Casimir elements from the Brauer–Schur–Weyl duality

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Abstract

We consider Casimir elements for the orthogonal and symplectic Lie algebras constructed with the use of the Brauer algebra. We calculate the images of these elements under the Harish-Chandra isomorphism and thus show that they (together with the Pfaffian-type element in the even orthogonal case) are algebraically independent generators of the centers of the corresponding universal enveloping algebras.

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1 Introduction

It is well-known that the Schur–Weyl duality can be used to get natural constructions of families of Casimir elements for the classical Lie algebras. For some particular choices of parameters the images of such elements under the Harish-Chandra isomorphism can be calculated in an explicit form. For the general linear Lie algebras $\mathfrak{gl}_N$ this leads to an explicit construction of a linear basis of the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. The basis elements are known as the quantum immanants and their Harish-Chandra images are the factorial (or shifted) Schur functions; see [14] and [15]. Constructing quantum immanant-type bases of the centers of the universal enveloping algebras $\mathrm{U}(\mathfrak{o}_N)$ and $\mathrm{U}(\mathfrak{sp}_N)$ for the orthogonal and symplectic Lie algebras remains an open problem; see, however [11], [13], [16] and [17] for some results in that direction.

In this paper we consider generators of the centers of $\mathrm{U}(\mathfrak{o}_N)$ and $\mathrm{U}(\mathfrak{sp}_N)$ obtained by an application of the Brauer–Schur–Weyl duality. They are associated with one-dimensional representations of the Brauer algebra and take the form of some versions of noncommutative determinants and permanents. We give explicit formulas for the Harish-Chandra images of these elements; the images turn out to coincide with the factorial (or double) complete and elementary symmetric functions.

In more detail, we regard $\mathbb{C}^N$ as the vector representation of each of the groups O_N and Sp_N (the latter with even N). The space of tensors

$$\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m} \tag{1.1}$$

carries the diagonal action of each group. By the Schur–Weyl duality, the centralizer of the action of the orthogonal group O_N or symplectic group Sp_N in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ is generated by the homomorphic image of the action of the respective Brauer algebra $\mathcal{B}_m(N)$ or $\mathcal{B}_m(-N)$. Consider multiple tensor products

$$\mathrm{U}(\mathfrak{g}_N) \otimes \underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{m}, \tag{1.2}$$

where $\mathfrak{g}_N$ denotes one of the Lie algebras $\mathfrak{o}_N$ or $\mathfrak{sp}_N$. We let $F = [F_{ij}]$ denote the $N \times N$ matrix whose entries F_{ij} are the standard generators of $\mathfrak{g}_N$; see definitions in Sec. 3. For each $a = 1, \ldots, m$ we denote by F_a the respective element of the algebra (1.2),

$$F_a = \sum_{i,j=1}^{N} F_{ij} \otimes 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)}, \tag{1.3}$$

where $e_{ij} \in \mathrm{End}\,\mathbb{C}^N$ denote the standard matrix units. We regard an arbitrary element C of the respective Brauer algebra $\mathcal{B}_m(N)$ or $\mathcal{B}_m(-N)$ as an operator in the space (1.1).

The adjoint action of the respective group $\mathrm{G}_N = \mathrm{O}_N$ or $\mathrm{G}_N = \mathrm{Sp}_N$ on the corresponding Lie algebra $\mathfrak{g}_N$ amounts to conjugations of the matrix F by elements of the group. Hence, since the action of the group G_N on the space (1.1) commutes with the action of the Brauer algebra, we find that the elements

$$\operatorname{tr} C\,(F_1 + u_1)\dots(F_m + u_m), \tag{1.4}$$

with the trace taken over all m copies of $\operatorname{End}\mathbb{C}^N$, belong to the subalgebra $\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ of G_N-invariants in $\mathrm{U}(\mathfrak{g}_N)$. We are using the notation $F+u$ to indicate the matrix $F+u1$. If $\mathrm{G}_N = \mathrm{O}_N$ with odd N or $\mathrm{G}_N = \mathrm{Sp}_N$ with even N, then the subalgebra $\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ coincides with the center of $\mathrm{U}(\mathfrak{g}_N)$. If $\mathrm{G}_N = \mathrm{O}_N$ with even N, then $\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ is a proper subalgebra of the center which contains an additional Pfaffian-type Casimir element.

Note that the quantum immanants of [14] are elements of the form (1.4), where F should be replaced with the matrix $E = [E_{ij}]$ formed by the basis elements E_{ij} of the general linear Lie algebra, C is a primitive idempotent of $\mathbb{C}[\mathfrak{S}_m]$ associated with a standard tableau and the u_i are contents of the tableau.

In a recent work [9] an explicit construction of generators of the center of the affine vertex algebra $V(\mathfrak{g}_N)$ at the critical level was given. Here $V(\mathfrak{g}_N)$ is the vacuum module over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_N$. Under the evaluation homomorphism the generators of the center of $V(\mathfrak{g}_N)$ get mapped into Casimir elements of the form (1.4). More precisely, in the orthogonal case the images of the central elements defined in [9] have the form of a differential operator

$$\operatorname{tr} S^{(m)}(-\partial_t + F_1 t^{-1})\dots(-\partial_t + F_m t^{-1}), \tag{1.5}$$

whose coefficients are Casimir elements, where $S^{(m)}$ denotes the symmetrizer in the Brauer algebra. In the symplectic case with $N = 2n$ the images of the central elements have the form of (1.5) with the additional factor $(n - m + 1)^{-1}$ and the values of m restricted to $1 \leqslant m \leqslant 2n$. Multiplying the operator (1.5) from the left by t^m we get an expression of the form (1.4) with $C = S^{(m)}$ and the parameters specialized as $u_i = u+m-i$ for $i = 1,\dots,m$, where $u = -t\,\partial_t$.

To identify this family of central elements we calculate their eigenvalues in highest weight representations of $\mathfrak{g}_N$ which is equivalent to finding their images under the Harish-Chandra isomorphism. The key starting point in the orthogonal case $\mathfrak{g}_N = \mathfrak{o}_N$ with $N = 2n$ or $N = 2n+1$ is Theorem 3.3 which implies that the Casimir elements

$$\operatorname{tr} S^{(2k)}(F_1 + k - 1)\dots(F_{2k} - k), \qquad k = 1,\dots,n \tag{1.6}$$

are algebraically independent generators of the algebra of invariants $\mathrm{U}(\mathfrak{o}_N)^{\mathrm{O}_N}$. Their eigenvalues in the highest weight representations with the highest weight $(\lambda_1,\dots,\lambda_n)$ coincide with the factorial complete symmetric functions $h_k(l_1^2,\dots,l_n^2\,|\,a)$, where the l_i are the labels of the highest weight shifted by the half-sum of the positive roots and $a = (a_i)$ is a sequence of parameters; see Sec. 3.1. In the symplectic case $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ the Casimir elements defined by (1.6) with a normalization factor are algebraically independent generators of the center of $\mathrm{U}(\mathfrak{sp}_{2n})$. Their eigenvalues in the highest weight representations coincide with the factorial elementary symmetric functions $e_k(l_1^2, \dots, l_n^2 \,|\, a)$; see Sec. 3.1.

