A categorical approach to classical and quantum Schur–Weyl duality

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Abstract

We use category theory to propose a unified approach to the Schur–Weyl dualities involving the general linear Lie algebras, their polynomial extensions and associated quantum deformations. We define multiplicative sequences of algebras exemplified by the sequence of group algebras of the symmetric groups and use them to introduce associated monoidal categories. Universal properties of these categories lead to uniform constructions of the Drinfeld functor connecting representation theories of the degenerate affine Hecke algebras and the Yangians and of its q-analogue. Moreover, we construct actions of these categories on certain (infinitesimal) braided categories containing a Hecke object.

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1 Introduction

Classically, starting with the vector representation V of $\mathfrak{gl}_N$ and taking its tensor powers $V^{\otimes n}$ one gets a sequence of the corresponding endomorphism algebras. In the limiting case $N \to \infty$, the sequence of the endomorphism algebras stabilizes and admits a precise description: this is a sequence of the group algebras of the symmetric groups S_n.

In a quantum version of the Schur–Weyl duality, the Lie algebra $\mathfrak{gl}_N$ is replaced by the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_N)$, and the corresponding stable sequence of the endomorphism algebras is a sequence of the Hecke algebras $\mathrm{H}_n(q)$. Similarly, the endomorphism algebras associated with the representations of the polynomial current and loop Lie algebras and their quantizations lead to sequences of group algebras of affine extensions of the symmetric groups and the sequences of affine Hecke algebras and their degenerate versions.

It is these sequences of algebras which we take as a starting point of our approach. We use them to construct certain categories (which we call the *Schur–Weyl categories*) modeling the parts of the corresponding representation categories generated by the vector representations. Each of the Schur–Weyl categories possesses a *universal property*: as a monoidal category, it is generated by a single object and a collection of endomorphisms. This property leads to a simple characterization of monoidal functors from these categories thus allowing us to reformulate the Schur–Weyl dualities as the existence and fullness properties of functors from the Schur–Weyl categories to the appropriate representation categories.

As another application of the universal property, we show that a localized category associated with the (degenerate) affine Hecke algebras is equivalent to a localized category associated with the (semi-) affine symmetric group algebras. Moreover, motivated by the work [23] we construct actions of the Schur–Weyl categories corresponding to the (degenerate) affine Hecke algebras on certain (infinitesimal) braided categories containing Hecke objects.

We will now explain our approach in more detail by using the classical Schur–Weyl duality as a model example. One consequence of this duality is the fact that there are no non-zero homomorphisms between the m-th and n-th tensor powers of the N-dimensional vector representation V of the general linear Lie algebra $\mathfrak{gl}_N$ for $m \neq n$:

$$\mathrm{Hom}_{\mathfrak{gl}_N}(V^{\otimes m}, V^{\otimes n}) = 0.$$

Moreover, the natural homomorphism

$$k[S_n] \to \mathrm{End}_{\mathfrak{gl}_N}(V^{\otimes n})$$

from the group algebra of the symmetric group is surjective (it is also injective if $n \leqslant N$) [15], [25] and [26]. Here and throughout the paper k denotes a field, and vector spaces and algebras are considered over k, unless stated otherwise.

At this point we want to shift the emphasis from the Lie algebra $\mathfrak{gl}_N$ to the sequence of the symmetric group algebras

$$k[S_*] = \{k[S_n] \mid n \geqslant 0\}$$

and to regard this sequence as the primary object of the duality. The sequence $k[S_*]$ comes equipped with the algebra homomorphisms

$$\mu_{m,n} : k[S_m] \otimes k[S_n] \longrightarrow k[S_{m+n}], \tag{1.1}$$

induced by the natural inclusions $S_m \times S_n \hookrightarrow S_{m+n}$. The homomorphisms $\mu_{m,n}$ satisfy a certain associativity property which we describe below in Sec. 2. The collection $k[S_*]$ together with these homomorphisms is an example of what we call a *multiplicative sequence of algebras*.

We can use the multiplicative sequence $k[S_*]$ to define a k-linear monoidal category $\overline{\mathcal{S}}$ whose objects $[n]$ are parameterized by the natural numbers $n \geqslant 0$ with no morphisms between the objects corresponding to different numbers and with the endomorphism algebra $\mathrm{End}_{\overline{\mathcal{S}}}([n])$ of the n-th object being $k[S_n]$. The tensor product on objects is defined by the addition of natural numbers, $[m] \otimes [n] = [m+n]$, while on the morphisms it is defined by the algebra homomorphisms (1.1). The category $\overline{\mathcal{S}}$ possesses a universal property: it is a free symmetric monoidal k-linear category generated by one object $X = [1]$.

We denote by $\mathcal{S}$ the category of k-linear functors from the opposite category $\overline{\mathcal{S}}^{op}$ to the category of vector spaces. Then $\mathcal{S}$ is a free symmetric abelian monoidal k-linear category generated by one object. More explicitly, $\mathcal{S}$ is the direct sum of the categories of right modules $\oplus_n \mathcal{M}od{-}k[S_n]$. The classical Schur–Weyl duality implies that the symmetric monoidal functor from $\mathcal{S}$ to the category of representations of $\mathfrak{gl}_N$ sending the generator X to V is full (surjective on morphisms). In other words, the universal enveloping algebra of $\mathfrak{gl}_N$ is Tannaka–Krein dual to the symmetric monoidal functor from $\mathcal{S}$ to the category of vector spaces, sending the generator X to V.

This construction can be generalized as follows. Let A be a unital associative algebra. One can start with a free symmetric monoidal category $\mathcal{S}(A)$ generated by one object X and with A as the algebra of its endomorphisms. The corresponding multiplicative sequence of algebras is now the sequence of cross-products $A^{\otimes n} * S_n$. We can associate with an N-dimensional vector space V the symmetric monoidal functor from $\mathcal{S}(A)$ to the category of vector spaces which sends the generator X to $V \otimes A$, considered as a vector space. Its Tannaka–Krein dual is the universal enveloping algebra of the Lie algebra $\mathrm{End}_A(V \otimes A)$ of A-linear endomorphisms of $V \otimes A$. Two examples of this situation will be particularly important for us due to their interesting quantizations. Namely, these are the cases where A is the algebra of polynomials in one variable, or its Laurent version. Then the Tannaka–Krein duals are respectively the universal enveloping algebras of the polynomial current Lie algebra $\mathfrak{gl}_N[t]$ and of the Lie algebra of Laurent polynomials $\mathfrak{gl}_N[t, t^{-1}]$. The corresponding (Schur–Weyl dual) multiplicative sequences of algebras are the sequences of group algebras of affine symmetric (semi-)groups.

More examples come via the Schur–Weyl duality from quantizations of the general linear Lie algebras or their current deformations. Namely, the multiplicative sequence of Hecke algebras arises from the endomorphism algebras of the tensor powers of the vector representation of the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_N)$. This gives rise to a monoidal category which we call the *Hecke category*. Furthermore, the quantum loop algebras and Yangians are respective quantizations of the algebras $\mathrm{U}(\mathfrak{gl}_N[t, t^{-1}])$ and $\mathrm{U}(\mathfrak{gl}_N[t])$. The Schur–Weyl dual sequence associated with the quantum loop algebras is formed by the affine Hecke algebras, while the dual sequence for the Yangians is formed by the sequence of degenerate affine Hecke algebras.

Now we outline the contents of the paper. We start by defining multiplicative sequences of algebras and the corresponding Schur–Weyl categories and list their basic properties (Sec. 2). In particular, we analyze a condition on an embedding of multiplicative sequences which guarantees that the right adjoint to the monoidal functor induced by the embedding (the restriction functor along the embedding) is also monoidal. This will be used later to construct fiber functors (i.e. monoidal functors to vector spaces) for the Schur–Weyl categories of (degenerate) affine Hecke algebras. We also study presentations of multiplicative sequences in terms of generators and relations and corresponding freeness properties of the Schur–Weyl categories.

In the subsequent sections we consider the families of multiplicative sequences of algebras which are Schur–Weyl dual to the general linear Lie algebra, its polynomial current versions and their quantum deformations: the quantized enveloping algebra, the Yangian and the quantum loop algebra. One of the advantages of our approach is a natural and unifying construction of the actions of the Hecke algebras (or their affine and degenerate versions) on tensor powers of the corresponding (polynomially extended) vector representation. In the case of the Yangians this action does not appear to have been previously described in the literature; cf. [1], [10]. In the quantum loop algebra case, the action of the affine Hecke algebras on the tensor powers of the vector representation provides an alternative to the construction previously given in [14, Theorem 4.9].

In the last section we prove equivalences of certain localized categories and construct categorical actions of the Schur–Weyl categories.

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2 Sequences of algebras and monoidal categories

Throughout the paper we will freely use the standard language of categories and functors; see e.g. [21]. All categories and functors will be linear over the ground field k, unless stated otherwise. All monoidal functors will be understood as strong monoidal. The set (or vector space) of morphisms between objects X, Y of a category $\mathcal{C}$ will be denoted by $\mathcal{C}(X, Y)$ and the endomorphism algebra $\mathcal{C}(X, X)$ of $X \in \mathcal{C}$ will be denoted by $\mathrm{End}_{\mathcal{C}}(X)$.

2.1 Multiplicative sequences of algebras and monoidal categories

We will deal with sequences of associative unital algebras $A_* = \{A_n \mid n \geqslant 0\}$ equipped with collections of (unital) algebra homomorphisms

$$\mu_{m,n} : A_m \otimes A_n \to A_{m+n}, \qquad m, n \geqslant 0,$$

satisfying the following associativity axiom: for any $l, m, n \geqslant 0$ the following diagram commutes:

$$\begin{array}{ccc} A_l \otimes A_m \otimes A_n & \xrightarrow{\mu_{l,m} \otimes I} & A_{l+m} \otimes A_n \\ {\scriptstyle I \otimes \mu_{m,n}} \downarrow & & \downarrow {\scriptstyle \mu_{l+m,n}} \\ A_l \otimes A_{m+n} & \xrightarrow{\mu_{l,m+n}} & A_{l+m+n}. \end{array}$$

We call such a sequence *multiplicative* and we will always assume that $A_0 = k$. A model example of a multiplicative sequence of algebras is provided by the following construction. Let $\mathcal{C}$ be a (strict) monoidal category such that $\mathrm{End}_{\mathcal{C}}(I) = k$, where I denotes the unit object of $\mathcal{C}$. Given an object X of $\mathcal{C}$, the sequence A_* with $A_n = \mathrm{End}_{\mathcal{C}}(X^{\otimes n})$ is multiplicative with respect to the homomorphisms $\mu_{m,n}$ given by the tensor product on morphisms

$$\mathrm{End}_{\mathcal{C}}(X^{\otimes m}) \otimes \mathrm{End}_{\mathcal{C}}(X^{\otimes n}) \to \mathrm{End}_{\mathcal{C}}(X^{\otimes m+n}).$$

Moreover, any multiplicative sequence can be obtained in this way. Indeed, starting with a multiplicative sequence A_*, define the category $\overline{\mathcal{C}}(A_*)$ with objects $[n]$ parameterized by natural numbers, with no morphisms between different objects and with the endomorphism algebras $\mathrm{End}_{\overline{\mathcal{C}}(A_*)}([n]) = A_n$. Define tensor product on the objects of $\overline{\mathcal{C}}(A_*)$ by $[m] \otimes [n] = [m+n]$. The multiplicative structure of the sequence A_* yields the tensor product on morphisms.

Let $f_* = \{f_n \mid n \geqslant 0\}$ be a sequence of algebra homomorphisms $f_n : A_n \to B_n$ between the corresponding algebras of two multiplicative sequences A_* and B_*. We call f_* a *homomorphism* of multiplicative sequences if for any m, n the following diagram commutes:

$$\begin{array}{ccc} A_m \otimes A_n & \xrightarrow{f_m \otimes f_n} & B_m \otimes B_n \\ {\scriptstyle \mu_{m,n}} \downarrow & & \downarrow {\scriptstyle \mu_{m,n}} \\ A_{m+n} & \xrightarrow{f_{m+n}} & B_{m+n}. \end{array}$$

We will say that f_* is an *epimorphism*, if all homomorphisms f_n are surjective.

Note that the construction of the category $\overline{\mathcal{C}}(A_*)$ is functorial with respect to homomorphisms of multiplicative sequences: a homomorphism $f_* : A_* \to B_*$ defines a monoidal functor $\overline{\mathcal{C}}(f_*) : \overline{\mathcal{C}}(A_*) \to \overline{\mathcal{C}}(B_*)$.

