

Existence, covolumes and infinite generation of lattices for Davis complexes

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Abstract

Let Σ be the Davis complex for a Coxeter system (W, S) . The automorphism group G of Σ is naturally a locally compact group, and a simple combinatorial condition due to Haglund–Paulin determines when G is nondiscrete. The Coxeter group W may be regarded as a uniform lattice in G . We show that many such G also admit a nonuniform lattice Γ , and an infinite family of uniform lattices with covolumes converging to that of Γ . It follows that the set of covolumes of lattices in G is nondiscrete. We also show that the nonuniform lattice Γ is not finitely generated. Examples of Σ to which our results apply include buildings and non-buildings, and many complexes of dimension greater than 2. To prove these results, we introduce a new tool, that of “group actions on complexes of groups”, and use this to construct our lattices as fundamental groups of complexes of groups with universal cover Σ .

1 Introduction

Let G be a locally compact topological group, with Haar measure μ . A discrete subgroup $\Gamma \leq G$ is a *lattice* if $\Gamma \backslash G$ carries a finite G -invariant measure, and is *uniform* if $\Gamma \backslash G$ is compact. Some basic questions are:

1. Does G admit a (uniform or nonuniform) lattice?
2. What is the set of *covolumes* of lattices in G , that is, the set of positive reals

$$\mathcal{V}(G) := \{\mu(\Gamma \backslash G) \mid \Gamma < G \text{ is a lattice}\}?$$

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3. Are lattices in G finitely generated?

These questions have been well-studied in classical cases. For example, suppose G is a reductive algebraic group over a local field K of characteristic 0. Then G admits a uniform lattice, constructed by arithmetic means (Borel–Harder [7]), and a nonuniform lattice only if K is archimedean (Tamagawa [28]). If G is a semisimple real Lie group, the set $\mathcal{V}(G)$ is in most cases discrete (see [23] and its references). If in addition G is simple and higher-rank, then G and hence its lattices have Kazhdan’s Property (T) (see, for example, [24]). Since countable groups with Property (T) are finitely generated, it follows that all lattices in G are finitely generated.

A nonclassical case is G the automorphism group of a locally finite tree T . The study of lattices in $G = \text{Aut}(T)$ was initiated by Bass and Lubotzky, and has yielded many surprising differences from classical results (see the survey [23] and the reference [3]). For example, the set $\mathcal{V}(G)$ is in many cases nondiscrete, and nonuniform tree lattices are never finitely generated.

In fact, the automorphism group G of any locally finite polyhedral complex X is naturally a locally compact group (see Section 2.1). For many X with $\dim(X) \geq 2$, there is greater rigidity than for trees, as might be expected in higher dimensions. For instance, Burger–Mozes [9] proved a ‘Normal Subgroup Theorem’ for products of trees (parallel to that of Margulis [24] for higher-rank semisimple Lie groups), and Bourdon–Pajot [6] and Xie [32] established quasi-isometric rigidity for certain Fuchsian buildings. On the other hand, lattices in $G = \text{Aut}(X)$ can exhibit the same flexibility as tree lattices. For example, the set $\mathcal{V}(G)$ is nondiscrete for certain right-angled buildings [30] and Fuchsian buildings [29]. Another example is density of commensurators of uniform lattices in G , proved by Haglund [17] for certain 2–dimensional Davis complexes, and by Haglund [14] and Kubena Barnhill–Thomas [21] for right-angled buildings. Apart from right-angled buildings, very little is known for X of dimension > 2 . Almost nothing is known for X not a building.

In this paper we consider Questions (1)–(3) above for lattices in $G = \text{Aut}(\Sigma)$, where Σ is the Davis complex for a Coxeter system (W, S) (see [11] and Section 2.2 below). The Davis complex is a locally finite, piecewise Euclidean CAT(0) polyhedral complex, and the Coxeter group W may be regarded as a uniform lattice in G . Our results are the Main Theorem and its Corollaries 1.1 and 1.2 below, which establish tree-like properties for lattices in many such G . After stating these results, we discuss how they apply to (barycentric subdivisions of) Fuchsian buildings and Platonic polygonal complexes, and to many Davis complexes Σ with $\dim(\Sigma) > 2$.

To state the Main Theorem, recall that for a Coxeter system (W, S) with $W = \langle S \mid (st)^{m_{st}} \rangle$, and any $T \subset S$, the *special subgroup* W_T is the subgroup of W generated by the elements $s \in T$. A special subgroup W_T is *spherical* if it is finite, and the set of spherical special subgroups of W is partially ordered by inclusion. The poset of nontrivial spherical special subgroups is an abstract simplicial complex L , called the *nerve* of (W, S) . We identify each generator $s \in S$ with the corresponding vertex $W_{\{s\}} = \langle s \rangle$ of L , and denote by A the group of *label-preserving automorphisms* of L , that is, the group of automorphisms α of L such that $m_{st} = m_{\alpha(s)\alpha(t)}$ for all $s, t \in S$. The group $G = \text{Aut}(\Sigma)$ is nondiscrete

if and only if there is a nontrivial $\alpha \in A$ such that α fixes the star in L of some vertex s (Haglund–Paulin [20]).

Main Theorem 1. *Let (W, S) be a Coxeter system, with nerve L and Davis complex Σ . Let A be the group of label-preserving automorphisms of L . Assume that there are vertices s_1 and s_2 of L , and nontrivial elements $\alpha_1, \alpha_2 \in A$, such that for $i = 1, 2$:*

1. α_i fixes the star of s_{3-i} in L ;
2. the subgroup $\langle \alpha_i \rangle$ of A acts freely on the $\langle \alpha_i \rangle$ -orbit of s_i , in particular $\alpha_i(s_i) \neq s_i$;
3. for all $t_i \neq s_i$ such that t_i is in the $\langle \alpha_i \rangle$ -orbit of s_i , $m_{s_i t_i} = \infty$; and
4. all spherical special subgroups W_T with $s_i \in T$ are halvable along s_i (see Definition 1.4 below).

Then $G = \text{Aut}(\Sigma)$ admits:

- a nonuniform lattice Γ ; and
- an infinite family of uniform lattices (Γ_n) , such that $\mu(\Gamma_n \backslash G) \rightarrow \mu(\Gamma \backslash G)$, where μ is Haar measure on G .

Corollary 1.1. *The set of covolumes of lattices in G is nondiscrete.*

Corollary 1.2. *The group G admits a lattice which is not finitely generated.*

Corollary 1.2 follows from the proof of the Main Theorem and Theorem 1.3 below. By the discussion above, Corollary 1.2 implies that the group G in the Main Theorem does not have Property (T). This was already known for $G = \text{Aut}(\Sigma)$, where Σ is any Davis complex (Haglund–Paulin [20]); our results thus provide an alternative proof of this fact in some cases.

We describe several infinite families of examples of Davis complexes Σ to which our results apply in Section 5 below. To establish these applications, we use properties of spherical buildings in [25], and some results of graph theory from [12]. In two dimensions, examples include the Fuchsian buildings considered in [29], and some of the highly symmetric Platonic polygonal complexes investigated by Świątkowski [27]. Platonic polygonal complexes are not in general buildings, and even the existence of lattices (other than the Coxeter group W) in their automorphism groups was not previously known. An example of a Platonic polygonal complex is the (unique) CAT(0) 2-complex with all 2-cells squares, and the link of every vertex the Petersen graph (Figure 1 below). The Main Theorem and its corollaries also apply to many higher-dimensional Σ , including both buildings and complexes which are not buildings.

To prove the Main Theorem, we construct the lattices Γ_n and Γ as fundamental groups of complexes of groups with universal covers Σ (see [8] and Section 2.3 below). The construction is given in Section 4 below, where we also prove Corollary 1.2.

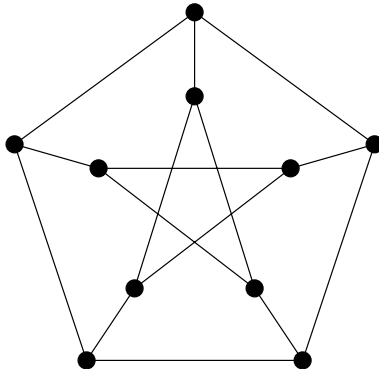


Figure 1: Petersen graph

Complexes of groups are a generalisation to higher dimensions of graphs of groups. Briefly, given a polyhedral complex Y , a (simple) complex of groups $G(Y)$ over Y is an assignment of a *local group* G_σ to each cell σ of Y , with monomorphisms $G_\sigma \rightarrow G_\tau$ whenever $\tau \subset \sigma$, so that the obvious diagrams commute. The action of a group G on a polyhedral complex X induces a complex of groups $G(Y)$ over $Y = G \backslash X$. A complex of groups is *developable* if it is isomorphic to a complex of groups induced in this way. A developable complex of groups $G(Y)$ has a simply-connected *universal cover* $\widetilde{G(Y)}$, equipped with a canonical action of the *fundamental group of the complex of groups* $\pi_1(G(Y))$.

A key difference from graphs of groups is that complexes of groups are not in general developable. In addition, even if $G(Y)$ is developable, with universal cover say X , it may be impossible to identify X of dimension ≥ 2 using only local data such as the links of its vertices (see Ballmann–Brin [1] and Haglund [18]). To ensure that our complexes of groups are developable with universal cover Σ , we use covering theory for complexes of groups (see [8] and [22], and Section 3.1 below). The main result needed is that if there is a covering of complexes of groups $G(Y) \rightarrow H(Z)$, then $G(Y)$ is developable if and only if $H(Z)$ is developable, and the universal covers of $G(Y)$ and $H(Z)$ are isometric (see Theorem 3.2 below).

The other main ingredient in the proof of the Main Theorem is Theorem 1.3 below, which introduces a theory of “group actions on complexes of groups”. This is a method of manufacturing new complexes of groups with a given universal cover, by acting on previously-constructed complexes of groups. Given a complex of groups $G(Y)$, and the action of a group H on Y , the H -action *extends to an action on $G(Y)$* if there is a homomorphism from H to $\text{Aut}(G(Y))$. Roughly, this means that for each cell σ of Y , each $h \in H$ induces a group isomorphism $G_\sigma \rightarrow G_{h \cdot \sigma}$, so that the obvious diagrams commute (see Section 3.1 below for definitions). In Section 3 below we prove:

Theorem 1.3. *Let $G(Y)$ be a (simple) complex of groups over Y , and suppose that the action of a group H on Y extends to an action on $G(Y)$. Then the H -action*

induces a complex of groups $H(Z)$ over $Z = H \backslash Y$ such that there is a covering of complexes of groups $G(Y) \rightarrow H(Z)$. Moreover there is a natural short exact sequence

$$1 \rightarrow \pi_1(G(Y)) \rightarrow \pi_1(H(Z)) \rightarrow H \rightarrow 1,$$

and if H fixes a vertex of Y , then

$$\pi_1(H(Z)) \cong \pi_1(G(Y)) \rtimes H.$$

Theorem 1.3 is also used in [21], and we expect this result to be of independent interest. To our knowledge, group actions on complexes of groups have not previously been considered. In [2], Bass–Jiang determined the structure of the full automorphism group of a graph of groups, but did not define or study the graph of groups induced by a group action on a graph of groups. A more precise statement of Theorem 1.3, including some additional facts about $H(Z)$, is given as Theorem 3.1 below.

The Main Theorem is proved as follows. The action of the Coxeter group W on Σ induces a complex of groups $G(Y_1)$ over $Y_1 = W \backslash \Sigma$, with local groups the spherical special subgroups of W . We then construct a family of finite complexes of groups $G(Y_n)$ and $H(Z_n)$, and two infinite complexes of groups $G(Y_\infty)$ and $H(Z_\infty)$, so that there are coverings of complexes of groups as sketched in Figure 2 below.

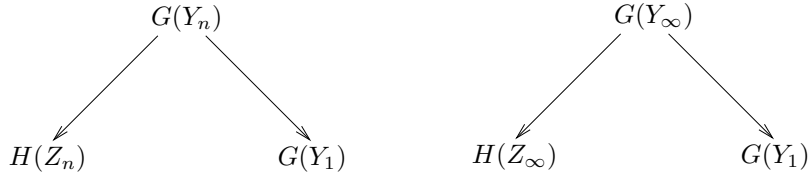


Figure 2: Coverings of complexes of groups

The fundamental groups of $H(Z_n)$ and $H(Z_\infty)$ are, respectively, the uniform lattices Γ_n , and the nonuniform lattice Γ , in $G = \text{Aut}(\Sigma)$. For the local groups of $G(Y_n)$ and $G(Y_\infty)$, we use Condition (4) in the Main Theorem to replace certain spherical special subgroups W_T by the subgroup $\text{half}_s(W_T)$, defined as follows:

Definition 1.4. *Let W_T be a spherical special subgroup of W , and suppose $s \in T$. Then W_T is halvable along s if the union*

$$(T - \{s\}) \cup \{sts \mid t \in T - \{s\}\}$$

generates an index 2 subgroup, denoted $\text{half}_s(W_T)$, of W_T .

The complexes of groups $H(Z_n)$ and $H(Z_\infty)$ are induced by group actions on, respectively, $G(Y_n)$ and $G(Y_\infty)$. To construct these group actions, we use the automorphisms α_1 and α_2 of L .

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2 Background

In this section we present brief background. In Section 2.1 we describe the natural topology on G the group of automorphisms of a locally finite polyhedral complex X , and characterise uniform and nonuniform lattices in G . Section 2.2 constructs the Davis complex Σ for a Coxeter system (W, S) , following [11]. In Section 2.3 we recall the basics of Haefliger's theory of complexes of groups (see [8]), and describe the complex of groups $G(Y_1)$ induced by the action of W on Σ .

2.1 Lattices in automorphism groups of polyhedral complexes

Let G be a locally compact topological group. We will use the following normalisation of Haar measure μ on G .