These results imply that the Casimir elements (1.6) respectively coincide, up to a constant factor, with those found previously in [10] as an application of the twisted Yangians. However, the expressions for those Casimir elements given in [10] are quite different from (1.6) so the coincidence appears to be surprising.

Our proofs are based on the characterization theorem for the factorial Schur polynomials (see [14] and [15]) as well as on the eigenvalues of the Jucys–Murphy elements for the Brauer algebra in the irreducible representations found in [7] and [12]. This approach extends to the Casimir elements of the form (1.4), where C is the anti-symmetrizer in the Brauer algebra. Although such elements are studied in the literature and their Harish-Chandra images are known (see e.g. [3], [5], [6], [10] and [18]), our arguments appear to be new and they apply uniformly to both families of elements constructed with the use of the symmetrizers and anti-symmetrizers.

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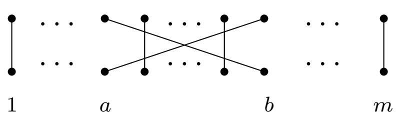
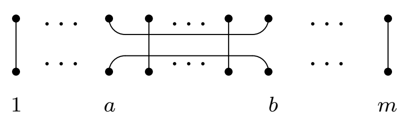
2 Brauer algebra

We let m be a positive integer and ω an indeterminate. An m-diagram d is a collection of $2m$ dots arranged into two rows with m dots in each row connected by m edges such that any dot belongs to only one edge. The product of two diagrams d_1 and d_2 is determined by placing d_1 above d_2 and identifying the vertices of the bottom row of d_1 with the corresponding vertices in the top row of d_2. Let s be the number of closed loops obtained in this placement. The product $d_1 d_2$ is given by ω^s times the resulting diagram without loops. The *Brauer algebra* $\mathcal{B}_m(\omega)$ [1] is defined as the $\mathbb{C}(\omega)$-linear span of the m-diagrams with the multiplication defined above. The dimension of the algebra is $1 \cdot 3 \cdots (2m-1)$. For $1 \leqslant a < b \leqslant m$ denote by s_{ab} and ϵ_{ab} the respective diagrams of the form



and



The subalgebra of $\mathcal{B}_m(\omega)$ generated over $\mathbb{C}$ by $s_{a\,a+1}$ with $a = 1, \dots, m-1$ is isomorphic to the group algebra of the symmetric group $\mathbb{C}[\mathfrak{S}_m]$ so that s_{ab} will be identified with the transposition $(a\,b)$. The Brauer algebra $\mathcal{B}_{m-1}(\omega)$ will be regarded as a natural subalgebra of $\mathcal{B}_m(\omega)$.

The *Jucys–Murphy elements* $x_1, \dots, x_m$ for the Brauer algebra $\mathcal{B}_m(\omega)$ are given by the formulas

$$x_1 = 0, \qquad x_b = \sum_{a=1}^{b-1} (s_{ab} - \epsilon_{ab}), \qquad b = 2, \dots, m; \tag{2.1}$$

see [7] and [12], where, in particular, the eigenvalues of the x_b in irreducible representations were calculated. The element x_m commutes with the subalgebra $\mathcal{B}_{m-1}(\omega)$. This implies that the elements $x_1, \dots, x_m$ of $\mathcal{B}_m(\omega)$ pairwise commute.

Irreducible representations of the algebra $\mathcal{B}_m(\omega)$ (over $\mathbb{C}(\omega)$) are parameterized by the set of partitions of the numbers $m - 2f$ with $f \in \{0, 1, \dots, \lfloor m/2 \rfloor\}$. We will identify partitions with their diagrams so that if the parts of λ are $\lambda_1, \lambda_2, \dots$ then the corresponding diagram is a left-justified array of rows of unit boxes containing λ_1 boxes in the top row, λ_2 boxes in the second row, etc. We will denote by $|\lambda|$ the number of boxes in the diagram and by $\ell(\lambda)$ its *length*, i.e., the number of rows. The box in row i and column j of a diagram will be denoted as the pair (i, j). An *updown λ-tableau* is a sequence $T = (\Lambda_1, \dots, \Lambda_m)$ of m diagrams such that for each $r = 1, \dots, m$ the diagram Λ_r is obtained from Λ_{r-1} by adding or removing one box, where $\Lambda_0 = \varnothing$ is the empty diagram and $\Lambda_m = \lambda$. To each updown tableau T we associate the corresponding sequence of *contents* $(c_1, \dots, c_m)$, $c_r = c_r(T)$, where

$$c_r = j - i \qquad \text{or} \qquad c_r = -\omega + 1 - j + i,$$

if Λ_r is obtained by adding the box (i, j) to Λ_{r-1} or by removing this box from Λ_{r-1}, respectively.

It is well-known that the Jucys–Murphy elements can be used to define the primitive idempotents $E_T = E_T^\lambda$; see e.g. [4] for explicit formulas. When λ runs over all partitions of $m, m-2, \dots$ and T runs over all updown λ-tableaux, the elements $\{E_T\}$ yield a complete set of pairwise orthogonal primitive idempotents for $\mathcal{B}_m(\omega)$. They have the properties

$$x_r\, E_T = E_T\, x_r = c_r(T)\, E_T, \qquad r = 1, \dots, m; \tag{2.2}$$

see [7] and [12]. In particular, if $\lambda = (m)$ is the single row diagram with m boxes then there is a unique updown λ-tableau which can also be regarded as the standard tableau obtained by writing the numbers $1, \dots, m$ into the boxes of λ from left to right. The corresponding primitive idempotent is the *symmetrizer* $S^{(m)} \in \mathcal{B}_m(\omega)$ given in terms of the Jucys–Murphy elements as

$$S^{(m)} = \prod_{r=2}^{m} \frac{(1 + x_r)(\omega + r - 3 + x_r)}{r\,(\omega + 2r - 4)}. \tag{2.3}$$

This element also admits an equivalent expression

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 + \frac{s_{ab}}{b-a} - \frac{\epsilon_{ab}}{\omega/2 + b - a - 1} \Big), \tag{2.4}$$

where the product is taken in the lexicographic order on the pairs $(a\, b)$; see [4] and [9] for some other equivalent formulas for $S^{(m)}$. We have the properties

$$s_{ab}\, S^{(m)} = S^{(m)}\, s_{ab} = S^{(m)} \qquad \text{and} \qquad \epsilon_{ab}\, S^{(m)} = S^{(m)}\, \epsilon_{ab} = 0 \tag{2.5}$$

for all $1 \leqslant a < b \leqslant m$.

