CONSTRUCTIONS OF COCOMPACT LATTICES IN CERTAIN HIGHER-RANK COMPLETE KAC–MOODY GROUPS

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ABSTRACT. Let G be a complete Kac–Moody group of rank $n \geq 2$ such that the Weyl group of G is a free product of cyclic groups of order 2. We construct new families of examples of cocompact lattices in G, many of which act transitively on the chambers of the building for G.

1. Introduction

Our main result, Theorem 1 below, constructs new cocompact lattices in certain complete Kac–Moody groups G of rank $n \geq 2$. By definition, a complete Kac–Moody group is the completion with respect to some topology of a minimal Kac–Moody group Λ over a finite field. We use the completion in the "building topology" (see [CaprRe]). Complete Kac–Moody groups are locally compact, totally disconnected topological groups, which act transitively on the chambers of their associated building Δ. For further background, see our earlier work [CapdTh], which considered complete Kac–Moody groups of rank $n = 2$.

We denote by B the standard Borel subgroup of G, which is the stabiliser in G of the standard chamber of Δ, and by P_i for $1 \leq i \leq n$ the standard parabolics, which are the stabilisers in G of the panels of the standard chamber of Δ. We denote by T a fixed maximal split torus of G with $T \leq B = \cap_{i=1}^{n} P_i$, and by $Z(G)$ the centre of G, which is finite and is the kernel of the action of G on Δ [CaprRe]. Each parabolic subgroup P_i has a Levi decomposition [CaprRe], with Levi complement denoted L_i.

Theorem 1. *Let G be a complete Kac-Moody group of rank $n \geq 2$ with generalised Cartan matrix $A = (A_{ij})$, defined over the finite field $\mathbb{F}_q$ of order q where $q = p^a$ with p a prime. Assume that $|A_{ij}| \geq 2$ for all $1 \leq i, j \leq n$.*

(1) *If $p = 2$, then G admits a chamber-transitive cocompact lattice Γ.*
(2) *If $q \equiv 3 \pmod 4$, then G admits a chamber-transitive cocompact lattice Γ and a cocompact lattice Γ' which has two orbits of chambers.*
(3) *If $q \equiv 1 \pmod 4$, then:*
    (a) *G admits a cocompact lattice Γ' which has two orbits of chambers; and*
    (b) *if in addition, for all $1 \leq i \leq n$ we have $L_i/Z(L_i) \cong \mathrm{PGL}_2(q)$, and for a non-split torus H_i of $[L_i, L_i]$ chosen among all the non-split tori of $[L_i, L_i]$ so that $N_T(H_i)$ is as big as possible, we have that $N_{T_0}(H_i) = N_{T_0}(H_j)$ for all $1 \leq i, j \leq n$ where $T_0 \in \mathcal{S}\mathrm{yl}_2(T)$, then G admits a chamber-transitive cocompact lattice Γ.*

Notice that the lattices in (1), (2) and (3a) do not require any special additional conditions on the L_i, thus for all G as in Theorem 1 we have constructed at least one cocompact lattice in G. The technical conditions in (3b) are precisely those required for our construction, and can hold under various simpler assumptions, for example if $L_i/Z(L_i) \cong \mathrm{PGL}_2(q)$ and $Z(L_i) \leq Z(G)$ for all $1 \leq i \leq n$, or for L_i as in [CapdTh, Theorem 1.1(3a)]. We will provide more explicit descriptions of the lattices Γ and Γ' in our proofs in Section 4 below.

As discussed in [CapdTh], it is interesting that the groups G we consider admit any cocompact lattices. In rank $n \geq 3$, the only previous constructions of cocompact lattices in non-affine complete Kac–Moody groups G that are known to us are as follows.