Theorem 2.1 (Serre, [26]). *Suppose that a locally compact group G acts on a set V with compact open stabilisers and a finite quotient $G \backslash V$. Then there is a normalisation of the Haar measure μ on G , depending only on the choice of G -set V , such that for each discrete subgroup Γ of G we have*

$$\mu(\Gamma \backslash G) = \text{Vol}(\Gamma \backslash V) := \sum_{v \in \Gamma \backslash V} \frac{1}{|\Gamma_v|} \leq \infty.$$

Suppose X is a connected, locally finite polyhedral complex. Let $G = \text{Aut}(X)$. With the compact-open topology, G is naturally a locally compact topological group, and the G -stabilisers of cells in X are compact and open. Hence if $G \backslash X$ is finite, there are several natural choices of sets V in Theorem 2.1 above. By the same arguments as for tree lattices ([3], Chapter 1), it can be shown (for any suitable set V) that a discrete subgroup $\Gamma \leq G$ is a lattice if and only if the series $\text{Vol}(\Gamma \backslash V)$ converges, and Γ is uniform if and only if this is a sum with finitely many terms. In Section 2.2 below we describe our choice of G -set V when G is the group of automorphisms of a Davis complex Σ .

2.2 Davis complexes

In this section we recall the construction of the Davis complex for a Coxeter system. We follow the reference [11].

A *Coxeter group* is a group W with a finite generating set S and presentation of the form

$$W = \langle s \in S \mid s^2 = 1 \forall s \in S, (st)^{m_{st}} = 1 \forall s, t \in S \text{ with } s \neq t \rangle$$

with m_{st} an integer ≥ 2 or $m_{st} = \infty$, meaning that st has infinite order. The pair (W, S) is called a *Coxeter system*.

Example 1: This example will be followed throughout this section, and also referred to in Sections 2.3 and 4 below. Let

$$W = \langle s_1, s_2, s_3, s_4, s_5 \mid s_i^2 = 1, (s_1s_4)^m = (s_2s_4)^m = (s_3s_4)^m = 1, \\ (s_1s_5)^{m'} = (s_2s_5)^{m'} = (s_3s_5)^{m'} = 1 \rangle$$

where m and m' are integers ≥ 2 .

Let (W, S) be a Coxeter system. A subset T of S is *spherical* if the corresponding special subgroup W_T is spherical, that is, W_T is finite. By convention, W_\emptyset is the trivial group. Denote by \mathcal{S} the set of all spherical subsets of S . The set \mathcal{S} is partially ordered by inclusion, and the poset $\mathcal{S}_{>\emptyset}$ is the nerve L of (W, S) (this is equivalent to the description of L in the introduction above). By definition, a nonempty set T of vertices of L spans a simplex σ_T in L if and only if T is spherical. We identify the generator $s \in S$ with the vertex $\{s\}$ of L . The vertices s and t of L are joined by an edge in L if and only if m_{st} is finite, in which case we label this edge by the integer m_{st} . The nerve L of Example 1 above, with its edge labels, is sketched in Figure 3 below.

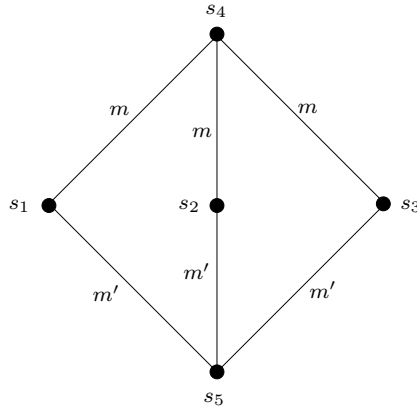


Figure 3: Nerve L of Example 1, with edge labels

We denote by K the geometric realisation of the poset \mathcal{S} . Equivalently, K is the cone on the barycentric subdivision of the nerve L of (W, S) . Note that K is compact and contractible, since it is the cone on a finite simplicial complex. Each

vertex of K has *type* a spherical subset of S , with the cone point having type \emptyset . For Example 1 above, K and the types of its vertices are sketched on the left of Figure 4.

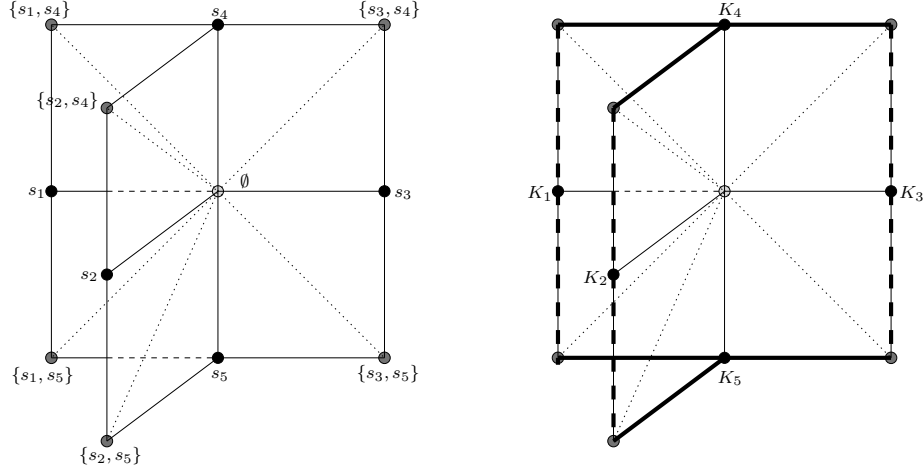


Figure 4: K and types of vertices (left) and mirrors (right) for Example 1

For each $s \in S$ let K_s be the union of the (closed) simplices in K which contain the vertex s but not the cone point. In other words, K_s is the closed star of the vertex s in the barycentric subdivision of L . Note that K_s and K_t intersect if and only if m_{st} is finite. The family $(K_s)_{s \in S}$ is a *mirror structure* on K , meaning that $(K_s)_{s \in S}$ is a family of closed subspaces of K , called *mirrors*. We call K_s the s -*mirror* of K . For Example 1 above, the mirrors $K_i = K_{s_i}$ are shown in heavy lines on the right of Figure 4.

For each $x \in K$, put

$$S(x) := \{s \in S \mid x \in K_s\}.$$

Now define an equivalence relation \sim on the set $W \times K$ by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. The *Davis complex* Σ for (W, S) is then the quotient space:

$$\Sigma := (W \times K) / \sim.$$

The types of vertices of K induce types of the vertices of Σ , and the natural W -action on $W \times K$ descends to a type-preserving action on Σ , with compact quotient K , so that the W -stabiliser of a vertex of Σ of type $T \in \mathcal{S}$ is a conjugate of the spherical special subgroup W_T .

We identify K with the subcomplex $(1, K)$ of Σ , and write wK for the translate (w, K) , where $w \in W$. Any wK is called a *chamber* of Σ . The mirrors K_s of K , or any of their translates by elements of W , are called the *mirrors* of Σ . Two distinct chambers of Σ are s -*adjacent* if their intersection is an s -mirror, and are *adjacent* if their intersection is an s -mirror for some $s \in S$. Note that the chambers wK and

$w'K$ are s -adjacent if and only if $w^{-1}w' = s$, equivalently $w' = ws$ and $w's = w$. For Example 1 above, part of the Davis complex Σ for (W, S) is shown in Figure 5 below. There are $2m$ copies of K glued around the vertices of types $\{s_i, s_4\}$, for $i = 1, 2, 3$, since $W_{\{s_i, s_4\}}$ has order $2m$. Similarly, there are $2m'$ copies of K glued around the vertices of types $\{s_i, s_5\}$, for $i = 1, 2, 3$.

The Davis complex Σ may be metrised with a piecewise Euclidean structure, such that Σ is a complete CAT(0) space (Moussong, see Theorem 12.3.3 of [11]). From now on we will assume that Σ is equipped with this metric.

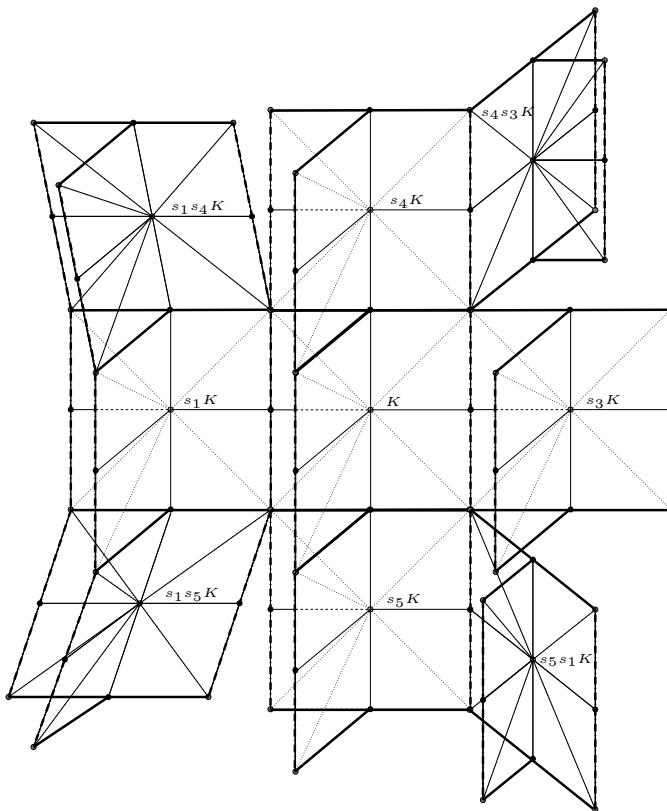


Figure 5: Part of Σ , for Example 1

Suppose that $G = \text{Aut}(\Sigma)$ is the group of automorphisms of a Davis complex Σ . Since W acts cocompactly on Σ , with finite stabilisers, it may be regarded as a uniform lattice in G . We take as the set V in Theorem 2.1 above the set of vertices of Σ of type \emptyset . Then the covolume of W is 1, since W acts simply transitively on V .

2.3 Complexes of groups

We now outline the basic theory of complexes of groups, following Chapter III.C of [8]. The definitions of the more involved notions of morphisms and coverings of complexes of groups are postponed to Section 3.1 below.

In the literature, a complex of groups $G(Y)$ is constructed over a space or set Y belonging to various different categories, including simplicial complexes, polyhedral complexes, or, most generally, *scwols* (small categories without loops). In each case there is a set of vertices, and a set of oriented edges with a composition law. The formal definition of a scwol is:

Definition 2.2. *A scwol X is the disjoint union of a set $V(X)$ of vertices and a set $E(X)$ of edges, with each edge a oriented from its initial vertex $i(a)$ to its terminal vertex $t(a)$, such that $i(a) \neq t(a)$. A pair of edges (a, b) is composable if $i(a) = t(b)$, in which case there is a third edge ab , called the composition of a and b , such that $i(ab) = i(b)$, $t(ab) = t(a)$ and if $i(a) = t(b)$ and $i(b) = t(c)$ then $(ab)c = a(bc)$ (associativity).*

We will always assume scwols are *connected* (see Section III.C.1.1 of [8]). Morphisms of scwols and group actions on scwols are defined as follows:

Definition 2.3. *Let X, Y and Z be scwols. A nondegenerate morphism $f : Y \rightarrow Z$ is a map that sends $V(Y)$ to $V(Z)$, sends $E(Y)$ to $E(Z)$ and is such that:*

1. *for each $a \in E(Y)$, we have $i(f(a)) = f(i(a))$ and $t(f(a)) = f(t(a))$;*
2. *for each pair of composable edges (a, b) in Y , we have $f(ab) = f(a)f(b)$; and*
3. *for each vertex $\sigma \in V(Y)$, the restriction of f to the set of edges with initial vertex σ is a bijection onto the set of edges of Z with initial vertex $f(\sigma)$.*

A morphism of scwols $f : Y \rightarrow Z$ is a functor from the category Y to the category Z (see Section III.C.A.1 of [8]). An automorphism of a scwol X is a morphism from X to X that has an inverse.

Definition 2.4. *An action of a group G on a scwol X is a homomorphism from G to the group of automorphisms of X such that for all $a \in E(X)$ and all $g \in G$:*

1. *$g \cdot i(a) \neq t(a)$; and*
2. *if $g \cdot i(a) = i(a)$ then $g \cdot a = a$.*

Suppose now that Σ is the Davis complex for a Coxeter system (W, S) , as defined in Section 2.2 above. Recall that each vertex $\sigma \in V(\Sigma)$ has type T a spherical subset of S . The edges $E(\Sigma)$ are then naturally oriented by inclusion of type. That is, if the edge a joins the vertex σ of type T to the vertex σ' of type T' , then $i(a) = \sigma$ and $t(a) = \sigma'$ exactly when $T \subsetneq T'$. It is clear that the sets $V(\Sigma)$ and $E(\Sigma)$ satisfy the properties of a scwol. Moreover, if Y is a subcomplex of Σ , then the sets $V(Y)$ and $E(Y)$ also satisfy Definition 2.2 above. By abuse of notation, we identify Σ and Y with the associated scwols.

We now define complexes of groups over scwols.

Definition 2.5. A complex of groups $G(Y) = (G_\sigma, \psi_a, g_{a,b})$ over a scwol Y is given by:

1. a group G_σ for each $\sigma \in V(Y)$, called the local group at σ ;
2. a monomorphism $\psi_a : G_{i(a)} \rightarrow G_{t(a)}$ along the edge a for each $a \in E(Y)$; and
3. for each pair of composable edges, a twisting element $g_{a,b} \in G_{t(a)}$, such that

$$\text{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$$

where $\text{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$, and for each triple of composable edges a, b, c the following cocycle condition holds

$$\psi_a(g_{b,c}) g_{a,bc} = g_{a,b} g_{ab,c}.$$

A complex of groups is *simple* if each $g_{a,b}$ is trivial.

Example: This example will be followed throughout this section, and used in the proof of the Main Theorem in Section 4 below. Let (W, S) be a Coxeter system with nerve L and let K be the cone on the barycentric subdivision of L , as in Section 2.2 above. Put $Y_1 = K$, with the orientations on edges discussed above. We construct a simple complex of groups $G(Y_1)$ over Y_1 as follows. Let $\sigma \in V(Y_1)$. Then σ has type a spherical subset T of S , and we define $G_\sigma = W_T$. All monomorphisms along edges of Y_1 are then natural inclusions, and all $g_{a,b}$ are trivial. For (W, S) as in Example 1 of Section 2.2 above, the complex of groups $G(Y_1)$ is shown in Figure 6 below. In this figure, D_{2m} and $D_{2m'}$ are the dihedral groups of orders $2m$ and $2m'$ respectively, and C_2 is the cyclic group of order 2.

