ has codimension 1 in V. Hence Lemma 4.6 follows. $\square$

To complete the proof of Proposition 4.5, we know (by Theorem 2.12(ii)) that $E_2 = xE_1$ for some $x \in G$. Write $x = r_1 r_2 \ldots r_m$ for some reflections $r_i \in G$. If for some i, r_i normalises $r_{i+1}r_{i+2} \ldots r_m E_1$, delete it from the expression for x. In other words, setting $\hat{x} = r_1 r_2 \ldots r_{i-1} r_{i+1} \ldots r_m$, it remains true that $E_2 = \hat{x}E_1$. Hence we can assume that in the expression for x, r_i does not normalise $r_{i+1}r_{i+2} \ldots r_m E_1$ for any i. Let $Y_1 = E_1$, $Y_{2i+1} = r_{m-i+1}r_{m-i+2} \ldots r_m E_1$ $(1 \leqslant i \leqslant m)$, and $Y_{2i} = Y_{2i-1} \cap Y_{2i+1}$ $(1 \leqslant i \leqslant m)$. Denote the product $r_{m-i+1}r_{m-i+2} \ldots r_m$ by x_i $(1 \leqslant i \leqslant m)$. Thus $x = x_m$.

It is clear that, for $1 \leqslant i \leqslant m$, $Y_{2i+1} = x_i E_1 = x_i V(\gamma g, \zeta) = V(x_i \gamma g x_i^{-1}, \zeta) \in \widetilde{\mathcal{S}}^V_\zeta(\gamma G)$, since $\gamma \in N_{GL(V)}(G)$. Also $\dim(Y_{2i+1}) = \dim(E_1)$, since x_i is invertible. Hence for $1 \leqslant i \leqslant m$, Y_{2i+1} is a maximal eigenspace. Also, by Corollary 3.2, $Y_{2i} = Y_{2i-1} \cap Y_{2i+1} \in \widetilde{\mathcal{S}}^V_\zeta(\gamma G)$. Note the requirement in the statement of Proposition 4.5 that $l(\widetilde{\mathcal{S}}^V_\zeta(\gamma G)) > 0$, since otherwise Y_{2i} is the zero space, and does not lie in $\widetilde{\mathcal{S}}^V_\zeta(\gamma G)$ by definition. Given that $l(\widetilde{\mathcal{S}}^V_\zeta(\gamma G)) > 0$, Lemma 4.6 guarantees that $Y_{2i} \neq \{0\}$, and that Y_{2i} covers both Y_{2i-1} and Y_{2i+1} $(1 \leqslant i \leqslant m)$.

Proposition 4.5 now follows. $\square$

To complete the proof that $(4.2) \Rightarrow (4.1)$, consider the case $l(\widetilde{\mathcal{S}}^V_\zeta(\gamma G)) = 1$. In this case, nonzero homology can exist only in degrees 0 and 1. By Proposition 4.5, $\widetilde{H}_0(\widetilde{\mathcal{S}}^V_\zeta(\gamma G)) = 0$, and hence Theorem 4.1 holds. Thus $(4.2) \Rightarrow (4.1)$, as claimed.

$(4.3) \Rightarrow (4.2)$. By Theorem 1.2, when $G = G(r, p, n)$ the poset $\mathcal{S}^V_\zeta(\gamma G(r, p, n))$ is Cohen-Macaulay over $\mathbb{Z}$. Hence so is $\widetilde{\mathcal{S}}^V_\zeta(\gamma G(r, p, n))$, by Corollary 2.3. Thus Theorem 4.1 holds immediately in these cases. This completes the proof of Theorem 4.4. $\square$

Hence, in order to prove Theorem 4.1, it suffices to prove Proposition 4.3. That is, we may assume in the statement of Theorem 4.1 that $l(\widetilde{\mathcal{S}}^V_\zeta(\gamma G)) \geqslant 2$ and that G is an exceptional irreducible reflection group.