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- Rémy–Ronan [RéRo, Section 4.B] constructed cocompact chamber-transitive lattices in certain groups defined using a twin root datum. In their construction, the finite ground fields were "mixed" (that is, of distinct characteristics). However under the additional conditions that all $L_i/Z(L_i) \cong \mathrm{PGL}_2(q)$ and all $Z(L_i) \leq Z(G)$, their construction can be carried out for fixed ground field $\mathbb{F}_q$, and in this case some of the lattices we obtain in (1), (2) and (3b) above are equivalent to the lattices that can be obtained via the construction of [RéRo, Section 4.B]. We explain this further in Section 4.1 below.
- Gramlich–Horn–Mühlherr [GHM, Section 7.3] showed that for certain complete Kac–Moody groups G the fixed point set of a quasi-flip in G must be a cocompact lattice in G.
- Carbone–Cobbs [CarbCobb, Lemma 21] constructed a cocompact chamber-transitive lattice in G as in Theorem 1 in the case $n = 3$ and $p = q = 2$. Their lattice, which has panel stabilisers cyclic of order three, is the same as one of the groups Γ in (1) above.[1]

It is also interesting that many of the lattices we construct are chamber-transitive, since for affine buildings of dimension ≥ 2 there exist very few chamber-transitive lattices [KLTi]. The chamber-transitive lattices we obtain in Theorem 1 above generalise many of the edge-transitive lattices in [CapdTh, Theorem 1.1].

To prove Theorem 1, we use the action of G on the $(n, q+1)$–biregular tree X which is the Davis geometric realisation of the building Δ for G. We recall this realisation and G–action in Section 2 below. Since G acts on X cocompactly with compact vertex stabilisers, a subgroup $\Gamma \leq G$ is a cocompact lattice in G if and only if Γ acts on X cocompactly with finite vertex stabilisers (see [BL]).

We construct the lattices in Theorem 1 as fundamental groups Γ or Γ' of finite graphs of finite groups with universal covering tree X. The theory of tree lattices then implies that Γ and Γ' are cocompact lattices in the full automorphism group $\mathrm{Aut}(X)$ of X. Since $\mathrm{Aut}(X)$ is much larger than the Kac–Moody group G, the key to proving Theorem 1 is to show that Γ and Γ' embed in G. For this, we use covering theory for graphs of groups (see [B]). Specifically, we generalise and simplify our earlier embedding criterion for cocompact lattices [CapdTh, Proposition 3.1], in Section 3 below. Our assumptions on the generalised Cartan matrix for G then allow us in Section 4 to generalise results from [CapdTh] concerning the actions of the finite subgroups of G on X, and hence apply our embedding criterion.

2. The Davis realisation of the building for G

Let G be as in Theorem 1 above. In this section we recall the construction of the Davis geometric realisation X of the building Δ for G, and describe how the action of G on X induces a graph of groups $\mathbb{G}$. For background on graphs of groups, see for example [B, BL].

We denote by C_k the cyclic group of order k. Our assumptions on the generalised Cartan matrix for G imply that its Weyl group is $W = \langle s_1, \ldots, s_n \mid s_i^2 = 1 \text{ for } 1 \leq i \leq n\rangle$, that is, a free product of n copies of C_2. For the general construction of the Davis realisation X of a building, see [D]. In our case, each chamber K of X is the star graph obtained by coning on n vertices, one for each of the Coxeter generators s_i. We will say that the cone point of K has type 0 and is coloured red, and that the other n vertices of K have types $1 \leq i \leq n$ respectively and are coloured blue.

The Davis realisation X is then the $(n, q + 1)$–biregular tree with alternating red and blue vertices, each red vertex having valence n and each blue vertex valence $(q + 1)$. More precisely, the tree X is a union of copies of K glued together along blue vertices, so that each blue vertex has a unique type $1 \leq i \leq n$ and is

[1]As acknowledged in [CarbCobb], the covering morphism used in the proof of [CarbCobb, Lemma 21] is due to Anne Thomas. However the claim in the statement of [CarbCobb, Lemma 21] that the resulting embedding is nondiscrete is incorrect, as we now explain. We recall in Section 2 that when $n = 3$ there is a realisation of the building for G as a hyperbolic building with ideal vertices. In particular, the G–stabilisers of the ideal vertices may contain infinite discrete subgroups, which invalidates the reason given for nondiscreteness in the proof of [CarbCobb, Lemma 21]. Indeed in [CarbCobb, Section 7], the tree X which is naturally embedded in this hyperbolic building is used to construct an embedding into G of the abstract group $\Gamma = C_3 * C_3 * C_3$ as a cocompact lattice. Although it is claimed in [CarbCobb] that a different embedding of Γ is obtained by using X, in fact the embedding of this abstract group in [CarbCobb, Section 7] the same as that in [CarbCobb, Lemma 21]. This shows that the embedding in [CarbCobb, Lemma 21] is discrete.

contained in exactly $(q+1)$ chambers (and each red vertex has type 0). We will sometimes refer to the blue vertices of X as its *panels*.