Suppose a group G acts on a scwol X , as in Definition 2.4 above. Then the quotient $Y = G \backslash X$ also has the structure of a scwol, and the action of G induces a complex of groups $G(Y)$ over Y (this construction is generalised in Section 3.2 below). Let $Y = G \backslash X$ with $p : X \rightarrow Y$ the natural projection. For each $\sigma \in V(Y)$, choose a lift $\bar{\sigma} \in V(X)$ such that $p(\bar{\sigma}) = \sigma$. The local group G_σ of $G(Y)$ is then defined to be the stabiliser of $\bar{\sigma}$ in G , and the monomorphisms ψ_a and elements $g_{a,b}$ are defined using further choices. The resulting complex of groups $G(Y)$ is unique up to isomorphism (see Definition 3.3 below).

A complex of groups is *developable* if it is isomorphic to a complex of groups $G(Y)$ induced, as just described, by such an action. Complexes of groups, unlike graphs of groups, are not in general developable. The complex of groups $G(Y_1)$ above is developable, since it is the complex of groups induced by the action of W on Σ , where for each $\sigma \in V(Y_1)$, with σ of type T , we choose $\bar{\sigma}$ in Σ to be the vertex of $(1, K) = K \subset \Sigma$ of type T .

Let $G(Y)$ be a complex of groups. The *fundamental group of the complex of groups* $\pi_1(G(Y))$ is defined so that if Y is simply connected and $G(Y)$ is simple, $\pi_1(G(Y))$ is isomorphic to the direct limit of the family of groups G_σ and monomorphisms ψ_a . For example, since $Y_1 = K$ is contractible and $G(Y_1)$ is a simple complex of groups, the fundamental group of $G(Y_1)$ is W .

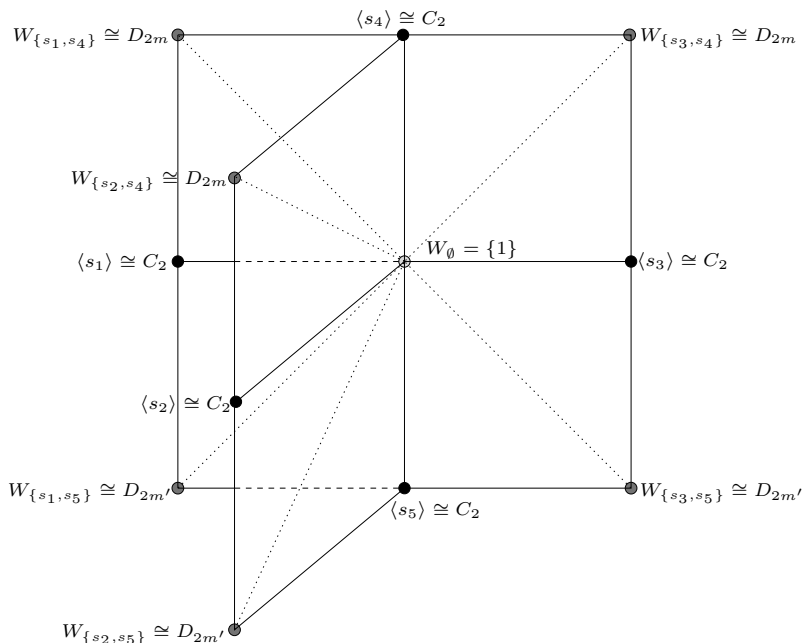


Figure 6: The complex of groups $G(Y_1)$, for Example 1 of Section 2.2

If $G(Y)$ is a developable complex of groups, then it has a *universal cover* $\widetilde{G(Y)}$. This is a connected, simply-connected scwol, equipped with an action of $\pi_1(G(Y))$, so that the complex of groups induced by the action of the fundamental group on the universal cover is isomorphic to $G(Y)$. For example, the universal cover of $G(Y_1)$ is Σ .

Let $G(Y)$ be a developable complex of groups over Y , with universal cover X and fundamental group Γ . We say that $G(Y)$ is *faithful* if the action of Γ on X is faithful, in which case we may identify Γ with its image in $\text{Aut}(X)$. If X is locally finite, then with the compact-open topology on $\text{Aut}(X)$, by the discussion in Section 2.1 above the subgroup $\Gamma \leq \text{Aut}(X)$ is discrete if and only if all local groups of $G(Y)$ are finite. Moreover, if $\text{Aut}(X)$ acts cocompactly on X , a discrete $\Gamma \leq \text{Aut}(X)$ is a uniform lattice in $\text{Aut}(X)$ if and only if $Y \cong \Gamma \backslash X$ is finite, and a discrete $\Gamma \leq \text{Aut}(X)$ is a nonuniform lattice if and only if $Y \cong \Gamma \backslash X$ is infinite and the series $\text{Vol}(\Gamma \backslash V)$ converges, for some $V \subset X$ on which $G = \text{Aut}(X)$ acts according to the hypotheses of Theorem 2.1 above.

3 Group actions on complexes of groups

In this section we introduce a theory of group actions on complexes of groups. The main result is Theorem 3.1 below, which makes precise and expands upon Theorem 1.3 of the introduction. The terms appearing in Theorem 3.1 which

have not already been discussed in Section 2.3 above will be defined in Section 3.1 below, where we also introduce some notation. In Section 3.2 below we construct the complex of groups induced by a group action on a complex of groups, and in Section 3.3 we construct the induced covering. Using these results, in Section 3.4 we consider the structure of the fundamental group of the induced complex of groups.

We will require only actions on *simple* complexes of groups $G(Y)$ by *simple* morphisms; this case is already substantially technical. If in addition the action on Y has a strict fundamental domain, it is possible to make choices so that the induced complex of groups is also simple, and many of the proofs in this section become much easier. However, in our applications, the group action will not necessarily have a strict fundamental domain.

Theorem 3.1. *Let $G(Y)$ be a simple complex of groups over a connected scwol Y , and suppose that the action of a group H on Y extends to an action by simple morphisms on $G(Y)$. Then the H -action induces a complex of groups $H(Z)$ over $Z = H \backslash Y$, with $H(Z)$ well-defined up to isomorphism of complexes of groups, such that there is a covering of complexes of groups*

$$G(Y) \rightarrow H(Z).$$

Moreover there is a natural short exact sequence

$$1 \rightarrow \pi_1(G(Y)) \rightarrow \pi_1(H(Z)) \rightarrow H \rightarrow 1,$$

and if H fixes a vertex of Y , then

$$\pi_1(H(Z)) \cong \pi_1(G(Y)) \rtimes H.$$

Finally, if $G(Y)$ is faithful and the H -action on $G(Y)$ is faithful then $H(Z)$ is faithful.

We will use the following general result on functoriality of coverings (which is implicit in [8], and stated and proved explicitly in [22]).

Theorem 3.2. *Let $G(Y)$ and $H(Z)$ be complexes of groups over scwols Y and Z . Suppose there is a covering of complexes of groups $\Phi : G(Y) \rightarrow H(Z)$. Then $G(Y)$ is developable if and only if $H(Z)$ is developable. Moreover, Φ induces a monomorphism of fundamental groups*

$$\pi_1(G(Y)) \hookrightarrow \pi_1(H(Z))$$

and an equivariant isomorphism of universal covers

$$\widetilde{G(Y)} \xrightarrow{\cong} \widetilde{H(Z)}.$$

3.1 Definitions and notation

We gather here definitions and notation needed for the statement and proof of Theorem 3.1 above. Throughout this section, Y and Z are scwols, $G(Y) = (G_\sigma, \psi_a)$ is a simple complex of groups over Y , and $H(Z) = (H_\tau, \theta_a, h_{a,b})$ is a complex of groups over Z .

Definition 3.3. Let $f : Y \rightarrow Z$ be a morphism of scwols (see Definition 2.3 above). A morphism $\Phi : G(Y) \rightarrow H(Z)$ over f consists of:

1. a homomorphism $\phi_\sigma : G_\sigma \rightarrow H_{f(\sigma)}$ for each $\sigma \in V(Y)$, called the local map at σ ; and
2. an element $\phi(a) \in H_{f(a)}$ for each $a \in E(Y)$, such that the following diagram commutes

$$\begin{array}{ccc} G_{i(a)} & \xrightarrow{\psi_a} & G_{t(a)} \\ \downarrow \phi_{i(a)} & & \downarrow \phi_{t(a)} \\ H_{f(i(a))} & \xrightarrow{\text{Ad}(\phi(a)) \circ \theta_{f(a)}} & H_{f(t(a))} \end{array}$$

and for all pairs of composable edges (a, b) in $E(Y)$,

$$\phi(ab) = \phi(a) \psi_a(\phi(b)) h_{f(a), f(b)}.$$

A morphism is *simple* if each element $\phi(a)$ is trivial. If f is an isomorphism of scwols, and each ϕ_σ an isomorphism of the local groups, then Φ is an *isomorphism of complexes of groups*.

We introduce the following, expected, definitions. An *automorphism* of $G(Y)$ is an isomorphism $\Phi : G(Y) \rightarrow G(Y)$. It is not hard to verify that the set of automorphisms of $G(Y)$ forms a group under composition, which we denote $\text{Aut}(G(Y))$ (see Section III.C.2.4 of [8] for the definition of composition of morphisms). We then say that a group H *acts on* $G(Y)$ if there is a homomorphism

$$H \rightarrow \text{Aut}(G(Y)).$$

Our notation is as follows. Suppose H acts on $G(Y)$. Then in particular, H acts on the scwol Y in the sense of Definition 2.4 above. We write the action of H on Y as $\sigma \mapsto h \cdot \sigma$ and $a \mapsto h \cdot a$, for $h \in H$, $\sigma \in V(Y)$ and $a \in E(Y)$. The element $h \in H$ induces the automorphism Φ^h of $G(Y)$. The data for Φ^h is a family $(\phi_\sigma^h)_{\sigma \in V(Y)}$ of group isomorphisms $\phi_\sigma^h : G_\sigma \rightarrow G_{h \cdot \sigma}$, and a family of elements $(\phi^h(a))_{a \in E(Y)}$ with $\phi^h(a) \in G_{t(h \cdot a)}$, satisfying the definition of morphism above (Definition 3.3).

We say that the H -action is *by simple morphisms* if each Φ^h is simple, that is, if each $\phi^h(a) \in G_{t(h \cdot a)}$ is the trivial element. Explicitly, for each $a \in E(Y)$ and each $h \in H$, the following diagram commutes.

$$\begin{array}{ccc} G_{i(a)} & \xrightarrow{\psi_a} & G_{t(a)} \\ \downarrow \phi_{i(a)}^h & & \downarrow \phi_{t(a)}^h \\ G_{h \cdot i(a)} & \xrightarrow{\psi_{h \cdot a}} & G_{h \cdot t(a)} \end{array}$$

We note also that the composition of simple morphisms $\Phi^{h'} \circ \Phi^h$ is the simple morphism $\Phi^{h'h}$ with local maps

$$\phi_\sigma^{h'h} = \phi_{h \cdot \sigma}^{h'} \circ \phi_\sigma^h. \quad (1)$$

Finally we recall the definition of a covering of complexes of groups.

Definition 3.4. *A morphism $\Phi : G(Y) \rightarrow H(Z)$ over a nondegenerate morphism of scwols $f : Y \rightarrow Z$ (see Definition 2.3 above) is a covering of complexes of groups if further:*

1. each ϕ_σ is injective; and
2. for each $\sigma \in V(Y)$ and $b \in E(Z)$ such that $t(b) = f(\sigma)$, the map on cosets

$$\Phi_{\sigma/b} : \left(\prod_{\substack{a \in f^{-1}(b) \\ t(a) = \sigma}} G_\sigma / \psi_a(G_{i(a)}) \right) \rightarrow H_{f(\sigma)} / \theta_b(H_{i(b)})$$

induced by $g \mapsto \phi_\sigma(g)\phi(a)$ is a bijection.

3.2 The induced complex of groups and its properties

Suppose that a group H acts by simple morphisms on a simple complex of groups $G(Y) = (G_\sigma, \psi_a)$. In this section we construct the complex of groups $H(Z)$ induced by this action, prove that $H(Z)$ is well-defined up to isomorphism of complexes of groups and discuss faithfulness.

Let Z be the quotient scwol $Z = H \backslash Y$ and let $p : Y \rightarrow Z$ be the natural projection. For each vertex $\tau \in V(Z)$ choose a representative $\bar{\tau} \in V(Y)$ such that $p(\bar{\tau}) = \tau$. Let $\text{Stab}_H(\bar{\tau})$ be the subgroup of H fixing $\bar{\tau}$ and let $G_{\bar{\tau}}$ be the local group of $G(Y)$ at $\bar{\tau}$. Since the H -action is by simple morphisms, by Equation (1) above there is a group homomorphism $\zeta : \text{Stab}_H(\bar{\tau}) \rightarrow \text{Aut}(G_{\bar{\tau}})$ given by $\zeta(h) = \phi_{\bar{\tau}}^h$. For each $\tau \in V(Z)$ we then define the local group H_τ to be the corresponding semidirect product of $G_{\bar{\tau}}$ by $\text{Stab}_H(\bar{\tau})$, that is,

$$H_\tau := G_{\bar{\tau}} \rtimes_\zeta \text{Stab}_H(\bar{\tau}) = G_{\bar{\tau}} \rtimes \text{Stab}_H(\bar{\tau}).$$

For each edge $a \in E(Z)$ with $i(a) = \tau$ there is, since H acts on Y in the sense of Definition 2.4 above, a unique edge $\bar{a} \in E(Y)$ such that $p(\bar{a}) = a$ and $i(\bar{a}) = \bar{i(a)} = \bar{\tau}$. For each $a \in E(Z)$ choose an element $h_a \in H$ such that $h_a \cdot t(\bar{a}) = t(a)$.

Lemma 3.5. *Let $g \in G_{i(\bar{a})} = G_{\bar{i(a)}}$ and $h \in \text{Stab}_H(\bar{i(a)})$. Then the map*

$$\theta_a : (g, h) \mapsto (\phi_{t(\bar{a})}^{h_a} \circ \psi_{\bar{a}}(g), h_a h h_a^{-1})$$

is a monomorphism $H_{i(a)} \rightarrow H_{t(a)}$.

