Feigin–Frenkel center in types B, C and D

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Abstract

For each simple Lie algebra $\mathfrak{g}$ consider the corresponding affine vertex algebra $V_{\text{crit}}(\mathfrak{g})$ at the critical level. The center of this vertex algebra is a commutative associative algebra whose structure was described by a remarkable theorem of Feigin and Frenkel about two decades ago. However, only recently simple formulas for the generators of the center were found for the Lie algebras of type A following Talalaev's discovery of explicit higher Gaudin Hamiltonians. We give explicit formulas for generators of the centers of the affine vertex algebras $V_{\text{crit}}(\mathfrak{g})$ associated with the simple Lie algebras $\mathfrak{g}$ of types B, C and D. The construction relies on the Schur–Weyl duality involving the Brauer algebra, and the generators are expressed as weighted traces over tensor spaces and, equivalently, as traces over the spaces of singular vectors for the action of the Lie algebra $\mathfrak{sl}_2$ in the context of Howe duality. This leads to an explicit construction of a commutative subalgebra of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}[t])$ and to higher order Hamiltonians in the Gaudin model associated with each Lie algebra $\mathfrak{g}$. We also introduce analogues of the Bethe subalgebras of the Yangians $\mathrm{Y}(\mathfrak{g})$ and show that their graded images coincide with the respective commutative subalgebras of $\mathrm{U}(\mathfrak{g}[t])$.

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1 Introduction

For each simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ the vacuum module $V_\kappa(\mathfrak{g})$ at the level $\kappa \in \mathbb{C}$ over the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ has a vertex algebra structure. The family of affine vertex algebras $V_\kappa(\mathfrak{g})$ has profound connections in geometry and mathematical physics; see e.g. [8], [15] and [24].

The critical value $\kappa = -h^\vee$, where $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$, plays a particular role, as the affine vertex algebra at the critical level contains a big center $\mathfrak{z}(\widehat{\mathfrak{g}})$. The structure of the center was described by a remarkable theorem of Feigin and Frenkel in [11] which states that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of polynomials in infinitely many variables associated with the algebra of invariants $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ in the symmetric algebra. The vertex algebra structure on the vacuum module brings up a few bridges connecting the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ with the representation theory of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$. In particular, the higher Sugawara operators associated with the elements of the center provide singular vectors in the Verma modules $M(\lambda)$ at the critical level which leads to a proof of the Kac–Kazhdan conjecture for the characters of the irreducible quotients $L(\lambda)$ (for the classical types the proofs were given in [17] and [18]). Moreover, the Sugawara operators act as scalars in the Wakimoto modules over $\widehat{\mathfrak{g}}$ thus providing an affine analogue of the Harish-Chandra homomorphism and leading to a description of the associated (Langlands dual) classical $\mathcal{W}$-algebra; see [14] for a detailed exposition of these results, and also [1] and [16] for the role of the center in the representation theory of $\widehat{\mathfrak{g}}$. A striking connection of $\mathfrak{z}(\widehat{\mathfrak{g}})$ and the Hamiltonians of the quantum Gaudin model was discovered in [12]; see also more recent work [13] for extensions to some generalized Gaudin models. This connection, together with a simple explicit construction of the higher Gaudin Hamiltonians by Talalaev [32] in type A, has produced equally simple formulas for generators of the Feigin–Frenkel center; see [6], [7]. A super-version of this construction was given in [27] together with simpler arguments in the purely even case.

The main result of this paper is an explicit construction of generators of the Feigin–Frenkel center for each simple Lie algebra $\mathfrak{g} = \mathfrak{g}_N$ of types B, C and D. The construction is based on the properties of the symmetrizer in the Brauer algebra which centralizes the action of the orthogonal or symplectic group on a tensor product of the vector representations in the context of the Schur–Weyl duality. The formulas for the central elements are also presented in the context of the Howe duality involving a representation of the Lie algebra $\mathfrak{g}_N$ on the symmetric and exterior powers of the vector space $\mathbb{C}^N$ and a dual action of the Lie algebra $\mathfrak{sl}_2$.

As a corollary of the main result we get explicit generators of the respective commutative subalgebras of the universal enveloping algebras $\mathrm{U}(\mathfrak{g}_N[t])$ which leads to an explicit construction of higher order Hamiltonians of the Gaudin model associated with $\mathfrak{g}_N$. Moreover, we use the symmetrizer in the Brauer algebra to produce commutative Bethe-type subalgebras in the Drinfeld Yangians $\mathrm{Y}(\mathfrak{g}_N)$ and show that their graded images coincide with the respective commutative subalgebras of $\mathrm{U}(\mathfrak{g}_N[t])$.

To describe the results in more detail, recall that the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \qquad \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], \tag{1.1}$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t. The vacuum module $V_{-h^\vee}(\mathfrak{g})$ at the critical level over $\widehat{\mathfrak{g}}$ is defined as the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K + h^\vee$. The center of the vertex algebra $V_{-h^\vee}(\mathfrak{g})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{S \in V_{-h^\vee}(\mathfrak{g}) \mid \mathfrak{g}[t]\, S = 0\}.$$

Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a *Segal–Sugawara vector*. The vertex algebra axioms imply that the center is a commutative associative algebra. As a vector space, the vacuum module $V_{-h^\vee}(\mathfrak{g})$ can be identified with the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_-)$, where $\mathfrak{g}_- = t^{-1}\mathfrak{g}[t^{-1}]$. Moreover, the multiplication in the commutative algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ coincides with the multiplication in $\mathrm{U}(\mathfrak{g}_-)$ so that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is naturally identified with a commutative subalgebra of $\mathrm{U}(\mathfrak{g}_-)$. For any element $S \in \mathrm{U}(\mathfrak{g}_-)$ denote by $\overline{S}$ its *symbol*, i.e., the image in the associated graded algebra $\mathrm{gr}\, \mathrm{U}(\mathfrak{g}_-) \cong \mathrm{S}(\mathfrak{g}_-)$. A set of elements $S_1, \dots, S_n \in \mathfrak{z}(\widehat{\mathfrak{g}})$, $n = \mathrm{rank}\, \mathfrak{g}$, is called a *complete set of Segal–Sugawara vectors* if the corresponding symbols $\overline{S}_1, \dots, \overline{S}_n$ coincide with the images of certain algebraically independent generators of the algebra of invariants $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ under the embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}(\mathfrak{g}_-)$ defined by the assignment $x \mapsto x\, t^{-1}$ for $x \in \mathfrak{g}$. According to Feigin and Frenkel [11], a complete set of Segal–Sugawara vectors exists for any $\mathfrak{g}$, and $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of polynomials in infinitely many variables,

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n, \quad r \geqslant 0], \tag{1.2}$$

where T denotes the translation operator on the vertex algebra; see Sec. 2.

Now let $\mathfrak{g} = \mathfrak{g}_N$ be the orthogonal Lie algebra $\mathfrak{o}_N$ or the symplectic Lie algebra $\mathfrak{sp}_N$. Recall that the degrees of algebraically independent generators of the algebra $\mathrm{S}(\mathfrak{g}_N)^{\mathfrak{g}_N}$ are $2, 4, \dots, 2n$ for $\mathfrak{g}_N = \mathfrak{o}_{2n+1}$ and $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ (types B and C, respectively), while for $\mathfrak{g}_N = \mathfrak{o}_{2n}$ (type D) the degrees are $2, 4, \dots, 2n-2, n$. In the latter case we let ϕ'_n denote a Segal–Sugawara vector whose symbol coincides with the image of the distinguished Pfaffian-type invariant of degree n in the symmetric algebra $\mathrm{S}(\mathfrak{o}_{2n})$. Introduce standard generators F_{ij} of $\mathfrak{g}_N$; see Sec. 2 for the definitions. Then $F_{ij}[r] = F_{ij}\, t^r$ with $r \in \mathbb{Z}$ are the corresponding elements of the loop algebra $\mathfrak{g}_N[t, t^{-1}]$. We will need the extended Lie algebra $\widehat{\mathfrak{g}}_N \oplus \mathbb{C}\tau$ with the element τ satisfying the commutation relations

$$\big[\tau, F_{ij}[r]\big] = -r\, F_{ij}[r-1], \qquad \big[\tau, K\big] = 0.$$

Introduce the matrix $\Phi = [\Phi_{ij}]$ with $\Phi_{ij} = \delta_{ij}\tau + F_{ij}[-1]$ and identify it with the element

$$\sum_{i,j=1}^{N} e_{ij} \otimes \Phi_{ij} \in \mathrm{End}\, \mathbb{C}^N \otimes \mathrm{U},$$

where the e_{ij} are the standard matrix units and U stands for the universal enveloping algebra of $\widehat{\mathfrak{g}}_N \oplus \mathbb{C}\tau$. For each $a \in \{1, \dots, m\}$ the element Φ_a of the algebra

$$\underbrace{\mathrm{End}\, \mathbb{C}^N \otimes \dots \otimes \mathrm{End}\, \mathbb{C}^N}_{m} \otimes \mathrm{U} \tag{1.3}$$

is defined by

$$\Phi_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes \Phi_{ij}.$$

Consider the *Brauer algebra* $\mathcal{B}_m(\omega)$ over $\mathbb{C}(\omega)$ and let $S^{(m)} \in \mathcal{B}_m(\omega)$ denote the *symmetrizer* in $\mathcal{B}_m(\omega)$ which is the primitive idempotent associated with the trivial representation of $\mathcal{B}_m(\omega)$. A few equivalent expressions for the symmetrizer are given in Sec. 3.1. The classical orthogonal and symplectic groups O_N and Sp_N act naturally on the tensor product space $(\mathbb{C}^N)^{\otimes m}$. By the results of Brauer [5], the centralizer of this action in the endomorphism algebra of the tensor product space coincides with the homomorphic image of the algebra $\mathcal{B}_m(\omega)$ with the parameter ω specialized to N and $-N$, respectively, in the orthogonal and symplectic case. Using the natural action of the Brauer algebra on $(\mathbb{C}^N)^{\otimes m}$ we will regard the symmetrizer $S^{(m)}$ as an element of the algebra (1.3) with the identity component in U. In the orthogonal case, taking trace over all m copies of $\mathrm{End}\,\mathbb{C}^N$, write the following element as a polynomial in τ with coefficients in the universal enveloping algebra $\mathrm{U}(t^{-1}\mathfrak{g}_N[t^{-1}])$:

$$\mathrm{tr}\, S^{(m)}\Phi_1 \dots \Phi_m = \phi_{m\,0}\, \tau^m + \phi_{m\,1}\, \tau^{m-1} + \dots + \phi_{m\,m}. \tag{1.4}$$

Defining analogues of the elements $\phi_{m\,k}$ in the symplectic case requires an extra care. The image of the element $S^{(m)} \in \mathcal{B}_m(-2n)$ in (1.3) with $N = 2n$ is well-defined for $m \leqslant n+1$ but it is zero for $m = n+1$, and the specialization of $S^{(m)} \in \mathcal{B}_m(\omega)$ at $\omega = -2n$ is not defined for $n+2 \leqslant m \leqslant 2n$. However, using a symmetry property of the matrix Φ, we may regard the product $\Phi_1 \dots \Phi_m$ as a polynomial in τ with coefficients in the algebra of the form (1.3) where each factor $\mathrm{End}\,\mathbb{C}^{2n}$ is replaced by the subspace of symplectic matrices or by the subspace of scalar matrices. Restricting the operator $S^{(m)}$ to the tensor products of these subspaces allows us to write

$$\frac{1}{n-m+1}\, \mathrm{tr}\, S^{(m)}\Phi_1 \dots \Phi_m = \phi_{m\,0}\, \tau^m + \phi_{m\,1}\, \tau^{m-1} + \dots + \phi_{m\,m}, \tag{1.5}$$

so that the elements $\phi_{m\,k}$ are well-defined for all values of the parameters with $1 \leqslant m \leqslant 2n$.