Similarly, if $\lambda = (1^m)$ is the single column diagram with m boxes then the unique updown λ-tableau can be regarded as the standard tableau obtained by writing the numbers $1, \dots, m$ into the boxes of λ from top to bottom. The corresponding primitive idempotent is the *anti-symmetrizer* $A^{(m)} \in \mathcal{B}_m(\omega)$. It is well-known that $A^{(m)}$ coincides with the anti-symmetrizer in the group algebra for the symmetric group $\mathfrak{S}_m$ and has the properties

$$s_{ab}\, A^{(m)} = A^{(m)}\, s_{ab} = -A^{(m)} \qquad \text{and} \qquad \epsilon_{ab}\, A^{(m)} = A^{(m)}\, \epsilon_{ab} = 0 \tag{2.6}$$

for all $1 \leqslant a < b \leqslant m$.

Consider now the action of the Brauer algebra on the tensor space (1.1). In the orthogonal case, $\omega = N$ and the generators of $\mathcal{B}_m(N)$ act by the rule

$$s_{ab} \mapsto P_{ab}, \qquad \epsilon_{ab} \mapsto Q_{ab}, \qquad a < b, \tag{2.7}$$

where P_{ab} is defined by

$$P_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(m-b)}, \tag{2.8}$$

while

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)} \tag{2.9}$$

and we use the involution on the set of indices $\{1, \dots, N\}$ defined by $i \mapsto i' = N - i + 1$.

In the symplectic case, $\omega = -N$ (where $N = 2n$ is even) and the action of $\mathcal{B}_m(-N)$ in the space (1.1) is now defined by

$$s_{ab} \mapsto -P_{ab}, \qquad \epsilon_{ab} \mapsto -Q_{ab}, \qquad a < b, \tag{2.10}$$

where P_{ab} is defined in (2.8), and

$$Q_{ab} = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j\, 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)} \tag{2.11}$$

with $\varepsilon_i = 1$ for $i = 1, \dots, n$ and $\varepsilon_i = -1$ for $i = n+1, \dots, 2n$.

3 Harish-Chandra images

Consider the Lie algebra $\mathfrak{gl}_N$ with its standard basis elements E_{ij}, $1 \leqslant i, j \leqslant N$. The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements $F_{ij} = E_{ij} - E_{j'i'}$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}_N$. Similarly, the Lie subalgebra of $\mathfrak{gl}_{2n}$ spanned by the elements $F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{j'i'}$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}_{2n}$. We will keep the notation $\mathfrak{g}_N$ for the Lie algebra $\mathfrak{o}_N$ (with $N = 2n$ or $N = 2n+1$) or $\mathfrak{sp}_N$ (with $N = 2n$).

Using the notation (1.3) we can write the defining relations of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ in the matrix form as

$$F_1 F_2 - F_2 F_1 = (P - Q) F_2 - F_2 (P - Q) \tag{3.1}$$

together with the relation $F + F' = 0$, where the prime denotes the matrix transposition defined by

$$(A')_{ij} = \begin{cases} A_{j'i'} & \text{in the orthogonal case,} \\ \varepsilon_i \varepsilon_j A_{j'i'} & \text{in the symplectic case.} \end{cases} \tag{3.2}$$

In the following lemma we identify the elements of the respective Brauer algebra $\mathcal{B}_m(N)$ or $\mathcal{B}_m(-N)$ with their images under the actions (2.7) or (2.10).

Lemma 3.1. *Let $u_1, \dots, u_m$ be complex parameters. For any permutations $\sigma, \tau \in \mathfrak{S}_m$ and for $C = S^{(m)}$ or $C = A^{(m)}$ we have*

$$\mathrm{tr}\,(F_{\sigma(1)} + u_{\tau(1)}) \dots (F_{\sigma(m)} + u_{\tau(m)})\, C = \mathrm{tr}\,(F_1 + u_1) \dots (F_m + u_m)\, C. \tag{3.3}$$

Proof. Let P_π denote the image of any element $\pi \in \mathfrak{S}_m$ in the algebra (1.2). Since $P_\pi F_a = F_{\pi(a)} P_\pi$, using the cyclic property of trace and applying the conjugation by the element $P_{\sigma^{-1}}$ in the left hand side of (3.3) we find that it suffices to verify the relation for the case where σ is the identity permutation. Note that by (3.1),

$$\begin{aligned}(F_a + u)(F_{a+1} + v) - (F_{a+1} + v)(F_a + u)& \\ = (P_{a\,a+1} - Q_{a\,a+1}) F_{a+1} &- F_{a+1} (P_{a\,a+1} - Q_{a\,a+1}),\end{aligned}$$

and by (2.5) and (2.6)

$$C\,(P_{a\,a+1} - Q_{a\,a+1}) = (P_{a\,a+1} - Q_{a\,a+1})\, C = \pm C.$$

Hence, the claim follows from the cyclic property of trace and the first part of the proof. □

Given any n-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n)$ the corresponding irreducible highest weight representation $L(\lambda)$ of the Lie algebra $\mathfrak{g}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$\begin{aligned} F_{ij}\, \xi &= 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and} \\ F_{ii}\, \xi &= \lambda_i\, \xi \qquad \text{for} \quad 1 \leqslant i \leqslant n. \end{aligned}$$

We will denote by G_N the orthogonal group O_N or the symplectic group Sp_N. Recall that finite-dimensional irreducible representations of the orthogonal group O_N are parameterized by all diagrams λ with the property $\lambda'_1 + \lambda'_2 \leqslant N$, where λ'_j denotes the number of boxes in the column j of λ. The corresponding representation will be denoted by $V(\lambda)$. Let λ^* be the diagram obtained from λ by replacing the first column with the column containing $N - \lambda'_1$ boxes. If $N = 2n + 1$ and $\lambda'_1 \leqslant n$ then the associated representation of the Lie algebra $\mathfrak{o}_N$ in the space $V(\lambda)$ is irreducible and isomorphic to the representation $L(\lambda)$ whose highest weight coincides with λ; if $\lambda'_1 > n$ then the associated representation of $\mathfrak{o}_N$ is isomorphic to $L(\lambda^*)$.

If $N = 2n$ and $\lambda'_1 < n$ then the associated representation of the Lie algebra $\mathfrak{o}_N$ in the space $V(\lambda)$ is irreducible and isomorphic to $L(\lambda)$, while for $\lambda'_1 > n$ the associated representation of $\mathfrak{o}_N$ is isomorphic to $L(\lambda^*)$. If $N = 2n$ and $\lambda'_1 = n$ then the associated representation of $\mathfrak{o}_N$ in $V(\lambda)$ is isomorphic to the direct sum of two irreducible representations $L(\lambda)$ and $L(\widetilde{\lambda})$ with $\widetilde{\lambda} = (\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$.

Finite-dimensional irreducible representations $V(\lambda)$ of the symplectic group Sp_N with $N = 2n$ are parameterized by partitions λ whose lengths do not exceed n. The associated representation of the Lie algebra $\mathfrak{sp}_N$ in $V(\lambda)$ is irreducible and isomorphic to $L(\lambda)$.