Proof. We will show that θ_a is a group homomorphism. Since $\phi_{t(\bar{a})}^{h_a}$, $\psi_{\bar{a}}$ and the conjugation $h \mapsto h_a h h_a^{-1}$ are all injective, the conclusion will then follow.

Let $g, g' \in G_{\overline{i(a)}}$ and $h, h' \in \text{Stab}_H(\overline{i(a)})$. Note that since h and h' fix $\overline{i(a)} = i(\overline{a})$, they fix the edge \overline{a} and hence fix the vertex $t(\overline{a})$ as well. We have

$$\begin{aligned}\theta_a((g, h)(g, h')) &= \theta_a(g\phi_{\overline{i(a)}}^h(g'), hh') \\ &= (\phi_{t(\overline{a})}^{h_a} \circ \psi_{\overline{a}}(g\phi_{\overline{i(a)}}^h(g')), h_a h h' h_a^{-1})\end{aligned}$$

while

$$\begin{aligned}\theta_a(g, h)\theta_a(g', h') &= (\phi_{t(\overline{a})}^{h_a} \circ \psi_{\overline{a}}(g), h_a h h_a^{-1})(\phi_{t(\overline{a})}^{h_a} \circ \psi_{\overline{a}}(g'), h_a h' h_a^{-1}) \\ &= (\phi_{t(\overline{a})}^{h_a} \circ \psi_{\overline{a}}(g)\phi_{\overline{t(a)}}^{h_a h h_a^{-1}} \circ \phi_{t(\overline{a})}^{h_a} \circ \psi_{\overline{a}}(g'), h_a h h' h_a^{-1}).\end{aligned}$$

After applying Equation (1) above to the map $\phi^{h_a h h_a^{-1}}$, and some cancellations, it remains to show that

$$\psi_{\overline{a}} \circ \phi_{\overline{i(a)}}^h(g') = \phi_{t(\overline{a})}^h \circ \psi_{\overline{a}}(g').$$

This follows from the fact that Φ^h is a simple morphism with $h \cdot \overline{a} = \overline{a}$. \square

To complete the construction of $H(Z)$, for each composable pair of edges (a, b) in $E(Z)$, define

$$h_{a,b} = h_a h_b h_{ab}^{-1}.$$

One checks that $h_{a,b} \in \text{Stab}_H(\overline{t(a)})$ hence $(1, h_{a,b}) \in H_{t(a)}$. By abuse of notation we write $h_{a,b}$ for $(1, h_{a,b})$.

Proposition 3.6. *The datum $H(Z) = (H_\sigma, \theta_a, h_{a,b})$ is a complex of groups.*

Proof. Given Lemma 3.5 above, it remains to show that for each pair of composable edges (a, b) in $E(Z)$,

$$\text{Ad}(h_{a,b}) \circ \theta_{ab} = \theta_a \circ \theta_b, \quad (2)$$

and that the cocycle condition holds. Let $(g, h) \in H_{i(b)} = G_{\overline{i(b)}} \rtimes \text{Stab}_H(\overline{i(b)})$. We compute

$$\text{Ad}(h_{a,b}) \circ \theta_{ab}(g, h) = (\phi_{\overline{t(ab)}}^{h_{a,b}} \circ \phi_{\overline{t(ab)}}^{h_{ab}} \circ \psi_{\overline{ab}}(g), h_{a,b} h_{ab} h h_{ab}^{-1} h_{a,b}^{-1})$$

while

$$\theta_a \circ \theta_b(g, h) = (\phi_{t(\overline{a})}^{h_a} \circ \psi_{\overline{a}} \circ \phi_{\overline{t(b)}}^{h_b} \circ \psi_{\overline{b}}(g), h_a h_b h h_b^{-1} h_a^{-1}).$$

By definition of $h_{a,b}$ it remains to show equality in the first component.

By Equation (1) and the definition of $h_{a,b}$,

$$\phi_{\overline{t(ab)}}^{h_{a,b}} = \phi_{\overline{t(a)}}^{h_a} \circ \phi_{\overline{t(ab)}}^{h_b} \circ \phi_{\overline{t(ab)}}^{h_{ab}^{-1}}.$$

Hence it suffices to prove

$$\phi_{\overline{t(ab)}}^{h_b} \circ \psi_{\overline{ab}} = \psi_{\overline{a}} \circ \phi_{\overline{t(b)}}^{h_b} \circ \psi_{\overline{b}}. \quad (3)$$

Since $G(Y)$ is a simple complex of groups, and \overline{ab} is the composition of the edges $h_b^{-1} \cdot \overline{a}$ and \overline{b} , we have

$$\psi_{\overline{ab}} = \psi_{h_b^{-1} \overline{a}} \circ \psi_{\overline{b}}.$$

Applying this, and the fact that $\phi_{t(\overline{b})}^{h_b}$ is a simple morphism on the edge $h_b^{-1} \overline{a}$, we have

$$\phi_{t(\overline{ab})}^{h_b} \circ \psi_{\overline{ab}} = \phi_{t(\overline{ab})}^{h_b} \circ \psi_{h_b^{-1} \overline{a}} \circ \psi_{\overline{b}} = \psi_{\overline{a}} \circ \phi_{t(\overline{b})}^{h_b} \circ \psi_{\overline{b}}.$$

Hence Equation (3) holds.

The cococycle condition follows from the definition of $h_{a,b}$. We conclude that $H(Z)$ is a complex of groups. \square

We now have a complex of groups $H(Z)$ induced by the action of H on $G(Y)$. This construction depended on choices of lifts $\overline{\tau}$ and of elements $h_a \in H$. We next show (in a generalisation of Section III.C.2.9(2) of [8]) that:

Lemma 3.7. *The complex of groups $H(Z)$ is well-defined up to isomorphism of complexes of groups.*

Proof. Suppose we made a different choice of lifts $\overline{\tau}'$ and elements h'_a , resulting in a complex of groups $H'(Z) = (H'_\tau, \theta'_a, h'_{a,b})$. An isomorphism $\Lambda = (\lambda_\sigma, \lambda(a))$ from $H(Z)$ to $H'(Z)$ over the identity map $Z \rightarrow Z$ is constructed as follows. For each $\tau \in V(Z)$, choose an element $k_\tau \in H$ such that $k_\tau \cdot \overline{\tau} = \overline{\tau}'$, and define a group isomorphism $\lambda_\tau : H_\tau \rightarrow H'_\tau$ by

$$\lambda_\tau(g, h) = (\phi_{\overline{\tau}'}^{k_\tau}(g), k_\tau h k_\tau^{-1}).$$

For each $a \in E(Z)$, define $\lambda(a) = (1, k_{t(a)} h_a k_{i(a)}^{-1} h'_a{}^{-1})$. Note that by III.C.2.9(2) of [8], $\lambda(a) \in H'_{t(a)}$.

The verification that $\Lambda = (\lambda_\sigma, \lambda(a))$ is an isomorphism of complexes of groups is straightforward. \square

We remind the reader that faithfulness of a complex of groups is defined in the final paragraph of Section 2.3 above.

Lemma 3.8. *If $G(Y)$ is faithful and the H -action on Y is faithful then $H(Z)$ is faithful.*

Proof. This follows from the construction of $H(Z)$, and the characterisation of faithful complexes of groups in Proposition 38 of [22]. \square

3.3 The induced covering

Suppose H acts by simple morphisms on a simple complex of groups $G(Y)$, inducing a complex of groups $H(Z)$ as in Section 3.2 above. In this section we construct a covering of complexes of groups $\Lambda : G(Y) \rightarrow H(Z)$ over the quotient map $p : Y \rightarrow Z$.

For $\sigma \in V(Y)$, the local maps $\lambda_\sigma : G_\sigma \rightarrow H_{p(\sigma)}$ are defined as follows. Recall that for each vertex $\tau \in V(Z)$ we chose a lift $\overline{\tau} \in V(Y)$. Now for each $\sigma \in V(Y)$, we

choose $k_\sigma \in H$ such that $k_\sigma \cdot \sigma = \overline{p(\sigma)}$. Hence $\phi_\sigma^{k_\sigma}$ is an isomorphism $G_\sigma \rightarrow G_{\overline{p(\sigma)}}$. The local map $\lambda_\sigma : G_\sigma \rightarrow H_{p(\sigma)}$ is then defined by

$$\lambda_\sigma : g \mapsto (\phi_\sigma^{k_\sigma}(g), 1).$$

Note that each λ_σ is injective.

For each edge $a \in E(Y)$, define

$$\lambda(a) = (1, k_{t(a)} k_{i(a)}^{-1} h_b^{-1})$$

where $p(a) = b \in E(Z)$. Note that, since H acts on Y in the sense of Definition 2.4 above, we have $k_{i(a)} \cdot a = \overline{b}$ hence $k_{t(a)} k_{i(a)}^{-1} h_b^{-1}$ fixes $\overline{t(b)}$. Thus $\lambda(a) \in H_{t(b)}$ as required.

Proposition 3.9. *The map $\Lambda = (\lambda_\sigma, \lambda(a))$ is a covering of complexes of groups.*

Proof. It may be checked that Λ is a morphism of complexes of groups. As noted, each of the local maps λ_σ is injective. It remains to show that for each $\sigma \in V(Y)$ and $b \in E(Z)$ such that $t(b) = p(\sigma) = \tau$, the map on cosets

$$\Lambda_{\sigma/b} : \left(\coprod_{\substack{a \in p^{-1}(b) \\ t(a) = \sigma}} G_\sigma / \psi_a(G_{i(a)}) \right) \rightarrow H_\tau / \theta_b(H_{i(b)})$$

induced by $g \mapsto \lambda_\sigma(g)\lambda(a) = (\phi_\sigma^{k_\sigma}(g), k_\sigma k_{i(a)}^{-1} h_b^{-1})$ is a bijection.

We first show that $\Lambda_{\sigma/b}$ is injective. Suppose a and a' are in $p^{-1}(b)$ with $t(a) = t(a') = \sigma$, and suppose $g, g' \in G_\sigma$ with g representing a coset of $\psi_a(G_{i(a)})$ in G_σ and g' a coset of $\psi_{a'}(G_{i(a')})$ in G_σ . Assume that $\lambda_\sigma(g)\lambda(a)$ and $\lambda_\sigma(g')\lambda(a')$ belong to the same coset of $\theta_b(H_{i(b)})$ in H_τ .

Looking at the second component of the semidirect product H_τ , it follows from the definition of θ_b (Lemma 3.5 above) that for some $h \in \text{Stab}_H(\overline{i(b)})$,

$$k_\sigma k_{i(a)}^{-1} h_b^{-1} = \left(k_\sigma k_{i(a')}^{-1} h_b^{-1} \right) (h_b h h_b^{-1}) = k_\sigma k_{i(a')}^{-1} h h_b^{-1}.$$

Thus $k_{i(a')} k_{i(a)}^{-1} = h$ fixes $\overline{i(b)}$. Hence $k_{i(a)}^{-1} k_{i(a')}$ fixes $k_{i(a)}^{-1} \overline{i(b)} = i(a)$, and so $k_{i(a)}^{-1} k_{i(a')}$ fixes a . Thus $k_{i(a')} \cdot a = k_{i(a)} \cdot a = \overline{b} = k_{i(a')} \cdot a'$, hence $a = a'$.

Looking now at the first component of $\lambda_\sigma(g)\lambda(a)$ and $\lambda_\sigma(g')\lambda(a') = \lambda_\sigma(g')\lambda(a)$ in the semidirect product H_τ , by definition of θ_b , for some $x \in G_{\overline{i(b)}}$ we have

$$\begin{aligned} \phi_\sigma^{k_\sigma}(g) &= \phi_\sigma^{k_\sigma}(g') \phi_{\overline{t(b)}}^{k_\sigma k_{i(a)}^{-1} h_b^{-1}} \circ \phi_{\overline{t(b)}}^{h_b} \circ \psi_{\overline{b}}(x) \\ &= \phi_\sigma^{k_\sigma}(g') \phi_\sigma^{k_\sigma} \circ \phi_{\overline{t(b)}}^{k_{i(a)}^{-1}} \circ \psi_{\overline{b}}(x). \end{aligned}$$

Since $\phi_\sigma^{k_\sigma}$ is an isomorphism, and $k_{i(a)}^{-1} \cdot \overline{b} = a$, this implies

$$(g')^{-1} g = \phi_{\overline{t(b)}}^{k_{i(a)}^{-1}} \circ \psi_{\overline{b}}(x) = \psi_a \circ \phi_{\overline{i(b)}}^{k_{i(a)}^{-1}}(x) \in \psi_a(G_{i(a)})$$

as required. Thus the map $\Lambda_{\sigma/b}$ is injective.

To show that $\Lambda_{\sigma/b}$ is surjective, let $g \in G_{\bar{\tau}}$ and $h \in \text{Stab}_H(\bar{\tau})$, so that $(g, h) \in H_{\bar{\tau}}$. Let a be the unique edge of Y with $t(a) = \sigma$ and such that $k_{\sigma} \cdot a = hh_b\bar{b}$. Let g' be the unique element of G_{σ} such that $\phi_{\sigma}^{k_{\sigma}}(g') = g \in G_{\bar{\tau}}$. We claim that $\lambda_{\sigma}(g')\lambda(a)$ lies in the same coset as (g, h) . Now

$$\lambda_{\sigma}(g')\lambda(a) = (\phi_{\sigma}^{k_{\sigma}}(g'), k_{\sigma}k_{i(a)}^{-1}h_b^{-1}) = (g, k_{\sigma}k_{i(a)}^{-1}h_b^{-1})$$

so it suffices to show that $k_{\sigma}k_{i(a)}^{-1}h_b^{-1} \in hh_b \text{Stab}_H(\overline{i(b)})h_b^{-1}$. Equivalently, we wish to show that $h_b^{-1}h^{-1}k_{\sigma}k_{i(a)}^{-1}$ fixes $\overline{i(b)}$. We have $k_{i(a)} \cdot i(a) = \overline{i(b)}$ by definition, and the result follows by our choice of a . Thus $\Lambda_{\sigma/b}$ is surjective.