Main Theorem. (i) *All elements $\phi_{m\,k}$ are Segal–Sugawara vectors for $\mathfrak{g}_N$.*
(ii) *$\{\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}\}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$.*
(iii) *$\{\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n-2\,2n-2}, \phi'_n\}$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n}$.*

The Segal–Sugawara vectors $\phi_{m\,k}$ defined by (1.4) and (1.5) admit an alternative presentation with the use of the respective Howe dual pair $(\mathfrak{sl}_2, \mathfrak{g}_N)$; see [19]. Namely, the operator $S^{(m)}$ projects the vector space $(\mathbb{C}^N)^{\otimes m}$ to a subspace of symmetric tensors in the orthogonal case and to a subspace of skew-symmetric tensors in the symplectic case. Using natural identifications of these subspaces with the respective homogeneous components of the symmetric algebra $\mathrm{S}(\mathbb{C}^N)$ and exterior algebra $\Lambda(\mathbb{C}^N)$, we find that the action of $S^{(m)}$ coincides with that of the extremal projector $p = p(\mathfrak{sl}_2)$ associated with the commuting action of the Lie algebra $\mathfrak{sl}_2$. Therefore, the traces on the left hand sides of (1.4) and (1.5) can be replaced by the respective traces $\operatorname{tr} p\, \Phi^{(m)}$ taken over the subspace of $\mathfrak{sl}_2$-singular vectors in the homogeneous components of degree m in $\mathrm{S}(\mathbb{C}^N)$ and $\Lambda(\mathbb{C}^N)$, where $\Phi^{(m)}$ denotes the restriction of the element $\Phi_1 \dots \Phi_m$ to this subspace.

I am grateful to Alexander Chervov and Eric Ragoucy for collaboration in [6] and [27] and discussions of examples and approaches to the Sugawara operators for the classical Lie algebras. I also thank Tomoyuki Arakawa, Joe Chuang, Pavel Etingof and Evgeny Mukhin for stimulating discussions.

2 Affine vertex algebras in types B, C and D

We will be working with a particular family of the affine vertex algebras $V_\kappa(\mathfrak{g}_N)$ associated with the classical Lie algebras $\mathfrak{g}_N$. To introduce necessary definitions, consider the vector space $\mathbb{C}^N$ with its canonical basis $e_1, \dots, e_N$ and equip it with a nondegenerate symmetric or skew-symmetric bilinear form

$$\langle e_i, e_j \rangle = g_{ij} \tag{2.1}$$

so that the matrix $G = [g_{ij}]$ is symmetric or skew-symmetric, respectively, $G^t = \pm G$, where t denotes the standard matrix transposition. The skew-symmetric case may only occur for even N. The respective classical group $\mathrm{G}_N = \mathrm{O}_N$ or $\mathrm{G}_N = \mathrm{Sp}_N$ is defined as the group of complex matrices preserving the form (2.1). For any $N \times N$ matrix A set $A' = G A^t G^{-1}$. For matrices with entries in $\mathbb{C}$ this is just the transpose with respect to the form determined by the inverse matrix $G^{-1} = [\overline{g}_{ij}]$. Denote by E_{ij}, $1 \leqslant i, j \leqslant N$, the standard basis vectors of the Lie algebra $\mathfrak{gl}_N$. Introduce the elements F_{ij} of $\mathfrak{gl}_N$ by the formulas

$$F_{ij} = E_{ij} - \sum_{k,l=1}^{N} g_{ik}\, \overline{g}_{lj}\, E_{lk}. \tag{2.2}$$

The Lie subalgebra of $\mathfrak{gl}_N$ spanned by the elements F_{ij} is isomorphic to the orthogonal Lie algebra $\mathfrak{o}_N$ associated with O_N in the symmetric case and to the symplectic Lie algebra $\mathfrak{sp}_N$ associated with Sp_N in the skew-symmetric case. Denote by F the $N \times N$ matrix whose (i, j) entry is F_{ij}. We shall also regard F as the element

$$F = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}(\mathfrak{g}_N).$$

The definition (2.2) can be written in the matrix form as $F = E - E'$, where $E = [E_{ij}]$. Consider the permutation operator

$$P = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N \tag{2.3}$$

and set

$$Q = G_1 P^t G_1^{-1} = G_2 P^t G_2^{-1}, \qquad P^t = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ij}, \tag{2.4}$$

where $G_1 = G \otimes 1$ and $G_2 = 1 \otimes G$. Note that the operators P and Q satisfy the relations

$$P^2 = 1, \qquad Q^2 = N\,Q \qquad PQ = QP = \begin{cases} Q & \text{in the symmetric case,} \\ -Q & \text{in the skew-symmetric case.} \end{cases} \tag{2.5}$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{g}_N)$ have the form

$$F_1\, F_2 - F_2\, F_1 = (P - Q)\, F_2 - F_2\, (P - Q) \tag{2.6}$$

together with the relation $F + F' = 0$, where both sides in (2.6) are regarded as elements of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{g}_N)$ and

$$F_1 = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes F_{ij}, \qquad F_2 = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes F_{ij}. \tag{2.7}$$

Now consider the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_N = \mathfrak{g}_N\,[t, t^{-1}] \oplus \mathbb{C}K$ and set $F_{ij}[r] = F_{ij}t^r$ for any $r \in \mathbb{Z}$. As with the Lie algebra $\mathfrak{g}_N$, introduce the matrix $F[r] = \big[F_{ij}[r]\big]$ and regard it as the element

$$F[r] = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij}[r] \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{U}(\widehat{\mathfrak{g}}_N).$$

By analogy with (2.6), the defining relations of the algebra $\mathrm{U}(\widehat{\mathfrak{g}}_N)$ can be written in the form (cf. [23]):

$$\begin{aligned} F[r]_1\, F[s]_2 - F[s]_2\, F[r]_1 &= (P - Q)\, F[r+s]_2 - F[r+s]_2\, (P - Q) \\ &\quad + \begin{cases} r\,\delta_{r,-s}\,(P - Q)\,K & \text{in the orthogonal case,} \\ 2r\,\delta_{r,-s}\,(P - Q)\,K & \text{in the symplectic case.} \end{cases} \end{aligned} \tag{2.8}$$

A *vertex algebra* V is a vector space with the additional data $(Y, T, \mathbf{1})$, where the state-field correspondence Y is a map $Y : V \to \mathrm{End}\,V[[z, z^{-1}]]$, the infinitesimal translation T is an operator $T : V \to V$, and $\mathbf{1}$ is a vacuum vector $\mathbf{1} \in V$. For the axioms satisfied by these data see e.g., [8], [15], [24]. For any element $a \in V$ we write

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}\,V.$$

The *center* of a vertex algebra V is its commutative vertex subalgebra spanned by all vectors $b \in V$ such that $a_{(n)}\, b = 0$ for all $a \in V$ and $n \geqslant 0$.

For any $\kappa \in \mathbb{C}$ define the *affine vertex algebra* $V_\kappa(\mathfrak{g}_N)$ as the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}}_N)$ by the left ideal generated by $\mathfrak{g}_N[t]$ and $K - \kappa$. The vacuum vector is 1 and the translation operator is determined by

$$T : 1 \mapsto 0, \quad \big[T, K\big] = 0 \quad \text{and} \quad \big[T, F_{ij}[r]\big] = -r\, F_{ij}[r-1].$$

The state-field correspondence Y is defined by setting $Y(1,z)=\mathrm{id}$, $Y(K,z)=K$,

$$Y(F_{ij}[-1],z)=F_{ij}(z):=\sum_{r\in\mathbb{Z}}F_{ij}[r]\,z^{-r-1}, \tag{2.9}$$

and then extending the map to the whole of $V_\kappa(\mathfrak{g}_N)$ with the use of normal ordering; see *loc. cit.*

3 Brauer algebra and Howe pairs

Let ω be an indeterminate and m a positive integer. The *Brauer algebra* $\mathcal{B}_m(\omega)$ over the field $\mathbb{C}(\omega)$ is generated by $2m-2$ elements $s_1,\dots,s_{m-1},\epsilon_1,\dots,\epsilon_{m-1}$ subject to the defining relations

$$\begin{aligned}
&s_i^2=1, \qquad \epsilon_i^2=\omega\,\epsilon_i, \qquad s_i\epsilon_i=\epsilon_i s_i=\epsilon_i, \qquad i=1,\dots,m-1,\\
&s_is_j=s_js_i, \qquad \epsilon_i\epsilon_j=\epsilon_j\epsilon_i, \qquad s_i\epsilon_j=\epsilon_js_i, \qquad |i-j|>1,\\
&s_is_{i+1}s_i=s_{i+1}s_is_{i+1}, \qquad \epsilon_i\epsilon_{i+1}\epsilon_i=\epsilon_i, \qquad \epsilon_{i+1}\epsilon_i\epsilon_{i+1}=\epsilon_{i+1},\\
&s_i\epsilon_{i+1}\epsilon_i=s_{i+1}\epsilon_i, \qquad \epsilon_{i+1}\epsilon_is_{i+1}=\epsilon_{i+1}s_i, \qquad i=1,\dots,m-2.
\end{aligned}$$

This algebra was originally defined by Brauer [5] in terms of diagrams and it is well-known it admits the above presentation. The subalgebra of $\mathcal{B}_m(\omega)$ generated over $\mathbb{C}$ by $s_1,\dots,s_{m-1}$ is isomorphic to the group algebra $\mathbb{C}[\mathfrak{S}_m]$ so that s_i can be identified with the transposition $(i,i+1)$. For any $1\leqslant i<j\leqslant m$ we will consider the transpositions $s_{ij}=(i,j)$ and the elements ϵ_{ij} defined in terms of the generators s_i and ϵ_i by

$$s_{ij}=s_i\,s_{i+1}\dots s_{j-2}\,s_{j-1}\,s_{j-2}\dots s_{i+1}\,s_i \qquad \text{and} \qquad \epsilon_{ij}=s_{i,j-1}\,\epsilon_{j-1}\,s_{i,j-1}.$$

The Brauer algebra $\mathcal{B}_{m-1}(\omega)$ can be regarded as a natural subalgebra of $\mathcal{B}_m(\omega)$.