Now we denote by $\mathcal{C}(A_*)$ the category $\mathcal{F}unct\big(\overline{\mathcal{C}}(A_*)^{op}, \mathcal{V}ect\big)$ of k-linear functors from the opposite category $\overline{\mathcal{C}}(A_*)^{op}$ to the category of vector spaces over k. As a category, $\mathcal{C}(A_*)$ is the direct sum of categories of right modules $\oplus_n \mathcal{M}od{-}A_n$. The monoidal structure on $\mathcal{C}(A_*)$ (induced by the monoidal structure on $\overline{\mathcal{C}}(A_*)$ via Day's convolution) is given by

$$M \otimes_{A_*} N = (M \otimes N) \otimes_{A_m \otimes A_n} A_{m+n}. \tag{2.1}$$

Here M is an A_m-module, N is an A_n-module, and A_{m+n} is considered as a left $A_m \otimes A_n$-module via the homomorphism $\mu_{m,n}$. We call the monoidal category $\mathcal{C}(A_*)$ the *Schur–Weyl category* corresponding to the multiplicative sequence of algebras A_*. The subscript A_* of the tensor product sign in (2.1) will usually be omitted, if no confusion is possible as to what multiplicative sequence is used or what monoidal category is considered.

Suppose that $\overline{\mathcal{C}}$ is a k-linear category and $\mathcal{C}$ is an abelian category together with a fully-faithful k-linear functor $F : \overline{\mathcal{C}} \to \mathcal{C}$. We call $\mathcal{C}$ an *abelian envelope* of $\overline{\mathcal{C}}$ if any object of $\mathcal{C}$ is a subquotient of a direct sum of objects from the image of F.

Lemma 2.1. *The category $\mathcal{C}(A_*)$ is an abelian envelope of $\overline{\mathcal{C}}(A_*)$.*

Proof. By its definition, the category $\mathcal{C}(A_*)$ is abelian. The category $\overline{\mathcal{C}}(A_*)$ is fully and faithfully embedded into $\overline{\mathcal{C}}(A_*)$ via the Yoneda functor $Y \mapsto \overline{\mathcal{C}}(A_*)(-, Y)$. Explicitly, the image of $[n]$ via this embedding is A_n considered as a module over itself. It is clear that any object in $\mathcal{M}od{-}A_n$ is a quotient of a direct sum of copies of A_n. □

Note that the construction of $\mathcal{C}(A_*)$ is functorial with respect to homomorphisms of multiplicative sequences: a homomorphism $f_* : A_* \to B_*$ of multiplicative sequences defines a functor

$$\mathcal{C}(f_*) : \mathcal{C}(A_*) \to \mathcal{C}(B_*), \qquad \mathcal{C}(f_*)(M) = M \otimes_{A_m} B_m,$$

if M is an A_m-module. Moreover, this functor is monoidal:

$$\begin{array}{ccc}
\mathcal{C}(f_*)(M \otimes_{A_*} N) & = & (M \otimes_{A_*} N) \otimes_{A_{m+n}} B_{m+n} \\
 & & \downarrow \\
 & & (M \otimes N) \otimes_{A_m \otimes A_n} B_{m+n} \\
 & & \downarrow \\
\mathcal{C}(f_*)(M) \otimes_{B_*} \mathcal{C}(f_*)(N) & = & \big((M \otimes_{A_m} B_m) \otimes (N \otimes_{A_n} B_n)\big) \otimes_{B_m \otimes B_n} B_{m+n}.
\end{array}$$

The following results will be important for proving monoidality of the right adjoint of $\mathcal{C}(f_*)$. We start with a technical definition. We will say that an algebra A admits a *multiplicative decomposition* $A = BC$, if B and C are subalgebras of A and any element of A can be uniquely written as a product of elements of B and C, i.e. the multiplication in A induces the isomorphism of vector spaces $B \otimes C \to A$.

Lemma 2.2. *Let A be an algebra with multiplicative decomposition $A = BC$ and let $A' \subset A$ be a subalgebra with multiplicative decomposition $A' = BC'$, where $C' \subset C$ is a subalgebra. Then for any A'-module M the natural homomorphism*

$$M \otimes_{C'} C \to M \otimes_{A'} A, \qquad m \otimes c \mapsto m \otimes c, \tag{2.2}$$

induced by the embedding $C \subset A$, is an isomorphism of C-modules.

Proof. Define the inverse to the homomorphism (2.2) as follows. For $a \in A$ write $a = bc$ for unique $b \in B$ and $c \in C$. Define the image of the element $m \otimes a = m \otimes bc \in M \otimes_{A'} A$ to be $mb \otimes c \in M \otimes_{C'} C$. It is straightforward to check that this is a well-defined map, which is inverse to the map (2.2). □

The next theorem gives a sufficient condition which guarantees that the restriction functor along an embedding of multiplicative sequences of algebras is monoidal.

Theorem 2.3. *Suppose that $C_n \subset A_n$, $n = 0, 1, \ldots$, is an embedding of multiplicative sequences of algebras satisfying the condition that for each n the algebra A_n admits a multiplicative decomposition $A_n = A^{\otimes n} C_n$ for a certain algebra A. Then the restriction functor $\mathcal{C}(A_*) \to \mathcal{C}(C_*)$ along the embedding $C_* \subset A_*$ is monoidal.*

Proof. Note that the algebra $A_m \otimes A_n$ has a multiplicative decomposition

$$A_m \otimes A_n = (A^{\otimes m} C_m) \otimes (A^{\otimes n} C_n) = A^{\otimes (m+n)}(C_m \otimes C_n).$$

Thus, by Lemma 2.2 the natural morphism

$$M \otimes_{C_*} N = (M \otimes N) \otimes_{C_m \otimes C_n} C_{m+n} \to (M \otimes N) \otimes_{A_m \otimes A_n} A_{m+n} = M \otimes_{A_*} N$$

is an isomorphism. The coherence condition for this isomorphism is straightforward from its definition; cf. [21, Ch. VII]. □

2.2 Generators and relations

We start by defining free multiplicative sequences of algebras. Let $V = \{V_l \mid l \geqslant 1\}$ be a collection of vector spaces V_l parameterized by positive integers. Define another collection of vector spaces $\{M(V)_m \mid m \geqslant 1\}$ by

$$M(V)_m = \bigoplus_{l=1}^{m} \Big(\bigoplus_{i=0}^{m-l} V_l(i) \Big),$$

where $V_l(i) = \{v(i) \mid v(i) \in V_l\}$ is a copy of the space V_l labeled by the index i. The components $M(V)_m$ are connected by linear maps

$$\alpha_{m,n} : M(V)_m \oplus M(V)_n \to M(V)_{m+n},$$

where for any $1 \leqslant l \leqslant m,\ 1 \leqslant k \leqslant n$ and $0 \leqslant i \leqslant m-l,\ 0 \leqslant j \leqslant n-k$ we have

$$\alpha_{m,n}(v(i), w(j)) = v(i) + w(m+j), \qquad v \in V_l, \quad w \in V_k.$$

By the definition of these maps, the following diagram commutes:

$$\begin{array}{ccc} M(V)_l \oplus M(V)_m \oplus M(V)_n & \xrightarrow{\alpha_{l,m} \oplus I} & M(V)_{l+m} \oplus M(V)_n \\ \downarrow {\scriptstyle I \oplus \alpha_{m,n}} & & \downarrow {\scriptstyle \alpha_{l+m,n}} \\ M(V)_l \oplus M(V)_{m+n} & \xrightarrow{\alpha_{l,m+n}} & M(V)_{l+m+n}. \end{array}$$

Now let $T\big(M(V)_m\big)$ denote the tensor algebra of the vector space $M(V)_m$. Denote by J_m the two-sided ideal of $T\big(M(V)_m\big)$ generated by all elements of the form $v(i)w(j) - w(j)v(i)$, where $v \in V_l,\ w \in V_k$ with $1 \leqslant l \leqslant m$, $1 \leqslant k \leqslant n$, and the indices i and j satisfy the conditions $0 \leqslant i \leqslant m-l$, $0 \leqslant j \leqslant n-k$ and

$$\{i+1, i+2, \dots, i+l\} \cap \{j+1, j+2, \dots, j+k\} = \varnothing.$$

For $m \geqslant 1$ denote by $A(V)_m$ the quotient algebra $T\big(M(V)_m\big)/J_m$ and set $A(V)_0 = k$. By the definition of the maps $\alpha_{m,n}$, the difference

$$\alpha_{m,n}(x, 0) - \alpha_{m,n}(0, y)$$

belongs to J_{m+n} for any $x \in M(V)_m$ and $y \in M(V)_n$. Hence, the natural homomorphisms

$$T(\alpha_{m,n}) : T\big(M(V)_m \oplus M(V)_n\big) \to T\big(M(V)_{m+n}\big)/J_{m+n}$$

induced by the linear maps $\alpha_{m,n}$ factor through $T\big(M(V)_m\big) \otimes T\big(M(V)_n\big)$. Moreover, the kernel of the homomorphism

$$T\big(M(V)_m\big) \otimes T\big(M(V)_n\big) \to T\big(M(V)_{m+n}\big)/J_{m+n}$$

contains $J_m \otimes 1 + 1 \otimes J_n$. Thus we have the algebra homomorphisms

$$\mu_{m,n} : A(V)_m \otimes A(V)_n \to A(V)_{m+n}$$

which turn $A(V)_* = \{A(V)_m \mid m \geqslant 0\}$ into a multiplicative sequence. The sequence $A(V)_*$ is the *free multiplicative sequence of algebras* generated by the collection V of vector spaces in the sense that given any multiplicative sequence of algebras $A_* = \{A_n \mid n \geqslant 0\}$, the homomorphisms of multiplicative sequences $A(V)_* \to A_*$ are in one-to-one correspondence with the sequences of vector space homomorphisms $V_l \to A_l$, $l \geqslant 1$.

Now we will use free multiplicative sequences to give the definition of a multiplicative sequence of algebras A_* *generated by a family* $\{a_i\}_{i\in I}$ *of elements* $a_i \in A_{n(i)}$, $n(i) \geqslant 1$. To this end, we let $\{\widehat{a}_i\}_{i\in I}$ be a set equipped with the degree function $\deg \widehat{a}_i = n(i)$. We will use this set to span a collection V of vector spaces over the field k by setting

$$V_l = \text{span of } \ \{\widehat{a}_i \mid \deg \widehat{a}_i = l\}. \tag{2.3}$$

We will say that a multiplicative sequence of algebras A_* is generated by a family $\{a_i\}_{i\in I}$ of elements $a_i \in A_{n(i)}$ if there is an epimorphism of multiplicative sequences $A(V)_* \to A_*$ such that $\widehat{a}_i \mapsto a_i$ for all $i \in I$.

Consider the collection of vector spaces $K_* = \{K_n \mid n \geqslant 1\}$, where K_n is the kernel of the epimorphism $A(V)_n \to A_n$. Suppose that $\{p_j\}_{j\in J}$ is a set of elements in the kernels, $p_j \in K_{m(j)}$. Let $P = \{P_l \mid l \geqslant 1\}$ be the collection of vector spaces spanned by this set so that

$$P_l = \text{span of } \ \{p_j \mid \deg p_j = l\}.$$

We will say that a multiplicative sequence of algebras A_* is generated by a set $\{a_i\}_{i\in I}$ of elements $a_i \in A_{n(i)}$ *subject to the relations*

$$p_j\big(\{a_i\}_{i\in I}\big) = 0, \quad j \in J, \tag{2.4}$$

if for any $n \geqslant 1$ the two-sided ideal K_n of $A(V)_n$ is generated by $M(P)_n$.

Furthermore, using free multiplicative sequences in a similar way, we can now give the definition of a *monoidal category* $\mathcal{C}$ *generated by its object* X *and a family* $\{a_i\}_{i\in I}$ *of endomorphisms* $a_i \in \mathrm{End}_{\mathcal{C}}(X^{\otimes n(i)})$ *subject to relations*

$$p_j\big(\{a_i\}_{i\in I}\big) = 0, \quad j \in J. \tag{2.5}$$

Namely, exactly as above, we let $\{\widehat{a}_i\}_{i\in I}$ be a set with the degree function $\deg \widehat{a}_i = n(i)$ and introduce a collection $V = \{V_l \mid l \geqslant 1\}$ of vector spaces by (2.3). The homomorphism of multiplicative sequences $A(V)_n \to \mathrm{End}_{\mathcal{C}}(X^{\otimes n})$, $\widehat{a}_i \mapsto a_i$, allows us to think of the elements of $A(V)_*$ as endomorphisms of the tensor powers of X. In particular, it allows us to say that the endomorphisms a_i satisfy a relation $p\big(\{a_i\}_{i\in I}\big) = 0$, where $p \in A(V)_*$.