Hence Λ is a covering of complexes of groups. \square

3.4 The fundamental group

Suppose H acts by simple morphisms on a simple complex of groups $G(Y)$, inducing a complex of groups $H(Z)$ as in Section 3.2 above. In this section we establish the short exact sequence of Theorem 3.1 above, and provide sufficient conditions for the fundamental group of $H(Z)$ to be the semidirect product of the fundamental group of $G(Y)$ by H .

Fix σ_0 a vertex of Y and let $p : Y \rightarrow Z$ be the natural projection. We refer the reader to Section III.C.3 of [8] for the definition of the *fundamental group of $G(Y)$ at σ_0* , denoted $\pi_1(G(Y), \sigma_0)$. We will use notation and results from that section in the following proof. Let $\pi_1(H(Z), p(\sigma_0))$ be the fundamental group of $H(Z)$ at $p(\sigma_0)$.

Proposition 3.10. *There is a natural short exact sequence*

$$1 \rightarrow \pi_1(G(Y), \sigma_0) \rightarrow \pi_1(H(Z), p(\sigma_0)) \rightarrow H \rightarrow 1.$$

Proof. To obtain a monomorphism $\pi_1(G(Y), \sigma_0) \rightarrow \pi_1(H(Z), p(\sigma_0))$, we use the morphism of complexes of groups $\Lambda : G(Y) \rightarrow H(Z)$ defined in Section 3.3 above. By Proposition III.C.3.6 of [8], Λ induces a natural homomorphism

$$\pi_1(\Lambda, \sigma_0) : \pi_1(G(Y), \sigma_0) \rightarrow \pi_1(H(Z), p(\sigma_0)).$$

Since Λ is a covering (Proposition 3.9 above), Theorem 3.2 above implies that this map $\pi_1(\Lambda, \sigma_0)$ is in fact injective.

We next define a surjection $\pi_1(H(Z), p(\sigma_0)) \rightarrow H$. The group H may be regarded as a complex of groups over a single vertex. There is then a canonical morphism of complexes of groups $\Phi : H(Z) \rightarrow H$, defined as follows. Recall that for each $\tau \in V(Z)$, the local group H_{τ} is given by $H_{\tau} = G_{\bar{\tau}} \rtimes \text{Stab}_H(\bar{\tau})$. The local map $\phi_{\tau} : H_{\tau} \rightarrow H$ in the morphism Φ is defined to be projection to the second factor $\text{Stab}_H(\bar{\tau}) \leq H$. For each edge b of Z , we define $\phi(b) = h_b$. It may then be checked that Φ is a morphism.

By Proposition III.C.3.6 of [8], the morphism Φ induces a homomorphism of fundamental groups

$$\pi_1(\Phi, p(\sigma_0)) : \pi_1(H(Z), p(\sigma_0)) \rightarrow H.$$

By III.C.3.14 and Corollary III.C.3.15 of [8], if $G(Y)$ were a complex of trivial groups, this map would be surjective. Since the image of $\pi_1(\Phi, p(\sigma_0))$ does not in fact depend on the local groups of $G(Y)$, we have that in all cases, $\pi_1(\Phi, p(\sigma_0))$ is surjective, as required.

It follows from definitions that the image of the monomorphism $\pi_1(\Lambda, \sigma_0)$ is the kernel of the surjection $\pi_1(\Phi, p(\sigma_0))$. Hence the sequence above is exact. \square

Corollary 3.11. *If H fixes a vertex of Y ,*

$$\pi_1(H(Z), p(\sigma_0)) \cong \pi_1(G(Y), \sigma_0) \rtimes H.$$

Proof. Suppose that H fixes the vertex σ of Y . We will construct a section $\iota : H \rightarrow \pi_1(H(Z), p(\sigma_0))$ for the surjective homomorphism $\pi_1(\Phi, p(\sigma_0)) : \pi_1(H(Z), p(\sigma_0)) \rightarrow H$ given in the proof of Proposition 3.10 above.

The vertex σ is the unique lift $\bar{\tau}$ of a vertex $p(\sigma) = \tau \in Z$. Hence

$$H_\tau = G_{\bar{\tau}} \rtimes \text{Stab}_H(\bar{\tau}) = G_\sigma \rtimes H.$$

By definition of the surjection $\pi_1(\Phi, p(\sigma_0)) : \pi_1(H(Z), p(\sigma_0)) \rightarrow H$, a section $\iota : H \rightarrow \pi_1(H(Z), p(\sigma_0))$ is then given by the inclusion $H \rightarrow H_\tau$. \square

This completes the proof of Theorem 3.1.

4 Proof of the Main Theorem

We now prove the Main Theorem and Corollary 1.2, stated in the introduction. Throughout this section, we adopt the notation of the Main Theorem, and assume that the vertices s_1 and s_2 of the nerve L , and the elements α_1 and α_2 of the group A of label-preserving automorphisms of L , satisfy Conditions (1)–(4) of its statement. In Section 4.1 we introduce notation, and construct a family of finite polyhedral complexes Y_n , for $n \geq 1$, and an infinite polyhedral complex Y_∞ . We then in Section 4.2 construct complexes of groups $G(Y_n)$ and $G(Y_\infty)$ over these spaces, and show that there are coverings of complexes of groups $G(Y_n) \rightarrow G(Y_1)$ and $G(Y_\infty) \rightarrow G(Y_1)$. In Section 4.3 we define the action of a finite group H_n on Y_n , and of an infinite group H_∞ on Y_∞ , and then in Section 4.4 we show that these actions extend to actions on the complexes of groups $G(Y_n)$ and $G(Y_\infty)$. In Section 4.5 we combine these results with Theorem 3.1 above to complete the proof of the Main Theorem. Corollary 1.2 is proved in Section 4.6.

4.1 The spaces Y_n and Y_∞

In this section we construct a family of finite polyhedral complexes Y_n and an infinite polyhedral complex Y_∞ .

We first set up some notation. For $i = 1, 2$, let $q_i \geq 2$ be the order of α_i . It will be convenient to put, for all $k \geq 0$, $s_{2k+1} = s_1$ and $s_{2k+2} = s_2$, and similarly $\alpha_{2k+1} = \alpha_1$, $\alpha_{2k+2} = \alpha_2$, $q_{2k+1} = q_1$ and $q_{2k+2} = q_2$. Conditions (1)–(4) of the Main Theorem then become:

1. for all $n \geq 1$, α_n fixes the star of s_{n+1} in L ;
2. for all $n \geq 1$, the subgroup $\langle \alpha_n \rangle$ of A acts freely on the $\langle \alpha_n \rangle$ -orbit of s_n , in particular $\alpha_n(s_n) \neq s_n$;
3. for all $n \geq 1$, and all $t_n \neq s_n$ such that t_n is in the $\langle \alpha_n \rangle$ -orbit of s_n , $m_{s_n t_n} = \infty$; and
4. for all $n \geq 1$, all spherical special subgroups of W which contain s_n are halvable along s_n .

We now use the sequences $\{s_n\}$ and $\{\alpha_n\}$ to define certain elements and subsets of W . Let w_1 be the trivial element of W and for $n \geq 2$ let w_n be the product

$$w_n = s_1 s_2 \cdots s_{n-1} \in W.$$

Denote by $W_{n,n}$ the one-element set $\{w_n\}$. For $n \geq 2$, and $1 \leq k < n$, in order to simplify notation, write $\alpha^{j_{n-1}, \dots, j_k}$ for the composition of automorphisms

$$\alpha^{j_{n-1}, \dots, j_k} = \alpha_{n-1}^{j_{n-1}} \cdots \alpha_k^{j_k}$$

where $0 \leq j_i < q_i$ for $k \leq i < n$. Let w_{j_{n-1}, \dots, j_k} be the element of W :

$$w_{j_{n-1}, \dots, j_k} = w_n \alpha^{j_{n-1}}(s_{n-1}) \alpha^{j_{n-1}, j_{n-2}}(s_{n-2}) \cdots \alpha^{j_{n-1}, \dots, j_{k+1}}(s_{k+1}) \alpha^{j_{n-1}, \dots, j_k}(s_k). \quad (4)$$

Now for $n \geq 2$ and $1 \leq k < n$, define

$$W_{k,n} = \{w_{j_{n-1}, \dots, j_k} \in W \mid 0 \leq j_i < q_i \text{ for } k \leq i < n\}.$$

Note that if $j_{n-1} = 0$ then $w_{j_{n-1}, \dots, j_k} \in W_{k, n-1}$.

Example: Let (W, S) be the Coxeter system in Example 1 of Section 2.2 above, with nerve L shown in Figure 3 above. For $i = 1, 2$, let $\alpha_i \in A$ be the automorphism of L which fixes the star of s_{3-i} in L and interchanges s_i and s_3 . Then if m and m' are both even, the Main Theorem applies to this example. (If $T = \{s\}$ then W_T is halvable along s with $\text{half}_s(W_T)$ the trivial group. If $T = \{s, t\}$ then W_T is the dihedral group of order $2m_{st}$, and W_T is halvable along s if and only if m_{st} is even, in which case $\text{half}_s(W_T)$ is the dihedral group of order m_{st} .) Note that $q_1 = q_2 = 2$, and so, for instance,

$$\begin{aligned} W_{1,3} &= \{1, s_1 \alpha_1(s_1), s_1 s_2 \alpha_2(s_2) \alpha_2(s_1), s_1 s_2 \alpha_2(s_2) \alpha_2 \alpha_1(s_1)\} \\ W_{2,3} &= \{s_1, s_1 s_2 \alpha_2(s_2)\} \\ W_{3,3} &= \{s_1 s_2\}. \end{aligned}$$

The following lemma establishes key properties of the sets $W_{k,n}$.

Lemma 4.1. *For all $n \geq 1$:*

1. the sets $W_{1,n}, W_{2,n}, \dots, W_{n,n}$ are pairwise disjoint; and

2. for all $1 \leq k < n$, if

$$w_{j_{n-1}, \dots, j_k} = w_{j'_{n-1}, \dots, j'_k}$$

(where $0 \leq j_i < q_i$ for $k \leq i < n$) then $j_k = j'_k$, $j_{k+1} = j'_{k+1}$, \dots , and $j_{n-1} = j'_{n-1}$.

Proof. Given $1 \leq k \leq k' < n$, with $0 \leq j_i < q_i$ for $k \leq i < n$ and $0 \leq j'_i < q_i$ for $k' \leq i < n$, suppose

$$w_{j_{n-1}, \dots, j_k} = w_{j'_{n-1}, \dots, j'_{k'}}. \quad (5)$$

Then

$$\begin{aligned} & \alpha^{j_{n-1}}(s_{n-1}) \alpha^{j_{n-1}, j_{n-2}}(s_{n-2}) \dots \alpha^{j_{n-1}, \dots, j_k, \dots, j_{k+1}}(s_{k+1}) \alpha^{j_{n-1}, \dots, j_{k'}, \dots, j_k}(s_k) \\ &= \alpha^{j'_{n-1}}(s_{n-1}) \alpha^{j'_{n-1}, j'_{n-2}}(s_{n-2}) \dots \alpha^{j'_{n-1}, \dots, j'_{k'+1}}(s_{k'+1}) \alpha^{j'_{n-1}, \dots, j'_{k'}}(s_{k'}). \end{aligned}$$

By Condition (1) above, for each $k \leq i < n$, the automorphism α_i fixes s_{i+1} , thus

$$\begin{aligned} \alpha^{j_{n-1}, \dots, j_{i+1}}(s_{i+1}) \alpha^{j_{n-1}, \dots, j_i}(s_i) &= \alpha^{j_{n-1}, \dots, j_{i+1}, j_i}(s_{i+1}) \alpha^{j_{n-1}, \dots, j_i}(s_i) \\ &= \alpha^{j_{n-1}, \dots, j_i}(s_{i+1} s_i). \end{aligned}$$

Also since α_i fixes the star of s_{i+1} but $\alpha_i(s_i) \neq s_i$, we have $m_{s_{i+1} s_i} = \infty$. Since $\alpha^{j_{n-1}, \dots, j_i}$ is a label-preserving automorphism, it follows that the product of the two generators

$$\alpha^{j_{n-1}, \dots, j_{i+1}}(s_{i+1}) \alpha^{j_{n-1}, \dots, j_i}(s_i)$$

has infinite order, for each $k \leq i < n$. Similarly for each $k' \leq i < n$. Thus the only way for Equation (5) to hold is if $k = k'$, and for each $k \leq i < n$, $\alpha_i^{j_i}(s_i) = \alpha_i^{j'_i}(s_i)$. Since $\langle \alpha_i \rangle$ acts freely on the $\langle \alpha_i \rangle$ -orbit of s_i and we specified $0 \leq j_i < q_i$, the result follows. \square

For $n \geq 1$, and $1 \leq k \leq n$, define $Y_{k,n}$ to be the set of chambers

$$Y_{k,n} := \{wK \mid w \in W_{k,n}\}.$$

Recall that we are writing wK for the pair (w, K) . By Lemma 4.1 above, for fixed n , the sets $Y_{1,n}, \dots, Y_{n,n}$ are pairwise disjoint. We now define Y_n to be the polyhedral complex obtained by “gluing together” the chambers in $Y_{1,n}, \dots, Y_{n,n}$, using the same relation \sim as in the Davis complex Σ for (W, S) . More precisely,

$$Y_n := \left(\prod_{k=1}^n Y_{k,n} \right) / \sim$$

where, for $x, x' \in K$, we have $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. Note that $Y_1 = Y_{1,1} = K$. To define Y_∞ , for each $k \geq 1$, noting that $W_{k,n}$ is only defined for $1 \leq k \leq n$, put

$$W_{k,\infty} := \bigcup_{n=k}^{\infty} W_{k,n}.$$

Then $Y_{k,\infty}$ is the set of chambers

$$Y_{k,\infty} := \{wK \mid w \in W_{k,\infty}\}.$$

Similarly to the finite case, the sets $Y_{1,\infty}, Y_{2,\infty}, \dots$ are pairwise disjoint, and we define

$$Y_\infty = \left(\prod_{k=1}^{\infty} Y_{k,\infty} \right) / \sim$$

for the same relation \sim . Note that there are natural strict inclusions as subcomplexes

$$Y_1 \subset Y_2 \subset \dots \subset Y_n \subset \dots \subset Y_\infty.$$

(In fact, Y_n and Y_∞ are subcomplexes of the Davis complex Σ , but we will not adopt this point of view.) We define a mirror of Y_n or Y_∞ to be an *interior mirror* if it is contained in more than one chamber.

