SECTIONS OF SURFACE BUNDLES

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Abstract. A bundle with base B and fibre F aspherical closed surfaces has a section if and only if the action : $\pi_1(B) \to Out(\pi_1(F))$ factors through $Aut(\pi_1(F))$ and a cohomology class is 0. We simplify and make more explicit the latter condition.

Let $p : E \to B$ be a bundle projection, with connected base B and fibre F, and let $\pi = \pi_1(E)$, $\beta = \pi_1(B)$ and $\phi = \pi_1(F)$. If the bundle has a section $s : B \to E$ with $ps = id_B$ then the exact sequence of homotopy for the projection gives an extension

$$\xi(p) : \quad 1 \to \phi \to \pi \to \beta \to 1,$$

and the projection $p_* : \pi \to \beta$ splits. In general, an epimorphism $\pi \to \beta$ splits if and only if the action $\theta : \beta \to Out(\phi)$ induced by conjugation in π factors through a homomorphism $\widetilde{\theta} : \beta \to Aut(\phi)$ and the cohomology class $[\xi] \in H^2(\beta; \zeta\phi)$ of the extension is 0. (Here $\zeta\phi$ is the centre of ϕ, considered as a $\mathbb{Z}[\beta]$-module via the action θ.) If so, then π is a semidirect product $\phi \rtimes_{\widetilde{\theta}} \beta$. If the base and fibre are aspherical surfaces, the bundle is determined by the extension (see Chapter V of [2]), and so it has a section if these conditions hold.

In this note we shall make the cohomological condition more explicit. We shall assume always that surfaces are compact and connected, and have no boundary.

1. extensions of groups

Let ζG, G' and $I(G)$ denote the centre, the commutator subgroup and the isolator subgroup of a group G, respectively. (Thus $G' \leq I(G)$ and $G/I(G)$ is the maximal torsion-free quotient of the abelianization $G^{ab} = G/G'$.) If H is a subgroup of G let $C_G(H)$ be the centralizer of H in G. Let c_g denote conjugation by g, for all $g \in G$.

There is a natural restriction homomorphism from $Aut(G)$ to $Aut(\zeta G)$, which factors through $Out(G)$. In particular, if $\theta : \beta \to Out(\phi)$ is

1991 *Mathematics Subject Classification.* 20J05, 57N13.

Key words and phrases. factor set, section, surface bundle.

a homomorphism then composition with restriction defines a natural $\mathbb{Z}[\beta]$-module structure on $\zeta\phi$. The extensions

$$1 \to \phi \to \pi \to \beta \to 1$$

with given action θ may be parametrized by $H^2(\beta;\zeta\phi)$. (In general, there is an obstruction in $H^3(\beta;\zeta\phi)$ for there to be such an extension, but this obstruction group is trivial when β is a surface group. See Chapter IV of [1].) If θ factors through $Aut(\phi)$ then the semidirect product corresponds to $0 \in H^2(\beta;\zeta\phi)$.

Lemma 1. *If $\zeta\phi = 1$ then $p_* : \pi \to \beta$ splits if and only if the action θ factors through $Aut(\phi)$.*

Proof. If ϕ has trivial centre then the extension is determined by the action, since $H^2(\beta;\zeta\phi) = 0$. Thus if the action factors π must be a semidirect product, i.e., p_* splits. The converse is clear. □

The exact sequence of low degree for the extension has the form

$$H_2(\pi;\mathbb{Z}) \to H_2(\beta;\mathbb{Z}) \to H_0(\beta;H_1(\phi;\mathbb{Z})) \to H_1(\pi;\mathbb{Z}) \to H_1(\beta;\mathbb{Z}) \to 0.$$

If the extension splits this gives an isomorphism

$$\pi^{ab} \cong (\phi^{ab}/(I-\theta^{ab})\phi^{ab}) \oplus \beta^{ab},$$

where θ^{ab} is the automorphism of ϕ^{ab} induced by θ.