Let V be a complex vector space of dimension n, G a reflection group acting irreducibly on V, and γG be a reflection coset in V. See §2.9 for a definition of the factors ε_i of (G, γ). Recall that for $\zeta \in \mathbb{C}^\times$ a root of unity, we define

$$A(\zeta) = \{i \mid \varepsilon_i \zeta^{d_i} = 1\},$$

where d_i is the degree of the basic invariant corresponding to ε_i. Note that there are exactly n basic invariants. Let $a(\zeta) = |A(\zeta)|$.

Proposition 4.7. *With the notation of the above paragraph, it suffices to check Proposition 4.3 for all (G, γ, V) such that G acts irreducibly on V, $a(\zeta) \geqslant 3$ and $a(\zeta) \neq n$.*

Proof of Proposition 4.7. If $l(\widetilde{\mathcal{S}}_\zeta^V(\gamma G)) \geqslant 2$, then the dimension of a maximal eigenspace is at least 3. By Theorem 2.12(i) the dimension of a maximal eigenspace is equal to $a(\zeta)$ and it follows immediately that $a(\zeta) \geqslant 3$..

If $a(\zeta) = n$, then applying Theorem 2.12(i) again, the full space V is the unique maximal ζ-eigenspace. Thus $V = V(\gamma g, \zeta)$ for some $g \in G$. But then for any $V(\gamma h, \zeta) \in \widetilde{\mathcal{S}}_\zeta^V(\gamma G)$, $V(\gamma h, \zeta) = V((\gamma g)^{-1}\gamma h, 1) = V(g^{-1}h, 1)$. Hence $\widetilde{\mathcal{S}}_\zeta^V(\gamma G) = \widetilde{\mathcal{S}}_1^V(G)$, which is a geometric lattice, and therefore is Cohen-Macaulay over $\mathbb{Z}$. Hence we can assume that $a(\zeta) \neq n$, and the proof of Proposition 4.7 is complete. $\square$

It remains to check all triples (G, γ, V) satisfying the conditions of Proposition 4.7. First consider the special case $\gamma = \mathrm{Id}$. In this case $\varepsilon_i = 1$ for all i.

A list of irreducible reflection groups and their degrees is given, for example, in [19, Table D.3, p.275]. An inspection of this table provides the following cases for consideration. Here $|\zeta|$ is the order of ζ, not its modulus.

(i) $G = K_5, |\zeta| = 3$
(ii) $G = K_5, |\zeta| = 6$
(iii) $G = E_6, |\zeta| = 2$
(iv) $G = E_6, |\zeta| = 3$
(v) $G = E_7, |\zeta| = 3$
(vi) $G = E_7, |\zeta| = 6$
(vii) $G = E_8, |\zeta| = 3$
(viii) $G = E_8, |\zeta| = 4$
(ix) $G = E_8, |\zeta| = 6$

We note immediately that for cases (i) & (ii), primitive 3rd roots of unity are precisely the negatives of primitive 6th roots of unity. Also $(-1)\,\mathrm{Id} \in K_5$, so in fact the posets $\widetilde{\mathcal{S}}_\zeta^V(G)$ are identical in these cases. The same argument applies to cases (v) & (vi), and (vii) & (ix).

As described in the Introduction, the reduced homology of these posets was computed using MAGMA ([5], [24]), the GAP package 'Simplicial Homology' ([13], [12], [10], [30]), and (for posets (vii), (viii) and (ix)), RedHom ([32], [8]). In particular, MAGMA was used to generate the posets under consideration, while GAP and Red-Hom actually performed the homology calculations. Here ω denotes a primitive 3rd

root of unity, while i denotes a primitive 4th root of unity.

$$\widetilde{H}_j(\widetilde{\mathcal{S}}_\omega^{\mathbb{C}^5}(K_5), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{364}, & \text{if } j = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\widetilde{H}_j(\widetilde{\mathcal{S}}_{-1}^{\mathbb{C}^6}(E_6), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{475}, & \text{if } j = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\widetilde{H}_j(\widetilde{\mathcal{S}}_\omega^{\mathbb{C}^6}(E_6), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{649}, & \text{if } j = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\widetilde{H}_j(\widetilde{\mathcal{S}}_\omega^{\mathbb{C}^7}(E_7), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{87\,751}, & \text{if } j = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\widetilde{H}_j(\widetilde{\mathcal{S}}_\omega^{\mathbb{C}^8}(E_8), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{723\,681} & \text{if } j = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\widetilde{H}_j(\widetilde{\mathcal{S}}_i^{\mathbb{C}^8}(E_8), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{21\,888\,721} & \text{if } j = 3 \\ 0, & \text{otherwise} \end{cases}$$