3.1 Symmetrizer in the Brauer algebra

Consider the one-dimensional representation of $\mathcal{B}_m(\omega)$ where each element s_{ij} acts as the identity operator and each element ϵ_{ij} acts as the zero operator. The corresponding idempotent $S^{(m)}\in\mathcal{B}_m(\omega)$ then has the properties

$$s_{ij}\,S^{(m)}=S^{(m)}\,s_{ij}=S^{(m)} \qquad \text{and} \qquad \epsilon_{ij}\,S^{(m)}=S^{(m)}\,\epsilon_{ij}=0 \tag{3.1}$$

for all i,j, and it can be given by several equivalent formulas. The first one relies on the Jucys–Murphy elements for the Brauer algebra and their eigenvalues in the irreducible representations found independently in [25] and [29]. The corresponding formula reads

$$S^{(m)}=\prod_{r=2}^{m}\frac{1}{r\,(\omega+2r-4)}\Big(1+\sum_{i=1}^{r-1}(s_{ir}-\epsilon_{ir})\Big)\Big(\omega+r-3+\sum_{i=1}^{r-1}(s_{ir}-\epsilon_{ir})\Big). \tag{3.2}$$

The ordering of the factors is irrelevant as they commute pairwise. Two more expressions for the symmetrizer are related to *fusion procedures* for the Brauer algebra; see [20], [21] and references therein. We have

$$S^{(m)} = \frac{1}{m!} \prod_{1\leqslant i<j\leqslant m} \Big(1 - \frac{\epsilon_{ij}}{\omega + i + j - 3}\Big) \prod_{1\leqslant i<j\leqslant m} \Big(1 + \frac{s_{ij}}{j-i}\Big) \tag{3.3}$$

and

$$S^{(m)} = \frac{1}{m!} \prod_{1\leqslant i<j\leqslant m} \Big(1 + \frac{s_{ij}}{j-i} - \frac{2\,\epsilon_{ij}}{\omega + 2j - 2i - 2}\Big), \tag{3.4}$$

where all products are taken in the lexicographic order on the pairs (i,j). It is well-known by Jucys [22] that the second product in (3.3) coincides with the symmetrizer in the group algebra $\mathbb{C}[\mathfrak{S}_m]$:

$$H^{(m)} := \frac{1}{m!} \prod_{1\leqslant i<j\leqslant m} \Big(1 + \frac{s_{ij}}{j-i}\Big) = \frac{1}{m!} \sum_{s\in\mathfrak{S}_m} s.$$

We will need yet another formula for $S^{(m)}$.

Proposition 3.1. *We have*

$$S^{(m)} = H^{(m)} \sum_{r=0}^{\lfloor m/2\rfloor} \frac{(-1)^r}{2^r\, r!} \binom{\omega/2 + m - 2}{r}^{-1} \sum_{i_a<j_a} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \dots \epsilon_{i_r j_r} \tag{3.5}$$

with the second sum taken over the (unordered) sets of disjoint pairs $\{(i_1,j_1),\dots,(i_r,j_r)\}$ *of indices from* $\{1,\dots,m\}$.

Proof. Note that for each r the second sum commutes with any element of $\mathfrak{S}_m$ and hence commutes with $H^{(m)}$. Now we apply (3.3) and write $H^{(m)} = H^{(m-1)}\, H^{(m)}$. Since $\epsilon_{im}\,\epsilon_{jm}\, H^{(m-1)} = \epsilon_{im}\, s_{ij}\, H^{(m-1)} = \epsilon_{im}\, H^{(m-1)}$ for distinct i and j, we get

$$\Big(1 - \frac{\epsilon_{1m}}{\omega + m - 2}\Big) \dots \Big(1 - \frac{\epsilon_{m-1\, m}}{\omega + 2m - 4}\Big) H^{(m-1)} = \Big(1 - \frac{\epsilon_{1m} + \dots + \epsilon_{m-1\, m}}{\omega + 2m - 4}\Big) H^{(m-1)}$$

via a straightforward calculation. Using induction on m we come to verifying that if (3.5) holds for m replaced by $m-1$, then

$$S^{(m-1)} \Big(1 - \frac{\epsilon_{1m} + \dots + \epsilon_{m-1\, m}}{\omega + 2m - 4}\Big) H^{(m)}$$

will coincide with the right hand side of (3.5). If none of the pairs (i_a, j_a) contains an index $l \in \{1,\dots,m-1\}$, then the product $\epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \dots \epsilon_{i_r j_r}\, \epsilon_{lm}$ of $r+1$ factors will contribute to the respective sum on the right hand side of (3.5). Otherwise, if $l = i_a$ or $l = j_a$ for some a, then $\epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \dots \epsilon_{i_r j_r}\, \epsilon_{lm}\, H^{(m)} = \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \dots \epsilon_{i_r j_r}\, H^{(m)}$ and the result follows by combining similar terms. $\square$

3.2 Dual pairs and extremal projector

Consider the natural action of the classical group $\mathrm{G}_N = \mathrm{O}_N$ or $\mathrm{G}_N = \mathrm{Sp}_N$ in the tensor product space

$$\underbrace{\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_m. \tag{3.6}$$

Label the copies of the vector space $\mathbb{C}^N$ with the numbers $1, \ldots, m$ and for each pair of indices $1 \leqslant i < j \leqslant m$ denote by P_{ij} and Q_{ij} the operators on the vector space (3.6) which act as the respective operators P and Q defined in (2.3) and (2.4) on the tensor product on the i-th and j-th copies of $\mathbb{C}^N$ and act as the identity operators on each of the remaining copies. The commuting action of the Brauer algebra $\mathcal{B}_m(N)$ in the orthogonal case is given by the assignment

$$s_{ij} \mapsto P_{ij}, \qquad \epsilon_{ij} \mapsto Q_{ij}, \qquad i < j; \tag{3.7}$$

see [5]. Similarly, in the symplectic case with $N = 2n$ the commuting action of the Brauer algebra $\mathcal{B}_m(-2n)$ on the space (3.6) is given by

$$s_{ij} \mapsto -P_{ij}, \qquad \epsilon_{ij} \mapsto -Q_{ij}, \qquad i < j. \tag{3.8}$$

We will need particular symmetric and skew-symmetric forms (2.1). We will be using the involution on the set of indices $\{1, \ldots, N\}$ defined by $i \mapsto i' = N - i + 1$ and take $G = [g_{ij}]$ with

$$g_{ij} = \begin{cases} \delta_{i,j'} & \text{in the symmetric case,} \\ \varepsilon_i \, \delta_{i,j'} & \text{in the skew-symmetric case,} \end{cases} \tag{3.9}$$

where $\varepsilon_i = 1$ for $i = 1, \ldots, n$ and $\varepsilon_i = -1$ for $i = n+1, \ldots, 2n$.

In the orthogonal case the element $H^{(m)} \in \mathcal{B}_m(N)$ acts as the symmetrizer on the space (3.6). Denote by $\mathcal{P}_N$ the space of polynomials in variables $z_1, \ldots, z_N$ and identify the image $H^{(m)}(\mathbb{C}^N)^{\otimes m}$ with the subspace $\mathcal{P}_N^m$ of homogeneous polynomials of degree m via the isomorphism

$$H^{(m)}(e_{i_1} \otimes \ldots \otimes e_{i_m}) \mapsto z_{i_1} \ldots z_{i_m}. \tag{3.10}$$

Proposition 3.1 allows us to consider $S^{(m)}$ as an operator on the space $\mathcal{P}_N^m$. This operator commutes with the action of the orthogonal group O_N on this space and so it can be expressed in terms of the action of the generators of the Lie algebra $\mathfrak{sl}_2$ provided by the Howe duality [19, Sec. 3.4]. Let $\{e, f, h\}$ be the standard basis of $\mathfrak{sl}_2$. The action of the basis elements on the space $\mathcal{P}_N$ is given by

$$e \mapsto -\frac{1}{2} \sum_{i=1}^N \partial_i \partial_{i'}, \qquad f \mapsto \frac{1}{2} \sum_{i=1}^N z_i z_{i'}, \qquad h \mapsto -\frac{N}{2} - \sum_{i=1}^N z_i \, \partial_i,$$

where ∂_i denotes the partial derivative over z_i. Recall that the *extremal projector* p for the Lie algebra $\mathfrak{sl}_2$ (see [4]) is given by the formula

$$p = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r}{r! \, (h+2) \ldots (h+r+1)} f^r e^r. \tag{3.11}$$

The projector p is an element of an extension of the universal enveloping algebra $\mathrm{U}(\mathfrak{sl}_2)$ and it possesses the properties $ep = pf = 0$. The basis element h of $\mathfrak{sl}_2$ acts on the space $\mathcal{P}_N^m$ as multiplication by the scalar $-N/2 - m$. Furthermore, this space is annihilated by the action of e^r with $r > m/2$. Hence, using (3.11) we may regard p as an operator on $\mathcal{P}_N$. The image of this operator coincides with the subspace of $\mathfrak{sl}_2$-singular vectors.