Any element $z \in \mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ acts in $V(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. In the case of $\mathfrak{g}_N = \mathfrak{o}_{2n}$ the eigenvalues of any element $z \in \mathrm{U}(\mathfrak{o}_{2n})^{\mathrm{O}_{2n}}$ in $L(\lambda)$ and $L(\widetilde{\lambda})$ coincide and they are equal to $\chi(z)$.

When regarded as a function of the highest weight, $\chi(z)$ is a symmetric polynomial in the variables $l_1^2, \dots, l_n^2$, where $l_i = \lambda_i + \rho_i$ and $\rho_i = n - i + \varepsilon$ with

$$\varepsilon = \begin{cases} 0 & \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n}, \\ \frac{1}{2} & \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n+1}, \\ 1 & \text{for} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}. \end{cases} \tag{3.4}$$

The mapping $z \mapsto \chi(z)$ defines an algebra isomorphism

$$\chi : \mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N} \to \mathbb{C}[l_1^2, \dots, l_n^2]^{\mathfrak{S}_n}$$

known as the Harish-Chandra isomorphism; see e.g. [2, Ch. 7].

3.1 Characterization properties for symmetric polynomials

Following [10], we will be using the factorial (or double) elementary and complete symmetric polynomials and their characterization properties; see [14] and [15] for more details. Here we recall the corresponding results to be used in the proofs below.

Consider the algebra of symmetric polynomials in the independent variables $z_1, \dots, z_n$ over $\mathbb{C}$ and fix a sequence $a = (a_1, a_2, \dots)$ of complex numbers. The *factorial elementary* and *complete symmetric polynomials* are defined by the respective formulas

$$e_k(z_1,\dots,z_n\,|\,a)=\sum_{1\leqslant p_1<\dots<p_k\leqslant n}(z_{p_1}-a_{p_1})(z_{p_2}-a_{p_2-1})\dots(z_{p_k}-a_{p_k-k+1}),\tag{3.5}$$

$$h_k(z_1,\dots,z_n\,|\,a)=\sum_{1\leqslant p_1\leqslant\dots\leqslant p_k\leqslant n}(z_{p_1}-a_{p_1})(z_{p_2}-a_{p_2+1})\dots(z_{p_k}-a_{p_k+k-1}),\tag{3.6}$$

so that $e_k(z_1,\dots,z_n\,|\,a)=0$ for $k>n$. These polynomials are particular cases of the factorial (or double) Schur polynomials; see e.g. [8]. When a is specialized to the sequence of zeros, then (3.5) and (3.6) become the elementary and complete symmetric polynomials $e_k(z_1,\dots,z_n)$ and $h_k(z_1,\dots,z_n)$.

For any partition $\lambda=(\lambda_1,\dots,\lambda_n)$ whose length $\ell(\lambda)$ does not exceed n introduce the n-tuple a_λ of complex numbers by

$$a_\lambda=(a_{\lambda_1+n},a_{\lambda_2+n-1},\dots,a_{\lambda_n+1}).$$

The polynomials (3.5) and (3.6) possess vanishing properties of the form:

$$\begin{aligned}&\text{if}\quad \ell(\lambda)<k &&\text{then}\quad e_k(a_\lambda\,|\,a)=0,\\ &\text{if}\quad \lambda_1<k &&\text{then}\quad h_k(a_\lambda\,|\,a)=0.\end{aligned}$$

We will need two particular cases of the characterization theorem for the factorial Schur polynomials [14]. Now we will be assuming that all elements a_i of the sequence a are distinct. Suppose that $f(z_1,\dots,z_n)$ is a symmetric polynomial of degree $\leqslant k$ whose component of degree k coincides with $e_k(z_1,\dots,z_n)$ or $h_k(z_1,\dots,z_n)$. If

$$f(a_\lambda)=0\qquad\text{for all }\lambda\text{ with }|\lambda|<k$$

then $f(z_1,\dots,z_n)$ equals $e_k(z_1,\dots,z_n\,|\,a)$ or $h_k(z_1,\dots,z_n\,|\,a)$, respectively.

From now on, we will work with particular sequences a defined by

$$a=(\varepsilon^2,(\varepsilon+1)^2,(\varepsilon+2)^2,\dots),\tag{3.7}$$

where ε is introduced in (3.4), so that $a_i=(\varepsilon+i-1)^2$. Furthermore, the n-tuple a_λ associated with the sequence (3.7) has the form

$$a_\lambda=\big((\lambda_1+n-1+\varepsilon)^2,\dots,(\lambda_n-1+\varepsilon)^2\big)=(l_1^2,\dots,l_n^2).$$

Note that any element $z\in\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ is uniquely determined by the eigenvalues $\chi(z)$ in the irreducible modules $L(\lambda)$, where $\lambda=(\lambda_1,\dots,\lambda_n)$ runs over the set of partitions with $\ell(\lambda)\leqslant n$. Hence, we come to the following characterization properties of Casimir elements; cf. [10, Corollary 2.5]. We use the canonical filtration on the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$.

Proposition 3.2. *Suppose that $z\in\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ is an element of degree $\leqslant 2k$ which vanishes in each representation $L(\lambda)$ with $|\lambda|<k$. If the homogeneous component of $\chi(z)$ of degree $2k$ coincides with $e_k(\lambda_1^2,\dots,\lambda_n^2)$ or $h_k(\lambda_1^2,\dots,\lambda_n^2)$ then $\chi(z)$ equals $e_k(l_1^2,\dots,l_n^2\,|\,a)$ or $h_k(l_1^2,\dots,l_n^2\,|\,a)$, respectively.*

3.2 Casimir elements for the orthogonal Lie algebras

As we pointed out in the Introduction, any element of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ of the form (1.4) belongs to the subalgebra of invariants $\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$. We will be concerned with two choices of the element $C \in \mathcal{B}_m(\omega)$; namely, $C = S^{(m)}$ and $C = A^{(m)}$. Moreover, the parameters u_a in (1.4) will be specialized accordingly so that all differences $u_a - u_{a+1}$ have the same value 1 or -1 for all $a = 1, \dots, m-1$.

We will be assuming here that $\mathfrak{g}_N = \mathfrak{o}_N$. For any $m \geqslant 0$ set

$$\alpha_m = \frac{N+2m-2}{N+m-2}. \tag{3.8}$$

We start by taking even values $m = 2k$ and particular specializations of the parameters.

Theorem 3.3. *For any $k \geqslant 1$ the image of the Casimir element*

$$\mathrm{tr}\,(F_1 + k - 1) \dots (F_{2k} - k)\, S^{(2k)} \in \mathrm{U}(\mathfrak{o}_N)^{\mathrm{O}_N} \tag{3.9}$$

under the Harish-Chandra isomorphism coincides with $\alpha_{2k}\, h_k(l_1^2, \dots, l_n^2\,|\,a)$.