We will say that a monoidal category $\mathcal{C}$ is generated by its object X and a set $\{a_i\}_{i\in I}$ of endomorphisms $a_i \in \mathrm{End}_{\mathcal{C}}(X^{\otimes n(i)})$ subject to the relations (2.5), if for any monoidal category $\mathcal{D}$ the assignment $F \mapsto (F(X), F(a_i)_{i\in I})$ defines an equivalence between the category of monoidal functors $\mathcal{C} \to \mathcal{D}$ and the category whose objects are collections $(Y, \{b_i\}_{i\in I})$, where Y is an object of $\mathcal{D}$ and $\{b_i\}_{i\in I}$ is a set of endomorphisms $b_i \in \mathrm{End}_{\mathcal{D}}(Y^{\otimes n(i)})$ such that $p_j\big(\{b_i\}_{i\in I}\big) = 0$ for all $j \in J$.

Theorem 2.4. *Let A_* be a multiplicative sequence of algebras generated by a set $\{a_i\}_{i\in I}$ of elements $a_i \in A_{n(i)}$ subject to certain relations* (2.4). *Then, as a monoidal category, the Schur–Weyl category $\mathcal{C}(A_*)$ is generated by the object $X = [1]$ and the set $\{a_i\}_{i\in I}$ of endomorphisms $a_i \in \mathrm{End}_{\mathcal{C}(A_*)}(X^{\otimes n(i)})$ subject to the relations* (2.5).

Proof. Suppose that $\mathcal{D}$ is a monoidal category and consider the category of monoidal functors $\mathcal{C}(A_*) \to \mathcal{D}$. This functor category is equivalent to the category with the objects (Y, g_*), where Y is an object of $\mathcal{D}$ and $g_* = \{g_n \mid n \geqslant 0\}$ is a homomorphism of multiplicative sequences

$$g_n : A_n \to \mathrm{End}_{\mathcal{D}}(Y^{\otimes n}), \qquad n \geqslant 0. \tag{2.6}$$

Now if the multiplicative sequence of algebras A_* is generated by a set $\{a_i\}_{i\in I}$ of elements $a_i \in A_{n(i)}$ subject to the relations (2.4), then the homomorphisms (2.6) are in a one-to-one correspondence with the families of endomorphisms $\{b_i\}_{i\in I}$ with $b_i \in \mathrm{End}_{\mathcal{D}}(Y^{\otimes n(i)})$ such that $p_j\big(\{b_i\}_{i\in I}\big) = 0$ for all $j \in J$. □

3 Symmetric groups and general linear Lie algebras

We start with the simplest case of the classical Schur–Weyl duality, namely the duality between the symmetric groups and the general linear Lie algebras.

3.1 Symmetric groups and their group algebras

Consider the standard presentation of the group S_n of permutations of the set $\{1, \dots, n\}$ so that S_n is generated by the elements $t_1, \dots, t_{n-1}$ subject to the relations:

$$t_i^2 = 1, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = t_j t_i \quad \text{for} \quad |i - j| > 1. \tag{3.1}$$

The assignments

$$t_i \otimes 1 \mapsto t_i, \qquad 1 \otimes t_j \mapsto t_{j+m},$$

define natural algebra homomorphisms

$$\mu_{m,n} : k[S_m] \otimes k[S_n] \to k[S_{m+n}]$$

which satisfy the associativity axiom; see Sec. 2.1. Thus we get a multiplicative sequence of algebras $k[S_*] = \{k[S_n] \mid n \geqslant 0\}$.

Proposition 3.1. *The multiplicative sequence $k[S_*]$ is generated by the element $t \in k[S_2]$ subject to the relations $t^2 = 1$ in $k[S_2]$ and*

$$\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) = \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) \qquad \text{in} \quad k[S_3].$$

Proof. This follows from the presentation of S_n; the element $t = t_1$ is the non-identity element of S_2. □

3.2 The free symmetric category

As an immediate consequence of Theorem 2.4 and Proposition 3.1 we have the following universal property of the category $\mathcal{C}(k[S_*])$.

Theorem 3.2. *The abelian monoidal category $\mathcal{S} = \mathcal{C}(k[S_*])$ is generated by an object X and an automorphism $c : X \otimes X \to X \otimes X$ subject to the relations*

$$c^2 = 1, \qquad (c \otimes 1_X)(1_X \otimes c)(c \otimes 1_X) = (1_X \otimes c)(c \otimes 1_X)(1_X \otimes c). \tag{3.2}$$

We will call X and c satisfying (3.2), an *involutive Yang–Baxter object* and an *involutive Yang–Baxter operator*, respectively. Theorem 3.2 states that $\mathcal{S}$ is a free abelian monoidal category generated by an involutive Yang–Baxter object.

We also formulate another universal property of $\mathcal{S}$, which will be used in the next section. Let $c_{m,n} \in S_{m+n}$ be the (m,n)-shuffle preserving the orders of the first m and last n letters. That is, under this permutation,

$$i \mapsto i+n, \quad i = 1, \ldots, m, \qquad j \mapsto j-m, \quad j = m+1, \ldots, m+n.$$

The collection $\{c_{m,n} \mid m, n \geqslant 0\}$ of the isomorphisms $c_{m,n} : X^{\otimes m} \otimes X^{\otimes n} \to X^{\otimes n} \otimes X^{\otimes m}$ defines a symmetry on the category $\mathcal{S}$; see [21]. The following freeness property of the category $\mathcal{S}$ is well-known; see [18].

Proposition 3.3. *The category $\mathcal{S}$ is a free abelian symmetric monoidal category generated by one object.*

3.3 Fiber functors and general linear Lie algebras

As before, we let $\mathcal{V}ect$ denote the category of vector spaces over k. A monoidal k-linear functor from a certain category to $\mathcal{V}ect$ will be called a *fiber functor*. Due to the freeness property of $\mathcal{S}$ as monoidal category, fiber functors $\mathcal{S} \to \mathcal{V}ect$ correspond to involutive Yang–Baxter operators. There are quite a number of them forming algebraic moduli spaces; see [8]. However, symmetric fiber functors are labeled by vector spaces (the associated Yang–Baxter operator is the standard permutation of the tensor factors in the tensor square). Thus, up to monoidal isomorphisms, symmetric fiber functors correspond to non-negative integers (the dimensions of the corresponding vector spaces). Denote by $F_N : \mathcal{S} \to \mathcal{V}ect$ the (isomorphism class of the) symmetric fiber functor labeled by N.

Now the classical Schur–Weyl duality can be interpreted as follows.

Proposition 3.4. *The functor F_N factors through the category $\mathcal{R}ep(\mathfrak{gl}_N)$ of representations of the general linear Lie algebra $\mathfrak{gl}_N$*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad F_N \quad} & \mathcal{V}ect \\ & {\scriptstyle SW_N}\searrow \quad \nearrow & \\ & \mathcal{R}ep(\mathfrak{gl}_N) & \end{array} \tag{3.3}$$

via a full monoidal functor $SW_N : \mathcal{S} \to \mathcal{R}ep(\mathfrak{gl}_N)$ and the forgetful functor $\mathcal{R}ep(\mathfrak{gl}_N) \to \mathcal{V}ect$.

Proof. Let V be the N-dimensional vector representation of $\mathfrak{gl}_N$. By the classical Schur–Weyl duality we have the homomorphisms

$$k[S_n] \to \mathrm{End}_{\mathfrak{gl}_N}(V^{\otimes n}) \tag{3.4}$$

which extend to a monoidal functor $SW_N : \mathcal{S} \to \mathcal{R}ep(\mathfrak{gl}_N)$ sending the generator X of $\mathcal{S}$ to V, so that $SW_N(X^{\otimes n}) = V^{\otimes n}$. Hence the functor SW_N fits into the commutative diagram (3.3). The fullness of SW_N follows from the surjectivity of the homomorphisms (3.4). □

We will call SW_N the *Schur–Weyl functor*. In what follows we will consider its quantum and affine analogues and establish similar properties.

4 Hecke algebras and quantized enveloping algebras

4.1 Braid groups and Hecke algebras

Recall that the *braid group* B_n is the group generated by elements $t_1, \dots, t_{n-1}$ subject to the relations

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = t_j t_i \quad \text{for} \quad |i - j| > 1.$$

The assignments

$$t_i \otimes 1 \mapsto t_i, \quad 1 \otimes t_j \mapsto t_{j+m},$$

define homomorphisms of group algebras

$$k[B_m] \otimes k[B_n] \to k[B_{m+n}].$$

These homomorphisms satisfy the associativity axiom so that we get a multiplicative sequence $k[B_*] = \{k[B_n] \mid n \geqslant 0\}$ as defined in Sec. 2.1. The following proposition is immediate from the presentations of the braid groups.

Proposition 4.1. *The multiplicative sequence $k[B_*]$ is generated by the element $t \in k[B_2]$ subject to the relation*

$$\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) = \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) \qquad in \quad k[B_3].$$

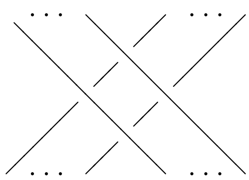
Theorem 2.4 and Proposition 4.1 imply a description of the monoidal category associated with the multiplicative sequence $k[B_*]$.

Theorem 4.2. *The abelian monoidal category $\mathcal{B} = \mathcal{C}(k[B_*])$ is generated by an object X and an automorphism $c : X \otimes X \to X \otimes X$ such that*

$$(c \otimes 1_X)(1_X \otimes c)(c \otimes 1_X) = (1_X \otimes c)(c \otimes 1_X)(1_X \otimes c). \tag{4.1}$$

An automorphism c satisfying (4.1) will be called a *Yang–Baxter operator*, and the corresponding object X – a *Yang–Baxter object*. Theorem 4.2 states that $\mathcal{B}$ is a free monoidal category generated by a Yang–Baxter object.

Let $c_{m,n} \in B_{m+n}$ be the braid whose geometric presentation in terms of strands is illustrated below; the first m strands pass on top of the remaining n strands:



The collection $\{c_{m,n} \mid m, n \geqslant 0\}$ of the isomorphisms $c_{m,n} : X^{\otimes m} \otimes X^{\otimes n} \to X^{\otimes n} \otimes X^{\otimes m}$ defines a braiding on the category $\mathcal{B}$. The following freeness property of the category $\mathcal{B}$ is well-known; see [18].

Proposition 4.3. *The category $\mathcal{B}$ is a free abelian braided monoidal category generated by one object.*

Suppose that q is a nonzero element of the field k. The *Hecke algebra* $\mathrm{H}_n(q)$ is the quotient of the group algebra $k[B_n]$ by the ideal generated by the elements $(t_i - q)(t_i + q^{-1})$ with $i = 1, \dots, n-1$. Equivalently, $\mathrm{H}_n(q)$ is the associative algebra generated by elements $t_1, \dots, t_{n-1}$ subject to the relations

$$(t_i - q)(t_i + q^{-1}) = 0, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = t_j t_i \quad \text{for} \quad |i - j| > 1.$$

By taking the quotients of the group algebras $k[B_n]$ and using the multiplicative structure on $k[B_*]$ we get a multiplicative sequence of algebras $\mathrm{H}_*(q) = \{\mathrm{H}_n(q) \mid n \geqslant 0\}$. Using the presentation of $\mathrm{H}_n(q)$ we come to the following.

Proposition 4.4. *The multiplicative sequence* $\mathrm{H}_*(q)$ *is generated by an element* $t \in \mathrm{H}_2(q)$ *subject to the relations*

$$(t_i - q)(t_i + q^{-1}) = 0 \qquad in \quad \mathrm{H}_2(q),$$
$$\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) = \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) \qquad in \quad \mathrm{H}_3(q).$$

We call the category $\mathcal{H}(q) = \mathcal{C}(\mathrm{H}_*(q))$ associated with the multiplicative sequence $\mathrm{H}_*(q)$ the *Hecke category.* The next theorem implied by Theorem 2.4 and Proposition 4.4 provides its description as a monoidal category.

Theorem 4.5. *The Hecke category is generated as a monoidal category by an object* X *and an automorphism* $c : X \otimes X \to X \otimes X$ *subject to the relations*

$$(c - q)(c + q^{-1}) = 0, \qquad (c \otimes 1_X)(1_X \otimes c)(c \otimes 1_X) = (1_X \otimes c)(c \otimes 1_X)(1_X \otimes c). \quad (4.2)$$

An automorphism c satisfying (4.2) will be called a *Hecke Yang–Baxter operator*, and the corresponding object X – a *Hecke Yang–Baxter object.* Theorem 4.5 states that $\mathcal{H}(q)$ is a free abelian monoidal category generated by a Hecke Yang–Baxter object. The following proposition is analogous to Proposition 3.3 and whose proof we also omit.