If ϕ is abelian then the transgression from $H_2(\beta;\mathbb{Z})$ to $H_0(\beta;H_1(\phi;\mathbb{Z}))$ in the exact sequence of low degree is the image of the extension class $[\xi]$ under the homomorphisms

$$H^2(\beta;\phi) \to H^2(\beta;H_0(\beta;\phi)) \to Hom(H_2(\beta;\mathbb{Z}), H_0(\beta;\phi))$$

given by change of coefficients and evaluation. (See Theorem 4 of [4] for the cohomological version.)

2. aspherical base

We shall assume henceforth that β is a PD_2-group. Let $\langle X \mid r\rangle$ be a 1-relator presentation for β, and let $q : F(X) \to \beta$ be the associated epimorphism. Let $w = w_1(\beta)$, let $\varepsilon_w : \mathbb{Z}[\beta] \to \mathbb{Z}$ be the w-twisted augmentation, defined by the linear extension of $w : \beta \to \mathbb{Z}^\times$, and let $J_w = \mathrm{Ker}(\varepsilon_w)$. Let $\partial_x : \mathbb{Z}[F(X)] \to \mathbb{Z}[\beta]$ be the composite of the Fox free derivative with the linear extension of q. Then the left ideal in $\mathbb{Z}[\beta]$ generated by $\{\partial_x r | x \in X\}$ is J_w, and $H^2(\beta;A) \cong H_0(\beta;\mathbb{Z}^w \otimes A) = A/J_w A$ for any $\mathbb{Z}[\beta]$-module A.

If $\theta : \beta \to Out(\phi)$ is a homomorphism a choice of lifts $\psi(x) \in Aut(\phi)$ for the values $\theta(q(x))$ determines a homomorphism $\psi : F(X) \to Aut(\phi)$ which lifts θq, and hence a semi-direct product $G = \phi \rtimes_\psi F(X)$. Since

$\theta(q(r)) = 1$, we must have $\psi(r) = c_g$, for some $g \in \phi$. Then $\pi = G/\langle\langle rg^{-1}\rangle\rangle$ is an extension of β by ϕ which realizes the action θ. (If $\zeta\phi = 1$ then g is uniquely determined by ψ.)

Suppose that θ lifts to a homomorphism $\widetilde{\theta} : \beta \to Aut(\phi)$. Then we may set $\psi = \widetilde{\theta}q$. For each $x \in X$ choose $s(x) \in \pi$ such that $p(s(x)) = q(x)$ and $c_{s(x)} = \psi(x)$. Then s extends to a homomorphism $s : F(X) \to \pi$ such that $ps = q$. Hence s extends to an epimorphism $S : G \to \pi$, giving a commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \phi & \longrightarrow & G & \longrightarrow & F(X) & \longrightarrow & 1 \\ \downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle S}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow \\ 1 & \longrightarrow & \phi & \longrightarrow & \pi & \xrightarrow{p} & \beta & \longrightarrow & 1. \end{array}$$

The image of r in G is in $C_G(\phi)$, since $q(r) = 1$. If $s' : F(X) \to \pi$ is another such homomorphism then $f(x) = s'(x)s(x)^{-1} \in \zeta\phi$, for all $x \in X$. Conversely, any function $f : X \to \zeta\phi$ may be realized as the difference $s's^{-1}$ of two such homomorphisms.

Lemma 2. *The element $s(r)$ is in $\zeta\phi$, and its image $[s(r)] \in \zeta\phi/J_w\zeta\phi$ is well defined. The epimorphism p_* splits if and only if $[s(r)] = 0$.*

Proof. The first assertion holds since $q(r) = 1$. If $s'(x) = \alpha s(x)$ (for some $\alpha \in \zeta\phi$) and $s'(y) = s(y)$ for $y \neq x$, then $s'(r) = s(r) + (\partial_x r)\alpha$. It follows easily that $[s(r)]$ is independent of the choice of s.