In each case, reduced homology is zero except in top possible dimension. This completes the proof of Proposition 4.3 in the case $\gamma = \mathrm{Id}$.

Now consider the case of arbitrary γ. In [19, Table D.5, p.278], there is a list of reflection cosets (G, γ_0). Recall (see the discussion following Lemma 3.15) that if (G, γ) is any reflection coset where G acts irreducibly, then either γ induces an inner automorphism, or else $(G, \gamma) = (G, \alpha\gamma_0)$, where (G, γ_0) appears in this table, and $\alpha \in \mathbb{C}^\times$. (See also [7, Proposition 3.13]).

If γ induces an inner automorphism, then as in Lemma 3.15 we may assume that γ is a scalar multiple of the identity. If γ is a scalar multiple of the identity, then $\widetilde{\mathcal{S}}_\zeta^V(\gamma G) = \widetilde{\mathcal{S}}_{\gamma^{-1}\zeta}(G)$. By the result for the $\gamma = \mathrm{Id}$ case, Proposition 4.3 holds.

Hence we may assume that $(G, \gamma) = (G, \alpha\gamma_0)$, where (G, γ_0) can be found in [19, Table D.5, p.278], and $\alpha \in \mathbb{C}^\times$. Since $\widetilde{\mathcal{S}}_\zeta^V(\alpha\gamma_0 G) = \widetilde{\mathcal{S}}_{\alpha^{-1}\zeta}^V(\gamma_0 G)$ as in the paragraph above, it suffices to consider the case $\alpha = 1$.

Applying Proposition 4.7, and inspecting [19, Table D.5, p.278], there are only two cases to consider:

$$\text{(i)} \quad G = G(2,2,4), \quad \gamma = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\text{(ii)} \quad G = G_{28}, \quad \gamma = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

The first of these cases is covered by Theorem 1.2, whose proof appears in [16]. For completeness, the proof is included here.

In case (i), we have $d_i = 2, 4, 4, 6$ and $\varepsilon_i = 1, \omega, \omega^2, 1$. For a counterexample in this case it would be necessary to find a ζ such that exactly 3 of $\{\zeta^2, \zeta^4\omega, \zeta^4\omega^2, \zeta^6\}$ are

equal to 1. Certainly $\zeta^4\omega$ and $\zeta^4\omega^2$ are always unequal. This forces $\zeta^2 = 1$, in which case neither $\zeta^4\omega$ nor $\zeta^4\omega^2$ are equal to 1. Hence no such counterexample exists in this case.

For (ii), we have $d_i = 2, 6, 8, 12$ and $\varepsilon_i = 1, -1, 1, -1$. For a counterexample here we would need to find a ζ such that exactly 3 of $\{\zeta^2, -\zeta^6, \zeta^8, -\zeta^{12}\}$ are equal to 1. Now if $\zeta^2 = 1$ then both $-\zeta^6$ and $-\zeta^{12}$ are equal to -1. Hence $-\zeta^6 = 1$ and $\zeta^8 = 1$. Thus $\zeta^2 = -1$. But then $-\zeta^{12} = -1$, and so at most two of $\{\zeta^2, -\zeta^6, \zeta^8, -\zeta^{12}\}$ are equal to 1. Thus no counterexample can be found here either. This completes the proof of Proposition 4.3 and Theorem 4.1. □

5. THE CM PROPERTY FOR $\mathcal{S}_\zeta^V(\gamma G)$

In this section we examine the Cohen-Macaulay property for the poset $\mathcal{S}_\zeta^V(\gamma G)$, and prove Theorem 1.1.