The action of G on X is chamber-transitive and preserves the types of panels. The stabiliser of the standard chamber is B and for $1 \leq i \leq n$, the stabiliser in G of the panel of type i of the standard chamber is the standard parabolic $P_i = B \sqcup Bs_iB$. We may thus describe G as the fundamental group of a graph of groups $\mathbb{G}$ over the star graph K, with the red vertex group and all edge groups being B, the blue vertex group of type i being P_i and all monomorphisms the natural inclusions.

In the case $n = 2$, the red vertices of the tree X may be omitted, so that the chamber K is a single edge, X is a $(q+1)$–regular tree with two types of (blue) vertices and $G = P_1 *_B P_2$ is the fundamental group of an edge of groups. In the case $n = 3$, the building for G may be realised as a hyperbolic building with chambers ideal hyperbolic triangles. The stabilisers in G of the (ideal) vertices of the standard chamber of this hyperbolic building are the amalgamated free products $P_i *_B P_j$ for $1 \leq i < j \leq 3$, which are noncompact and so may contain infinite discrete subgroups. For all $n \geq 3$, the building Δ for G may be realised as a simplicial complex with chambers simplices on n vertices, so that the G–stabilisers of the vertices of Δ are noncompact and the Davis realisation X embeds naturally in the barycentric subdivision of Δ. The reason we consider the action of G on X rather than on Δ is that, as explained in the introduction, actions on X provide a straightforward characterisation of cocompact lattices in G.

3. Embedding criterion

In this section we establish our embedding criterion, Proposition 2 below. The definitions and results from covering theory for graphs of groups that we will need are recalled in [CapdTh, Section 2.2].

We continue all assumptions and notation from Section 2 above. In addition, we denote by $x_1, \ldots, x_n$ the (blue) vertices of X which have stabilisers $P_1, \ldots, P_n$ respectively, and by x_0 the (red) vertex of X which has stabiliser B. For $1 \leq i \leq n$ let f_i be the oriented edge of X with initial vertex x_i and terminal vertex x_0, so that $\overline{f_i}$ is the edge of X from x_0 to x_i. Given a vertex x of X, we denote by $E_X(x)$ the set of oriented edges of X with initial vertex x. We let $K_{m,n}$ be the complete bipartite graph with m red vertices and n blue vertices. The case $m = 1$ is the chamber K, that is, a star graph on n vertices. We denote the n blue vertices of $K_{m,n}$ by $a_1, \ldots, a_n$, and the m red vertices by $b_1, \ldots, b_m$. For $1 \leq i \leq n$ and $1 \leq k \leq m$, the oriented edge of $K_{m,n}$ from a_i to b_k will be denoted by $e_{i,k}$, and that from b_k to a_i by $\overline{e_{i,k}}$.

Proposition 2. *Let $m \geq 1$ be an integer. Suppose that for all $1 \leq i \leq n$ there are finite groups $A_i \leq P_i$ such that:*

1. *the group A_i has m orbits of equal size on $E_X(x_i)$;*
2. *for all $1 \leq j \leq n$ with $i \neq j$, we have $A_i \cap A_j = A_0 := \cap_{i=1}^n A_i$;*
3. *there are representatives $f_{i,1} = f_i$, $f_{i,2}$, ..., $f_{i,m}$ of the orbits of A_i on $E_X(x_i)$ and elements $g_{i,1} = 1$, $g_{i,2}$, ..., $g_{i,m} \in P_i$ such that for all $1 \leq k \leq m$:*
   (a) *$g_{i,k} \cdot f_{i,1} = f_{i,k}$; and*
   (b) *$A_i \cap B^{g_{i,k}} = A_0^{g_{i,k}}$.*

Let $\mathbb{A}$ be the graph of groups over $K_{m,n}$ with:

- *blue vertex groups $\mathcal{A}_{a_i} = A_i$ for $1 \leq i \leq n$, and red vertex groups and edge groups A_0;*
- *the monomorphism $\alpha_{e_{i,k}}$ from the edge group $\mathcal{A}_{e_{i,k}} = A_0$ into A_i inclusion composed with $\mathrm{ad}(g_{i,k})$, and all other monomorphisms inclusions.*

Then the fundamental group of the graph of groups $\mathbb{A}$ is a cocompact lattice in G, with quotient $K_{m,n}$.