Proposition 4.6. *The category* $\mathcal{H}(q)$ *is a free abelian braided monoidal category generated by a Hecke Yang–Baxter object.*

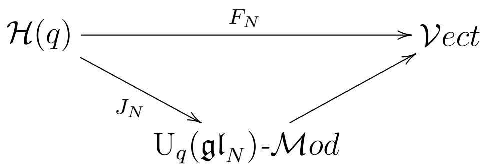
4.2 Fiber functors and quantized enveloping algebras

Let V be an N-dimensional vector space with the basis $e_1, \ldots, e_N$. Following [17], define the linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ by

$$R(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{if} \quad i < j, \\ q\, e_j \otimes e_i & \text{if} \quad i = j, \\ e_j \otimes e_i + (q - q^{-1})\, e_i \otimes e_j & \text{if} \quad i > j. \end{cases} \quad (4.3)$$

Label the copies of V in the tensor product $V^{\otimes n}$ by $1, \ldots, n$, respectively. Let $R_{l,l+1}$ denote the operator in this tensor product space which acts as R in the tensor product of the copies of V labeled by l and $l+1$ and acts as the identity operator in the remaining copies of V. By the results of [17], the mapping $t_l \mapsto R_{l,l+1}$ defines a representation of the Hecke algebra $\mathrm{H}_n(q)$ in $V^{\otimes n}$ and the image of $\mathrm{H}_n(q)$ in the endomorphism algebra of $V^{\otimes n}$ commutes with the image of an action of the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_N)$ associated with $\mathfrak{gl}_N$. Thus we get a monoidal functor $J_N : \mathcal{H}(q) \to \mathrm{U}_q(\mathfrak{gl}_N)$-$\mathcal{M}od$ sending the generator X of $\mathcal{H}(q)$ to V, so that $J_N(X^{\otimes n}) = V^{\otimes n}$. We call J_N the *Jimbo functor*. It is a quantum analogue of the Schur–Weyl functor SW_N; cf. Sec. 3.3. The following is a respective analogue of Proposition 3.4, where $F_N : \mathcal{H}(q) \to \mathcal{V}ect$ denotes the fiber functor labeled by N.

Proposition 4.7. *The functor F_N factors through the category $\mathrm{U}_q(\mathfrak{gl}_N)$-$\mathcal{M}od$ of representations of the quantized enveloping algebra*



via a monoidal functor $J_N : \mathcal{H}(q) \to \mathrm{U}_q(\mathfrak{gl}_N)$-$\mathcal{M}od$ and the forgetful functor $\mathrm{U}_q(\mathfrak{gl}_N)$-$\mathcal{M}od \to \mathcal{V}ect$.

As follows from [17], the Jimbo functor J_N is full under some additional conditions ($k = \mathbb{C}$ and q is not a root of unity). On the other hand, if k and q are arbitrary, then an analogous property of J_N can be established by replacing the quantized enveloping algebra with an appropriate integral version; see [12].

5 Affine symmetric groups and loop algebras

Let A be a unital associative algebra. Define the sequence of algebras $SA_n = A^{\otimes n} * S_n$ which are the *cross-products* of the symmetric group algebras with the tensor powers of A with respect to the natural permutation actions of S_n on $A^{\otimes n}$. As a vector space, the cross-product $A^{\otimes n} * S_n$ coincides with the tensor product $A^{\otimes n} \otimes k[S_n]$. However, the algebra structure of $A^{\otimes n} * S_n$ is different to the tensor product algebra. We will emphasize this fact by using the notation $(a_1 \otimes \cdots \otimes a_n) * \sigma$ for the element of $A^{\otimes n} * S_n$ corresponding to $(a_1 \otimes \cdots \otimes a_n) \otimes \sigma \in A^{\otimes n} \otimes k[S_n]$. The multiplication in $A^{\otimes n} * S_n$ is given by the following rule:

$$\big((a_1 \otimes \cdots \otimes a_n) * \sigma\big)\big((b_1 \otimes \cdots \otimes b_n) * \tau\big) = \big(a_1 b_{\sigma^{-1}(1)} \otimes \cdots \otimes a_n b_{\sigma^{-1}(n)}\big) * \sigma\tau. \tag{5.1}$$

The multiplicative structure on the sequence of symmetric group algebras extends to a multiplicative structure on SA_n:

$$\mu_{m,n} : SA_m \otimes SA_n \to SA_{m+n}, \qquad m, n \geqslant 0$$

with

$$\mu_{m,n}\big(((a_1 \otimes \cdots \otimes a_m) * \sigma) \otimes ((b_1 \otimes \cdots \otimes b_n) * \tau)\big) \\ = (a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n) * \mu_{m,n}(\sigma \otimes \tau).$$

Proposition 5.1. *The multiplicative sequence SA_* is generated by $t \in S_2 \subset SA_2$ and the elements of $SA_1 = A$ subject to the relations:*

$$t^2 = 1 \qquad in \quad SA_2,$$
$$t\,\mu_{1,1}(u \otimes 1) = \mu_{1,1}(1 \otimes u)\,t \qquad in \quad SA_2,$$

for any $u \in SA_1$, and

$$\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) = \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) \quad in \quad SA_3.$$

Proof. This follows from the presentation for SA_n where all elements of $A^{\otimes n}$ and the elements $t_1, \ldots, t_{n-1}$ are taken as generators. The defining relations are obtained by using the multiplication rule (5.1) and the standard presentation of S_n. □

We let $\mathcal{S}(A)$ denote the Schur–Weyl category $\mathcal{C}(SA_*)$ associated with the multiplicative sequence SA_*. The following theorem is immediate from Theorem 2.4 and Proposition 5.1.

Theorem 5.2. *The category $\mathcal{S}(A)$ is a monoidal category generated by an object X such that $\mathrm{End}_{\mathcal{S}(A)}(X) = A$ and an automorphism $c : X \otimes X \to X \otimes X$ subject to the relations*

$$c^2 = 1, \qquad (c \otimes 1_X)(1_X \otimes c)(c \otimes 1_X) = (1_X \otimes c)(c \otimes 1_X)(1_X \otimes c) \tag{5.2}$$

and

$$(a \otimes b)\,c = c\,(b \otimes a), \qquad a, b \in A.$$

In other words, the category $\mathcal{S}(A)$ is a free symmetric category generated by an object X such that $\mathrm{End}_{\mathcal{S}(A)}(X) = A$. □

Example 5.3. *Monoidal autoequivalences of $\mathcal{S}(A)$.* Let f be an automorphism of the algebra A. By Theorem 5.2, the assignment $(F(X), F(a), F(c)) := (X, f(a), c)$ for all $a \in A$ defines a monoidal functor $T_f : \mathcal{S}(A) \to \mathcal{S}(A)$. The composition $T_f \circ T_g$ is canonically isomorphic (as a monoidal functor) to T_{fg}. Clearly, T_1 is canonically isomorphic to the identity functor. Thus, T_f is a monoidal autoequivalence of $\mathcal{S}(A)$ for any automorphism f of A.

Note that in the particular case when $A = k[u]$, the automorphism group $\mathrm{Aut}(k[u])$ is isomorphic to the semi-direct product $k^* \ltimes k$: any automorphism of $k[u]$ has the form $\phi_{a,b}$ with $a \in k^*$ and $b \in k$, and $\phi_{a,b}(u) = au + b$. Later on we will consider quantum deformations of the category $\mathcal{S}(k[u])$; namely, the categories corresponding to multiplicative sequences of the affine Hecke algebras and their degenerate versions. In those cases, the automorphism group $\mathrm{Aut}(k[u])$ will be reduced to one-parameter subgroups with $b = 0$ and $a = 1$, respectively; see also Examples 6.3 and 7.6. □

5.1 Affine symmetric groups and free affine symmetric category

Here we consider two particular cases of the general construction described above. We will take A to be the algebra of Laurent polynomials $k[u^{\pm 1}] = k[u, u^{-1}]$ in a variable u and the algebra of polynomials $k[u]$.

Let AS_n be the cross-product algebra $k[u_1^{\pm 1}, \ldots, u_n^{\pm 1}] * S_n$ with respect to the natural permutation action of S_n on $k[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$. In other words, AS_n is generated by the elements $t_1, \ldots, t_{n-1}$ and invertible elements $u_1, \ldots, u_n$ subject to the relations (3.1) together with

$$t_i u_i t_i = u_{i+1}, \qquad u_i u_j = u_j u_i.$$

Similarly, we let SAS_n denote the cross-product algebra $k[u_1, \ldots, u_n] * S_n$ with respect to the natural permutation action of S_n on $k[u_1, \ldots, u_n]$. The algebra SAS_n can be presented in the same way as AS_n, with the invertibility condition on the u_i omitted.

Now define the *affine symmetric category* $\mathcal{AS}$ to be the monoidal category $\mathcal{C}(AS_*)$ corresponding to the multiplicative sequence $AS_* = \{AS_n \mid n \geqslant 0\}$. Similarly, define the *semi-affine symmetric category* $\mathcal{SAS}$ to be the monoidal category $\mathcal{C}(SAS_*)$ corresponding to the multiplicative sequence $SAS_* = \{SAS_n \mid n \geqslant 0\}$.

Using the presentations of the algebras AS_n and SAS_n we obtain the following.

Proposition 5.4. *The multiplicative sequence AS_* is generated by $t \in AS_2$ and an invertible element $u \in AS_1$ subject to the relations*

$$\begin{aligned} t^2 &= 1 && \text{in} \quad AS_2, \\ t\mu_{1,1}(u \otimes 1) &= \mu_{1,1}(1 \otimes u)t && \text{in} \quad AS_2, \\ \mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) &= \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) && \text{in} \quad AS_3. \end{aligned}$$

Moreover, the multiplicative sequence SAS_* admits the same presentation with the invertibility condition on $u \in SAS_1$ omitted.

The next corollary is immediate from Theorem 5.2.

Corollary 5.5. *The affine symmetric category $\mathcal{AS}$ is a free symmetric monoidal category generated by an object X and an automorphism $x : X \to X$. Moreover, the semi-affine symmetric category $\mathcal{SAS}$ is a free symmetric monoidal category generated by an object X and an endomorphism $x : X \to X$.* □

5.2 Fiber functors from $\mathcal{S}(A)$

We now return to the Schur–Weyl category $\mathcal{S}(A)$ associated with an arbitrary unital associative algebra A as defined in the beginning of Sec. 5.

The natural embeddings $k[S_n] \hookrightarrow SA_n$ define a homomorphism of multiplicative sequences $k[S_*] \to SA_*$. This gives rise to a monoidal functor $\mathcal{S} \to \mathcal{S}(A)$ which sends the generator $X \in \mathcal{S}$ to the generator $X \in \mathcal{S}(A)$ and sends the Yang–Baxter operator $c \in \mathrm{End}_{\mathcal{S}}(X^{\otimes 2})$ to $c \in \mathrm{End}_{\mathcal{S}(A)}(X^{\otimes 2})$. The algebras $A^{\otimes n} * S_n$ admit natural multiplicative decompositions with the components $A^{\otimes n}$ and $k[S_n]$. Hence, applying Theorem 2.3 we derive that the right adjoint $F : \mathcal{S}(A) \to \mathcal{S}$ is also monoidal. By Theorem 5.2, the functor F is determined by its values on the generating object $X \in \mathcal{S}(A)$ together with its values on the generating morphisms $a \in A = \mathrm{End}_{\mathcal{S}(A)}(X)$ and $c \in \mathrm{End}_{\mathcal{S}(A)}(X^{\otimes 2})$. The object $F(X)$ is the tensor product $X \otimes A$ of the generator $X \in \mathcal{S}$ with the underlying vector space of the algebra A. For any element $a \in A$ the endomorphism $F(a)$ is the morphism induced by the linear map $A \to A$, which is the left multiplication by a. The automorphism $F(c)$ is $c \otimes t$, where we identify $\mathrm{End}_{\mathcal{S}}((X \otimes A)^{\otimes 2})$ with the tensor product $\mathrm{End}_{\mathcal{S}}(X^{\otimes 2}) \otimes \mathrm{End}(A^{\otimes 2})$ and $t : A^{\otimes 2} \to A^{\otimes 2}$ is the transposition $a \otimes b \mapsto b \otimes a$.

Composing the monoidal functor $\mathcal{S}(A) \to \mathcal{S}$ with a fiber functor $\mathcal{S} \to \mathcal{V}ect$ we get a fiber functor $\mathcal{S}(A) \to \mathcal{V}ect$. In particular, taking the symmetric fiber functor $F_N : \mathcal{S} \to \mathcal{V}ect$ we get a fiber functor $F_N : \mathcal{S}(A) \to \mathcal{V}ect$. Applying Theorem 5.2 as above, we find that F_N is determined by its values $F_N(X)$, $F_N(c)$ and $F_N(a)$ for all $a \in A$. Introducing the vector space $V = k^N$ we find that the object $F_N(X)$ is the tensor product $V \otimes A$. For any $a \in A$ the endomorphism $F_N(a)$ is the morphism, induced by the linear map $A \to A$ which is the left multiplication by a. The automorphism $F_N(c)$ is the transposition

$$t : (V \otimes A) \otimes (V \otimes A) \to (V \otimes A) \otimes (V \otimes A), \quad (v \otimes a) \otimes (u \otimes b) \mapsto (u \otimes b) \otimes (v \otimes a).$$

The following is an analogue of Proposition 3.4 providing an affine version of the Schur–Weyl functor SW_N and it is verified in the same way.