Example: Let (W, S) , α_1 and α_2 be as in the previous example of this section. To indicate the construction of Y_n and Y_∞ in this case, Figure 7 below depicts the dual graph for Y_4 , that is, the graph with vertices the chambers of Y_4 , and edges joining adjacent chambers. The edges are labelled with the type of the corresponding interior mirror. Figure 8 sketches the dual graph for Y_∞ .

We now describe features of Y_n and Y_∞ which will be needed below. The first lemma follows from the construction of Y_n and Y_∞ and Lemma 4.1 above.

Lemma 4.2. *Let $w = w_{j_{n-1}, \dots, j_k} \in W_{k,n}$. All of the chambers of Y_n to which $wK \in Y_{k,n}$ is adjacent are described by the following.*

1. *For $n \geq 1$ and $1 \leq k < n$, the chamber wK is adjacent to exactly one chamber of $Y_{k+1,n}$, namely it is $\alpha^{j_{n-1}, \dots, j_k}(s_k)$ -adjacent to the chamber $w_{j_{n-1}, \dots, j_{k+1}}K$ of $Y_{k+1,n}$.*
2. *For $n \geq 2$ and $1 \leq k \leq n$, the chamber wK is adjacent to exactly q_{k-1} distinct chambers of $Y_{k-1,n}$, namely for each $0 \leq j_{k-1} < q_{k-1}$, the chamber wK is $\alpha^{j_{n-1}, \dots, j_k, j_{k-1}}(s_{k-1})$ -adjacent to the chamber $w_{j_{n-1}, \dots, j_k, j_{k-1}}K$ of $Y_{k-1,n}$.*

Similarly for Y_∞ .

Corollary 4.3. *1. Any vertex of Y_n is contained in at most two distinct chambers of Y_n , and similarly for Y_∞ .*

2. *Any two interior mirrors of Y_n or Y_∞ are disjoint.*

Proof. Suppose σ is a vertex of Y_n , contained in the chamber wK , where w is as in Lemma 4.2 above. If σ is contained in more than one chamber of Y_n or Y_∞ , then σ is contained in an interior mirror K_s , for some $s \in S$. By the construction of Y_n and Lemma 4.2 above, s is either an image of s_k , or one of q_{k-1} distinct images of s_{k-1} , under some element of A . Suppose s is in the image of s_k . Condition (1) of the Main Theorem implies that $m_{s_k s_{k-1}} = \infty$. Hence the mirror K_s is disjoint from each of the q_{k-1} mirrors of types the q_{k-1} images of s_{k-1} . Therefore the only

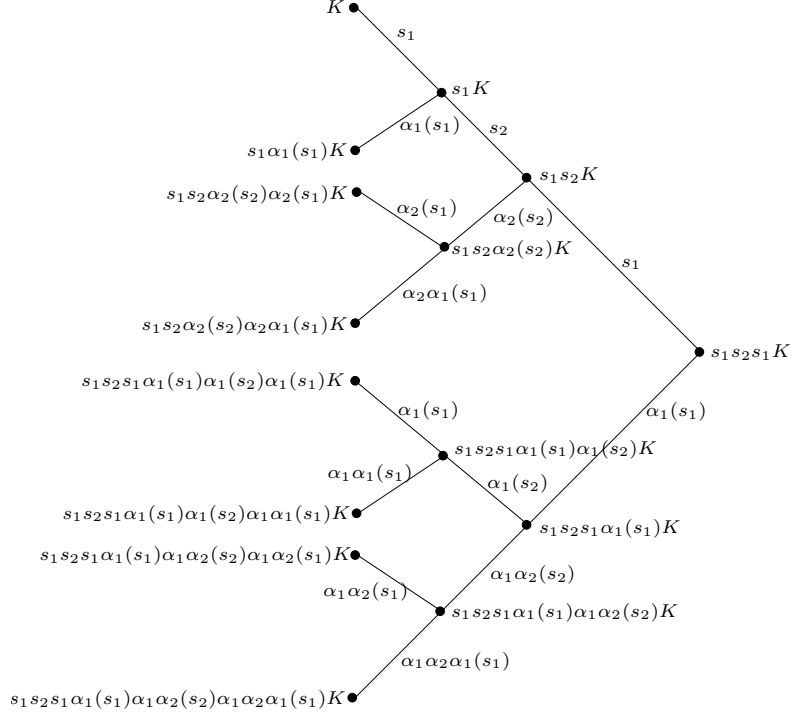


Figure 7: Dual graph for Y_4 , with vertices and edges labelled

chambers of Y_n which contain σ are the two chambers wK and wsK . Now suppose s is one of the q_{k-1} images of s_{k-1} under some element of A . Condition (3) of the Main Theorem implies that the mirrors of types each of these images are pairwise disjoint, and so again σ is contained in only two distinct chambers of Y_n . Similarly, any two interior mirrors of Y_n or Y_∞ are disjoint. \square

Corollary 4.4. *For all $n \geq 2$, there are q_{n-1} disjoint subcomplexes of Y_n , denoted $Y_{n-1}^{j_{n-1}}$ for $0 \leq j_{n-1} < q_{n-1}$, each isomorphic to Y_{n-1} , and with $Y_{n-1}^0 = Y_{n-1} \subset Y_n$. For each $0 \leq j_{n-1} < q_{n-1}$, the subcomplex $Y_{n-1}^{j_{n-1}}$ is attached to the chamber $w_n K = s_1 s_2 \cdots s_{n-1} K$ of Y_n along its mirror of type $\alpha^{j_{n-1}}(s_{n-1})$. An isomorphism*

$$F^{j_{n-1}} : Y_{n-1} \rightarrow Y_{n-1}^{j_{n-1}}$$

is given by sending the chamber

$$w_{j_{n-2}, \dots, j_k} K \in Y_{k, n-1}$$

to the chamber

$$w_{j_{n-1}, j_{n-2}, \dots, j_k} K \in Y_{k, n},$$

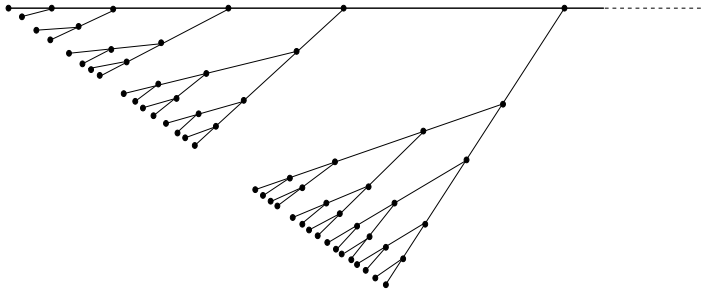


Figure 8: Dual graph for Y_∞

and the vertex of $w_{j_{n-2}, \dots, j_k} K$ of type T to the vertex of $w_{j_{n-1}, j_{n-2}, \dots, j_k} K$ of type $\alpha^{j_{n-1}}(T)$, for each spherical subset T of S .

Proof. By induction on n , using Lemma 4.2 and Corollary 4.3 above. \square

4.2 Complexes of groups $G(Y_n)$ and $G(Y_\infty)$

We now construct complexes of groups $G(Y_n)$ over each Y_n , and $G(Y_\infty)$ over Y_∞ , and show that there are coverings $G(Y_n) \rightarrow G(Y_1)$ and $G(Y_\infty) \rightarrow G(Y_1)$. To simplify notation, write Y for Y_n or Y_∞ .

To define the local groups of $G(Y)$, let σ be a vertex of Y , of type T . By Corollary 4.3 above, σ is contained in at most two distinct chambers of Y . If σ is only contained in one chamber of Y , put $G_\sigma = W_T$. If σ is contained in two distinct chambers of Y , then by Corollary 4.3 above σ is contained in a unique interior mirror K_s , with $s \in T$. By the construction of Y , s is in the A -orbit of some s_n , $n \geq 1$. By Condition (4) of the Main Theorem, it follows that the group W_T is halvable along s . We define the local group at σ to be $G_\sigma = \text{half}_s(W_T)$.

The monomorphisms between local groups are defined as follows. Let a be an edge of Y , with $i(a)$ of type T and $t(a)$ of type T' , so that $T \subsetneq T'$. If both of the vertices $i(a)$ and $t(a)$ are contained in a unique chamber of Y , then the monomorphism ψ_a along this edge is defined to be the natural inclusion $W_T \hookrightarrow W_{T'}$. If $i(a)$ is contained in two distinct chambers, then $i(a)$ is contained in a unique interior mirror K_s , with $s \in T$. Thus $s \in T'$ as well, and so $t(a)$ is also contained in the mirror K_s . From the definitions of $\text{half}_s(W_T)$ and $\text{half}_s(W_{T'})$, it follows that there is a natural inclusion $\text{half}_s(W_T) \hookrightarrow \text{half}_s(W_{T'})$, and we define ψ_a to be this inclusion. Finally suppose $i(a)$ is contained in a unique chamber of Y but $t(a)$ is contained in two distinct chambers of Y . Then for some $k \geq 1$, $i(a)$ is in a chamber of $Y_{k,n}$ (respectively, $Y_{k,\infty}$), and $t(a)$ is either in $Y_{k-1,n}$ or in $Y_{k+1,n}$ (respectively, in $Y_{k-1,\infty}$ or $Y_{k+1,\infty}$). Moreover $t(a)$ is contained in a unique interior mirror K_s , with $s \in T' - T$. If $t(a)$ is in $Y_{k-1,n}$ (respectively, $Y_{k-1,\infty}$), then we define ψ_a to be the natural inclusion $W_T \hookrightarrow \text{half}_s(W_{T'})$. If $t(a)$ is in $Y_{k+1,n}$ (respectively, $Y_{k+1,\infty}$), then we define ψ_a to be the monomorphism defined on the generators $t \in T$ of W_T by $\psi_a(t) := sts \in \text{half}_s(W_{T'})$, that is, $\psi_a = \text{Ad}(s)$.

It is not hard to verify that for all pairs of composable edges (a, b) in Y , $\psi_{ab} = \psi_a \circ \psi_b$. Hence we have constructed simple complexes of groups $G(Y_n)$ and $G(Y_\infty)$ over Y_n and Y_∞ respectively. Note that these complexes of groups are faithful, since by construction the local group at each vertex of type \emptyset is trivial. Note also that $G(Y_1)$ is the same complex of groups as constructed in Section 2.3 above, which has fundamental group W and universal cover Σ .

Example: Let (W, S) , α_1 and α_2 be as in the examples in Section 4.1 above. The complex of groups $G(Y_2)$ is sketched in Figure 9. From left to right, the three chambers here are K , s_1K and $s_1\alpha_1(s_1)K$. We denote by D_{2m} the dihedral group of order $2m$, with D_m the dihedral group of order m , and similarly for $D_{2m'}$ and $D_{m'}$ (recall that m and m' are even).

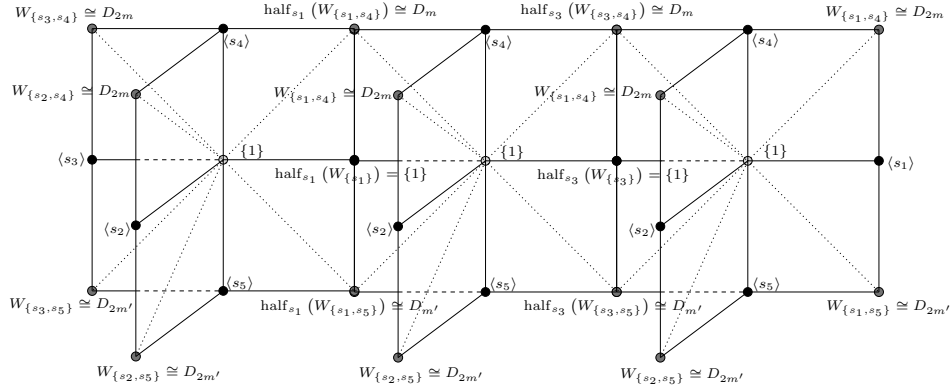


Figure 9: Complex of groups $G(Y_2)$

Proposition 4.5. *There are coverings of complexes of groups $G(Y_n) \rightarrow G(Y_1)$ and $G(Y_\infty) \rightarrow G(Y_1)$.*

Proof. Let $f_n : Y_n \rightarrow Y_1$ and $f_\infty : Y_\infty \rightarrow Y_1$ be the maps sending each vertex of Y_n or Y_∞ of type T to the unique vertex of $Y_1 = K$ of type T . Then by construction of Y_n and Y_∞ , the maps f_n and f_∞ are nondegenerate morphisms of scwols. We define coverings $\Phi_n : G(Y_n) \rightarrow G(Y_1)$ and $\Phi_\infty : G(Y_\infty) \rightarrow G(Y_1)$ over f_n and f_∞ respectively. To simplify notation, write Y for respectively Y_n or Y_∞ , f for respectively f_n or f_∞ , and Φ for respectively Φ_n or Φ_∞ .

Let σ be a vertex of Y , of type T . If the local group at σ is $G_\sigma = W_T$ then the map of local groups $\phi_\sigma : G_\sigma \rightarrow W_T$ is the identity map. If the local group at σ is $\text{half}_s(W_T)$, for some $s \in T$, then $\phi_\sigma : \text{half}_s(W_T) \rightarrow W_T$ is the natural inclusion as an index 2 subgroup. To define elements $\phi(a)$, if the monomorphism ψ_a in $G(Y)$ is natural inclusion, define $\phi(a) = 1$. If ψ_a is $\text{Ad}(s)$, then define $\phi(a) = s$. It is then easy to check that, by construction, Φ is a morphism of complexes of groups.