Note that when $m = 1$, so that each A_i acts transitively on $E_X(x_i)$, Condition (3) reduces to the requirement that $A_i \cap B = A_0$.

Proof. We construct a covering of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$, where $\mathbb{G}$ is the graph of groups for G constructed in Section 2 above. Since $K_{m,n}$ is a finite graph and $A_0, A_1, \ldots, A_n$ are finite, the result follows.

Let $\theta : K_{m,n} \to K$ be the graph morphism given by $\theta(a_i) = x_i$ for $1 \leq i \leq n$, $\theta(b_k) = x_0$ for $1 \leq k \leq m$ and $\theta(e_{i,k}) = f_i$ and $\theta(\overline{e_{i,k}}) = \overline{f_i}$ for $1 \leq i \leq n$ and $1 \leq k \leq m$. We construct a morphism of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$ over θ as follows. All of the local maps $\phi_{a_i} : A_i \to P_i$, $\phi_{b_k} : A_0 \to B$ and $\phi_{e_{i,k}} : A_0 \to B$

are natural inclusions. We put $\phi(e_{i,k}) = g_{i,k}$ and $\phi(\overline{e_{i,k}}) = 1$. Then it is easy to check that Φ is a morphism of graphs of groups.

To show that Φ is a covering, we first show that for $1 \leq i \leq n$ the map

$$\Phi_{a_i/e_{i,k}} : \coprod_{k=1}^{m} \mathcal{A}_{a_i}/\alpha_{e_{i,k}}(\mathcal{A}_{e_{i,k}}) \to P_i/B$$

induced by $g \mapsto \phi_{a_i}(g)\phi(e_{i,k}) = gg_{i,k}$, for g representing a coset of $\alpha_{e_{i,k}}(\mathcal{A}_{e_{i,k}}) = A_0^{g_{i,k}}$ in $\mathcal{A}_{a_i} = A_i$, is a bijection. For this, we note that since the edges $f_{i,k} = g_{i,k} \cdot f_{i,1} = g_{i,k} \cdot f_i$ represent pairwise distinct A_i–orbits on $E_X(x_i)$, for all $g, h \in A_i$ and all $1 \leq k \neq k' \leq m$ the cosets $gg_{i,k}B$ and $hg_{i,k'}B$ are pairwise distinct. The conclusion that $\Phi_{a_i/e_{i,k}}$ is a bijection then follows from the hypothesis that $A_i \cap B^{g_{i,k}} = A_0^{g_{i,k}}$.

It remains to show that for $1 \leq i \leq n$ and $1 \leq k \leq m$ the map

$$\Phi_{b_k/\overline{e_{i,k}}} : \mathcal{A}_{b_k}/\alpha_{\overline{e_{i,k}}}(\mathcal{A}_{\overline{e_{i,k}}}) \to B/B$$

is a bijection, which is immediate since this is a map $A_0/A_0 \to B/B$. We conclude that $\Phi : \mathbb{A} \to \mathbb{G}$ is a covering of graphs of groups, as desired. □

4. Constructions of lattices

We now complete the proof of Theorem 1, by applying Proposition 2 above to construct the chamber-transitive lattices Γ in Section 4.1 and the lattices Γ' which have two orbits of chambers in Section 4.2. Recall that for each $1 \leq i \leq n$, the Levi complement L_i factors as $L_i = M_iT$, where $M_i \cong A_1(q)$ is normalised by T. We denote by H_i a non-split torus of M_i such that $N_T(H_i)$ is as big as possible.

4.1. **Chamber-transitive case.**

We will apply Proposition 2 with $m = 1$.

We first consider the case $p = 2$. As explained in [CapdTh, Section 3.2.1], in this case $H_i \cong C_{q+1}$. The edges $E_X(x_i)$ may be identified with the cosets of B in P_i, and then the same proof as for [CapdTh, Lemma 3.2] shows that H_i acts simply transitively on $E_X(x_i)$. Let A_0 be any subgroup of $Z(G)$ and for $1 \leq i \leq n$ put $A_i := A_0 \times H_i$. The conditions of Proposition 2 are then easily verified, using the fact that $H_i \cap B$ is trivial. Thus we have constructed a cocompact lattice $\Gamma \leq G$ which acts on X with quotient the standard chamber K, that is, a chamber-transitive lattice $\Gamma \leq G$.