If $\sigma : \beta \to \pi$ splits p_* then we may take $s = \sigma q$, and so $s(r) = 1$ in ϕ. Hence $[s(r)] = 0$. Conversely, if $[s(r)] = 0$ then we may choose s so that $s(r) = 1$, and so p_* splits. □

In fact $[s(r)] = [\xi(p)]$. Although we shall not need to know this, we sketch an argument which holds for any group β with a finite presentation $\langle X|R\rangle$. After introducing new generators x' and new relators $x'x$, if necessary, we may assume that the exponents of the generators in each relator are all positive. The presentation determines a Fox-Lyndon partial resolution C_*^{FL} for the augmentation β-module $\mathbb{Z}$. Let C_*^{bar} be the normalized bar resolution for $\mathbb{Z}$, and let $h_* : C_*^{FL} \to C_*^{bar}$ be the chain morphism given by the identity on $C_0^{FL} = \mathbb{Z}[\beta] = C_0^{bar}$, the natural inclusion of $C_1^{FL} = \mathbb{Z}[\beta]^X$ into C_1^{bar}, and which sends the generator e_r of C_2^{FL} corresponding to the relator $r \in R$ to $\Sigma_{x\in X}[\partial_x r|x] \in C_2^{bar}$. (See Exercises II.5.3 and II.5.4 of [1]. If $c.d.\beta \leq 2$ then C_*^{FL} is a resolution and h is a chain homotopy equivalence.))

Let $\sigma : \beta \to \pi$ be a set-theoretic section such that $\sigma(1) = 1$, and let $s : F(X) \to \pi$ be the homomorphism defined by $s(x) = \sigma(q(x))$ for all $x \in X$. The class $[\xi(p)]$ is represented by the 2-cocycle f defined by $\sigma(g)\sigma(h) = f(g,h)\sigma(gh)$ for all $g, h \in \beta$. (See Chapter IV.3 of [1].)

Let $r = \Pi_{i=1}^{c} x_i$ and let $I_k = \Pi_{i=1}^{k-1} x_i$, for $1 \leq k \leq c$. (There may be repetitions amongst the generators x_i.) Then $\partial_x r = \Sigma_{x_i=x} I_i$, for all $x \in X$, and so

$$f(h_2(e_r)) = f(\Sigma_{x\in X}[\partial_x r|x]) = f(\Sigma_{i=1}^{c}[I_i|x_i]) = \Sigma_{i=1}^{c} f(I_i, x_i).$$

On the other hand,

$$s(r) = \Pi_{i=1}^{c} s(x_i) = \Pi_{i=1}^{c} \sigma(q(x_i)).$$

A simple induction shows that this is

$$\Pi_{i=1}^{c} f(I_i, x_i)\sigma(q(r)) = \Pi_{i=1}^{c} f(I_i, x_i)\sigma(1) = \Pi_{i=1}^{c} f(I_i, x_i).$$

In additive notation, this is just $f(h_2(e_r))$. With a little more effort, we could avoid the assumption that the exponents in the relators are all positive. In particular, we may conclude that if β is a PD_2-group with a 1-relator presentation $\langle X|r\rangle$ then $h^*[\xi(p)] = [s(r)]$.

Lemma 3. *Let G be a group with a finitely generated abelian normal subgroup A such that $\beta = G/A$ is a PD_2^+-group. Then the canonical projection from G to β has a section if and only if*

$$G^{ab} \cong A/[G, A] \oplus \beta^{ab}.$$

Proof. Let $\overline{A} = A/[G, A]$ and $\overline{G} = G/[G, A]$. Then $\overline{G}$ is a central extension of β by $\overline{A}$, and $G^{ab} = \overline{G}^{ab}$. Since $c.d.\beta = 2$, the epimorphism from A to $\overline{A}$ induces an epimorphism from $H^2(\beta; A)$ to $H^2(\beta; \overline{A})$. Since β is a PD_2^+-group, $H^2(\beta; A) \cong H_0(\beta; A)$ and $H^2(\beta; \overline{A}) \cong H_0(\beta; \overline{A})$. These are each isomorphic to $\overline{A}$, and so the natural homomorphism from $H^2(\beta; A)$ to $H^2(\beta; \overline{A})$ is an isomorphism. Therefore G splits as a semidirect product if and only if the same is true for $\overline{G}$. Since $\overline{A}$ is central in $\overline{G}$, this is so if and only if $\overline{G} \cong \overline{A} \times \beta$, and this is equivalent to $\overline{G}^{ab} \cong \overline{A} \oplus \beta^{ab}$. □

(Since $H_2(\beta; \mathbb{Z}) \cong \mathbb{Z}$, this lemma also follows from the observation at the end of §1 above.)