To show that Φ is a covering of complexes of groups, we first observe that each of the local maps ϕ_σ is injective. Now fix σ a vertex of Y , of type T' , and b an

edge of $Y_1 = K$ such that $t(b) = f(\sigma)$, with $i(b)$ of type T (hence $T \subsetneq T'$). We must show that the map

$$\Phi_{\sigma/b} : \coprod_{\substack{a \in f^{-1}(b) \\ t(a) = \sigma}} G_\sigma / \psi_a(G_{i(a)}) \rightarrow W_{T'} / W_T$$

induced by $g \mapsto \phi_\sigma(g)\phi(a)$ is a bijection, where G_σ and $G_{i(a)}$ are the local groups of $G(Y)$.

First suppose that σ is contained in a unique chamber of Y . Then by construction, there is a unique edge a of Y with $i(a)$ of type T and $t(a) = \sigma$, hence a unique edge $a \in f^{-1}(b)$ with $t(a) = \sigma$. Moreover, $G_\sigma = W_{T'}$, $G_{i(a)} = W_T$, the monomorphism ψ_a is natural inclusion hence $\phi(a) = 1$, and $\phi_\sigma : G_\sigma \rightarrow W_{T'}$ is the identity map. Hence $\Phi_{\sigma/b}$ is a bijection in this case.

Now suppose that σ is contained in two distinct chambers of Y . Then σ is contained in a unique interior mirror K_s of Y , with $s \in T'$. Assume first that $s \in T$ as well. Then there is a unique edge a of Y with $i(a)$ of type T and $t(a) = \sigma$. This edge is also contained in the mirror K_s . Hence there is a unique $a \in f^{-1}(b)$ with $t(a) = \sigma$. By construction, we have $G_\sigma = \text{half}_s(W_{T'})$, the map $\phi_\sigma : G_\sigma \rightarrow W_{T'}$ is natural inclusion as an index 2 subgroup, $G_{i(a)} = \text{half}_s(W_T)$, the map ψ_a is natural inclusion, and $\phi(a)$ trivial. Since the index $[W_{T'} : W_T] = [\text{half}_s(W_{T'}) : \text{half}_s(W_T)]$ is finite, it is enough to verify that the inclusion $\text{half}_s(W_{T'}) \rightarrow W_{T'}$ induces an injective map on cosets

$$\text{half}_s(W_{T'}) / \text{half}_s(W_T) \rightarrow W_{T'} / W_T.$$

For this, suppose that $w, w' \in \text{half}_s(W_{T'})$ and that $wW_T = w'W_T$ in $W_{T'}$. Then $w^{-1}w' \in W_T \cap \text{half}_s(W_{T'})$. By definitions, it follows that $w^{-1}w' \in \text{half}_s(W_T)$, as required.

Now assume that σ is contained in the interior mirror K_s , with $s \notin T$. There are then two edges $a_1, a_2 \in f^{-1}(b)$ such that $t(a_1) = t(a_2) = \sigma$. Without loss of generality, ψ_{a_1} is natural inclusion $W_T \rightarrow \text{half}_s(W_{T'})$ and $\phi(a_1) = 1$, while $\psi_{a_2}(g) = sgs$ with $\phi(a_2) = s$. Since the index $[\text{half}_s(W_{T'}) : W_T] = \frac{1}{2}[W_{T'} : W_T]$ is finite, it is enough to show that the map on cosets $\Phi_{\sigma/b}$ is surjective. Let $w \in W_{T'}$. If $w \in \text{half}_s(W_{T'}) \leq W_{T'}$, then the image of the coset $w\psi_{a_1}(G_{i(a_1)}) = wW_T$ in G_σ is the coset wW_T in $W_{T'}$. If $w \notin \text{half}_s(W_{T'})$, then since $\text{half}_s(W_{T'})$ has index 2 in $W_{T'}$, and $s \notin \text{half}_s(W_{T'})$, there is a $w' \in \text{half}_s(W_{T'}) \leq W_{T'}$ such that $w = w's$. The image of the coset $w'\psi_{a_2}(G_{i(a_2)}) = w'(sW_Ts)$ in $\text{half}_s(W_{T'})$ is then the coset $w'\phi(a_2)W_T = w'sW_T = wW_T$ in $W_{T'}$. Thus $\Phi_{\sigma/b}$ is surjective, as required.

We conclude that Φ is a covering of complexes of groups. \square

4.3 Group actions on Y_n and Y_∞

In this section we construct the action of a finite group H_n on Y_n in the sense of Definition 2.4 above, and that of an infinite group H_∞ on Y_∞ .

We first define the groups H_n and H_∞ . For each $n \geq 1$, let C_{q_n} denote the cyclic group of order q_n . Note that $C_{q_n} \cong \langle \alpha_n \rangle$. We define H_1 to be the trivial

group and $H_2 = C_{q_1}$. For $n \geq 3$, we define H_n to be the wreath product

$$\begin{aligned} H_n &= H_{n-1} \wr C_{q_{n-1}} \\ &= (\cdots ((C_{q_1} \wr C_{q_2}) \wr C_{q_3}) \wr \cdots) \wr C_{q_{n-1}} \\ &= C_{q_1} \wr C_{q_2} \wr \cdots \wr C_{q_{n-1}}, \end{aligned}$$

that is, H_n is the semidirect product by $C_{q_{n-1}}$ of the direct product of q_{n-1} copies of H_{n-1} , where $C_{q_{n-1}}$ acts on this direct product by cyclic permutation of coordinates. Note that H_n is a finite group of order

$$|H_n| = q_1^{q_2 q_3 \cdots q_{n-1}} q_2^{q_3 \cdots q_{n-1}} \cdots q_{n-2}^{q_{n-1}} q_{n-1}. \quad (6)$$

We define H_∞ to be the infinite iterated (unrestricted) wreath product

$$H_\infty := C_{q_1} \wr C_{q_2} \wr \cdots \wr C_{q_{n-1}} \wr \cdots$$

We then have natural inclusions

$$H_1 < H_2 < \cdots < H_n < \cdots < H_\infty.$$

The following lemma will be needed for the proof of Corollary 1.2 in Section 4.6 below.

Lemma 4.6. *The group H_∞ is not finitely generated.*

Proof. By definition of H_∞ , for any nontrivial $h \in H_\infty$ there is an $n \geq 1$ such that $h \in H_n$. \square

We now define the actions of H_n and H_∞ on Y_n and Y_∞ respectively. This uses the label-preserving automorphisms $\alpha_n \in A$. Note that the action of A on the nerve L extends to the chamber K , fixing the vertex of type \emptyset . This action does not in general have a strict fundamental domain. Inconveniently, this action also does not satisfy Condition (2) of Definition 2.4 above, since for any nontrivial $\alpha \in A$, there is an edge a of K with $i(a)$ of type \emptyset but $\alpha(a) \neq a$. However, to satisfy Definition 2.4, it suffices to define actions on Y_n and Y_∞ , and then extend in the obvious way to the scwols which are the barycentric subdivisions of these spaces, with naturally oriented edges.

For each $n \geq 1$ fix a generator a_n for the cyclic group C_{q_n} . Recall that $\alpha_n \in A$ has order q_n . Thus for any $\alpha \in A$, there is a faithful representation $C_{q_n} \rightarrow A$, given by $a_n \mapsto \alpha \alpha_n \alpha^{-1}$. Recall also that α_n fixes the star in L of the vertex s_{n+1} , and that $\langle \alpha_n \rangle$ acts freely on the $\langle \alpha_n \rangle$ -orbit of s_n . Hence $a_n \mapsto \alpha \alpha_n \alpha^{-1}$ induces an action of C_{q_n} on the chamber K , which fixes pointwise the mirror of type $\alpha(s_{n+1})$, and permutes cyclically the set of mirrors of types $\alpha \alpha_n^{j_n}(s_n)$, for $0 \leq j_n < q_n$.

We define the action of H_n on Y_n inductively, as follows. The group H_1 is trivial. For $n \geq 2$, assume that the action of H_{n-1} on Y_{n-1} has been given. The subgroup $C_{q_{n-1}}$ of H_n then fixes the chamber $w_n K = s_1 s_2 \cdots s_{n-1} K$ of Y_n setwise, and acts on this chamber via $a_{n-1} \mapsto \alpha_{n-1}$. By the discussion above, this action fixes pointwise the mirror of type s_n of $w_n K$, and permutes cyclically the q_{n-1}

mirrors of types $\alpha_{n-1}^{j_{n-1}}(s_{n-1})$, with $0 \leq j_{n-1} < q_{n-1}$, along which (by Lemma 4.4 above), q_{n-1} disjoint subcomplexes of Y_n , each isomorphic to Y_{n-1} , are attached.

By induction, a copy of H_{n-1} in H_n acts on each of these copies of Y_{n-1} in Y_n . More precisely, for $0 \leq j_{n-1} < q_{n-1}$, the j_{n-1} st copy of H_{n-1} in H_n acts on the subcomplex $Y_{n-1}^{j_{n-1}}$ of Lemma 4.4 above. This action is given by conjugating the (inductively defined) action of H_{n-1} on $Y_{n-1} \subset Y_n$ by the isomorphism $F^{j_{n-1}} : Y_{n-1} \rightarrow Y_{n-1}^{j_{n-1}}$ in Lemma 4.4. By definition, the action of $C_{q_{n-1}}$ cyclically permutes the subcomplexes $Y_{n-1}^{j_{n-1}}$, and so we have defined an action of H_n on Y_n . The action of H_∞ on Y_∞ is similar.

We now describe the fundamental domains for these actions. For each $n \geq 1$ and each $1 \leq k \leq n$, observe that H_n acts transitively on the set of chambers $Y_{k,n}$. Let $K_1 = K$, and for $n \geq 2$ let K_n be the quotient of the chamber $w_n K = s_1 s_2 \cdots s_{n-1} K$ by the action of $C_{q_{n-1}} \leq H_n$ as defined above. In K_n , the mirrors of types $\alpha_{n-1}^{j_{n-1}}(s_{n-1})$, for $0 \leq j_{n-1} < q_{n-1}$, have been identified. By abuse of notation, we refer to these identified mirrors as the mirror of type s_{n-1} of K_n . Note also that $C_{q_{n-1}} \leq H_n$ fixes pointwise the mirror of type s_n of $w_n K$, and so we may speak of the mirror of type s_n of K_n . Then a fundamental domain for the action of H_n on Y_n is the finite complex

$$Z_n := (K_1 \cup K_2 \cup \cdots \cup K_n) / \sim,$$

where \sim means we identify the s_{i-1} -mirrors of K_{i-1} and K_i , for $1 \leq i < n$. Similarly, a fundamental domain for the action of H_∞ on Y_∞ is the infinite complex

$$Z_\infty := (K_1 \cup K_2 \cup \cdots \cup K_n \cup \cdots) / \sim.$$

Finally we describe the stabilisers in H_n and H_∞ of the vertices of Y_n and Y_∞ . Let wK be a chamber of Y_n or Y_∞ . Then there is a smallest $k \geq 1$ such that $wK \in Y_k$. By construction, it follows that the stabiliser in H_n or H_∞ of any vertex in the chamber wK is a subgroup of the finite group H_k . Hence H_n and H_∞ act with finite stabilisers. Note also that for every $n \geq 1$, the action of H_n fixes the vertex of type \emptyset in the chamber $w_n K$. We may thus speak of the vertex of type \emptyset in the quotient K_n defined above. In fact, in the fundamental domains Z_n and Z_∞ defined above, the vertex of type \emptyset in K_n , for $n \geq 1$, has a lift in Y_n or Y_∞ with stabiliser the finite group H_n . We observe also that the actions of H_n and H_∞ are faithful, since the stabiliser of the vertex of type \emptyset of $K_1 = K$ is the trivial group H_1 . Figure 10 shows Z_∞ and the stabilisers of (lifts of) its vertices of type \emptyset for the example in Section 4.1 above.

4.4 Group actions on $G(Y_n)$ and $G(Y_\infty)$

In this section we show that the actions of H_n and H_∞ on Y_n and Y_∞ , defined in Section 4.3 above, extend to actions (by simple morphisms) on the complexes of groups $G(Y_n)$ and $G(Y_\infty)$. To simplify notation, write H for H_n or H_∞ , Y for Y_n or Y_∞ , and Z for Z_n or Z_∞ . Technically, instead of working with $G(Y)$, we work with the corresponding naturally defined complex of groups over the barycentric

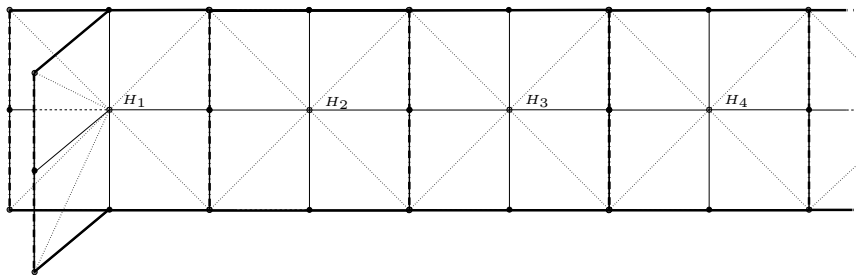


Figure 10: Fundamental domain Z_∞

subdivision of Y , so that the action of H satisfies Definition 2.4 above. By abuse of notation we will however continue to write $G(Y)$.

Recall that for σ a vertex of Y of type T , the local group G_σ is either W_T or $\text{half}_s(W_T)$, and the latter occurs if and only if σ is contained in an interior s -mirror of Y with $s \in T$. Let wK be a chamber of Y and let $h \in H$. By definition of the H -action, there is an $\alpha \in A$ such that for each vertex σ in wK , with σ of type T , the vertex $h \cdot \sigma$ of $h \cdot wK$ has type $\alpha(T)$. Moreover, if σ is contained in an interior s -mirror then $h \cdot \sigma$ is contained in an interior $\alpha(s)$ -mirror. We may thus define the local map $\phi_\sigma^h : G_\sigma \rightarrow G_{h \cdot \sigma}$ by $\phi_\sigma^h(t) = \alpha(t)$ for each $t \in T$, and (if $G_\sigma = \text{half}_s(W_T)$), $\phi_\sigma^h(sts) = \alpha(s)\alpha(t)\alpha(s)$. Then ϕ_σ^h is an isomorphism either $W_T \rightarrow W_{\alpha(T)}$, or $\text{half}_s(W_T) \rightarrow \text{half}_{\alpha(s)}(W_{\alpha(T)})$, as appropriate. It is not hard to verify that these local maps define an action of H on $G(Y)$ by simple morphisms.