If in the case $p = 2$, we have in addition that $Z(L_i) \leq Z(G)$ for all $1 \leq i \leq n$, then the lattice Γ just constructed would be the lattice obtained via the construction of Rémy–Ronan [RéRo, Section 4.B].

Suppose now that $q \equiv 3 \pmod 4$. Let $T_0 \in \mathcal{S}\mathrm{yl}_2(T)$. For each $1 \leq i \leq n$, consider the group $N_{L_i}(H_i)$. If $L_i/Z(L_i) \cong \mathrm{PSL}_2(q)$, then $N_{L_i}(H_i) = N_{M_i}(H_i)C_T(L_i)$ where $C_T(L_i) \cap N_{M_i}(H_i) = Z(M_i)$ and $[C_T(L_i), N_{M_i}(H_i)] = 1$. In this case put $A_i := N_{M_i}(H_i)T_0Z_0$ where Z_0 is any subgroup of $Z(G)$. By the same proof as for [CapdTh, Lemma 3.3], the group $N_{M_i}(H_i)$ acts transitively on $E_X(x_i)$, and therefore so does A_i. Moreover, $A_i \cap T = T_0Z_0$. If on the other hand $L_i/Z(L_i) \cong \mathrm{PGL}_2(q)$, then $N_{L_i}(H_i) = H_iQ_i'T_0C_T(L_i)$ where $Q_i' \in \mathcal{S}\mathrm{yl}_2(C_{L_i}(H_i))$, $C_T(L_i) \cap H_iQ_i'T_0 = C_{T_0}(L_i)$ and $[C_T(L_i), H_iQ_i'T_0] = 1$. This time put $A_i := H_iQ_i'T_0Z_0$ where Z_0 is again any subgroup of $Z(G)$. By the same proof as for [CapdTh, Lemma 3.4], it follows that the group $A_i := H_iQ_i'T_0Z_0$ acts transitively on $E_X(x_i)$. Moreover, $A_i \cap T = T_0Z_0$. If we now fix a subgroup $Z_0 \leq Z(G)$, then independently of whether each $L_i/Z(L_i)$ is isomorphic to $\mathrm{PSL}_2(q)$ or to $\mathrm{PGL}_2(q)$, we obtain that $A_i \cap A_j = T_0Z_0$ for all $1 \leq i \neq j \leq n$. It follows that if $A_0 := T_0Z_0$ then $A_i \cap B = A_i \cap T = T_0Z_0 = A_0$ for $1 \leq i \leq n$. The existence of a chamber-transitive lattice $\Gamma \leq G$ in this case then follows from Proposition 2.

Notice that if $L_i/Z(L_i) \cong \mathrm{PGL}_2(q)$ and $Z(L_i) \leq Z(G)$ for all $1 \leq i \leq n$, we could simply take $A_i := H_iQ_i'Z_0$, since H_iQ_i' acts transitively on $E_X(x_i)$ as explained in [CapdTh, Lemma 3.4], and $A_i \cap B = H_iQ_i'Z_0 \cap T \leq Z(L_i) \leq Z(G)$. This would generalise (3)(b)(ii) of [CapdTh, Theorem 1.1] to construct another chamber-transitive lattice Γ in G. This would also be the lattice obtained by the construction of [RéRo, Section 4.B].

Finally, suppose that $q \equiv 1 \pmod 4$, that $L_i/Z(L_i) \cong \mathrm{PGL}_2(q)$ for all $1 \leq i \leq n$ and that $N_{T_0}(H_i) = N_{T_0}(H_j)$ for $1 \leq i, j \leq n$ where $T_0 \in \mathcal{S}\mathrm{yl}_2(T)$. Take any $1 \leq i \leq n$. Notice that $H_i \cap T = Z(M_i)$ and $H_i = H_i' \times Z(M_i)$ with $H_i' \cong C_{\frac{q+1}{2}}$. Let $Q_i' \in \mathcal{S}\mathrm{yl}_2(C_{L_i}(H_i))$. Then $|Q_i' : Q_i' \cap T| = |Q_i' : Q_i' \cap T_0| = 2$, $Q_i' \cap T = C_{T_0}(H_i)$ and by the same proof as for [CapdTh, Lemma 3.5], it follows that the group H_iQ_i' acts transitively on $E_X(x_i)$. Take $A_i := H_i Q_i' N_{T_0}(H_i)$. Then A_i acts transitively on $E_X(x_i)$ and $A_i \cap A_j = N_{T_0}(H_i) = N_{T_0}(H_j)$. Putting $A_0 := N_{T_0}(H_i)$ for any i, we then have $A_i \cap B = A_i \cap T = N_{T_0}(H_i) = A_0$ for all i. The existence of a chamber-transitive lattice $\Gamma \leq G$ in this case then follows from Proposition 2.