3. Surface bundles with aspherical base and flat fibre

The group $\kappa = \pi_1(Kb)$ has a presentation $\langle x, y \mid xyx^{-1} = y^{-1}\rangle$, and $\zeta\kappa$ is generated by the image of x^2. Let α and γ be the automorphisms determined by $\alpha(x) = x^{-1}$, $\gamma(x) = xy$ and $\alpha(y) = \gamma(y) = y$. Then $Aut(\kappa)$ is generated by α, γ and c_x, and $\gamma^2 = c_y$. It is easily verified that $\alpha\gamma = \gamma\alpha$, and so $Out(\kappa) \cong (Z/2Z)^2$ is the image of an abelian subgroup of $Aut(\kappa)$.

Let $\beta_+ = \mathrm{Ker}(w)$ and let B^+ be the associated orientable covering space of B. If $p : E \to B$ is a bundle, let $p^+ : E^+ \to B^+$ be the induced bundle, and let $\pi^+ = \mathrm{Ker}(wp_*)$.

Theorem 4. *Let $p : E \to B$ be a bundle with base B an aspherical surface and fibre $F = T$ or Kb. Then p has a section if and only if θ factors through $Aut(\phi)$ and $[s(r)] = 0$. In particular, p has a section if either*

(1) $F = T$, B is orientable and

$$H_1(E;\mathbb{Z}) \cong H_0(B; H_1(F;\mathbb{Z})) \oplus H_1(B;\mathbb{Z}); or$$

(2) $F = Kb$, θ factors through $Aut(\kappa)$, β acts on $\zeta\phi$ through $w_1(\beta)$, and $\beta_1(\pi^+) = \beta_1(\beta^+) + 1$.

The ϕ-conjugacy classes of sections are parametrized by $H^1(\beta;\zeta\phi)$.

Proof. The first assertion follows from Lemmas 1 and 2.

If $F = T$ then ϕ is abelian, and so $Aut(\phi) = Out(\phi)$. Hence p_* splits if and only if $[s(r)] = 0$. If B is orientable then Lemma 3 gives the more explicit criterion of (1).

If $F \cong Kb$ then $\phi \cong \kappa$, and $\zeta\phi \cong \mathbb{Z}$, and p_* splits if and only if the action factors through $Aut(\kappa)$ and $[s(r)] = 0$. If β acts on $\zeta\phi$ through $w_1(\beta)$ we can make this more explicit. For then $H^2(\beta;\zeta\phi)$ maps injectively to $H^2(\beta^+;\zeta\phi) \cong \mathbb{Z}$ under passage to β^+. Thus p_* splits if and only if θ factors through $Aut(\kappa)$ and the restriction to $p_*^{-1}(\beta^+)$ splits. (If β is orientable then β/β' is a free abelian group, and so every homomorphism $\theta : \beta \to Out(\kappa)$ factors through $Aut(\kappa)$.) Since $\zeta\phi$ maps injectively to $\phi/I(\phi)$, $H^2(\beta^+;\zeta\phi)$ in turn maps injectively to a subgroup of index 2 in $H^2(\beta^+;\phi/I(\phi)) \cong \mathbb{Z}$. The image of $[\xi(p)]$ is the class of the extension

$$1 \to \phi/I(\phi) \to \pi^+/I(\phi) \to \beta^+ \to 1,$$

and so (2) follows from Lemma 3, since $I(\phi) < \pi^{+\prime}$, $\phi/I(\phi) \cong \mathbb{Z}$ and $\beta^+/\beta^{+\prime}$ is free abelian.