On the other hand, for each value of $r = 0, \dots, \lfloor m/2 \rfloor$ the image of the second sum in (3.5) under the action (3.7) coincides with the operator

$$\sum Q_{i_1 j_1} Q_{i_2 j_2} \dots Q_{i_r j_r} \tag{3.12}$$

which preserves the subspace $H^{(m)}(\mathbb{C}^N)^{\otimes m}$. Regarding (3.12) as an operator on $\mathcal{P}_N^m$ via the isomorphism (3.10), we verify that it coincides with the action of the element $(-2)^r f^r e^r / r!$. Comparing (3.5) and (3.11) we come to the following result.

Proposition 3.2. *The operator $S^{(m)}$ on the space $\mathcal{P}_N^m$ coincides with the restriction of the action of the extremal projector p to this subspace.* □

In the symplectic case the element $H^{(m)} \in \mathcal{B}_m(-2n)$ acts as the anti-symmetrizer on (3.6). We will regard the exterior algebra $\Lambda_{2n} = \Lambda(\mathbb{C}^{2n})$ as the space of polynomials in the anti-commuting variables $\zeta_1, \dots, \zeta_{2n}$ and denote by Λ_{2n}^m the subspace of homogeneous polynomials of degree m. We identify the image $H^{(m)}(\mathbb{C}^{2n})^{\otimes m}$ with Λ_{2n}^m via the isomorphism

$$H^{(m)}(e_{i_1} \otimes \dots \otimes e_{i_m}) \mapsto \zeta_{i_1} \wedge \dots \wedge \zeta_{i_m}. \tag{3.13}$$

If the condition $m \leqslant n$ holds, then $S^{(m)}$ is a well-defined operator on Λ_{2n}^m by Proposition 3.1. This operator commutes with the action of the symplectic group Sp_{2n} on this space and can be expressed in terms of the action of the generators of the Lie algebra $\mathfrak{sl}_2$ provided by the skew Howe duality [19, Sec. 3.8]. The action of the basis elements of $\mathfrak{sl}_2$ on the space Λ_{2n} is given by

$$e \mapsto \sum_{i=1}^{n} \partial_i \wedge \partial_{i'}, \qquad f \mapsto -\sum_{i=1}^{n} \zeta_i \wedge \zeta_{i'}, \qquad h \mapsto n - \sum_{i=1}^{n} \big(\zeta_i \wedge \partial_i + \zeta_{i'} \wedge \partial_{i'}\big),$$

where ∂_i denotes the (left) partial derivative over ζ_i and $i' = 2n - i + 1$. Note that the basis element h acts as multiplication by the scalar $n - m$ on Λ_{2n}^m. Since the representation Λ_{2n} of $\mathfrak{sl}_2$ is finite-dimensional, the subspace Λ_{2n}^m can contain singular vectors only if $n - m \geqslant 0$ which is consistent with our condition on the parameters. Under this condition the extremal projector p defined in (3.11) can be regarded as an operator on Λ_{2n}^m projecting to a subspace of singular vectors. Using again Proposition 3.1 we come to the following.

Proposition 3.3. *If $m \leqslant n$ then the operator $S^{(m)}$ on the space Λ_{2n}^m coincides with the restriction of the action of the extremal projector p to this subspace.* □

3.3 Noncommutative characteristic maps

Extend the notation (2.7) to multiple tensor products

$$\underbrace{\mathrm{End}\,\mathbb{C}^N\otimes\ldots\otimes\mathrm{End}\,\mathbb{C}^N}_{m}\otimes\mathrm{U}(\mathfrak{g}_N),\tag{3.14}$$

and for an arbitrary element C of the respective Brauer algebra $\mathcal{B}_m(N)$ or $\mathcal{B}_m(-N)$ consider the element $\mathrm{tr}\, C\, F_1\ldots F_m$ of $\mathrm{U}(\mathfrak{g}_N)$, where the trace is taken over all m copies of $\mathrm{End}\,\mathbb{C}^N$, and the element C is identified with its image in the algebra (3.14) acting as the identity operator on $\mathrm{U}(\mathfrak{g}_N)$. Observe that $\mathrm{tr}\, C\, F_1\ldots F_m$ belongs to the subalgebra $\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ of G_N-invariants in the universal enveloping algebra, as follows by noting that for any $\mathbf{h}\in\mathrm{G}_N$

$$\begin{aligned}\mathrm{tr}\, C\,\mathbf{h}_1F_1\mathbf{h}_1^{-1}\ldots\mathbf{h}_mF_m\mathbf{h}_m^{-1}&=\mathrm{tr}\, C\,\mathbf{h}_1\ldots\mathbf{h}_m\,F_1\ldots F_m\mathbf{h}_1^{-1}\ldots\mathbf{h}_m^{-1}\\&=\mathrm{tr}\,\mathbf{h}_1^{-1}\ldots\mathbf{h}_m^{-1}C\,\mathbf{h}_1\ldots\mathbf{h}_m\,F_1\ldots F_m=\mathrm{tr}\, C\, F_1\ldots F_m,\end{aligned}$$

where we used the cyclic property of trace and the fact that the action of the element C commutes with the action of G_N. On the other hand, the algebra of invariants $\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ is isomorphic to the algebra of symmetric polynomials $\mathbb{C}[l_1^2,\ldots,l_n^2]^{\mathfrak{S}_n}$ in n variables, if $N=2n+1$ or $N=2n$ via the Harish-Chandra isomorphism; see e.g. [9, Sec. 7.4]. Here we use basis elements $h_1,\ldots,h_n$ of a Cartan subalgebra $\mathfrak{h}_N$ of $\mathfrak{g}_N$ and $l_i=h_i+\rho_i$ for $i=1,\ldots,n$ are their shifts by constants ρ_i determined by the half-sum of the positive roots. This gives (noncommutative) Brauer algebra versions of the well-known characteristic map associated with the symmetric groups:

$$\mathrm{ch}\,:\mathcal{B}_m(\omega)\to\mathbb{C}[l_1^2,\ldots,l_n^2]^{\mathfrak{S}_n},\qquad\omega=N,-N.\tag{3.15}$$

We will need to calculate the leading terms of the symmetric polynomials $\mathrm{ch}\,(S^{(m)})$. To this end, we can work with the corresponding commutative version of the characteristic map (3.15) by replacing the algebra of invariants $\mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ with the associated graded algebra $\mathrm{S}(\mathfrak{g}_N)^{\mathrm{G}_N}$ which is isomorphic to the algebra of the symmetric polynomials $\mathbb{C}[h_1^2,\ldots,h_n^2]^{\mathfrak{S}_n}$; see [9, Sec. 7.3]. Furthermore, we fix the respective bilinear form on $\mathbb{C}^N$ defined in (3.9). The leading term of the symmetric polynomial $\mathrm{ch}\,(S^{(m)})$ (with the condition $m\leqslant n$ in the symplectic case) can now be found as the trace

$$\mathrm{tr}\, S^{(m)}X_1\ldots X_m,\tag{3.16}$$

taken over all copies of $\mathrm{End}\,\mathbb{C}^N$ in the algebra

$$\underbrace{\mathrm{End}\,\mathbb{C}^N\otimes\ldots\otimes\mathrm{End}\,\mathbb{C}^N}_{m}\otimes\mathrm{S}(\mathfrak{h}_N),$$

where X is the diagonal matrix

$$X=\mathrm{diag}[h_1,\ldots,h_n,0,-h_n,\ldots,-h_1]\qquad\text{or}\qquad X=\mathrm{diag}[h_1,\ldots,h_n,-h_n,\ldots,-h_1]$$

for $N=2n+1$ or $N=2n$, respectively, with entries in $\mathrm{S}(\mathfrak{h}_N)$.

Proposition 3.4. *In the orthogonal case, the trace* (3.16) *is zero if m is odd, while for even $m = 2k$ we have*

$$\operatorname{tr} S^{(2k)} X_1 \dots X_{2k} = \frac{N + 4k - 2}{N + 2k - 2} \sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_k \leqslant n} h_{i_1}^2 \dots h_{i_k}^2.$$

Proof. We will write the diagonal entries of the matrix X as $h_1, \dots, h_N$ by setting $h_{i'} = -h_i$ so that $h_{n+1} = 0$ if $N = 2n + 1$. The left hand side of (3.16) then takes the form

$$\sum_{i_1, \dots, i_m = 1}^{N} h_{i_1} \dots h_{i_m} a_{i_1, \dots, i_m}, \tag{3.17}$$

where the coefficients $a_{i_1, \dots, i_m}$ are defined as the diagonal entries in the expansion

$$S^{(m)}(e_{i_1} \otimes \dots \otimes e_{i_m}) = a_{i_1, \dots, i_m}(e_{i_1} \otimes \dots \otimes e_{i_m}) + \dots$$

in terms of the basis vectors of $(\mathbb{C}^N)^{\otimes m}$. Note that the values $i_a = n + 1$ in the case $N = 2n+1$ can be excluded in the sum in (3.17) as $h_{n+1} = 0$. Now we use Proposition 3.2. We have

$$\frac{f^r}{r!} \mapsto \sum \frac{(z_1 z_{1'})^{a_1}}{a_1!} \dots \frac{(z_n z_{n'})^{a_n}}{a_n!}, \qquad \frac{(-e)^r}{r!} \mapsto \sum \frac{(\partial_1 \partial_{1'})^{a_1}}{a_1!} \dots \frac{(\partial_n \partial_{n'})^{a_n}}{a_n!},$$

where each sum is taken over all tuples of nonnegative integers a_i such that $a_1 + \dots + a_n = r$ and for $N = 2n+1$ we restrict the action to the subspace of polynomials of degree 0 in the variable z_{n+1}. For any monomial $z_1^{b_1} z_{1'}^{b_{1'}} \dots z_n^{b_n} z_{n'}^{b_{n'}}$ of degree m we then have the diagonal coefficient in the expansion

$$\frac{f^r}{r!} \frac{(-e)^r}{r!} z_1^{b_1} z_{1'}^{b_{1'}} \dots z_n^{b_n} z_{n'}^{b_{n'}} = \sum \binom{b_1}{a_1} \binom{b_{1'}}{a_1} \dots \binom{b_n}{a_n} \binom{b_{n'}}{a_n} z_1^{b_1} z_{1'}^{b_{1'}} \dots z_n^{b_n} z_{n'}^{b_{n'}} + \dots$$

with the sum over all tuples of nonnegative integers a_i such that $a_1 + \dots + a_n = r$. Since $h_{i'} = -h_i$, calculating the coefficient of the monomial $h_1^{c_1} \dots h_n^{c_n}$ in (3.17) we will need to evaluate the sums

$$\sum_{b_i + b_{i'} = c_i} (-1)^{b_{i'}} \binom{b_i}{a_i} \binom{b_{i'}}{a_i}.$$