Proposition 5.6. *The monoidal functor $F_N : \mathcal{S}(A) \to \mathcal{V}ect$ factors through the category $\mathcal{R}ep(\mathfrak{gl}_N(A))$ of representations of the general linear Lie algebra over A*

$$\begin{array}{ccc} \mathcal{S}(A) & \xrightarrow{\quad F_N \quad} & \mathcal{V}ect \\ & {}_{SW_N}\searrow \quad \nearrow & \\ & \mathcal{R}ep(\mathfrak{gl}_N(A)) & \end{array}$$

via a monoidal functor $SW_N : \mathcal{S}(A) \to \mathcal{R}ep(\mathfrak{gl}_N(A))$ and the forgetful functor $\mathcal{R}ep(\mathfrak{gl}_N(A)) \to \mathcal{V}ect$.

Remark 5.7. We believe that the functor SW_N is full. Although we do not have a proof of this conjecture.

6 Yangians and degenerate affine Hecke algebras

6.1 Degenerate affine Hecke algebras

The *degenerate affine Hecke algebra* Λ_n is the unital associative algebra generated by elements $t_1, \ldots, t_{n-1}$ and $y_1, \ldots, y_n$ subject to the relations

$$t_i^2 = 1, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = t_j t_i \quad \text{for} \quad |i-j| > 1,$$
$$y_i t_i - t_i y_{i+1} = 1, \qquad y_i y_j = y_j y_i.$$

The assignments

$$t_i \otimes 1 \mapsto t_i, \qquad 1 \otimes t_j \mapsto t_{j+m},$$
$$y_i \otimes 1 \mapsto y_i, \qquad 1 \otimes y_j \mapsto y_{j+m}$$

define algebra homomorphisms

$$\Lambda_m \otimes \Lambda_n \to \Lambda_{m+n}.$$

It is easy to see that these homomorphisms satisfy the associativity axiom thus giving rise to the multiplicative sequence of algebras $\Lambda_* = \{\Lambda_n \mid n \geqslant 0\}$; see Sec. 2.1. Hence we get a monoidal category $\mathcal{L} = \mathcal{C}(\Lambda_*)$ which we call the *degenerate affine Hecke category.*

Proposition 6.1. *The multiplicative sequence Λ_* is generated by elements $y \in \Lambda_1$ and $t \in \Lambda_2$ subject to the relations*

$$t^2 = 1 \qquad in \quad \Lambda_2,$$
$$t\,\mu_{1,1}(y \otimes 1) - \mu_{1,1}(1 \otimes y)\,t = 1 \qquad in \quad \Lambda_2,$$
$$\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) = \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) \qquad in \quad \Lambda_3.$$

Proof. This follows from the presentation for Λ_n. □

Theorem 6.2. *The degenerate affine Hecke category $\mathcal{L}$ is a free monoidal category generated by one object X, an endomorphism $x : X \to X$ and an involutive Yang–Baxter operator $c : X^{\otimes 2} \to X^{\otimes 2}$ subject to the relations*

$$c^2 = 1, \qquad (c \otimes 1_X)(1_X \otimes c)(c \otimes 1_X) = (1_X \otimes c)(c \otimes 1_X)(1_X \otimes c),$$

and

$$(x \otimes 1)\,c - c\,(1 \otimes x) = 1. \tag{6.1}$$

Proof. This is immediate from Theorem 2.4 and Proposition 6.1. □

Theorem 6.2 implies that monoidal functors from $\mathcal{L}$ to a monoidal category $\mathcal{C}$ correspond to triples (V, x, c), where V is an object in $\mathcal{C}$, $x \in \mathrm{End}_{\mathcal{C}}(V)$ is its endomorphism, and $c \in \mathrm{End}(V^{\otimes 2})$ is an involutive Yang–Baxter operator satisfying (6.1).

Example 6.3. *Monoidal autoequivalences of* $\mathcal{L}$. Let u be an element of the basic field k. The triple $(X, x + uI, c)$ defines a monoidal functor $T_u : \mathcal{L} \to \mathcal{L}$. The composition $T_u \circ T_v$ is canonically isomorphic (as a monoidal functor) to T_{u+v}. Hence T_u is a monoidal autoequivalence of $\mathcal{L}$. □

Due to [20], the degenerate affine Hecke algebra Λ_n admits multiplicative decompositions

$$\Lambda_n = k[y_1, \ldots, y_n]\, k[S_n] = k[S_n]\, k[y_1, \ldots, y_n]. \tag{6.2}$$

In the particular case $n = 2$ the decompositions (6.2) follow from the relation

$$t f(y_1, y_2) = f(y_2, y_1) t + \frac{f(y_1, y_2) - f(y_2, y_1)}{y_1 - y_2},$$

where $f(y_1, y_2)$ is a polynomial in y_1, y_2.

The natural embeddings $k[S_n] \hookrightarrow \Lambda_n$ define a homomorphism of multiplicative sequences $k[S_*] \to \Lambda_*$. Hence, we get a monoidal functor $\mathcal{S} \to \mathcal{L}$ which sends the generator $X \in \mathcal{S}$ to the generator $X \in \mathcal{L}$ and sends the Yang–Baxter operator $c \in \mathrm{End}_{\mathcal{S}}(X^{\otimes 2})$ to $c \in \mathrm{End}_{\mathcal{L}}(X^{\otimes 2})$. By Theorem 2.3, its right adjoint $F : \mathcal{L} \to \mathcal{S}$ is also monoidal. Furthermore, due to Theorem 6.2, the functor F is determined by its values on the generating object $X \in \mathcal{L}$ together with its values on the generating morphisms $x \in \mathrm{End}_{\mathcal{L}}(X)$ and $c \in \mathrm{End}_{\mathcal{L}}(X^{\otimes 2})$. We now describe these values. The object $F(X)$ is the tensor product $X \otimes k[y]$ of the generator $X \in \mathcal{S}$ with the vector space of polynomials $k[y]$. The endomorphism $F(x)$ is the morphism induced by the linear map $k[y] \to k[y]$ which is the multiplication by y. To describe the automorphism $F(c)$, identify $\mathrm{End}_{\mathcal{S}}((X \otimes k[y])^{\otimes 2})$ with the tensor product

$$\mathrm{End}_{\mathcal{S}}(X^{\otimes 2}) \otimes \mathrm{End}(k[y_1, y_2]) \simeq k[S_2] \otimes \mathrm{End}(k[y_1, y_2]).$$

Here $\mathrm{End}(k[y_1, y_2])$ is the algebra of k-endomorphisms of the vector space $k[y_1, y_2] \simeq k[y]^{\otimes 2}$. Then we have $F(c) = \partial + t\tau$, where $t \in S_2$ is the involution, $\partial \in \mathrm{End}(k[y_1, y_2])$ is the divided difference operator

$$\partial f(y_1, y_2) = \frac{f(y_1, y_2) - f(y_2, y_1)}{y_1 - y_2} \tag{6.3}$$

and τ is the algebra automorphism of $k[y_1, y_2]$ defined on the generators by $\tau(y_1) = y_2$ and $\tau(y_2) = y_1$.

6.2 Fiber functors and Yangians

Composing the monoidal functor $\mathcal{L} \to \mathcal{S}$ with a fiber functor $\mathcal{S} \to \mathcal{V}ect$ we get a fiber functor $\mathcal{L} \to \mathcal{V}ect$. In particular, taking the symmetric fiber functor $F_N : \mathcal{S} \to \mathcal{V}ect$ we get a fiber functor $F_N : \mathcal{L} \to \mathcal{V}ect$ (for which we keep the same notation). By Theorem 6.2,

F_N is determined by its values on the generating object $X \in \mathcal{L}$ together with its values on the generating morphisms $x \in \mathrm{End}_{\mathcal{L}}(X)$ and $c \in \mathrm{End}_{\mathcal{L}}(X^{\otimes 2})$. Now we describe the triple $(F_N(X), F_N(x), F_N(c))$. The object $F_N(X)$ is the tensor product $V \otimes k[y]$, where $V = k^N$. The endomorphism $F_N(x)$ is induced by the operator of multiplication by y in $k[y]$ so that

$$F_N(x)(v \otimes y^s) = v \otimes y^{s+1}, \qquad v \in V. \tag{6.4}$$

To describe $F_N(c)$, we identify $F_N(X) \otimes F_N(X)$ with $V \otimes V \otimes k[y_1, y_2]$ by

$$(v \otimes y^r) \otimes (w \otimes y^s) \mapsto v \otimes w \otimes y_1^r y_2^s.$$

Then for any $f \in k[y_1, y_2]$,

$$F_N(c)(v \otimes w \otimes f(y_1, y_2)) = w \otimes v \otimes f(y_2, y_1) + v \otimes w \otimes \partial f(y_1, y_2). \tag{6.5}$$

In particular, it follows from the definition that the homomorphisms $F_N(x)$ and $F_N(c)$ satisfy the relations

$$(F_N(x) \otimes 1)F_N(c) - F_N(c)(1 \otimes F_N(x)) = 1,$$
$$(F_N(c) \otimes 1)(1 \otimes F_N(c))(F_N(c) \otimes 1) = (1 \otimes F_N(c))(F_N(c) \otimes 1)(1 \otimes F_N(c)).$$

Now we recall some basic facts about the *Yangian*; see e.g. [5, Ch. 12] and [22, Ch. 1] for more details. The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is the unital associative algebra generated by elements $t_{ij}^{(r)}$ with $1 \leqslant i, j \leqslant N$ and $r = 1, 2, \ldots$ subject to the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}, \tag{6.6}$$

where $r, s \geqslant 0$ and $t_{ij}^{(0)} = \delta_{ij}$. The Yangian is a Hopf algebra with the coproduct defined by

$$\Delta(t_{ij}^{(r)}) = \sum_{k=1}^{N} \sum_{s=0}^{r} t_{ik}^{(s)} \otimes t_{kj}^{(r-s)}. \tag{6.7}$$

The Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N[y])$ in the class of Hopf algebras. As before, we let $\{e_1, \ldots, e_N\}$ denote a basis of an N-dimensional vector space V. Then the vector representation of $\mathfrak{gl}_N$ in V extends to a representation of $\mathfrak{gl}_N[y]$ on the vector space $V \otimes k[y]$, and it gives rise to the representation of $\mathrm{Y}(\mathfrak{gl}_N)$ on this space defined by

$$t_{ij}^{(r)}(e_k \otimes y^s) = \delta_{jk} e_i \otimes y^{r+s-1}. \tag{6.8}$$

Proposition 6.4. *The fiber functor $F_N : \mathcal{L} \to \mathcal{V}ect$ factors through the category of representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$-$\mathcal{M}od$,*

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\quad F_N \quad} & \mathcal{V}ect \\ & {\scriptstyle D_N}\searrow \quad \nearrow & \\ & \mathrm{Y}(\mathfrak{gl}_N)\text{-}\mathcal{M}od & \end{array}$$

via a monoidal functor $D_N : \mathcal{L} \to \mathrm{Y}(\mathfrak{gl}_N)$-$\mathcal{M}od$ and the forgetful functor $\mathrm{Y}(\mathfrak{gl}_N)$-$\mathcal{M}od \to \mathcal{V}ect$.