Notice that for $n = 2$, under the technical conditions on G listed in (3)(a)(i) of [CapdTh, Theorem 1.1], the conclusion of (3)(a)(i) of [CapdTh, Theorem 1.1] coincides with our current conclusion. We could describe analogous conditions for larger n, but have omitted them since they would be tedious.

Finally, if $L_i/Z(L_i) \cong \mathrm{PGL}_2(q)$ and $Z(L_i) \leq Z(G)$ for all $1 \leq i \leq n$, we could also generalise (3)(a)(ii) of [CapdTh, Theorem 1.1] to construct another chamber-transitive lattice Γ in G, which would be the lattice that can be obtained in this case via the construction of [RéRo, Section 4.B].

4.2. **Construction of Γ'.** We will apply Proposition 2 with $m = 2$.

Suppose first that $q \equiv 1 \pmod 4$. Then $q + 1 = 2r$ where r is odd and $H_i \cong C_2 \times C_r$. Take $A_i \leq H_i$ such that $A_i \cong C_r$. That is, A_i is the unique subgroup of H_i of index 2 and $|A_i| = \frac{q+1}{2}$ is odd. By the same arguments as in [CapdTh, Section 3.3.2], each A_i has 2 orbits of equal size $\frac{1}{2}(q+1)$ on $E_X(x_i)$, and we may choose an edge $f_{i,2} \in E_X(x_i)$ so that f_i and $f_{i,2}$ represent these two orbits (notice that these orbits are exactly the same as the orbits of N_i from [CapdTh, Section 3.3.2]). Since L_i acts transitively on the set $E_X(x_i)$, we may choose an element $g_{i,2} \in L_i$ such that $f_{i,2} = g_{i,2} \cdot f_i$. As $A_i \cap A_j = 1$ for $1 \leq i \neq j \leq n$, let A_0 be the trivial group. Since $(|A_i|, |T|) = 1$ for all $1 \leq i \leq n$, we have $A_i \cap T^g = 1$ for all $g \in G$, and so $A_i \cap B = 1 = A_0$ and $A_i \cap B^{g_{i,2}} = 1 = A_0^{g_{i,2}}$ for $1 \leq i \leq n$. We may now apply Proposition 2 to obtain a cocompact lattice $\Gamma' \leq G$ which acts with two orbits of chambers.

Finally suppose $q \equiv 3 \pmod 4$. Notice that if $T_0 \in \mathcal{S}\mathrm{yl}_2(T)$, then T_0 is a group of exponent 2 and T_0 normalises H_i. Take $A_i := H_i T_0$ for $1 \leq i \leq n$ and put $A_0 := T_0$. Since A_i intersects every Borel subgroup of L_i in T_0, by the same arguments as in [CapdTh, Section 3.3.2], each A_i has 2 orbits of equal size $\frac{q+1}{2}$ on $E_X(x_i)$. Moreover, we may choose an edge $f_{i,2} \in E_X(x_i)$ so that f_i and $f_{i,2}$ represent these two orbits and moreover f_i and $f_{i,2}$ are the only two edges of $E_X(x_i)$ which are fixed by T. Since $A_0 \neq 1$, this time we must choose the element $g_{i,2}$ a bit more carefully. Take $g_{i,2}$ to be an element of $N_{P_i}(T)$ which represents the Weyl group generator s_i. It then follows that $f_{i,2} = g_{i,2} \cdot f_i$. We also have $A_i \cap B = T_0 = A_0$ and, as T_0 is a characteristic subgroup of T, $A_i \cap B^{g_{i,2}} = T_0 = A_0 = A_0^{g_{i,2}}$ for $1 \leq i \leq n$. We may now again apply Proposition 2 to obtain a cocompact lattice $\Gamma' \leq G$ which acts with two orbits of chambers.

This completes the proof of Theorem 1.

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