If c_i is odd then the sum is zero, while for $c_i = 2d_i$ the sum equals $(-1)^{a_i} \binom{d_i}{a_i}$. This implies the first part of the proposition. Now suppose that $m = 2k$ is even. Then the coefficient of the monomial $h_1^{2d_1} \dots h_n^{2d_n}$ in (3.17) equals

$$\sum_{r=0}^{k} (-1)^r \binom{N/2 + 2k - 2}{r}^{-1} \sum (-1)^{a_1} \binom{d_1}{a_1} \dots (-1)^{a_n} \binom{d_n}{a_n}$$

$$= \sum_{r=0}^{k} \binom{N/2 + 2k - 2}{r}^{-1} \binom{k}{r} = \frac{N/2 + 2k - 1}{N/2 + k - 1}$$

thus completing the proof. $\square$

Proposition 3.5. *In the symplectic case with $m \leqslant n$, the trace* (3.16) *is zero if m is odd, while for even values $m = 2k$ we have*

$$\operatorname{tr} S^{(2k)} X_1 \dots X_{2k} = (-1)^k \frac{n-2k+1}{n-k+1} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} h_{i_1}^2 \dots h_{i_k}^2.$$

Proof. Now we use the formulas (3.8) for the action of the Brauer algebra $\mathcal{B}_m(-N)$ with $N = 2n$ to calculate the corresponding sum (3.17). Since $H^{(m)}$ now acts as the anti-symmetrizer, the condition $m \leqslant n$ implies that a basis in the vector space $H^{(m)}(\mathbb{C}^N)^{\otimes m}$ is formed by the vectors

$$H^{(m)}(e_{c_1} \otimes e_{c_1'} \otimes \dots \otimes e_{c_p} \otimes e_{c_p'} \otimes e_{b_1} \otimes \dots \otimes e_{b_{m-2p}}), \qquad p = 0, 1, \dots, \lfloor m/2 \rfloor,$$

where $1 \leqslant c_1 < \dots < c_p \leqslant n$ and $1 \leqslant b_1 < \dots < b_{m-2p} \leqslant 2n$ with the condition that $b_i + b_j \neq 2n+1$ for all i and j. Applying the corresponding operator (3.12) to a basis vector of this form we find that the coefficient of this vector in the expansion equals $r! \binom{p}{r}$. Since $h_{i'} = -h_i$, the contributions of the basis vectors with $m - 2p > 0$ in the respective sum (3.17) add up to zero. This proves the first part of the proposition. Now suppose that $m = 2k$ is even. Then the coefficient of the monomial $h_{i_1}^2 \dots h_{i_k}^2$ with $1 \leqslant i_1 < \dots < i_k \leqslant n$ in the respective sum (3.17) equals

$$(-1)^k \sum_{r=0}^{k} \binom{-n+2k-2}{r}^{-1} \binom{k}{r} = (-1)^k \frac{-n+2k-1}{-n+k-1},$$

completing the proof. $\square$

Now we will extend Proposition 3.5 to the values $n+1 \leqslant m \leqslant 2n$.

Proposition 3.6. *For $m \leqslant n$ suppose that $A, \dots, B, C$ are m square matrices of size $2n$ with entries in $\mathbb{C}$. Then the following recurrence relations hold:*

$$\operatorname{tr} S^{(m)} A_1 \dots B_{m-1} C_m = \frac{(n-m+1)(2n-m+3)}{m(n-m+2)} \operatorname{tr} S^{(m-1)} A_1 \dots B_{m-1}, \tag{3.18}$$

if C is the identity matrix, and

$$\operatorname{tr} S^{(m)} A_1 \dots B_{m-1} C_m = -\frac{n-m+1}{m(n-m+2)} \sum_{i=1}^{m-1} \operatorname{tr} S^{(m-1)} A_1 \dots B_{m-1} C_i, \tag{3.19}$$

if C satisfies the symmetry condition $C + C' = 0$.

Proof. Expression (3.2) for the symmetrizer gives the recurrence relation

$$S^{(m)} = \frac{1}{m(\omega+2m-4)} \Big(1 + \sum_{i=1}^{m-1} (s_{im} - \epsilon_{im})\Big) \Big(\omega + m - 3 + \sum_{i=1}^{m-1} (s_{im} - \epsilon_{im})\Big) S^{(m-1)} \tag{3.20}$$

in the Brauer algebra $\mathcal{B}_m(\omega)$. Consider the corresponding operator on the space (3.6) with $\omega = -2n$ and the action (3.8). Taking trace over the m-th copy of $\operatorname{End}\mathbb{C}^{2n}$ and using the relations $\operatorname{tr}_m P_{im} = \operatorname{tr}_m Q_{im} = 1$, $i < m$, we obtain

$$\operatorname{tr}_m S^{(m)} = \frac{(n-m+1)(2n-m+3)}{m(n-m+2)}\, S^{(m-1)}$$

which gives (3.18). Similarly, if $C + C' = 0$ then

$$\begin{aligned}
&\operatorname{tr} S^{(m)} A_1 \dots B_{m-1} C_m \\
&= \frac{1}{m(-2n+2m-4)} \operatorname{tr} S^{(m-1)} A_1 \dots B_{m-1} C_m \Big(-1 + \sum_{i=1}^{m-1} \Phi_{im}\Big)\Big(2n - m + 3 + \sum_{i=1}^{m-1} \Phi_{im}\Big),
\end{aligned}$$

where we used the cyclic property of trace and set $\Phi_{im} = P_{im} - Q_{im}$. For the partial trace we calculate

$$\operatorname{tr}_m C_m \Big(-1 + \sum_{i=1}^{m-1} \Phi_{im}\Big)\Big(2n - m + 3 + \sum_{i=1}^{m-1} \Phi_{im}\Big) = 2(n-m+1) \sum_{i=1}^{m-1} C_i$$

thus proving (3.19). □

Proposition 3.6 implies that if each of the m matrices $A, \dots, C$ is either the identity matrix or satisfies the symmetry condition, then for any $1 \leqslant l \leqslant \lfloor m/2 \rfloor$ the expression

$$\frac{1}{n-m+1} \operatorname{tr} S^{(m)} A_1 \dots C_m \tag{3.21}$$

can be written as a linear combination of traces of the form

$$\frac{1}{n-m+l+1} \operatorname{tr} S^{(m-l)} D_1 \dots E_{m-l},$$

where each of the matrices $D, \dots, E$ is a product of some of the matrices $A, \dots, C$. Now observe that if $n + 1 \leqslant m \leqslant 2n$ then the symmetrizer $S^{(m-l)}$ is well-defined for $l = m - n$ so that the expression (3.21) may be *defined* as being equal to this linear combination. To see that this evaluation of (3.21) is well-defined, it suffices to note that various linear combinations obtained by different applications of the recurrence relations (3.18) and (3.19) agree for any fixed m and infinitely many values of n with $n \geqslant m$.

With this interpretation of (3.21), relation (1.5) defines the elements ϕ_{mk} for all values $1 \leqslant m \leqslant 2n$. Moreover, we can now extend Proposition 3.5 to all these values of the parameters.

Corollary 3.7. *In the symplectic case with $1 \leqslant m \leqslant 2n$, the expression*

$$\frac{1}{n-m+1} \operatorname{tr} S^{(m)} X_1 \dots X_m$$

is zero if m is odd, while for even values $m = 2k$ we have

$$\frac{1}{n-2k+1} \operatorname{tr} S^{(2k)} X_1 \dots X_{2k} = \frac{(-1)^k}{n-k+1} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} h_{i_1}^2 \dots h_{i_k}^2 .$$

Proof. For a fixed value of m regard n as an integer parameter. By Proposition 3.5 the relations hold for infinitely many values of n with $n \geqslant m$, and so they hold for the values $m/2 \leqslant n < m$ as well. $\square$

4 Proof of the Main Theorem

Recall that the dual Coxeter number for the Lie algebra $\mathfrak{g}_N$ is given by

$$h^\vee = \begin{cases} N-2 & \text{for } \mathfrak{g}_N = \mathfrak{o}_N, \\ n+1 & \text{for } \mathfrak{g}_N = \mathfrak{sp}_{2n}. \end{cases}$$

From now on we will assume that $\kappa = -h^\vee$ so that κ is at the *critical level*. To prove the first part of the theorem (with the assumption $m \leqslant n$ in the symplectic case) it will be sufficient to verify that for all i, j

$$F_{ij}[0] \operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_m = F_{ij}[1] \operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_m = 0 \tag{4.1}$$

in the $\widehat{\mathfrak{g}}_N$-module $V_{-h^\vee}(\mathfrak{g}_N) \otimes \mathbb{C}[\tau]$. Consider the tensor product algebra

$$\operatorname{End} \mathbb{C}^N \otimes \dots \otimes \operatorname{End} \mathbb{C}^N \otimes \mathrm{U} \tag{4.2}$$

with $m+1$ copies of $\operatorname{End} \mathbb{C}^N$ labeled by $0, 1, \dots, m$, where U stands for the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}}_N \oplus \mathbb{C}\tau)$. Relations (4.1) can now be written in the equivalent form

$$F[0]_0 \operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_m = 0 \qquad \text{and} \qquad F[1]_0 \operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_m = 0 \tag{4.3}$$

modulo the left ideal of U generated by $\mathfrak{g}_N[t]$ and $K + h^\vee$. To verity the first relation, note that by (2.8) we have

$$[F[0]_0, \Phi_i] = (P_{0i} - Q_{0i})\, \Phi_i - \Phi_i\, (P_{0i} - Q_{0i}). \tag{4.4}$$

Hence

$$\begin{aligned} F[0]_0 \operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_m &= \sum_{i=1}^{m} \operatorname{tr} S^{(m)} \Phi_1 \dots \Big((P_{0i} - Q_{0i})\, \Phi_i - \Phi_i\, (P_{0i} - Q_{0i}) \Big) \dots \Phi_m \\ &= \operatorname{tr} S^{(m)} \Big(\sum_{i=1}^{m} (P_{0i} - Q_{0i}) \Big) \Phi_1 \dots \Phi_m - \operatorname{tr} S^{(m)}\, \Phi_1 \dots \Phi_m \, \Big(\sum_{i=1}^{m} (P_{0i} - Q_{0i}) \Big). \end{aligned}$$

The sum $\sum_{i=1}^{m} (P_{0i} - Q_{0i})$ commutes with the action of the symmetrizer $S^{(m)}$ so that the expression is equal to zero by the cyclic property of trace.