Proof. We need to show that the maps $F_N(x)$ and $F_N(c)$ defined in (6.4) and (6.5) are morphisms of $\mathrm{Y}(\mathfrak{gl}_N)$-modules. This is obviously true for $F_N(x)$. Furthermore, writing the action (6.8) in terms of the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r}$$

we get

$$t_{ij}(u)(v \otimes y^s) = \Big(\delta_{ij} + \frac{e_{ij}}{u-y}\Big)(v \otimes y^s),$$

where the $e_{ij} \in \mathrm{End}(V)$ denote the standard matrix units. By the coproduct formula (6.7), we have

$$t_{ij}(u)\big(v \otimes w \otimes f(y_1, y_2)\big) = \sum_{k=1}^{N} \Big(\delta_{ik} + \frac{e_{ik}}{u-y_1}\Big)v \otimes \Big(\delta_{kj} + \frac{e_{kj}}{u-y_2}\Big)w \otimes f(y_1, y_2).$$

Hence, applying (6.5), we get

$$\begin{aligned} F_N(c)\,t_{ij}(u)\big(v \otimes w \otimes f(y_1, y_2)\big) &= \sum_{k=1}^{N} \Big(\delta_{kj} + \frac{e_{kj}}{u-y_1}\Big)w \otimes \Big(\delta_{ik} + \frac{e_{ik}}{u-y_2}\Big)v \otimes f(y_2, y_1) \\ &+ \delta_{ij}\big(v \otimes w \otimes \partial f(y_1, y_2)\big) + \big(e_{ij}v \otimes w \otimes \partial \frac{f(y_1, y_2)}{u-y_1}\big) \\ &+ \big(v \otimes e_{ij}w \otimes \partial \frac{f(y_1, y_2)}{u-y_2}\big) + \sum_{k=1}^{N} \Big(e_{ik}v \otimes e_{kj}w \otimes \partial \frac{f(y_1, y_2)}{(u-y_1)(u-y_2)}\Big). \end{aligned}$$

On the other hand,

$$\begin{aligned} t_{ij}(u)\,F_N(c)\big(v \otimes w \otimes f(y_1, y_2)\big) &= \sum_{k=1}^{N} \Big(\delta_{ik} + \frac{e_{ik}}{u-y_1}\Big)w \otimes \Big(\delta_{kj} + \frac{e_{kj}}{u-y_2}\Big)v \otimes f(y_2, y_1) \\ &+ \sum_{k=1}^{N} \Big(\delta_{ik} + \frac{e_{ik}}{u-y_1}\Big)v \otimes \Big(\delta_{kj} + \frac{e_{kj}}{u-y_2}\Big)w \otimes \partial f(y_1, y_2). \end{aligned}$$

In order to compare these two expressions, note that

$$\begin{aligned} \partial \frac{f(y_1, y_2)}{u-y_1} &= \frac{f(y_2, y_1)}{(u-y_1)(u-y_2)} + \frac{1}{u-y_1}\,\partial f(y_1, y_2), \\ \partial \frac{f(y_1, y_2)}{u-y_2} &= -\frac{f(y_2, y_1)}{(u-y_1)(u-y_2)} + \frac{1}{u-y_2}\,\partial f(y_1, y_2), \end{aligned}$$

and

$$\partial\,\frac{f(y_1,y_2)}{(u-y_1)(u-y_2)}=\frac{1}{(u-y_1)(u-y_2)}\,\partial f(y_1,y_2).$$

Therefore,

$$\begin{aligned}&F_N(c)\,t_{ij}(u)\big(v\otimes w\otimes f(y_1,y_2)\big)-t_{ij}(u)\,F_N(c)\,\big(v\otimes w\otimes f(y_1,y_2)\big)\\&\qquad=\Big(e_{ij}v\otimes w-v\otimes e_{ij}w+\sum_{k=1}^{N}\big(e_{kj}w\otimes e_{ik}v-e_{ik}w\otimes e_{kj}v\big)\Big)\otimes\frac{f(y_2,y_1)}{(u-y_1)(u-y_2)}.\end{aligned}$$

Taking the basis vectors $v=e_a$ and $w=e_b$ we find that the expression in the brackets equals

$$\delta_{ja}e_i\otimes e_b-\delta_{jb}e_a\otimes e_i+\delta_{jb}e_a\otimes e_i-\delta_{ja}e_i\otimes e_b=0,$$

thus completing the proof. □

The functor D_N is called the *Drinfeld functor*. The following property of D_N was essentially established in [1], [10].

Proposition 6.5. *The Drinfeld functor $D_N:\mathcal{L}\to\mathrm{Y}(\mathfrak{gl}_N)$-$\mathcal{M}od$ is full.*

Proof. By the construction, D_N sends the generator X of $\mathcal{L}$ to the vector representation $V[y]=V\otimes k[y]$ of $\mathrm{Y}(\mathfrak{gl}_N)$. As D_N is a monoidal functor, we have $D_N(X^{\otimes n})=(V[y])^{\otimes n}$, while the effect of D_N on morphisms amounts to the collection of homomorphisms

$$\Lambda_n\to\mathrm{End}_{\mathrm{Y}(\mathfrak{gl}_N)}\big(V[y]\otimes V[y]\otimes\ldots\otimes V[y]\big).$$

Note that in this description we actually work in the category $\overline{\mathcal{C}}(\Lambda_*)$ rather than $\mathcal{L}=\mathcal{C}(\Lambda_*)$. Extending now D_N to $\mathcal{L}$ we come to the formula

$$D_N(M)=M\otimes_{\Lambda_n}(V[y])^{\otimes n},\qquad M\in\mathcal{M}od\text{-}\Lambda_n.$$

The multiplicative decomposition $\Lambda_n=k[S_n]\,k[y]^{\otimes n}$ allows us to identify

$$M\otimes_{\Lambda_n}(V[y])^{\otimes n}\simeq M\otimes_{k[S_n]}V^{\otimes n}.$$

Due to the results of [1], [10], the functors

$$\mathcal{M}od\text{-}\Lambda_n\longrightarrow\mathrm{Y}(\mathfrak{gl}_N)\text{-}\mathcal{M}od,\qquad M\mapsto M\otimes_{k[S_n]}V^{\otimes n}$$

are full. □

7 Quantum affine algebras and affine Hecke algebras

7.1 Affine braid groups and affine Hecke algebras

The *affine braid group* $\widetilde{B}_n$ is the group with generators $t_1, \ldots, t_{n-1}$ and $y_1, \ldots, y_n$ subject to the defining relations

$$\begin{aligned} t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, & t_i t_j &= t_j t_i \quad \text{for} \quad |i-j| > 1, \\ t_i y_i t_i &= y_{i+1}, & y_i y_j &= y_j y_i. \end{aligned}$$

The assignments

$$\begin{aligned} t_i \otimes 1 &\mapsto t_i, & 1 \otimes t_j &\mapsto t_{j+m}, \\ y_i \otimes 1 &\mapsto y_i, & 1 \otimes y_j &\mapsto y_{j+m} \end{aligned}$$

define algebra homomorphisms

$$k[\widetilde{B}_m] \otimes k[\widetilde{B}_n] \to k[\widetilde{B}_{m+n}]$$

These homomorphisms satisfy the associativity axiom and so give rise to the multiplicative sequence of algebras $k[\widetilde{B}_*] = \{k[\widetilde{B}_n] \mid n \geqslant 0\}$; see Sec. 2.1. Hence we get a monoidal category $\widetilde{\mathcal{B}} = \mathcal{C}(k[\widetilde{B}_*])$ which we call the *affine braid category.*

Proposition 7.1. *The multiplicative sequence* $k[\widetilde{B}_n]$ *is generated by elements* $y \in k[\widetilde{B}_1]$ *and* $t \in k[\widetilde{B}_2]$ *subject to the relations*

$$\begin{aligned} t\,\mu_{1,1}(y \otimes 1)\,t &= \mu_{1,1}(1 \otimes y) && \text{in} \quad k[\widetilde{B}_2], \\ \mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) &= \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) && \text{in} \quad k[\widetilde{B}_3]. \end{aligned}$$

Proof. This follows from the presentation for $\widetilde{B}_n$. □

Theorem 7.2. *The affine braid category is a free monoidal category generated by one object* X*, an endomorphism* $x : X \to X$ *and a Yang–Baxter operator* $c : X^{\otimes 2} \to X^{\otimes 2}$ *subject to the relations*

$$(c \otimes 1_X)(1_X \otimes c)(c \otimes 1_X) = (1_X \otimes c)(c \otimes 1_X)(1_X \otimes c)$$

and

$$c(x \otimes 1)c = 1 \otimes x.$$

Proof. This follows from Theorem 2.4 and Proposition 7.1. □

Now we define certain quotients of the affine braid group algebras. Fix a nonzero element $q \in k$. The *affine Hecke algebra* $\widetilde{\mathrm{H}}_n(q)$ is the associative algebra generated by elements $t_1, \ldots, t_{n-1}$ and invertible elements $y_1, \ldots, y_n$ subject to the relations

$$(t_i - q)(t_i + q^{-1}) = 0, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = t_j t_i \quad \text{for} \quad |i-j| > 1,$$
$$t_i y_i t_i = y_{i+1}, \qquad y_i y_j = y_j y_i.$$

The assignments

$$t_i \otimes 1 \mapsto t_i, \qquad 1 \otimes t_j \mapsto t_{j+m},$$
$$y_i \otimes 1 \mapsto y_i, \qquad 1 \otimes y_j \mapsto y_{j+m}$$

define algebra homomorphisms

$$\mu_{m,n} : \widetilde{\mathrm{H}}_m(q) \otimes \widetilde{\mathrm{H}}_n(q) \to \widetilde{\mathrm{H}}_{m+n}(q)$$

making the sequence $\widetilde{\mathrm{H}}_*(q) = \{\widetilde{\mathrm{H}}_n(q) \mid n \geqslant 0\}$ into a multiplicative sequence of algebras; see Sec. 2.1. This gives rise to a monoidal category $\widetilde{\mathcal{H}}(q) = \mathcal{C}(\widetilde{\mathrm{H}}_*(q))$, which we call the *affine Hecke category.*

Proposition 7.3. *The multiplicative sequence $\widetilde{\mathrm{H}}_*(q)$ is generated by elements $y \in \widetilde{\mathrm{H}}_1(q)$ and $t \in \widetilde{\mathrm{H}}_2(q)$ subject to the relations*

$$(t-q)(t+q^{-1}) = 0 \qquad \textit{in} \quad \widetilde{\mathrm{H}}_2(q),$$
$$t\,\mu_{1,1}(y \otimes 1)\,t = \mu_{1,1}(1 \otimes y) \qquad \textit{in} \quad \widetilde{\mathrm{H}}_2(q),$$
$$\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1) = \mu_{1,2}(1 \otimes t)\mu_{2,1}(t \otimes 1)\mu_{1,2}(1 \otimes t) \qquad \textit{in} \quad \widetilde{\mathrm{H}}_3(q).$$

Proof. This follows from the presentation of $\widetilde{\mathrm{H}}_n(q)$. □

Theorem 7.4. *The affine Hecke category is a free monoidal category generated by one object X, an endomorphism $x : X \to X$ and a Hecke Yang–Baxter operator $c : X^{\otimes 2} \to X^{\otimes 2}$ subject to the relations*

$$(c-q)(c+q^{-1}) = 0, \qquad (c \otimes 1_X)(1_X \otimes c)(c \otimes 1_X) = (1_X \otimes c)(c \otimes 1_X)(1_X \otimes c)$$

and

$$c(x \otimes 1)c = 1 \otimes x.$$

Proof. This follows from Theorem 2.4 and Proposition 7.3. □

Corollary 7.5. *Monoidal functors from the affine braid category (resp., from the affine Hecke category) to a monoidal category $\mathcal{C}$ are determined by triples (V, x, c), where V is an object in $\mathcal{C}$, $x \in \mathrm{End}_{\mathcal{C}}(V)$ is its endomorphism, and $c \in \mathrm{End}(V^{\otimes 2})$ is a (Hecke) Yang–Baxter operator such that*

$$c(x \otimes 1)c = 1 \otimes x. \tag{7.1}$$

Example 7.6. *Monoidal autoequivalences of* $\widetilde{\mathcal{H}}(q)$. Let u be an invertible element of the basic field k. The triple (V, ux, c) satisfies the conditions of Corollary 7.5 and so it defines a monoidal functor $T_u : \widetilde{\mathcal{H}}(q) \to \widetilde{\mathcal{H}}(q)$. The composition $T_u \circ T_v$ is canonically isomorphic (as a monoidal functor) to T_{uv}. Hence T_u is a monoidal autoequivalence of $\widetilde{\mathcal{H}}(q)$. □

Due to [20, Lemma 3.4], affine Hecke algebras admit multiplicative decompositions

$$\widetilde{\mathrm{H}}_n(q) = k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]\, \mathrm{H}_n(q) = \mathrm{H}_n(q)\, k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]. \tag{7.2}$$

For $n = 2$ the decomposition (7.2) follows from the relation

$$t\, f(y_1, y_2) = f(y_2, y_1)t - (q - q^{-1})\, y_2\, \frac{f(y_1, y_2) - f(y_2, y_1)}{y_1 - y_2},$$

where $f(y_1, y_2)$ is an arbitrary Laurent polynomial in y_1, y_2.