Proof. Denote the Casimir element (3.9) by D_k. We will use Proposition 3.2 and start by showing that D_k vanishes in all representations $L(\lambda)$ of $\mathfrak{o}_N$, where the partitions λ satisfy $|\lambda| < k$. We employ a realization of $L(\lambda)$ in tensor spaces as follows. Let $r = |\lambda|$. Consider the action of the Lie algebra $\mathfrak{o}_N$ on $\mathbb{C}^N$ defined by

$$F_{ij} \mapsto -e_{ji} + e_{i'j'}, \qquad 1 \leqslant i, j \leqslant N,$$

so that this representation is isomorphic to $L(1, 0, \dots, 0)$. The space of tensors $(\mathbb{C}^N)^{\otimes r}$ then also becomes a representation of $\mathfrak{o}_N$. Using the matrix notation (1.3), under the corresponding homomorphism

$$\varphi : \mathrm{U}(\mathfrak{o}_N) \otimes \mathrm{End}\,(\mathbb{C}^N)^{\otimes 2k} \to \mathrm{End}\,(\mathbb{C}^N)^{\otimes r} \otimes \mathrm{End}\,(\mathbb{C}^N)^{\otimes 2k}$$

we have

$$\varphi(F_a) = \sum_{b=1}^{r} (-P_{b\, r+a} + Q_{b\, r+a}), \qquad a = 1, \dots, 2k, \tag{3.10}$$

where we use the operators (2.8) and (2.9). The decomposition of $(\mathbb{C}^N)^{\otimes r}$ into a direct sum of irreducible representations of $\mathfrak{o}_N$ contains $L(\lambda)$ with a nonzero multiplicity. Hence, the desired vanishing condition of Proposition 3.2 will follow if we show that

$$\big(\varphi(F_1) + k - 1\big) \dots \big(\varphi(F_{2k}) - k\big)\, S'^{(2k)} = 0, \tag{3.11}$$

where $S'^{(2k)}$ denotes the image of the symmetrizer in the Brauer algebra $\mathcal{B}_{2k}(N)$ acting of the last $2k$ copies of the tensor product space $(\mathbb{C}^N)^{\otimes(r+2k)}$. Due to (3.10) and relations (2.5), the desired identity (3.11) can be written in the form

$$(-x_{r+1} + k - 1) \dots (-x_{r+2k} + k - 1)\, S'^{(2k)} = 0, \tag{3.12}$$

where $x_{r+1}, \dots, x_{r+2k}$ denote the images of the Jucys–Murphy elements (2.1) of the Brauer algebra $\mathcal{B}_{r+2k}(N)$ under its action in $(\mathbb{C}^N)^{\otimes(r+2k)}$. To prove (3.12) we note that the two operators $(-x_{r+1}+k-1)\dots(-x_{r+2k}+k-1)$ and $S'^{(2k)}$ on the vector space $(\mathbb{C}^N)^{\otimes(r+2k)}$ commute with the action of O_N and show that their images have zero intersection. To describe the image of the first operator, represent the vector space as the direct sum of irreducible representations of O_N,

$$(\mathbb{C}^N)^{\otimes(r+2k)} = \sum_{l=0}^{\lfloor \frac{r}{2}\rfloor+k} \sum_{\nu\vdash r+2k-2l} \sum_U E_U (\mathbb{C}^N)^{\otimes(r+2k)},$$

where the last sum is taken over all updown tableaux $U = (\Lambda_1, \dots, \Lambda_{r+2k})$ of shape ν associated with the Brauer algebra $\mathcal{B}_{r+2k}(N)$. We claim that if U is an updown tableau of shape $\nu = \Lambda_{r+2k}$ with $\nu_1 \geqslant k$ then

$$(-x_{r+1}+k-1)\dots(-x_{r+2k}+k-1)\, E_U = 0. \tag{3.13}$$

Indeed, since $r < k$ there exists a pair of diagrams $(\Lambda_{r+a}, \Lambda_{r+a+1})$ with $a \in \{0, 1, \dots, 2k-1\}$ with the property that the second diagram is obtained from the first by adding the box $(1, k)$. The content of this box is $k-1$ so that (3.13) follows from relations (2.2). Thus, as a representation of O_N the image

$$(-x_{r+1}+k-1)\dots(-x_{r+2k}+k-1)\,(\mathbb{C}^N)^{\otimes(r+2k)}$$

is contained in a direct sum of representations $V(\nu)$ with $\nu_1 < k$.

On the other hand, the operator $S'^{(2k)}$ projects the vector space $(\mathbb{C}^N)^{\otimes(r+2k)}$ onto the tensor product $(\mathbb{C}^N)^{\otimes r} \otimes V(2k, 0, \dots, 0)$ of representations of O_N. For the irreducible decomposition of the tensor product of representations of O_N we have

$$\mathbb{C}^N \otimes V(\mu) \cong \bigoplus_{\widetilde{\mu}} V(\widetilde{\mu}),$$

where $\widetilde{\mu}$ is obtained from μ by adding or removing one box. Hence, as a representation of O_N, the image $S'^{(2k)}(\mathbb{C}^N)^{\otimes(r+2k)}$ is contained in a direct sum of representations $V(\nu)$ with $\nu_1 \geqslant k+1$. This completes the proof of (3.12) and hence (3.11).

The leading term of the symmetric polynomial $\chi(D_k)$ was calculated in [9] and [11]. It coincides with $\alpha_{2k}\, h_k(\lambda_1^2, \dots, \lambda_n^2)$ so that the proof is completed by the application of Proposition 3.2. □

Remark 3.4. (i) Theorem 3.3 together with Theorem 3.8 below show that the elements (3.9) and (3.19) are proportional to the respective Casimir elements given in [10, Theorems 3.2 and 3.3] by completely different formulas. It would be interesting to find a direct argument connecting these families.

(ii) It is possible to give an independent proof of Theorems 3.3 without relying on the calculation of the leading term of the symmetric polynomial $\chi(D_k)$ in [9] and [11]. To this end, we could use a different version of the characterization theorem from [14] where the assumption $|\lambda| < k$ is replaced by $\lambda_1 < k$. The vanishing condition is verified in a similar way and it determines the polynomial $\chi(D_k)$, up to a constant factor. The latter is calculated by finding the leading term in the case where λ is the one box diagram. $\square$

Now we turn to more general Casimir elements of the form (1.4) and (1.5). Let u be a variable. Consider the polynomials in u whose coefficients are Casimir elements for $\mathfrak{o}_N$ given by

$$D_m(u) = \mathrm{tr}\Big(F_1 + u + \frac{m-1}{2}\Big)\Big(F_2 + u + \frac{m-3}{2}\Big)\dots\Big(F_m + u - \frac{m-1}{2}\Big)\, S^{(m)}.$$

We use notation (3.8).