For the proof of the second relation in (4.3) use the following consequence of (2.8):

$$[F[1]_0, \Phi_i] = F[0]_0 + (P_{0i} - Q_{0i})\, F[0]_i - F[0]_i\, (P_{0i} - Q_{0i}) + (P_{0i} - Q_{0i})\, K',$$

where $K' = K$ in the orthogonal case and $K' = 2K$ in the symplectic case. We have

$$F[1]_0 \operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_m = \sum_{i=1}^{m} \operatorname{tr} S^{(m)} \Phi_1 \dots \Phi_{i-1}$$
$$\times \Big(F[0]_0 + (P_{0i} - Q_{0i})\, F[0]_i - F[0]_i\, (P_{0i} - Q_{0i}) + (P_{0i} - Q_{0i})\, K' \Big) \dots \Phi_m$$

and applying (4.4) we can write this expression as

$$\sum_{1 \leqslant i<j \leqslant m} \operatorname{tr} S^{(m)} \Phi_1 \dots \widehat{\Phi}_i \dots \Big((P_{0j} - Q_{0j})\, \Phi_j - \Phi_j\, (P_{0j} - Q_{0j}) \Big) \dots \Phi_m$$
$$+ \sum_{1 \leqslant i<j \leqslant m} \operatorname{tr} S^{(m)} (P_{0i} - Q_{0i})\, \Phi_1 \dots \widehat{\Phi}_i \dots \Big((P_{ij} - Q_{ij})\, \Phi_j - \Phi_j\, (P_{ij} - Q_{ij}) \Big) \dots \Phi_m$$
$$- \sum_{1 \leqslant i<j \leqslant m} \operatorname{tr} S^{(m)} \Phi_1 \dots \widehat{\Phi}_i \dots \Big((P_{ij} - Q_{ij})\, \Phi_j - \Phi_j\, (P_{ij} - Q_{ij}) \Big) \dots \Phi_m\, (P_{0i} - Q_{0i})$$
$$+ K' \sum_{i=1}^{m} \operatorname{tr} S^{(m)} (P_{0i} - Q_{0i})\, \Phi_1 \dots \widehat{\Phi}_i \dots \Phi_m,$$

where hats indicate that the corresponding symbols should be skipped. Now we simplify it by using the cyclic property of trace, conjugations by elements of $\mathfrak{S}_m$ and relations (3.1). In the orthogonal case, for the first sum we have

$$\sum_{1 \leqslant i<j \leqslant m} \operatorname{tr} S^{(m)} \Phi_1 \dots \widehat{\Phi}_i \dots (P_{0j} - Q_{0j})\, \Phi_j \dots \Phi_m$$
$$= \sum_{1 \leqslant i<j \leqslant m} \operatorname{tr} S^{(m)} P_{m-1,m} \dots P_{i,i+1}\, \Phi_1 \dots \widehat{\Phi}_i \dots (P_{0j} - Q_{0j})\, \Phi_j \dots \Phi_m$$
$$= \sum_{1 \leqslant i<j \leqslant m} \operatorname{tr} S^{(m)} (P_{0\, j-1} - Q_{0\, j-1})\, \Phi_1 \dots \Phi_{m-1}\, P_{m-1,m} \dots P_{i,i+1}$$
$$= \sum_{i=1}^{m-1} i \operatorname{tr} S^{(m)} (P_{0i} - Q_{0i})\, \Phi_1 \dots \Phi_{m-1}.$$

In the symplectic case, $S^{(m)} = -S^{(m)}\, P_{j\, j+1}$ so that the final expression in this chain of equalities will have the same form. In both cases, applying similar transformations to all remaining sums we will get

$$\begin{aligned}F[1]_0 \operatorname{tr} S^{(m)}\Phi_1 \dots \Phi_m = &\sum_{i=1}^{m-1} i \operatorname{tr} S^{(m)}(P_{0i} - Q_{0i})\, \Phi_1 \dots \Phi_{m-1} \\ &- \sum_{i=1}^{m-1} i \operatorname{tr} (P_{0i} - Q_{0i})\, S^{(m)}\Phi_1 \dots \Phi_{m-1} \\ &+ \sum_{i=1}^{m-1} i \operatorname{tr} S^{(m)}(P_{0m} - Q_{0m})(P_{im} - Q_{im})\, \Phi_1 \dots \Phi_{m-1} \\ &+ \sum_{i=1}^{m-1} i \operatorname{tr} (P_{im} - Q_{im})\, (P_{0m} - Q_{0m})\, S^{(m)}\Phi_1 \dots \Phi_{m-1} \\ &+ \big(mK' \mp m(m-1)\big) \operatorname{tr} S^{(m)}(P_{0m} - Q_{0m})\, \Phi_1 \dots \Phi_{m-1},\end{aligned}$$

where the upper sign in the double sign corresponds to the orthogonal case and lower sign to the symplectic case. Furthermore, observe that

$$S^{(m)}(P_{0m} - Q_{0m})\, P_{im} = S^{(m)} P_{im}\, (P_{0i} - Q_{0i}) = \pm S^{(m)}(P_{0i} - Q_{0i}),$$

and that

$$S^{(m)}(P_{0m} - Q_{0m})\, Q_{im} = 0 \tag{4.5}$$

since

$$\begin{aligned}S^{(m)}(P_{0m} - Q_{0m})\, Q_{im} &= \pm S^{(m)}(P_{0m} - Q_{0m})\, P_{im}\, Q_{im} \\ &= S^{(m)}(P_{0i} - Q_{0i})\, Q_{im} = -S^{(m)}(P_{0m} - Q_{0m})\, Q_{im}.\end{aligned}$$

Hence, simplifying the expressions further we get

$$\begin{aligned}F[1]_0 \operatorname{tr} S^{(m)}\Phi_1 \dots \Phi_m = &\sum_{i=1}^{m-1} 2\, i \operatorname{tr} S^{(m)}(P_{0i} - Q_{0i})\, \Phi_1 \dots \Phi_{m-1} \\ &+ \big(mK' - m(m-1)\big) \operatorname{tr} S^{(m)}(P_{0m} - Q_{0m})\, \Phi_1 \dots \Phi_{m-1}\end{aligned}$$

in the orthogonal case, and

$$\begin{aligned}F[1]_0 \operatorname{tr} S^{(m)}\Phi_1 \dots \Phi_m = &-\sum_{i=1}^{m-1} 2\, i \operatorname{tr} (P_{0i} - Q_{0i})\, S^{(m)}\Phi_1 \dots \Phi_{m-1} \\ &+ \big(mK' + m(m-1)\big) \operatorname{tr} S^{(m)}(P_{0m} - Q_{0m})\, \Phi_1 \dots \Phi_{m-1}\end{aligned}$$

in the symplectic case. As a next step, calculate the partial trace tr_m on the right hand sides with respect to the m-th copy of $\operatorname{End} \mathbb{C}^N$ in (4.2) with the use of the following lemma.

Lemma 4.1. *We have*

$$\mathrm{tr}_m\, S^{(m)} = \pm\, \frac{(\omega+m-3)(\omega+2m-2)}{m\,(\omega+2m-4)}\, S^{(m-1)}$$

and

$$\mathrm{tr}_m\, S^{(m)}(P_{0m}-Q_{0m}) = \pm\, \frac{\omega+2m-2}{m\,(\omega+2m-4)}\, S^{(m-1)} \sum_{i=1}^{m-1} (P_{0i}-Q_{0i}),$$

where ω equals N and $-N$ in the orthogonal and symplectic case, respectively.

Proof. This is derived from the recurrence relation (3.20) in the same way as in the proof of Proposition 3.6. □

Applying Lemma 4.1 we come to the relations

$$\begin{aligned} F[1]_0\, \mathrm{tr}\, S^{(m)} \Phi_1 \dots \Phi_m = {} & \frac{N+2m-2}{m\,(N+2m-4)} \\ & \times \Big((N+m-3) \sum_{i=1}^{m-1} 2\, i\, \mathrm{tr}\, S^{(m-1)} (P_{0i}-Q_{0i})\, \Phi_1 \dots \Phi_{m-1} \\ & \qquad + m\big(K'-m+1\big) \sum_{i=1}^{m-1} \mathrm{tr}\, S^{(m-1)} (P_{0i}-Q_{0i})\, \Phi_1 \dots \Phi_{m-1} \Big) \end{aligned}$$

in the orthogonal case, and

$$\begin{aligned} F[1]_0\, \mathrm{tr}\, S^{(m)} \Phi_1 \dots \Phi_m = {} & -\frac{N-2m+2}{m\,(N-2m+4)} \\ & \times \Big((N-m+3) \sum_{i=1}^{m-1} 2\, i\, \mathrm{tr}\, (P_{0i}-Q_{0i})\, S^{(m-1)} \Phi_1 \dots \Phi_{m-1} \\ & \qquad + m\big(K'+m-1\big) \sum_{i=1}^{m-1} \mathrm{tr}\, (P_{0i}-Q_{0i})\, S^{(m-1)} \Phi_1 \dots \Phi_{m-1} \Big) \end{aligned}$$

in the symplectic case, where the traces on the right hand sides are now taken over the $m-1$ copies of $\mathrm{End}\,\mathbb{C}^N$.

Lemma 4.2. *For any $1 \leqslant i < j \leqslant m$ we have*

$$\mathrm{tr}\, S^{(m)} (P_{0i}-Q_{0i})\, \Phi_1 \dots \Phi_m = \mathrm{tr}\, S^{(m)} (P_{0j}-Q_{0j})\, \Phi_1 \dots \Phi_m$$

in the orthogonal case, and

$$\mathrm{tr}\, (P_{0i}-Q_{0i})\, S^{(m)} \Phi_1 \dots \Phi_m = \mathrm{tr}\, (P_{0j}-Q_{0j})\, S^{(m)} \Phi_1 \dots \Phi_m$$

in the symplectic case, where the traces are taken over all m copies of $\mathrm{End}\,\mathbb{C}^N$.