The monoidal functor $\mathcal{H}(q) \to \widetilde{\mathcal{H}}(q)$ defined by the natural homomorphism of multiplicative sequences $\mathrm{H}_*(q) \to \widetilde{\mathrm{H}}_*(q)$, sends the generator $X \in \mathcal{H}(q)$ to the generator $X \in \widetilde{\mathcal{H}}(q)$ and sends $c \in \mathrm{End}_{\mathcal{H}(q)}(X^{\otimes 2})$ to $c \in \mathrm{End}_{\widetilde{\mathcal{H}}(q)}(X^{\otimes 2})$. By Theorem 2.3, its right adjoint $F : \widetilde{\mathcal{H}}(q) \to \mathcal{H}(q)$ is also monoidal. By Corollary 7.5, the functor F is determined by its values on the generating object $X \in \widetilde{\mathcal{H}}(q)$ together with its values on the generating morphisms $x \in \mathrm{End}_{\widetilde{\mathcal{H}}(q)}(X)$ and $c \in \mathrm{End}_{\widetilde{\mathcal{H}}(q)}(X^{\otimes 2})$. Now we describe these values. The object $F(X)$ is the tensor product $X \otimes k[y^{\pm 1}]$ of the generator $X \in \mathcal{H}(q)$ with the vector space $k[y^{\pm}]$ of Laurent polynomials. The automorphism $F(x)$ is the morphism induced by the linear map $k[y^{\pm 1}] \to k[y^{\pm 1}]$, which is multiplication by y. To describe the automorphism $F(c)$, identify $\mathrm{End}_{\widetilde{\mathcal{H}}(q)}((X \otimes k[y^{\pm 1}])^{\otimes 2})$ with the tensor product

$$\mathrm{End}_{\mathcal{H}(q)}(X^{\otimes 2}) \otimes \mathrm{End}(k[y_1^{\pm}, y_2^{\pm}]) \simeq \mathrm{H}_2(q) \otimes \mathrm{End}(k[y_1^{\pm}, y_2^{\pm}]).$$

Here $\mathrm{End}(k[y_1^{\pm}, y_2^{\pm}])$ is the algebra of k-endomorphisms of the vector space of Laurent polynomials $k[y_1^{\pm}, y_2^{\pm}] \simeq k[y^{\pm}]^{\otimes 2}$. We have $F(c) = d + t\,\tau$, where $t = t_1$ is the generator of $\mathrm{H}_2(q)$, $\tau \in \mathrm{End}(k[y_1^{\pm}, y_2^{\pm}])$ is the algebra automorphism $\tau(f)(y_1, y_2) = f(y_2, y_1)$ and $d \in \mathrm{End}(k[y_1^{\pm}, y_2^{\pm}])$ is defined by $d(f) = -(q - q^{-1})\, y_2\, (y_1 - y_2)^{-1}(f - \tau(f))$.

7.2 Fiber functors and quantum affine algebras

Let us compose the monoidal functor $F : \widetilde{\mathcal{H}}(q) \to \mathcal{H}(q)$ with the monoidal fiber functor $F_N : \mathcal{H}(q) \to \mathcal{V}ect$ considered in Sec. 4.2. We get a fiber functor $F_N : \widetilde{\mathcal{H}}(q) \to \mathcal{V}ect$ which we denote by the same symbol. By Corollary 7.5, F_N is determined by its values on the generating object $X \in \widetilde{\mathcal{H}}(q)$ together with its values on the generating morphisms $x \in \mathrm{End}_{\widetilde{\mathcal{H}}(q)}(X)$ and $c \in \mathrm{End}_{\widetilde{\mathcal{H}}(q)}(X^{\otimes 2})$. The object $F_N(X)$ is the tensor product $V \otimes k[y^{\pm 1}]$, where $V = k^N$. The endomorphism $F_N(x)$ is the morphism, induced by the multiplication by y in $k[y^{\pm 1}]$,

$$F_N(x)(v \otimes y^s) = v \otimes y^{s+1}. \tag{7.3}$$

The map $F_N(c) : F_N(X) \otimes F_N(X) \to F_N(X) \otimes F_N(X)$ is given as follows. Identifying $F_N(X) \otimes F_N(X)$ with $V \otimes V \otimes k[y_1^{\pm 1}, y_2^{\pm 1}]$ by

$$(v \otimes y^r) \otimes (w \otimes y^s) \mapsto v \otimes w \otimes y_1^r y_2^s,$$

we can write $F_N(c)$ as

$$F_N(c)\big(v \otimes w \otimes f(y_1, y_2)\big) = R(v \otimes w) \otimes f(y_2, y_1) - (q - q^{-1}) v \otimes w \otimes y_2 \, \partial f(y_1, y_2), \quad (7.4)$$

where the operator R is defined in (4.3), and ∂ is the divided difference operator (6.3) extended to Laurent polynomials.

We will now formulate an analogue of Proposition 6.4, where the role of the Yangian is played by the *quantum affine algebra* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ (with the trivial center charge), also known as the *quantum loop algebra*. The role of the vector representation is now played by the space $V \otimes k[y^{\pm 1}]$. Explicit formulas for the action of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ on this space are analogous to (6.8) and they can be found in [14].

Proposition 7.7. *The fiber functor $F_N : \widetilde{\mathcal{H}}(q) \to \mathcal{V}ect$ factors through the category of representations $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$-$\mathcal{M}od$ of the quantum affine algebra*

$$\begin{array}{ccc} \widetilde{\mathcal{H}}(q) & \xrightarrow{\quad F_N \quad} & \mathcal{V}ect \\ & {\scriptstyle GRV_N} \searrow \qquad \nearrow & \\ & \mathrm{U}_q(\widehat{\mathfrak{gl}}_N)\text{-}\mathcal{M}od & \end{array}$$

via a monoidal functor $GRV_N : \widetilde{\mathcal{H}}(q) \longrightarrow \mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$-$\mathcal{M}od$ and the forgetful functor $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$-$\mathcal{M}od \longrightarrow \mathcal{V}ect$.

The proof is quite similar to that of Proposition 6.4 and amounts to checking that $F_N(x)$ and $F_N(c)$ are morphisms of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$-modules. We omit the details; see also [14].

We call GRV_N the *Ginzburg–Reshetikhin–Vasserot functor*, as the following version of the Schur–Weyl duality for the quantum loop algebras was proved in [14]; see also [4].

Proposition 7.8. *The functor $GRV_N : \widetilde{\mathcal{H}}(q) \longrightarrow \mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$-$\mathcal{M}od$ is full.*

8 Localizations and categorical actions

8.1 Localizations with respect to discriminants

Here we discuss some applications of the universal properties of the affine Hecke category $\widetilde{\mathcal{H}}(q)$ and its degenerate version $\mathcal{L}$. We will regard $\widetilde{\mathcal{H}}(q)$ and $\mathcal{L}$ as respective quantum deformations of the affine symmetric category $\mathcal{AS}$ and the semi-affine symmetric category $\mathcal{SAS}$ and we will show that these deformations are trivial away from some discriminant-type loci.

To formulate the precise statement, let $\Delta \subset \mathcal{M}or\mathcal{AS}$ be the monoidally and multiplicatively closed set of morphisms generated by $x\otimes 1 - 1\otimes x$; see Sec. 5.1. In other words, for each n we consider the multiplicatively closed set of morphisms generated by

$$\Delta_n \in k[y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_n} \subset k[y_1^{\pm 1}, \dots, y_n^{\pm 1}] * S_n = \mathrm{End}_{\mathcal{AS}}(X^{\otimes n}),$$

where $\Delta_n = \prod_{i\neq j}(y_i - y_j)$ is the *discriminant polynomial*. Note that the algebra of symmetric polynomials $k[y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_n}$ coincides with the center of the endomorphism algebra $\mathrm{End}_{\mathcal{AS}}(X^{\otimes n})$ so that Δ_n commutes with all morphisms in $\mathcal{AS}$. Denote by $\mathcal{AS}[\Delta^{-1}]$ the category of fractions with respect to Δ; see e.g. [13] for the definition. This category has the form $\mathcal{C}(A_*)$, where $A_* = \{A_n \mid n \geqslant 0\}$ is the multiplicative sequence of the localized algebras $A_n = \big(k[y_1^{\pm 1}, \dots, y_n^{\pm 1}] * S_n\big)[\Delta_n^{-1}]$. Therefore, the category $\mathcal{AS}[\Delta^{-1}]$ is monoidal. A similar argument shows that the category $\mathcal{SAS}[\Delta^{-1}]$ is also monoidal. Moreover, the localization functors $\mathcal{AS} \to \mathcal{AS}[\Delta^{-1}]$ and $\mathcal{SAS} \to \mathcal{SAS}[\Delta^{-1}]$ are monoidal.

It is well known from [3] that the center of the affine Hecke algebra $\widetilde{\mathrm{H}}_n(q)$ for generic q coincides with the algebra of symmetric Laurent polynomials $k[y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_n}$, while the center of the degenerate affine Hecke algebra Λ_n coincides with the algebra of symmetric polynomials $k[y_1, \dots, y_n]^{S_n}$. Therefore, it is unambiguous to define the respective categories of fractions as

$$\widetilde{\mathcal{H}}(q)[\Delta^{-1}] = \mathcal{C}(\widetilde{\mathrm{H}}_*(q)[\Delta_*^{-1}]) \qquad \text{and} \qquad \mathcal{L}[\Delta^{-1}] = \mathcal{C}(\Lambda_*[\Delta_*^{-1}]).$$

Proposition 8.1. *The assignment* $(X, x, c) \mapsto (X, x, \widetilde{c})$*, where*

$$\widetilde{c} = (x\otimes 1 - 1\otimes x)^{-1}\big((q^{-1} - q)(1\otimes x) + (q(x\otimes 1) - q^{-1}(1\otimes x))c\big) \tag{8.1}$$

defines a monoidal functor $\widetilde{\mathcal{H}}(q)[\Delta^{-1}] \to \mathcal{AS}[\Delta^{-1}]$.

Moreover, the assignment $(X, x, c) \mapsto (X, x, \widetilde{c})$*, where*

$$\widetilde{c} = (x\otimes 1 - 1\otimes x)^{-1}\big(1\otimes 1 + (x\otimes 1 - 1\otimes x - 1\otimes 1)c\big) \tag{8.2}$$

defines a monoidal functor $\mathcal{L}[\Delta^{-1}] \to \mathcal{SAS}[\Delta^{-1}]$.

Proof. In the affine Hecke category case, write $\widetilde{c} = a + (q-a)t$ as an element of $k[x_1, x_2] * S_2$ with $t = c$ and $a = (q^{-1} - q)x_2(x_1 - x_2)^{-1}$, where $x_1 = x\otimes 1,\ x_2 = 1\otimes x$. Now we verify the relations

$$(\widetilde{c} - q)(\widetilde{c} + q^{-1}) = 0, \qquad (x\otimes 1)\widetilde{c} - \widetilde{c}(1\otimes x) = 1, \tag{8.3}$$

$$(\widetilde{c}\otimes 1_X)(1_X\otimes \widetilde{c})(\widetilde{c}\otimes 1_X) = (1_X\otimes \widetilde{c})(\widetilde{c}\otimes 1_X)(1_X\otimes \widetilde{c}) \tag{8.4}$$

by direct computations in $k[x_1,x_2]*S_2$ and $k[x_1,x_2,x_3]*S_3$, respectively, where in the latter case we interpret the variables as $x_1 = x\otimes 1\otimes 1$, $x_2 = 1\otimes x\otimes 1$ and $x_3 = 1\otimes 1\otimes x$. Noting that $ta = (q-q^{-1}-a)t$ we get

$$\begin{aligned}&(a-q+(q-a)\,t)(a+q^{-1}+(q-a)\,t)\\ &\qquad = (a-q)(a+q^{-1})+(a-q)(q-a)\,t-(q-a)(a-q)\,t+(q-a)(q^{-1}+a)\,t^2 = 0\end{aligned}$$

and

$$\begin{aligned}&(a+(q-a)t)\,x_1(a-(q-a)t)\\ &\qquad = a^2x_1+(q-a)x_2(q-q^{-1}-a)\,t+ax_1(q-a)\,t-(q-a)(q^{-1}+a)\,t^2\\ &\qquad = x_2+a\big(a(x_1-x_2)-(q^{-1}-q)x_2\big)+(q-a)\big(a(x_1-x_2)-(q^{-1}-q)x_2\big)\,t = x_2,\end{aligned}$$

thus proving (8.3). To verify (8.4), note that the relation is equivalent to

$$\begin{aligned}&(a_{12}+(q-a)\,t_{12})(a_{23}+(q-a)\,t_{23})(a_{12}+(q-a)\,t_{12})\\ &\qquad = (a_{23}+(q-a)\,t_{23})(a_{12}+(q-a)\,t_{12})(a_{23}+(q-a)\,t_{23}),\end{aligned} \tag{8.5}$$

where we used the notation $a_{12} = a\otimes 1$, $a_{23} = 1\otimes a$ and $a_{13} = t_1a_{23}t_1 = t_2a_{12}t_2$. Now compare the coefficients of the elements of S_3 on both sides of (8.5). They obviously equal for the elements t_1t_2, t_2t_1 and $t_1t_2t_1 = t_2t_1t_2$, while for 1, t_1 and t_2 we need to check the following relations, respectively:

$$\begin{aligned}a_{12}a_{23}a_{12}+(q-a_{12})a_{13}(q^{-1}+a_{12}) &= a_{23}a_{12}a_{23}+(q-a_{23})a_{13}(q^{-1}+a_{23}),\\ a_{12}a_{23}(q-a_{12})+(q-a_{12})a_{13}(q-q^{-1}-a_{12}) &= a_{23}(q-a_{12})a_{13},\\ a_{12}(q-a_{23})a_{13} &= a_{23}a_{12}(q-a_{23})+(q-a_{23})a_{13}(q-q^{-1}-a_{23}).\end{aligned}$$

However, all of them follow from the identity

$$a_{12}a_{23}-a_{12}a_{13}-a_{13}a_{23}+(q-q^{-1})a_{13} = 0,$$

which is verified directly by substituting the expressions for a_{12}, a_{23} and a_{13} in terms of x_1, x_2 and x_3.