4.5 Conclusion

In this section we combine the results of Sections 4.1–4.4 above to complete the proof of the Main Theorem.

Recall that $G(Y_1)$ is developable with universal cover Σ (see Section 2.3). By Proposition 3.9 and Theorem 3.2 above, it follows that the complexes of groups $G(Y_n)$ and $G(Y_\infty)$ are developable with universal cover Σ . Let $H(Z_n)$ be the complex of groups induced by H_n acting on $G(Y_n)$, and $H(Z_\infty)$ that induced by H_∞ acting on $G(Y_\infty)$. By Theorem 3.1 above, there are coverings of complexes of groups $G(Y_n) \rightarrow H(Z_n)$ and $G(Y_\infty) \rightarrow H(Z_\infty)$. Hence (by Theorem 3.2 above) each $H(Z_n)$ and $H(Z_\infty)$ is developable with universal cover Σ .

Let Γ_n be the fundamental group of $H(Z_n)$ and Γ the fundamental group of $H(Z_\infty)$. Since the complexes of groups $G(Y_n)$ and $G(Y_\infty)$ are faithful, and the actions of H_n and H_∞ are faithful, Theorem 3.1 above implies that $H(Z_n)$ and $H(Z_\infty)$ are faithful complexes of groups. Thus Γ_n and Γ may be identified with subgroups of $G = \text{Aut}(\Sigma)$. Now $G(Y_n)$ and $G(Y_\infty)$ are complexes of finite groups, and the H_n - and H_∞ -actions have finite vertex stabilisers. Hence by construction, $H(Z_n)$ and $H(Z_\infty)$ are complexes of finite groups. Therefore Γ_n and Γ are discrete subgroups of G . Since the fundamental domain Z_n is finite, it follows that each Γ_n is a uniform lattice. To show that Γ is a nonuniform lattice,

we use the normalisation of Haar measure μ on $G = \text{Aut}(\Sigma)$ defined in Section 2.1 above, with the G -set V the set of vertices of Σ of type \emptyset . Since the local groups of $H(Z_\infty)$ at the vertices of type \emptyset in Z_∞ are H_1, H_2, \dots , we have

$$\mu(\Gamma \backslash G) = \sum_{n=1}^{\infty} \frac{1}{|H_n|}.$$

This series converges (see Equation (6) above for the order of H_n , and note that each $q_n \geq 2$). We conclude that Γ is a nonuniform lattice in G . Moreover, as the covolumes of the uniform lattices Γ_n are the partial sums of this series, we have $\mu(\Gamma_n \backslash G) \rightarrow \mu(\Gamma \backslash G)$, as required. This completes the proof of the Main Theorem.

4.6 Proof of Corollary 1.2

The nonuniform lattice Γ is the fundamental group of the complex of groups $H(Z_\infty)$ induced by the action of H_∞ on $G(Y_\infty)$. By the short exact sequence in Theorem 3.1 above, there is a surjective homomorphism $\Gamma \rightarrow H_\infty$. Since H_∞ is not finitely generated (Lemma 4.6 above), we conclude that Γ is not finitely generated.

5 Examples

In this section we describe several infinite families of examples to which the Main Theorem applies. By the *dimension* of the Davis complex Σ for a Coxeter system (W, S) , we mean the maximum cardinality of a spherical subset of S . We note that there may be maximal spherical special subgroups W_T with $|T|$ strictly less than $\dim(\Sigma)$.

5.1 Two-dimensional examples

If $\dim(\Sigma) = 2$ then the nerve of the Coxeter system (W, S) is a graph L with vertex set S and two vertices s and t joined by an edge if and only if m_{st} is finite. Assume for simplicity that for some integer $m \geq 2$ all finite $m_{st} = m$. Then Σ is the barycentric subdivision of a polygonal complex X , with all 2-cells of X regular Euclidean $2m$ -gons, and the link of every vertex of X the graph L . Such an X is called a $(2m, L)$ -complex. Condition (4) of the Main Theorem can hold only if m is even, and so we also assume this. It is then not hard to find graphs L so that, for some pair s_1 and s_2 of non-adjacent vertices of L , and for some nontrivial elements $\alpha_1, \alpha_2 \in \text{Aut}(L)$, Conditions (1), (2) and (3) of the Main Theorem also hold. We present three infinite families of examples.

5.1.1 Buildings with complete bipartite links

Let L be the complete bipartite graph $K_{q,q'}$, with $q, q' \geq 2$. If $q \geq 3$ then there are (nonadjacent) vertices s_1 and s_2 of L , and nontrivial elements α_1 and α_2 of $\text{Aut}(L)$, so that the Main Theorem applies.

If $m = 2$ then Σ is the barycentric subdivision of the product of trees $T_q \times T_{q'}$, where T_q is the q -regular tree. In particular, if $m = m' = 2$ in Example 1 of Section 2.2 above, then Σ is the barycentric subdivision of $T_3 \times T_2$. If $m \geq 4$, then by Theorem 12.6.1 of [11] the complex Σ may be metrised as a piecewise hyperbolic CAT(-1) polygonal complex. With this metric, if $p = 2m$ and $q = q'$ then Σ is the barycentric subdivision of Bourdon's building $I_{p,q}$ (studied in, for example, [5] and [6]), which is the unique 2-complex with all 2-cells regular right-angled hyperbolic p -gons P , and the link of every vertex the complete bipartite graph $K_{q,q}$. Bourdon's building is a right-angled hyperbolic building, of type (W', S') where W' is the Coxeter group generated by the set of reflections S' in the sides of P .

5.1.2 Fuchsian buildings

A Fuchsian building is a 2-dimensional hyperbolic building. Bourdon's building $I_{p,q}$ is a (right-angled) Fuchsian building. For Fuchsian buildings which are not right-angled see, for example, [4] and [13].

To show that the Main Theorem applies to certain Fuchsian buildings which are not right-angled, let L be the finite building of rank 2 associated to a Chevalley group \mathcal{G} (see [25]). Then L is a bipartite graph, with vertex set say $S = S_1 \sqcup S_2$, and for some $k \in \{3, 4, 6, 8\}$, L has girth $2k$ and diameter k . Figure 11 depicts the building L for the group $\mathcal{G} = GL(3, \mathbb{F}_2) = GL(3, 2)$, for which $k = 3$. The white vertices of this building may be identified with the set of one-dimensional subspaces of the vector space $V = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$, and the black vertices with the set of two-dimensional subspaces of V . Two vertices are joined by an edge if those two subspaces are incident.

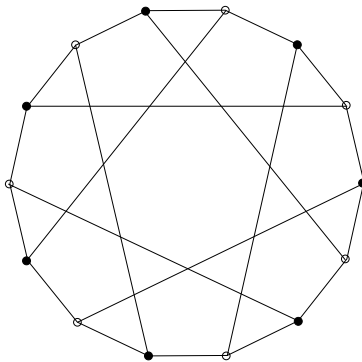


Figure 11: The building L for $\mathcal{G} = GL(3, 2)$

The group \mathcal{G} acts on L , preserving the type of vertices, with quotient an edge. Suppose $s_1 \in S_1$, and let $s_2 \in S_2$ be a vertex at distance k from s_1 . Since L is a thick building, there is more than one such vertex s_2 . For $i = 1, 2$, the stabiliser P_i of s_i in \mathcal{G} acts transitively on the set of vertices of L at distance k from s_i . Now,

by Theorem 6.18 of [25], P_i has a Levi decomposition

$$P_i = U_i \rtimes L_i$$

where L_i is the subgroup of P_i fixing the vertex s_{3-i} . Moreover, by Lemma 6.5 of [25], U_i fixes the star of s_i in L . Hence we may find elements $\alpha_{3-i} \in U_i$ for which Conditions (1) and (2) of the Main Theorem hold. Condition (3) of the Main Theorem follows since L is bipartite and the action of \mathcal{G} preserves the type of vertices. For example, for L as in Figure 11, if s_1 is the vertex $\{(1, 0, 0)\}$, we may choose s_2 to be the vertex $\{(0, 1, 0), (0, 0, 1), (0, 1, 1)\}$, and then choose

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose now that L as above is the nerve of a Coxeter system (W, S) . By Theorem 12.6.1 of [11], since L has girth ≥ 6 , the corresponding Davis complex Σ may also be metrised as a piecewise hyperbolic $\text{CAT}(-1)$ polygonal complex. With this metrisation, Σ is then the barycentric subdivision of a Fuchsian building, with the link of every vertex L and all 2-cells regular hyperbolic $2m$ -gons (of vertex angle $\frac{\pi}{k}$). We call such a building a $(2m, L)$ -building. In general, there may be uncountably many isomorphism classes of $(2m, L)$ -buildings (see for instance [13]). In fact, the Davis complex Σ is the barycentric subdivision of the unique locally reflexive $(2m, L)$ -building with trivial holonomy (see Haglund [17]).

5.1.3 Platonic polygonal complexes

A polygonal complex X is *Platonic* if $\text{Aut}(X)$ acts transitively on the set of flags (vertex, edge, face) in X . Any Platonic polygonal complex is a (k, L) -complex, with $k \geq 3$ and L a graph such that $\text{Aut}(L)$ acts transitively on the set of oriented edges in L . In [27], Świątkowski studied $\text{CAT}(0)$ Platonic polygonal complexes X , where L is a trivalent graph. Such complexes are not in general buildings.

A graph L is said to be *n -arc regular*, for some $n \geq 1$, if $\text{Aut}(L)$ acts simply transitively on the set of edge paths of length n in L . For example, the Petersen graph in Figure 1 above is 3-arc regular. Any finite, connected, trivalent graph L , with $\text{Aut}(L)$ transitive on the set of oriented edges of L , is n -arc regular for some $n \in \{1, 2, 3, 4, 5\}$ (Tutte [31]). Świątkowski [27] showed that if $n \in \{3, 4, 5\}$, then for all $k \geq 4$ there is a unique (k, L) -complex X , with X Platonic. Thus if $k = 2m$ is even, the barycentric subdivision of X is the Davis complex Σ for (W, S) , where (W, S) has nerve L and all finite $m_{st} = m$.

Now suppose L is a finite, connected, trivalent, n -arc regular graph with $n \in \{3, 4, 5\}$. Choose vertices s_1 and s_2 of L at distance two in L if $n = 3, 4$, and at distance three in L if $n = 5$. Then by Propositions 3–5 of Djoković–Miller [12], for $i = 1, 2$ there are involutions $\alpha_i \in \text{Aut}(L)$ such that α_i fixes the star of s_{3-i} in L , and $\alpha_i(s_i) \neq s_i$ is not adjacent to s_i . Thus if m is even, the Main Theorem applies to $G = \text{Aut}(\Sigma)$.

5.2 Higher-dimensional examples

We now discuss examples in dimension > 2 to which the Main Theorem applies. The construction of the building Σ below was suggested by an anonymous referee (our own examples were just for W right-angled).

We first discuss when Condition (4) in the Main Theorem can hold. Suppose W_T is a spherical special subgroup of W , with $k = |T| > 2$. If W_T is irreducible, then from the classification of spherical Coxeter groups (see, for example, [11]), it is not hard to verify that W_T is halvable along $s \in T$ if and only if W_T is of type B_k , with $s \in T$ the unique generator so that $m_{st} \in \{2, 4\}$ for all $t \in T - \{s\}$; in this case $\text{half}_s(W_T)$ is of type D_k . If W_T is reducible, then so long as s is contained in a direct factor $W_{T'}$, $T' \subsetneq T$, such that either $W_{T'} = \langle s \rangle \cong C_2$, $W_{T'}$ is an even dihedral group, or $W_{T'}$ is of type B_j with $j < k$ and s the particular generator described above, then W_T will be halvable along s .

Now let L be a thick spherical building of rank $k > 2$. A reducible example is L the join of k sets of points, with each set having cardinality at least 3. An irreducible example is L the building for a Chevalley group \mathcal{G} of rank k over a finite field, such as $GL(k+1, 2)$.

Define a Coxeter group W with nerve L as follows. Fix Δ a chamber of L . Then Δ is a simplex on k vertices. Let $p : L \rightarrow \Delta$ be the projection onto this chamber. Label the edges of Δ by the m_{st} for a finite Coxeter group V on k generators, such that V is a product of cyclic groups of order 2, even dihedral groups and copies of B_j , $j < k$. For example, when V is right-angled all $m_{st} = 2$. Pull the edge labels of Δ back via p to obtain a labelling of the edges of L . This defines a Coxeter group W with nerve L , so that each maximal spherical special subgroup of W is isomorphic to V .

The Davis complex Σ for W is tiled by copies of the barycentric subdivision of the Coxeter polytope P associated to V . For example, when V is right-angled, P is a k -cube. The link of each vertex of P is L . Applying the metric criterion of Charney–Lytchak [10], it follows that Σ is the barycentric subdivision of a building. Note that $\dim(\Sigma) = k > 2$.

Choose vertices s_1 and s_2 in L which are *opposite* (see [25]). By the same arguments as in Section 5.1.2 above, there are (type-preserving) elements $\alpha_1, \alpha_2 \in \text{Aut}(L)$ so that Conditions (1)–(3) of the Main Theorem hold. A careful choice of V , such that s_1 and s_2 if contained in some copy of B_j are both the required generators, then guarantees that Condition (4) of the Main Theorem holds. Hence the Main Theorem applies to many examples of buildings of dimension > 2 .

We do not know of any *hyperbolic* buildings of dimension > 2 to which the Main Theorem applies. For the 3-dimensional constructions of Haglund–Paulin in [19], certain of the m_{st} must be equal to 3, so Condition (4) of the Main Theorem will not hold.

A slight modification of the above construction, for example by adding a vertex s to L with $m_{st} = \infty$ for all $t \in S - \{s\}$, produces nerves which are not buildings, hence examples of Σ of dimension > 2 which are not buildings.

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