Proof. We may assume that $m = 2$. In the orthogonal case we have

$$\operatorname{tr} S^{(2)}\,(P_{01} - Q_{01} - P_{02} + Q_{02})\,\Phi_1\Phi_2 = -\operatorname{tr} S^{(2)}\,(P_{01} - Q_{01} - P_{02} + Q_{02})\,\Phi_2\Phi_1,$$

and by (2.8)

$$\begin{aligned}\operatorname{tr} S^{(2)}\,(P_{01} - Q_{01} - P_{02} + Q_{02})\,[\Phi_1, \Phi_2] &= \operatorname{tr} S^{(2)}\,(P_{01} - Q_{01} - P_{02} + Q_{02}) \\ &\times \Big((F[-2]_1 - F[-2]_2)(P_{12} - 1) + Q_{12}\,F[-2]_1 - F[-2]_1\,Q_{12}\Big)\end{aligned}$$

which is zero by (3.1), (4.5) and the cyclic property of trace. The calculation in the symplectic case is similar; it relies on the fact that the trace

$$\begin{aligned}&\operatorname{tr} S^{(2)}[\Phi_1, \Phi_2]\,(P_{01} - Q_{01} - P_{02} + Q_{02}) \\ &= \operatorname{tr} S^{(2)}\Big((P_{12}+1)(F[-2]_2 - F[-2]_1) + Q_{12}\,F[-2]_1 - F[-2]_1\,Q_{12}\Big)\,(P_{01} - Q_{01} - P_{02} + Q_{02})\end{aligned}$$

is zero. □

By Lemma 4.2, up to a numerical factor, $F[1]_0 \operatorname{tr} S^{(m)}\Phi_1 \dots \Phi_m$ equals

$$(K' + N \mp 2) \sum_{i=1}^{m-1} \operatorname{tr} S^{(m-1)}(P_{0i} - Q_{0i})\,\Phi_1 \dots \Phi_{m-1}.$$

At the critical level, $K' + N - 2 = K + N - 2 = 0$ and $K' + N + 2 = 2(K + n + 1) = 0$ in the orthogonal and symplectic case, respectively. This completes the proof of the first part of the Main Theorem in the orthogonal case, while in the symplectic case the proof is complete under the assumption $m \leqslant n$.

Now we continue the argument in the symplectic case. Recalling that $\Phi = \tau + F[-1]$, we can write $\Phi_1 \dots \Phi_m$ as a polynomial in τ whose coefficients are linear combinations of products of the form $F[r_1]_{p_1} \dots F[r_s]_{p_s}$ with the conditions $1 \leqslant p_1 < \dots < p_s \leqslant m$ and $r_i = -1, -2, \dots$. Using the cyclic property of trace and applying conjugations by elements of $\mathfrak{S}_m$ we find that for $n \geqslant m$ the element $\operatorname{tr} S^{(m)} F[r_1]_{p_1} \dots F[r_s]_{p_s}$ coincides with $\operatorname{tr} S^{(m)} F[r_1]_1 \dots F[r_s]_s$. Since the matrix $F[r]$ satisfies $F[r] + F[r]' = 0$, it can be written in the form

$$F[r] = \sum_{i,j=1}^{2n} e_{ij} \otimes F_{ij}[r] = \frac{1}{2} \sum_{i,j=1}^{2n} e_{ij} \otimes (F_{ij}[r] - F_{ij}[r]') = \frac{1}{2} \sum_{i,j=1}^{2n} f_{ij} \otimes F_{ij}[r],$$

where f_{ij} is defined by

$$f_{ij} = e_{ij} - \sum_{k,l=1}^{2n} \overline{g}_{ik}\, g_{lj}\, e_{lk} \in \operatorname{End} \mathbb{C}^{2n}.$$

Hence

$$\frac{1}{n-m+1}\operatorname{tr} S^{(m)} F[r_1]_1 \dots F[r_s]_s =$$
$$\sum_{i_1,j_1,\dots,i_s,j_s} \Big(\frac{1}{2^s(n-m+1)}\operatorname{tr} S^{(m)} f_{i_1 j_1}\otimes\dots\otimes f_{i_s j_s}\otimes 1\otimes\dots\otimes 1\Big)\otimes F_{i_1 j_1}[r_1]\dots F_{i_s j_s}[r_s].$$

Note that the expression in the brackets has the form (3.21) so it is well-defined for the values of m with $n+1 \leqslant m \leqslant 2n$. To show that the coefficients $\phi_{m\,k}$ defined in (1.5) are Segal–Sugawara vectors for $\mathfrak{sp}_{2n}$, note that for each fixed value of m the relations (4.3) hold for infinitely many values $n \geqslant m$ of the parameter n. This implies that they hold for all values $m/2 \leqslant n < m$.

To prove the remaining parts of the Main Theorem, note that the symbols of the elements $\phi_{2k\,2k}$ with $k = 1,\dots,n$ defined in (1.4) and (1.5) respectively coincide with the images of the leading terms of the invariants $\operatorname{tr} S^{(2k)} F_1 \dots F_{2k} \in \mathrm{U}(\mathfrak{g}_N)^{\mathrm{G}_N}$ introduced in Sec. 3.3 (with the additional factor $(n-2k+1)^{-1}$ in the symplectic case), under the embedding $F_{ij} \mapsto F_{ij}[-1]$. Due to Proposition 3.4 and Corollary 3.7, the images of the leading terms in the algebra of symmetric polynomials in $h_1^2,\dots,h_n^2$ coincide, up to numerical factors, with the complete and elementary symmetric functions, respectively. The proof of the theorem is complete.

Now we use the results of Sec. 3.2 to derive expressions for the Segal–Sugawara vectors $\phi_{m\,k}$ based on the use of the Howe dual pairs $(\mathfrak{sl}_2, \mathfrak{g}_N)$. In the orthogonal case, using the notation as in (4.2), for each m introduce the element $\Phi^{(m)} \in \operatorname{End}\mathcal{P}_N^m \otimes \mathrm{U}$ by setting

$$\Phi^{(m)} : z_{j_1}\dots z_{j_m} \mapsto \sum_{i_1\leqslant\dots\leqslant i_m} z_{i_1}\dots z_{i_m}\otimes \Phi^{i_1,\dots,i_m}_{j_1,\dots,j_m}$$

where

$$\Phi^{i_1,\dots,i_m}_{j_1,\dots,j_m} = \frac{1}{\alpha_1!\dots\alpha_N!\,m!}\sum_{\sigma,\pi\in\mathfrak{S}_m} \Phi_{i_{\sigma(1)}j_{\pi(1)}}\dots\Phi_{i_{\sigma(m)}j_{\pi(m)}}$$

and α_i is the multiplicity of i in the multiset $\{i_1,\dots,i_m\}$. The Main Theorem and Proposition 3.2 imply the following corollary.

Corollary 4.3. *The Segal–Sugawara vectors $\phi_{m\,k}$ can be found from the expansion*

$$\operatorname{tr} p\,\Phi^{(m)} = \phi_{m\,0}\,\tau^m + \phi_{m\,1}\,\tau^{m-1} + \dots + \phi_{m\,m}$$

with the trace taken over the subspace of $\mathfrak{sl}_2$-singular vectors in $\mathcal{P}_N^m$. □

Similarly, in the symplectic case introduce the element $\Phi^{(m)} \in \operatorname{End}\Lambda_{2n}^m \otimes \mathrm{U}$ by

$$\Phi^{(m)} : \zeta_{j_1}\wedge\dots\wedge\zeta_{j_m} \mapsto \sum_{i_1<\dots<i_m} \zeta_{i_1}\wedge\dots\wedge\zeta_{i_m}\otimes\Phi^{i_1,\dots,i_m}_{j_1,\dots,j_m},$$

where

$$\Phi^{i_1,\dots,i_m}_{j_1,\dots,j_m} = \frac{1}{m!} \sum_{\sigma,\pi\in\mathfrak{S}_m} \operatorname{sgn}\sigma\pi\, \Phi_{i_{\sigma(1)}j_{\pi(1)}} \dots \Phi_{i_{\sigma(m)}j_{\pi(m)}}.$$

Applying the Main Theorem and Proposition 3.3 we get the following corollary.

Corollary 4.4. *The Segal–Sugawara vectors $\phi_{m\,k}$ with $m \leqslant n$ can be found from the expansion*

$$\frac{1}{n-m+1} \operatorname{tr} p\, \Phi^{(m)} = \phi_{m\,0}\,\tau^m + \phi_{m\,1}\,\tau^{m-1} + \dots + \phi_{m\,m}$$

with the trace taken over the subspace of $\mathfrak{sl}_2$-singular vectors in Λ^m_{2n}. Moreover, for any fixed value of m the vectors $\phi_{m\,k}$ are well-defined for the specializations of n to all values $m/2 \leqslant n < m$. □

5 Gaudin Hamiltonians and Bethe subalgebras

Apply the state-field correspondence map Y to the Segal–Sugawara vectors provided by the Main Theorem. Introduce the matrix $F(z) = [F_{ij}(z)]$, where the fields $F_{ij}(z)$ are defined in (2.9). Then in the orthogonal case all Fourier coefficients of the fields $f_{m\,i}(z)$ defined by the decompositions of the normally ordered trace

$$:\operatorname{tr} S^{(m)}\big(\partial_z + F_1(z)\big)\dots\big(\partial_z + F_m(z)\big): \; = f_{m\,0}(z)\,\partial_z^m + f_{m\,1}(z)\,\partial_z^{m-1} + \dots + f_{m\,m}(z) \quad (5.1)$$

belong to the center of the local completion of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}}_N)$; that is, they are *Sugawara operators* for $\widehat{\mathfrak{g}}_N$; see [11]. The same holds in the symplectic case, where the left hand side of (5.1) should get the factor $(n-m+1)^{-1}$ and the values of m restricted to $1 \leqslant m \leqslant 2n$. By the vacuum axiom of a vertex algebra, the application of the fields $f_{m\,i}(z)$ to the vacuum vector yields formal power series in z with coefficients in $\mathrm{U}(t^{-1}\mathfrak{g}_N[t^{-1}])$. All coefficients of these formal power series belong to the center $\mathfrak{z}(\widehat{\mathfrak{g}}_N)$ of the vertex algebra $V_{-h^\vee}(\mathfrak{g}_N)$. Moreover, by the Main Theorem, the center is generated by these coefficients (together with the additional series corresponding to the Segal–Sugawara vector ϕ'_n in type D). Thus we get another family of generators of this commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{g}_N[t^{-1}])$; see a general result in [31] which implies that this subalgebra is maximal commutative. Explicitly, the coefficients of the series are given by the same expansions as in (5.1) by omitting the normal ordering signs and by replacing $F(z)$ with the matrix $F(z)_+ = [F_{ij}(z)_+]$, where

$$F_{ij}(z)_+ = \sum_{r=0}^{\infty} F_{ij}[-r-1]\,z^r.$$

The same argument as in [27, Sec. 3.2] then yields a corresponding family of commuting elements in $\mathrm{U}(\mathfrak{g}_N[t])$. Introduce the matrix $F(z)_- = [F_{ij}(z)_-]$ and set $L(z) = \partial_z - F(z)_-$, where

$$F_{ij}(z)_- = \sum_{r=0}^{\infty} F_{ij}[r]\,z^{-r-1}.$$

Corollary 5.1. *The coefficients of all series $l_{mi}(z)$ with $m = 1, 2, \dots$ defined by the decompositions*

$$\operatorname{tr} S^{(m)} L_1(z) \dots L_m(z) = l_{m0}(z)\, \partial_z^m + l_{m1}(z)\, \partial_z^{m-1} + \dots + l_{mm}(z), \tag{5.2}$$

where in the symplectic case $m \leqslant 2n$ and the left hand side gets the factor $(n-m+1)^{-1}$, generate a commutative subalgebra of $\mathrm{U}(\mathfrak{g}_N[t])$. □