Corollary 3.5. *For the images under the Harish-Chandra isomorphism we have*

$$\chi : D_m(u) \mapsto \alpha_m \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \binom{N+m-2}{m-2r} h_r(l_1^2, \dots, l_n^2 \,|\, a) \prod_{i=0}^{m-2r-1} \Big(u - \frac{m-1}{2} + r + i\Big)$$

and

$$\chi : \mathrm{tr}\,(-\partial_t + F_1 t^{-1}) \dots (-\partial_t + F_m t^{-1})\, S^{(m)} \mapsto \\ \alpha_m \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \binom{N+m-2}{m-2r} h_r(l_1^2, \dots, l_n^2 \,|\, a)\, t^{-2r}\, (-\partial_t + r\, t^{-1})^{m-2r}.$$

Proof. Observe that by the property of trace, $D_m(u)$ is stable under the transposition (3.2) applied simultaneously to each of the m copies of the algebra $\mathrm{End}\,\mathbb{C}^N$. The symmetrizer $S^{(m)}$ is also stable under this transposition. On the other hand, since $F' = -F$, using Lemma 3.1, for the image of $D_m(u)$ we find

$$\begin{aligned} D_m(u) &= \mathrm{tr}\Big(-F_1 + u + \frac{m-1}{2}\Big)\dots\Big(-F_m + u - \frac{m-1}{2}\Big)\, S^{(m)} \\ &= (-1)^m\, \mathrm{tr}\Big(F_1 - u - \frac{m-1}{2}\Big)\dots\Big(F_m - u + \frac{m-1}{2}\Big)\, S^{(m)} \\ &= (-1)^m\, \mathrm{tr}\Big(F_1 - u + \frac{m-1}{2}\Big)\dots\Big(F_m - u - \frac{m-1}{2}\Big)\, S^{(m)} = (-1)^m\, D_m(-u). \end{aligned}$$

This shows that the polynomials $D_{2k}(u)$ are even, while the polynomials $D_{2k-1}(u)$ are odd. In particular, $D_{2k-1}(0) = 0$, while the value $\chi(D_{2k}(1/2)) = \chi(D_{2k}(-1/2))$ is found from Theorem 3.3. These values agree with the Harish-Chandra images provided in the statement of the corollary. Hence, the proof will be completed if we show that the polynomials

$D_m(u)$ and the polynomials which are claimed to be their Harish-Chandra images satisfy the same recurrence relations. We show first that

$$D_m(u+1/2) - D_m(u-1/2) = \frac{(N+m-3)(N+2m-2)}{N+2m-4}\, D_{m-1}(u), \qquad m \geqslant 1, \quad (3.14)$$

where we set $D_0(u) = 1$. By Lemma 3.1,

$$D_m(u+1/2) = \operatorname{tr}\Big(F_1 + u + \frac{m}{2} - 1\Big) \dots \Big(F_{m-1} + u - \frac{m}{2} - 1\Big) \Big(F_m + u + \frac{m}{2}\Big)\, S^{(m)},$$

so that

$$D_m(u+1/2) - D_m(u-1/2) = m \operatorname{tr}\Big(F_1 + u + \frac{m}{2} - 1\Big) \dots \Big(F_{m-1} + u - \frac{m}{2} - 1\Big)\, S^{(m)}.$$

By [9, Lemma 4.1], the partial trace of the symmetrizer is found by

$$\operatorname{tr}_m S^{(m)} = \frac{(N+m-3)(N+2m-2)}{m\,(N+2m-4)}\, S^{(m-1)}$$

thus verifying (3.14). A simple calculation shows that the same relation is satisfied by the polynomials which are claimed to be the images $\chi(D_m(u))$ as stated in the corollary.

To prove the second part, note that by the relation

$$t^m\,(-\partial_t + F_1 t^{-1}) \dots (-\partial_t + F_m t^{-1}) = (-t\partial_t + F_1 + m - 1) \dots (-t\partial_t + F_m)$$

the polynomial $D_m(u)$ can be written in the form

$$t^m \operatorname{tr}(-\partial_t + F_1 t^{-1}) \dots (-\partial_t + F_m t^{-1})\, S^{(m)}$$

after the subsequent replacement of $-t\partial_t$ with $u - (m-1)/2$. Similarly, for any $k \geqslant 0$,

$$t^k\,(-\partial_t + r t^{-1})^k = (-t\partial_t + r + k - 1) \dots (-t\partial_t + r),$$

so that the second relation follows from the first. □

In what follows we state analogues of Theorem 3.3 and Corollary 3.5, where the role of the symmetrizer $S^{(m)}$ is taken by the anti-symmetrizer $A^{(m)}$. The arguments are quite similar so we only indicate the key steps in the proofs. The corresponding Casimir elements turn out to coincide with those already appeared in the literature; cf. e.g. [5], [10] and [18].

Theorem 3.6. *For any $1 \leqslant k \leqslant n$ the image of the Casimir element*

$$\operatorname{tr}(F_1 - k + 1) \dots (F_{2k} + k)\, A^{(2k)} \in \mathrm{U}(\mathfrak{o}_N)^{\mathrm{O}_N} \qquad (3.15)$$

under the Harish-Chandra isomorphism coincides with $(-1)^k\, e_k(l_1^2, \dots, l_n^2\,|\,a)$.

Proof. Denote the Casimir element (3.15) by C_k. We use Proposition 3.2 and show that C_k vanishes in all representations $L(\lambda)$ of $\mathfrak{o}_N$, where the partitions λ satisfy $|\lambda| < k$. Using the same realization of $L(\lambda)$ as in the proof of Theorem 3.3, we come to showing the following analogue of (3.11):

$$\big(\varphi(F_1) - k + 1\big) \dots \big(\varphi(F_{2k}) + k\big)\, A'^{(2k)} = 0. \tag{3.16}$$

Here the homomorphism φ is defined in (3.10) and $A'^{(2k)}$ denotes the image of the anti-symmetrizer in the Brauer algebra $\mathcal{B}_{2k}(N)$ acting of the last $2k$ copies of the tensor product space $(\mathbb{C}^N)^{\otimes(r+2k)}$ with $r = |\lambda|$. By (2.6), the identity (3.16) can be written in the form

$$(-x_{r+1} - k + 1) \dots (-x_{r+2k} - k + 1)\, A'^{(2k)} = 0.$$

To verify this relation, we show exactly as in the proof of Theorem 3.3 that the image of the operator $(-x_{r+1} - k + 1) \dots (-x_{r+2k} - k + 1)$ on the vector space $(\mathbb{C}^N)^{\otimes(r+2k)}$ is contained in a direct sum of representations $V(\nu)$ of O_N with $\ell(\nu) < k$, while the image of the operator $A'^{(2k)}$ is contained in a direct sum of representations $V(\nu)$ with $\ell(\nu) \geqslant k+1$.