The first step in the proof of Theorem 1.1 is a reduction to the case where V is irreducible as a G-module. Although Theorem 1.1 concerns reflection groups, we will make use of the following lemma for reflection cosets. G, γ and V all have their usual meaning.

Lemma 5.1. *If $\mathcal{S}_\zeta^V(\gamma G)$ is Cohen-Macaulay over $\mathbb{Z}$ for all (G, γ, V) such that V is irreducible as a $\widehat{G} = \langle G, \gamma \rangle$-module, then $\mathcal{S}_\zeta^V(\gamma G)$ is Cohen-Macaulay over $\mathbb{Z}$ for all (G, γ, V).*

Proof. Suppose $V = V_1 \oplus \ldots \oplus V_k$, where each V_i is an irreducible $\langle G, \gamma \rangle$-module. Then $G = G_1 \times \cdots \times G_k$, and $\gamma = \gamma_1 \oplus \cdots \oplus \gamma_k$, where G_i acts a reflection group in V_i and trivially on V_j for $j \neq i$, and $\gamma_i \in GL(V_i)$ normalises G_i. Suppose $g = (g_1, g_2, \ldots, g_k) \in G$, and $v = v_1 + \cdots + v_k \in V$. Denote by γ_i' the restriction of γ_i to V_i. Then $v \in V(\gamma g, \zeta)$ if and only if $v_i \in V_i(\gamma_i' g_i, \zeta)$ for each i. Hence $\mathcal{S}_\zeta^V(\gamma G) \cong \mathcal{S}_\zeta^{V_1}(\gamma_1 G_1) \times \cdots \times \mathcal{S}_\zeta^{V_k}(\gamma_k G_k)$. The result now follows from [2, Theorem 7.1] (thanks to Alex Miller for pointing out this theorem[1]). □

Noting that when $\gamma = \text{Id}$, the proof of Lemma 5.1 never leaves the class of reflection groups, Theorem 1.1 will follow if we can show that $\mathcal{S}_\zeta^V(G)$ is Cohen-Macaulay over $\mathbb{Z}$ whenever V is irreducible as a G-module.

By Theorem 1.2, it suffices to show that $\mathcal{S}_\zeta^V(G)$ is Cohen-Macaulay over $\mathbb{Z}$ whenever G is an exceptional irreducible reflection group.

By Corollary 2.3, it is equivalent to show that $\widetilde{\mathcal{S}}_\zeta^V(G)$ is Cohen-Macaulay over $\mathbb{Z}$ for all such G.

To do this, we make use of Lemma 2.1. The statement that $\widetilde{H}_i(\widetilde{\mathcal{S}}_\zeta^V(G), \mathbb{Z}) = 0$ for $i \neq l(\widetilde{\mathcal{S}}_\zeta^V(G))$ is just the case $\gamma = \text{Id}$ of Theorem 4.1. Now choose $X = V(h, \zeta) \in \widetilde{\mathcal{S}}_\zeta^V(G)$ and consider $\text{lk}(X) = \widetilde{\mathcal{S}}_\zeta^V(G)_{<X} * \widetilde{\mathcal{S}}_\zeta^V(G)_{>X}$. By Lemma 2.1 it will suffice to show that $\text{lk}(X)$ is CM. By Proposition 2.2 it is enough to show that each of $\widetilde{\mathcal{S}}_\zeta^V(G)_{<X}$ and $\widetilde{\mathcal{S}}_\zeta^V(G)_{>X}$ is CM.

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[1]Theorem 1.1 was originally proved over $\mathbb{F}$. The use of this theorem of Baclawski affords a proof over $\mathbb{Z}$.

Examining $\widetilde{\mathcal{S}}_\zeta^V(G)_{>X}$ first, we know that X is contained in some maximal eigenspace E. Hence $\mathcal{S}_\zeta^V(G)_{\geqslant X}$ is a principal upper order ideal of $[E, \hat{1}]$, which is a geometric lattice by Corollary 3.5. Hence $\mathcal{S}_\zeta^V(G)_{\geqslant X}$ is a geometric lattice too (see [37, Lecture 3]), and so is certainly CM. Removing unique minimal and maximal elements from a poset does not affect whether the poset is CM or not, by Corollary 2.3. Thus $\widetilde{\mathcal{S}}_\zeta^V(G)_{>X}$ is CM too.