If p_* splits and s and s' are two sections determining the same lift $\widetilde{\theta}$ then $s'(g)s(g)^{-1}$ is in $\zeta\phi$, for all $g \in \beta$. Therefore the sections are parametrized (up to conjugation by an element of ϕ) by $H^1(\beta;\zeta\phi)$. (See Proposition IV.2.3 of [1] for the cases with ϕ abelian.) □

If p has a section then so does p^+. The converse also holds if $F = T$ and $H^2(\beta;\phi) \cong H_0(\beta;\mathbb{Z}^w \otimes \phi)$ has no 2-torsion. For then restriction to $H^2(\beta^+;\phi)$ is injective, since composition with the transfer is multiplication by 2. (See §9 of Chapter III of [1].)

Examples. Let π be a discrete cocompact subgroup of $Nil^3 \times \mathbb{R}$. Then $\zeta\pi \cong \mathbb{Z}^2$ and $\pi/\zeta\pi \cong \mathbb{Z}^2$, and so the coset space $E = \pi\backslash Nil^3 \times \mathbb{R}$ is the total space of a T-bundle over T. The action is trivial, and so the split extension is the product $\mathbb{Z}^4$. Thus the bundle projection for this coset space has no section. (In fact, π/π' has rank 2, rather than 4, and so the criterion of (1) fails.) Similarly, coset spaces of discrete cocompact subgroups of Nil^4 are T-bundles over T without sections.

The group with presentation

$$\langle u, v, x, y \mid u, v \leftrightharpoons x, y,\ [u, v] = x^2,\ xyx^{-1} = y^{-1}\rangle$$

is the group of a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold which fibres over T with fibre Kb. The base group acts trivially on the fibre, but $\beta_1(\pi) = 2$, rather than 3, and so the bundle does not have a section.

The group with presentation

$$\langle u, v, x, y \mid u \leftrightharpoons x, y,\ vxv^{-1} = x^{-1},\ vy = yv,\ [u, v] = x^2,\ xyx^{-1} = y^{-1}\rangle$$

is the group of a flat 4-manifold which fibres over T with fibre Kb. In this case $H^2(\beta; \zeta\phi) = Z/2Z$, but $[s(r)] \neq 0$, and so the bundle does not have a section.

4. Bundles with hyperbolic fibre

If $\chi(F) < 0$ then p has a section if and only if the action θ factors through $Aut(\phi)$, and the section is unique up to conjugation by an element of ϕ. If p has a section then $H^*(\beta; R)$ is a retract of $H^*(\pi; R)$, for any coefficient ring R, and all quotients of β by terms of the lower central series (or, more generally, by verbal subgroups) are retracts of corresponding quotients of π. In particular, π/ϕ' must split as a semidirect product $(\phi/\phi') \rtimes \beta$.

When F is orientable, this is close to a result of Morita [6]. Suppose F has genus $g \geq 2$, and fix an orientation. There is an associated flat bundle $j(p) : J \to B$, with fibre the $2g$-torus, called the Jacobian bundle of p, and a fibre-preserving inclusion $E \subset J$ which induces an isomorphism on H_1, for each fibre. Let $\mathcal{M}_g = Out(\phi)$. Morita defines a universal class $\mu \in H^2(\mathcal{M}_g; H^1(\phi; \mathbb{Z}))$, and shows that $j(p)$ has a section if and only if $\theta^*\mu = 0$. This is clearly a necessary condition for p itself to have a section. Examining his construction, we see that if f is the 2-cocycle with values in ϕ^{ab} associated to a set-theoretic section $\sigma : \beta \to \pi/\phi'$, as in §1 above, then $\theta^*\mu$ is the image of $[f]$ under the change of coefficient isomorphism induced by the Poincaré duality isomorphism $\phi^{ab} \cong H^1(\phi; \mathbb{Z})$. Thus $j(p)$ has a section if and only if $[s(r)] = 0$ in $\phi^{ab}/(I - \theta^{ab})\phi^{ab}$, where $s : F(X) \to \pi/\phi'$ is as in §1 above. Since $J \simeq K(\pi/\phi', 1)$, this holds if and only if $\pi/\phi' \cong$

$(\phi/\phi') \rtimes \beta$. If, moreover, B is also orientable then this is so if and only if $\pi^{ab} \cong (\phi/[\pi,\phi]) \oplus \beta^{ab}$, by Lemma 3.