It was shown in [9] and [11] that the leading term of the symmetric polynomial $\chi(C_k)$ coincides with $(-1)^k e_k(\lambda_1^2, \dots, \lambda_n^2)$. □

Consider the polynomials in u whose coefficients are Casimir elements for $\mathfrak{o}_N$ given by

$$C_m(u) = \mathrm{tr}\Big(F_1 + u + \frac{m-1}{2}\Big)\Big(F_2 + u + \frac{m-3}{2}\Big) \dots \Big(F_m + u - \frac{m-1}{2}\Big)\, A^{(m)}.$$

Corollary 3.7. *For the images under the Harish-Chandra isomorphism we have*

$$\chi : C_m(u) \mapsto \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{N-2r}{m-2r}\, e_r(l_1^2, \dots, l_n^2\,|\,a) \prod_{i=0}^{m-2r-1} \Big(u - \frac{m-1}{2} + r + i\Big)$$

and

$$\chi : \mathrm{tr}\,(-\partial_t + F_1 t^{-1}) \dots (-\partial_t + F_m t^{-1})\, A^{(m)} \mapsto$$
$$\sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{N-2r}{m-2r}\, e_r(l_1^2, \dots, l_n^2\,|\,a)\, t^{-2r}\, (-\partial_t + r\, t^{-1})^{m-2r}.$$

Proof. We argue as in the proof of Corollary 3.5. Using Lemma 3.1 we show first that $C_m(u) = (-1)^m\, C_m(-u)$. In particular, $C_{2k-1}(0) = 0$ and $\chi(C_{2k}(1/2)) = \chi(C_{2k}(-1/2))$ is found from Theorem 3.6. As the next step, we verify that

$$C_m(u + 1/2) - C_m(u - 1/2) = (N - m + 1)\, C_{m-1}(u), \qquad m \geqslant 1, \tag{3.17}$$

where $C_0(u) = 1$. This follows easily from Lemma 3.1, and the calculation of the partial trace of the anti-symmetrizer which is found by

$$\mathrm{tr}_m\, A^{(m)} = \frac{N - m + 1}{m}\, A^{(m-1)}$$

thus verifying (3.17). The same relation is satisfied by the polynomials which are claimed to be the images $\chi(C_m(u))$ as stated in the corollary. The second part follows from the first; see the proof of Corollary 3.5. $\square$

3.3 Casimir elements for the symplectic Lie algebras

Now take $\mathfrak{g}_N = \mathfrak{sp}_{2n}$. Consider the action of the Brauer algebra $\mathcal{B}_m(-2n)$ on the space (1.1) defined by (2.10). Note that the image of the symmetrizer $S^{(m)}$ (see (2.3) and (2.4)) under this action is well-defined for $m \leqslant n+1$ and it is zero for $m = n+1$, while the specialization of $S^{(m)}$ at $\omega = -2n$ is not defined for $n+2 \leqslant m \leqslant 2n$. Nevertheless, the expression

$$\frac{1}{n-m+1}\,\mathrm{tr}\, S^{(m)}(F_1+u_1)\dots(F_m+u_m) \tag{3.18}$$

still defines a Casimir element for $\mathfrak{sp}_{2n}$ for all $1 \leqslant m \leqslant 2n$, where $u_1,\dots,u_m$ are arbitrary complex numbers. Indeed, assuming that $m \leqslant n$, using (2.3) and calculating first the partial trace tr_m in (3.18) over the m-th copy of $\mathrm{End}\,\mathbb{C}^{2n}$ we get an expression involving the symmetrizer $S^{(m-1)}$ with the extra factor $(n-m+1)/(n-m+2)$. The latter expression is well-defined for $m \leqslant n+1$ thus allowing us to extend the value of (3.18) to $m = n+1$. Continuing with a similar calculation and taking further partial traces we extend the definition of (3.18) to all values $m \leqslant 2n$; see [9, Sec. 3.3] for more details.

Recall that the factorial elementary symmetric polynomials $e_r(l_1^2,\dots,l_n^2\,|\,a)$ with the sequence a defined as in (3.7) with $\varepsilon = 1$ are algebraically independent generators of the algebra $\mathbb{C}[l_1^2,\dots,l_n^2]^{\mathfrak{S}_n}$. Now we let m be fixed and let n run over integer values $\geqslant m/2$. Set $k = \lfloor m/2 \rfloor$. It is clear from the above argument and from the explicit formulas for the symmetrizer $S^{(m)}$ that the Harish-Chandra image of the Casimir element (3.18) can be written as the following linear combination

$$\sum_{\pi} c_{\pi}^{(m)}(u_1,\dots,u_m)\prod_{r=1}^{k} e_r(l_1^2,\dots,l_n^2\,|\,a)^{p_r},$$

where π runs over the k-tuples of nonnegative integers $\pi = (p_1,\dots,p_k)$ satisfying the condition $p_1 + 2p_2 + \dots + kp_k \leqslant m/2$. Moreover, the $c_{\pi}^{(m)}(u_1,\dots,u_m)$ are symmetric polynomials in $u_1,\dots,u_m$ whose coefficients are rational functions in n. Since a rational function is uniquely determined by its values at infinitely many points, the Harish-Chandra image of the Casimir element (3.18) is uniquely determined by its values at infinitely many values of n.

Theorem 3.8. *For any $1 \leqslant k \leqslant n$ the image of the Casimir element*

$$\frac{n-k+1}{n-2k+1}\,\mathrm{tr}\,(F_1-k+1)\dots(F_{2k}+k)\,S^{(2k)} \in \mathrm{U}(\mathfrak{sp}_{2n})^{\mathrm{Sp}_{2n}} \tag{3.19}$$

under the Harish-Chandra isomorphism coincides with $(-1)^k\, e_k(l_1^2,\dots,l_n^2\,|\,a)$.

Proof. As explained above, it will be sufficient to prove the statement for any fixed k under the assumption that $n \geqslant 2k$. We will use the same argument as in the proof of Theorem 3.3. Let D_k denote the Casimir element (3.19). Relying on Proposition 3.2 we show that D_k vanishes in all representations $V(\lambda)$ of Sp_{2n}, where the partitions λ satisfy $|\lambda| < k$. Set $r = |\lambda|$ and consider the action of the Lie algebra $\mathfrak{sp}_{2n}$ on $\mathbb{C}^{2n}$ defined by

$$F_{ij} \mapsto -e_{ji} + \varepsilon_i \varepsilon_j e_{i'j'}, \qquad 1 \leqslant i, j \leqslant 2n.$$