Note that given any $N \times N$ matrix $B = [b_{ij}]$ over $\mathbb{C}$ with the condition $B + B' = 0$, the commutation relations between the series $F_{ij}(z)_-$ remain valid after the replacement $F_{ij}(z)_- \mapsto b_{ij} + F_{ij}(z)_-$ thus yielding more general families of commutative subalgebras of $\mathrm{U}(\mathfrak{g}_N[t])$. Suppose now that M is a finite-dimensional $\mathfrak{g}_N$-module and $a \in \mathbb{C}$. Define the corresponding evaluation $\mathfrak{g}_N[t]$-module M_a such that the action of the elements of the Lie algebra is given by $F_{ij}[r] \mapsto F_{ij}\, a^r$ for $r \geqslant 0$, that is, $F_{ij}(z)_- \mapsto F_{ij}/(z-a)$. Given finite-dimensional $\mathfrak{g}_N$-modules $M^{(1)}, \dots, M^{(k)}$ and complex numbers $a_1, \dots, a_k$, the tensor product of the evaluation modules $M^{(1)}_{a_1} \otimes \dots \otimes M^{(k)}_{a_k}$ becomes a $\mathfrak{g}_N[t]$-module such that the images of the entries of the matrix $L(z)$ are found by

$$\ell_{ij}(z) = \delta_{ij} \partial_z - b_{ij} - \sum_{r=1}^{k} \frac{F_{ij}^{(r)}}{z - a_r},$$

where $F_{ij}^{(r)}$ denotes the image of F_{ij} in the $\mathfrak{g}_N$-module $M^{(r)}$. Replacing $L(z)$ by the matrix $\mathcal{L}(z) = [\ell_{ij}(z)]$ in (5.2) we obtain a family of commuting operators in the tensor product module, thus producing higher Gaudin Hamiltonians associated with $\mathfrak{g}_N$; cf. [7], [12], [13], [28]. Namely, the coefficients of all series $\mathcal{H}_{mi}(z)$, $m = 1, 2, \dots$ (with $m \leqslant 2n$ and additional factor $(n-m+1)^{-1}$ on the left hand side in the symplectic case) defined by the decompositions

$$\operatorname{tr} S^{(m)} \mathcal{L}_1(z) \dots \mathcal{L}_m(z) = \mathcal{H}_{m0}(z)\, \partial_z^m + \mathcal{H}_{m1}(z)\, \partial_z^{m-1} + \dots + \mathcal{H}_{mm}(z),$$

generate a commuting family of operators in the module $M^{(1)}_{a_1} \otimes \dots \otimes M^{(k)}_{a_k}$.

Consider the rational R-matrix $R(z) = 1 - P z^{-1} + Q(z - \varkappa)^{-1}$ associated with the Lie algebra $\mathfrak{g}_N$, where $\varkappa$ equals $N/2 - 1$ or $N/2 + 1$ in the orthogonal and symplectic case, respectively, and P and Q are defined in (2.3) and (2.4); see [33]. The *extended Yangian* $\mathrm{X}(\mathfrak{g}_N)$ is defined as an associative algebra with generators $t_{ij}^{(r)}$, where $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \dots$, satisfying certain quadratic relations; see [10], [30]. To write them down, set

$$T(z) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(z), \qquad t_{ij}(z) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, z^{-r} \in \mathrm{X}(\mathfrak{g}_N)[[z^{-1}]].$$

The defining relations for the algebra $\mathrm{X}(\mathfrak{g}_N)$ are written as the RTT relation

$$R(z-v)\, T_1(z)\, T_2(v) = T_2(v)\, T_1(z)\, R(z-v), \tag{5.3}$$

with the subscripts indicating the corresponding copies of $\mathrm{End}\,\mathbb{C}^N$ in the tensor product algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{X}(\mathfrak{g}_N)$. The *Yangian* $\mathrm{Y}(\mathfrak{g}_N)$ is defined as the quotient of the extended Yangian $\mathrm{X}(\mathfrak{g}_N)$ by the relation $T'(z+\varkappa)\,T(z)=1$; see [2], [10].

Fix an arbitrary $N\times N$ matrix C over $\mathbb{C}$ with the property $CC'=1$. Consider the tensor product

$$\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{g}_N)[[z^{-1}]]$$

with m copies of $\mathrm{End}\,\mathbb{C}^N$ and introduce the formal series $\tau_m(z,C)$ with coefficients in the Yangian by

$$\tau_m(z,C)=\mathrm{tr}\, S^{(m)} C_1\ldots C_m\, T_1(z)\ldots T_m(z+m-1)$$

in the orthogonal case, and by

$$\tau_m(z,C)=\mathrm{tr}\, S^{(m)} C_1\ldots C_m\, T_1(z)\ldots T_m(z-m+1)$$

in the symplectic case with $N=2n$ and $m\leqslant n$.

Proposition 5.2. *The coefficients of the series $\tau_m(z,C)$, $m\geqslant 1$, generate a commutative subalgebra of* $\mathrm{Y}(\mathfrak{g}_N)$.

Proof. The argument is based on the fact that the symmetrizer $S^{(m)}$ admits a multiplicative presentation (3.4) and uses standard R-matrix techniques; see e.g. [26, Sec. 1.14]. First suppose that $C=1$. The RTT-relation (5.3) implies

$$R(z_1,\ldots,z_{m+l})\,T_1(z_1)\ldots T_{m+l}(z_{m+l})=T_{m+l}(z_{m+l})\ldots T_1(z_1)\,R(z_1,\ldots,z_{m+l}), \tag{5.4}$$

where the z_i are formal variables and

$$R(z_1,\ldots,z_{m+l})=\prod_{1\leqslant i<j\leqslant m+l} R_{ij}(z_i-z_j) \tag{5.5}$$

with the product taken in the lexicographic order on the pairs (i,j). Now specialize the variables by setting

$$z_i=z+i-1,\quad i=1,\ldots,m\qquad\text{and}\qquad z_{m+j}=v+j-1,\quad j=1,\ldots,l$$

in the orthogonal case, and by

$$z_i=z-i+1,\quad i=1,\ldots,m\qquad\text{and}\qquad z_{m+j}=v-j+1,\quad j=1,\ldots,l$$

in the symplectic case, where z and v are formal variables. The product (5.5) then becomes $R(z_1,\ldots,z_{m+l})=\widetilde{R}(z,v)\,S^{(m)}S^{(l)\prime}$, where $S^{(l)\prime}$ is the image of the Brauer algebra symmetrizer in the tensor product of the copies of $\mathrm{End}\,\mathbb{C}^N$ labeled by $m+1,\ldots,m+l$, and $\widetilde{R}(z,v)$ is the product of the R-matrix factors of the form $R_{ij}(z_i-z_j)$ with $1\leqslant i\leqslant m$ and $m+1\leqslant j\leqslant m+l$. Since all these factors are invertible, using (5.4) and taking the trace over all copies of $\mathrm{End}\,\mathbb{C}^N$ we find that

$$\tau_m(z)\,\tau_l(v)=\mathrm{tr}\,\widetilde{R}(z,v)^{-1}\tau_l(v)\,\tau_m(z)\,\widetilde{R}(z,v)=\tau_l(v)\,\tau_m(z).$$

The RTT relation (5.3) will hold if $T(z)$ is replaced with the matrix $CT(z)$. Therefore the above arguments also apply to this matrix instead of $T(z)$. $\square$

Using (5.4) with $l = 0$ together with the multiplicative formula (3.4), we obtain that, up to a shift $z \mapsto z + \text{const}$, both in the orthogonal and symplectic case, the series $\tau_m(z, C)$ can be written as

$$\operatorname{tr} S^{(m)} C_1 T_1(z) \dots C_m T_m(z - m + 1) = \operatorname{tr} S^{(m)} C_1 T_1(z) e^{-\partial_z} \dots C_m T_m(z) e^{-\partial_z} e^{m\partial_z}.$$

The Yangian $\mathrm{Y}(\mathfrak{g}_N)$ admits a filtration defined on the generators by $\deg t_{ij}^{(r)} = r - 1$. The associated graded algebra $\operatorname{gr} \mathrm{Y}(\mathfrak{g}_N)$ is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N[t])$; see [3, Theorem 3.6]. Following [28, Sec. 10], extend the filtration to the algebra $\mathrm{Y}(\mathfrak{g}_N)[[z^{-1}, \partial_z]]$ by setting $\deg z^{-1} = \deg \partial_z = -1$, and regard the matrix C as an element of an extended filtered algebra such that $C - 1$ has degree $\leqslant -1$ with the image B in the component of degree -1 of the associated graded algebra. Then the matrix B satisfies $B + B' = 0$. The entries of the matrix $1 - e^{-\partial_z} C T(z)$ have degree -1 and its image in the graded algebra coincides with the matrix $L(z) - B$. Therefore, the commutative subalgebra in the Yangian $\mathrm{Y}(\mathfrak{g}_N)$ provided by Proposition 5.2 gives rise to a commutative subalgebra in the associated graded algebra $\mathrm{U}(\mathfrak{g}_N[t])$ by the argument originated in [32]; see also [28, Sec. 10] for more details. Moreover, in the orthogonal case this subalgebra coincides with the one obtained in Corollary 5.1, while in the symplectic case this subalgebra is smaller due to the restriction $m \leqslant n$.

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