We may construct the extension corresponding to an action θ as in §1. However it does not seem easy to construct potential examples with no section. Is there an example for which the Jacobian bundle has no section? In particular, are there such examples with $B = T$?

There is a related, perhaps easier question (Problem 2.17 of [5]). A *multi-section* of p is a surface $C \subset E$ such that $p|_C : C \to B$ is a finite covering projection. In terms of groups, p has a multisection if β has a subgroup γ of finite index such that $\theta|_\gamma$ factors through $Aut(\phi)$. Does every bundle with hyperbolic base and fibre and θ injective admits a multi-section?

5. BUNDLES WITH BASE S^2 OR RP^2

In this final section we consider bundles with aspherical fibre but with spherical base. If $B = S^2$ and $\chi(F) < 0$ or $\chi(F) = 0$ and $\phi \cong \pi$ then p is trivial, and so has a section. (See Theorem 5.19 of [2].)

The characterization of bundles over RP^2 with sections is based on a study of S^2-orbifold bundles. (See [3].)

Theorem 5. *Let F be an aspherical surface. A closed orientable* 4*-manifold M is homotopy equivalent to the total space of an F-bundle over RP^2 with a section if and only if $\pi = \pi_1(M)$ has an element of order* 2*, $\pi_2(M) \cong Z$ and* $\mathrm{Ker}(u) \cong \phi = \pi_1(F)$*, where u is the natural action of π on $\pi_2(M)$.*

Proof. The conditions are clearly necessary. Suppose that they hold. If $\pi \cong \phi \times Z/2Z$ then the bundle is trivial. Thus we may assume that π is not a direct product, and so M is not homotopy equivalent to an RP^2-bundle space. Hence it is homotopy equivalent to the total space E of an S^2-orbifold bundle over a 2-orbifold B [3]. The involution η of F corresponding to the orbifold covering has non-empty fixed point set, since π has torsion. Let $M_{st} = S^2 \times F/\sim$, where $(s,f) \sim (-s,\eta(f))$. Then M_{st} is the total space of an F-bundle over RP^2, and the fixed points of η determine sections of this bundle.

The double cover of E corresponding to κ is an S^2-bundle over F. Since M is orientable and κ acts trivially on $\pi_2(M)$, F must also be orientable and the covering involution of F over B must be orientation-reversing. Since π has torsion ΣB is a non-empty union of reflector curves, and since F is orientable these are "untwisted". Therefore $M \simeq M_{st}$, by Corollary 4.8 of [3]. □

Orientability is used here mainly to ensure that the base orbifold has an untwisted reflector curve.

References

[1] Brown, K. S. *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer-Verlag, Berlin – Heidelberg – New York (1982).

[2] Hillman, J. A. *Four-Manifolds, Geometries and Knots*, Geometry and Topology Monographs 5, Geometry and Topology Publications (2002). (Revision 2007).

[3] Hillman, J. A. S^2-bundles over 2-orbifolds, J. London Math. Soc. 87 (2013), 69–86.

[4] Hochschild, G. and Serre, J.-P. Cohomology of group extensions, Trans. Amer. Math. Soc. 74 (1953), 110–134.

[5] Kirby, R.C. Problems in low-dimensional topology, in *Geometric Topology* (edited by W. Kazez) vol.2, AMS/IP Studies in Advanced Mathematics, American Mathematical Society, Providence (1997), 35–473.

[6] Morita, S. Families of Jacobian manifolds and characteristic classes of surface bundles II, Math. Proc. Cambridge Phil. Soc. 105 (1989), 79–101.

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