As for $\widetilde{\mathcal{S}}_\zeta^V(G)_{\leqslant X}$, a similar argument to that used in Lemma 3.19 shows that $\widetilde{\mathcal{S}}_\zeta^V(G)_{\leqslant X} \cong \widetilde{\mathcal{S}}_\zeta^V(hG_X) \cong \widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$. The last isomorphism holds because every element of hG_X actis by multiplication by ζ on X. Note that hG_X is a genuine reflection coset, not just a refection group, since g acts by multiplication by ζ on X.

Now if $l(\widetilde{\mathcal{S}}_\zeta^V(G)_{\leqslant X}) \leqslant 1$ then $\widetilde{\mathcal{S}}_\zeta^V(G)_{<X}$ is trivially Cohen-Macaulay.

If $l(\widetilde{\mathcal{S}}_\zeta^V(G)_{\leqslant X}) = 2$ then $l(\widetilde{\mathcal{S}}_\zeta^V(G)_{<X}) = 1$, and $\widetilde{\mathcal{S}}_\zeta^V(G)_{<X}$ is Cohen-Macaulay since it is connected, by Proposition 4.5.

A list of irreducible reflection groups and their degrees is given in [19, Table D.3, p.275]. An inspection of this table reveals that the only other possibility is $l(\widetilde{\mathcal{S}}_\zeta^{V/X}(G)_{\leqslant X}) = 3$, when X is an eigenspace of dimension 1, and maximal eigenspaces have dimension 4. This occurs for only three posets: $\widetilde{\mathcal{S}}_{-1}^{\mathbb{C}^6}(E_6)$, $\widetilde{\mathcal{S}}_\omega^{\mathbb{C}^8} E_8)$ and $\widetilde{\mathcal{S}}_i^{\mathbb{C}^8}(E_8)$, where ω and i are respectively primitive 3rd and 4th roots of unity.

Now if V/X is reducible as a $\widehat{G} = \langle G, h \rangle$-module, then Lemma 5.1 shows that $\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$ is the product of smaller posets of length less than or equal to 2, which must be Cohen Macaulay by the arguments used above. Hence $\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$ is Cohen-Macaulay, by [2, Theorem 7.1]. Thus we may assume that V/X is irreducible as a $\widehat{G} = \langle G, h \rangle$-module.

If V/X is irreducible as a G-module, then $\widetilde{H}_i(\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X), \mathbb{Z}) = 0$ if $i \neq l(\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X))$, by Theorem 4.1. Lower order ideals of $\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$ have length less than or equal to 2, and hence are Cohen-Macaulay by the arguments used above. Upper order ideals of $\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$ are also Cohen-Macaulay by Corollary 3.5. Hence if V/X is irreducible as a G-module, then $\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$ is Cohen-Macaulay as required.

So assume $\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$ is reducible as a G-module. For the three cases we are considering, V/X has dimenion either 5 or 7. Since V/X is the direct sum of isomorphic vector spaces V_i, each of these vector spaces must have dimension 1.

But then by Lemma 3.14. maximal eigenspaces in $\widetilde{\mathcal{S}}_\zeta^{V/X}(hG_X)$ must have dimension at most 1, which contradicts the assumption that $l(\widetilde{\mathcal{S}}_\zeta^{V/X}(G)_{\leqslant X}) = 3$.

This completes the proof of Theorem 1.1.

Remark 5.2. The question as to whether $\mathcal{S}_\zeta^V(\gamma G)$ is Cohen-Macaulay over $\mathbb{Z}$ for all reflection cosets γG is open. By Lemma 5.1, it would suffice to prove this result for all (G, γ, V) such that V is irreducible as a $\widehat{G} = \langle G, \gamma \rangle$-module, but this is unknown at this stage.

6. Acknowledgements

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