Under the corresponding representation in the tensor space

$$\varphi : \mathrm{U}(\mathfrak{sp}_{2n}) \otimes \mathrm{End}\,(\mathbb{C}^{2n})^{\otimes 2k} \to \mathrm{End}\,(\mathbb{C}^{2n})^{\otimes r} \otimes \mathrm{End}\,(\mathbb{C}^{2n})^{\otimes 2k}$$

we have

$$\varphi(F_a) = \sum_{b=1}^{r} (-P_{b\,r+a} + Q_{b\,r+a}), \qquad a = 1, \dots, 2k, \tag{3.20}$$

where we use the operators (2.8) and (2.11). The vanishing condition of Proposition 3.2 will be verified if we show that if $|\lambda| < k$ then

$$\big(\varphi(F_1) - k + 1\big) \dots \big(\varphi(F_{2k}) + k\big)\, S'^{(2k)} = 0, \tag{3.21}$$

where $S'^{(2k)}$ denotes the image of the symmetrizer in the Brauer algebra $\mathcal{B}_{2k}(-2n)$ acting of the last $2k$ copies of the tensor product space $(\mathbb{C}^{2n})^{\otimes (r+2k)}$. Due to (3.20) and relations (2.5), the desired identity (3.21) can be written in the form

$$(x_{r+1} - k + 1) \dots (x_{r+2k} - k + 1)\, S'^{(2k)} = 0,$$

where $x_{r+1}, \dots, x_{r+2k}$ denote the images of the Jucys–Murphy elements (2.1) of the Brauer algebra $\mathcal{B}_{r+2k}(-2n)$ under its action (2.10) in $(\mathbb{C}^{2n})^{\otimes (r+2k)}$. It is verified by the same argument as in the proof of Theorem 3.3. The leading term of $\chi(D_k)$ coincides with $(-1)^k e_k(\lambda_1^2, \dots, \lambda_n^2)$ as shown in [9] and [11]. $\square$

For $1 \leqslant m \leqslant 2n$ consider the polynomials in a variable u whose coefficients are Casimir elements for $\mathfrak{sp}_{2n}$ given by

$$D_m(u) = \frac{n - m/2 + 1}{n - m + 1}\, \mathrm{tr}\Big(F_1 + u + \frac{m-1}{2}\Big)\Big(F_2 + u + \frac{m-3}{2}\Big) \dots \Big(F_m + u - \frac{m-1}{2}\Big)\, S^{(m)}.$$

Corollary 3.9. *For the images under the Harish-Chandra isomorphism we have*

$$\chi : D_m(u) \mapsto \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^r \binom{2n - 2r + 1}{m - 2r}\, e_r(l_1^2, \dots, l_n^2\,|\,a) \prod_{i=0}^{m-2r-1} \Big(u - \frac{m-1}{2} + r + i\Big)$$

and

$$\chi : \frac{n-m/2+1}{n-m+1}\,\mathrm{tr}\,(-\partial_t+F_1t^{-1})\dots(-\partial_t+F_mt^{-1})\,S^{(m)}\mapsto$$
$$\sum_{r=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^r\binom{2n-2r+1}{m-2r}\,e_r(l_1^2,\dots,l_n^2\,|\,a)\,t^{-2r}\,(-\partial_t+r\,t^{-1})^{m-2r}.$$

Proof. Exactly as in the proof of Corollary 3.5, we use Lemma 3.1 to verify the relations $D_m(u)=(-1)^m\,D_m(-u)$ and

$$D_m(u+1/2)-D_m(u-1/2)=(2n-m+2)\,D_{m-1}(u),\qquad m\geqslant 1,$$

with $D_0(u)=1$. Since the same relation is satisfied by the polynomials which are claimed to be the images $\chi(D_m(u))$, the statement follows from Theorem 3.8. □

Finally, we obtain analogues of Theorem 3.8 and Corollary 3.9, where $S^{(m)}$ is replaced by $A^{(m)}$. We keep the notation $A^{(m)}$ for the image of the anti-symmetrizer under the action (2.10). Note that due to the minus signs in that formula, the operator $A^{(m)}$ acts in the tensor space (1.1) as the *symmetrization operator*. The corresponding Casimir elements coincide with those constructed in [6] and [10].

Theorem 3.10. *For any $k\geqslant 1$ the image of the Casimir element*

$$\mathrm{tr}\,(F_1+k-1)\dots(F_{2k}-k)\,A^{(2k)}\in\mathrm{U}(\mathfrak{sp}_{2n})^{\mathrm{Sp}_{2n}}\tag{3.22}$$

under the Harish-Chandra isomorphism coincides with $h_k(l_1^2,\dots,l_n^2\,|\,a)$.

Proof. Denote the Casimir element (3.22) by C_k. We use Proposition 3.2 and show that C_k vanishes in all representations $V(\lambda)$ of Sp_{2n}, where the partitions λ satisfy $|\lambda|<k$. Using the same realization of $V(\lambda)$ as in the proof of Theorem 3.8, we come to showing the following analogue of (3.21):

$$\big(\varphi(F_1)+k-1\big)\dots\big(\varphi(F_{2k})-k\big)\,A'^{(2k)}=0.$$

We verify that the image of the operator $(-x_{r+1}+k-1)\dots(-x_{r+2k}+k-1)$ on the vector space $(\mathbb{C}^{2n})^{\otimes(r+2k)}$ is contained in a direct sum of representations $V(\nu)$ of Sp_{2n} with $\nu_1<k$, while the image of the operator $A'^{(2k)}$ is contained in a direct sum of representations $V(\nu)$ with $\nu_1\geqslant k+1$. An easy calculation shows that the leading term of $\chi(C_k)$ coincides with $h_k(\lambda_1^2,\dots,\lambda_n^2)$. □

Now consider the polynomials in u given by

$$C_m(u)=\mathrm{tr}\Big(F_1+u+\frac{m-1}{2}\Big)\Big(F_2+u+\frac{m-3}{2}\Big)\dots\Big(F_m+u-\frac{m-1}{2}\Big)\,A^{(m)}.$$

Their coefficients are Casimir elements for $\mathfrak{sp}_{2n}$.

Corollary 3.11. *For the images under the Harish-Chandra isomorphism we have*

$$\chi : C_m(u) \mapsto \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \binom{2n+m-1}{m-2r} h_r(l_1^2, \dots, l_n^2 \,|\, a) \prod_{i=0}^{m-2r-1} \Big(u - \frac{m-1}{2} + r + i \Big)$$

and

$$\chi : \mathrm{tr}\, (-\partial_t + F_1 t^{-1}) \dots (-\partial_t + F_m t^{-1}) \, A^{(m)} \mapsto \\ \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \binom{2n+m-1}{m-2r} h_r(l_1^2, \dots, l_n^2 \,|\, a) \, t^{-2r} \, (-\partial_t + r \, t^{-1})^{m-2r}.$$

Proof. Lemma 3.1 implies $C_m(u) = (-1)^m \, C_m(-u)$ so that for the odd values $m = 2k-1$ we have $C_{2k-1}(0) = 0$, while $\chi(C_{2k}(1/2)) = \chi(C_{2k}(-1/2))$ is found from Theorem 3.10. Furthermore, we verify that

$$C_m(u+1/2) - C_m(u-1/2) = (2n+m-1) \, C_{m-1}(u), \qquad m \geqslant 1, \tag{3.23}$$

where $C_0(u) = 1$. This follows from Lemma 3.1 and the relation for the partial trace of the operator $A^{(m)}$ found by

$$\mathrm{tr}_m \, A^{(m)} = \frac{2n+m-1}{m} \, A^{(m-1)}$$

thus verifying (3.23). The same relation is satisfied by the polynomials which are claimed to be the images $\chi(C_m(u))$ as stated in the corollary. □

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