In the case of the degenerate affine Hecke category, the argument is quite similar. We write $\widetilde{c} = a+(1-a)\,t$ as an element of $k[x_1,x_2]*S_2$ with $t=c$ and $a = (x_1-x_2)^{-1}$, where $x_1 = x\otimes 1$, $x_2 = 1\otimes x$. Now the relations

$$\begin{aligned}&\widetilde{c}^2 = 1, \qquad (x\otimes 1)\,\widetilde{c}-\widetilde{c}\,(1\otimes x) = 1,\\ &(\widetilde{c}\otimes 1_X)(1_X\otimes\widetilde{c})(\widetilde{c}\otimes 1_X) = (1_X\otimes\widetilde{c})(\widetilde{c}\otimes 1_X)(1_X\otimes\widetilde{c})\end{aligned}$$

are verified directly by computations in $k[x_1,x_2]*S_2$ and $k[x_1,x_2,x_3]*S_3$, respectively, exactly as in the Hecke category case. □

Let $\Delta(q) \subset \mathcal{M}or\mathcal{AS}$ be a monoidally and multiplicatively closed set of morphisms generated by $q(x \otimes 1) - q^{-1}(1 \otimes x)$. That is, for each n we consider the multiplicatively closed set of morphisms generated by

$$\Delta(q)_n \in k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]^{S_n} \subset k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] * S_n = \mathrm{End}_{\mathcal{AS}}(X^{\otimes n}),$$

where $\Delta(q)_n = \prod_{i \neq j}(qy_i - q^{-1}y_j)$ is the *quantum discriminant polynomial.* We can also consider Δ_q as a set of morphisms of $\widetilde{\mathcal{H}}(q)$. Similarly, let $\widetilde{\Delta} \subset \mathcal{M}or\mathcal{SAS}$ be a monoidally and multiplicatively closed set of morphisms generated by $x \otimes 1 - 1 \otimes x - 1 \otimes 1$, so that for each n we consider the multiplicatively closed set of morphisms generated by

$$\widetilde{\Delta}_n \in k[y_1, \ldots, y_n]^{S_n} \subset k[y_1 \ldots, y_n] * S_n = \mathrm{End}_{\mathcal{SAS}}(X^{\otimes n}),$$

where $\widetilde{\Delta}_n = \prod_{i \neq j}(y_i - y_j - 1)$ is the *degenerate quantum discriminant polynomial.* Similar to the above, we can consider $\widetilde{\Delta}$ as a set of morphisms of $\mathcal{L}$.

The proof of the following is completely analogous to and partly follows from the proof of Proposition 8.1.

Proposition 8.2. *The assignment* $(X, x, c) \mapsto (X, x, \widetilde{c})$*, where*

$$\widetilde{c} = (q(x \otimes 1) - q^{-1}(1 \otimes x))^{-1}\big((x \otimes 1 - 1 \otimes x)\, c + (q^{-1} - q)(1 \otimes x)\big) \tag{8.6}$$

defines a monoidal functor $\mathcal{AS}[\Delta(q)^{-1}] \to \widetilde{\mathcal{H}}(q)[\Delta(q)^{-1}]$.

Moreover, the assignment $(X, x, c) \mapsto (X, x, \widetilde{c})$*, where*

$$\widetilde{c} = (x \otimes 1 - 1 \otimes x - 1 \otimes 1)^{-1}\big((x \otimes 1 - 1 \otimes x)\, c + 1 \otimes 1\big) \tag{8.7}$$

defines a monoidal functor $\mathcal{SAS}[\widetilde{\Delta}^{-1}] \to \mathcal{L}[\widetilde{\Delta}^{-1}]$.

Combining Propositions 8.1 and 8.2, we come to the following theorem, where we let $D(q) = \Delta\Delta(q)$ denote the set of compositions of morphisms from Δ and $\Delta(q)$, and let $D = \Delta\widetilde{\Delta}$ denote the set of compositions of morphisms from Δ and $\widetilde{\Delta}$.

Theorem 8.3. *The monoidal categories* $\mathcal{AS}[D(q)^{-1}]$ *and* $\widetilde{\mathcal{H}}(q)[D(q)^{-1}]$ *are equivalent.*

Moreover, the monoidal categories $\mathcal{SAS}[D^{-1}]$ *and* $\mathcal{L}[D^{-1}]$ *are equivalent.*

Proof. Both statements follow from the observation that each pair of constructions (8.1) and (8.6), as well as (8.2) and (8.7), are inverse to each other. □

Remark 8.4. By the Galois theory, $k[y_1, \ldots, y_n][\Delta_n^{-1}] * S_n$ is isomorphic to the algebra

$$M_{n!}\big(k[y_1, \ldots, y_n]^{S_n}[\Delta_n^{-1}]\big)$$

of $n! \times n!$ matrices with coefficients in the localization $k[y_1, \dots, y_n]^{S_n}[\Delta_n^{-1}]$ of the algebra of symmetric polynomials. Theorem 8.3 implies that the algebra $\widetilde{\mathrm{H}}_n(q)[(\Delta_n \Delta(q)_n)^{-1}]$ is isomorphic to the matrix algebra

$$M_{n!}\big(k[y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_n}[(\Delta_n \Delta(q)_n)^{-1}]\big).$$

Similarly, the algebra $\Lambda_n[(\Delta_n \widetilde{\Delta}_n)^{-1}]$ is isomorphic to the matrix algebra

$$M_{n!}\big(k[y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_n}[(\Delta_n \widetilde{\Delta}_n)^{-1}]\big).$$

8.2 Orellana–Ram and Cherednik–Arakawa–Suzuki functors

Here we give a construction, turning a braided monoidal category with a Hecke object into a module category over the affine Hecke category; see e.g. [16], [24] for the definition of a module category. Our construction has been motivated by the work of Orellana and Ram [23].

Let $\mathcal{C}$ be a braided monoidal category and let $c_{X,Y}$ denote the braiding

$$c_{X,Y} : X \otimes Y \to Y \otimes X.$$

Fix an object X of $\mathcal{C}$ and let $O : \mathcal{C} \to \mathcal{C}$ be the functor of tensoring by X from the right: $O(Y) = O_X(Y) = Y \otimes X$. Note that O, as an object of the monoidal category of endofuctors $\mathcal{F}unct(\mathcal{C},\mathcal{C})$, possesses a Yang–Baxter operator c, which is (as a morphism in the functor category $\mathcal{F}unct(\mathcal{C},\mathcal{C})$) the natural transformation $O_{c_{X,X}}$:

$$O \circ O(Y) = Y \otimes X^{\otimes 2} \xrightarrow{1_Y \otimes c_{X,X}} Y \otimes X^{\otimes 2} = O \circ O(Y)\ .$$

Define an endomorphism $x : O \to O$ (a natural transformation) as the composition:

$$O(Y) = Y \otimes X \xrightarrow{c_{Y,X}} X \otimes Y \xrightarrow{c_{X,Y}} Y \otimes X = O(Y).$$

Lemma 8.5. *The triple (O, x, c) defines a monoidal functor $OR_X : \widetilde{\mathcal{B}} \to \mathcal{F}unct(\mathcal{C},\mathcal{C})$.*

Proof. The following commutative diagram (which is a joint of two coherence diagrams for the braiding) guarantees that c and x satisfy the condition (7.1):

$$\begin{array}{ccccc} Y \otimes X^{\otimes 2} & \xrightarrow{c_{Y\otimes X,X}} & X \otimes Y \otimes X & \xrightarrow{c_{X,Y\otimes X}} & Y \otimes X^{\otimes 2} \\ \big\downarrow{\scriptstyle 1_Y \otimes c_{X,X}} & & \big\| & & \big\uparrow{\scriptstyle 1_Y \otimes c_{X,X}} \\ Y \otimes X^{\otimes 2} & \xrightarrow{c_{Y\otimes X,X}} & X \otimes Y \otimes X & \xrightarrow{c_{X,Y\otimes X}} & Y \otimes X^{\otimes 2} \end{array}$$

thus proving the claim. □

We call OR_X the *Orellana–Ram* functor corresponding to $X \in \mathcal{C}$. In the particular case $\mathcal{C} = \mathcal{B}$ we get a monoidal functor $OR : \widetilde{\mathcal{B}} \to \mathcal{F}unct(\mathcal{B}, \mathcal{B})$ corresponding to the generating object of $\mathcal{B}$. It is easy to see that this functor is faithful (injective on morphisms).

An object X of a braided monoidal category $\mathcal{C}$ will be called a *Hecke object*, if the braiding $c_{X,X} \in \mathrm{End}_{\mathcal{C}}(X^{\otimes 2})$ satisfies the equation $(c_{X,X} - q\, 1_{X \otimes X})(c_{X,X} + q^{-1}\, 1_{X \otimes X}) = 0$ for some non-zero scalar $q \in k$.

Proposition 8.6. *If X be a Hecke object of $\mathcal{C}$, then the functor OR_X factors through the affine Hecke category $\mathcal{H}(q)$, giving rise to a functor $OR_X : \widetilde{\mathcal{H}}(q) \to \mathcal{F}unct(\mathcal{C}, \mathcal{C})$.*

Proof. By definition, the Yang–Baxter operator c on the functor O satisfies the equation $(c - q)(c + q^{-1}) = 0$. □

Now we describe a degenerate analog of the Orellana–Ram functors. The construction (a special case of which was studied in [2] and [6]) requires an infinitesimal version of the notion of braided category. The notion of *chorded categories* was virtually defined by Drinfeld [11] and was studied in [19] under the name *infinitesimal symmetric categories*. Due to their relation with Kontsevich's chord diagrams we will call them chorded categories.

Let $\mathcal{C}$ be a symmetric monoidal category with the symmetry $c_{X,Y}$. A *chording* on $\mathcal{C}$ is a natural collection of morphisms $h_{X,Y} : X \otimes Y \to X \otimes Y$ satisfying the conditions:

$$c_{X,Y} h_{Y,X} = h_{X,Y} c_{X,Y} \tag{8.8}$$

and

$$h_{X,Y \otimes Z} = h_{X,Y} \otimes 1_Z + (1_X \otimes c_{Y,Z})(h_{X,Z} \otimes 1_Y)(1_X \otimes c_{Y,Z})^{-1}. \tag{8.9}$$

A symmetric monoidal category with a chording will be called a *chorded category*.

A symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between chorded categories is *chorded* if

$$F_{X,Y} F(h_{X,Y}) = h_{F(X),F(Y)} F_{X,Y},$$

where

$$F_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$$

is the monoidal constraint of F.

Now we describe a construction, which endows the category of representations of a Lie algebra with a chorded structure. A Lie algebra $\mathfrak{g}$ will be called a *Casimir Lie algebra*, if it is equipped with a symmetric and $\mathfrak{g}$-invariant element $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$.

Proposition 8.7. *Let $(\mathfrak{g}, \Omega)$ be Casimir Lie algebra and let $M, N \in \mathcal{R}ep(\mathfrak{g})$. The relation*

$$h_{M,N}(m \otimes n) = \Omega(m \otimes n), \qquad m \in M, \quad n \in N,$$

defines a chorded structure on the category of representations $\mathcal{R}ep(\mathfrak{g})$ of $\mathfrak{g}$.

Proof. The $\mathfrak{g}$-invariance of Ω implies that $h_{M,N}$ is a homomorphism of representations of $\mathfrak{g}$; i.e., a morphism in $\mathcal{R}ep(\mathfrak{g})$. The condition (8.8) follows from the symmetry property of Ω. Finally, we have the identity

$$(1 \otimes \Delta)(\Omega) = \Omega_{12} + \Omega_{13},$$

where $\Delta(y) = y \otimes 1 + 1 \otimes y$ for $y \in \mathfrak{g}$, which verifies the condition (8.9). $\square$

An object X of a chorded category $\mathcal{C}$ will be called a *degenerate Hecke object*, if the chording is proportional to the commutativity morphism, $h_{X,X} = \lambda\, c_{X,X}$ for some non-zero scalar $\lambda \in k$. Let $\mathcal{C}$ be a chorded monoidal category and let X be a degenerate Hecke object of $\mathcal{C}$. Let $O : \mathcal{C} \to \mathcal{C}$ be a functor of tensoring by X from the right, $O(Y) = O_X(Y) = Y \otimes X$. Note that O, as an object of the monoidal category of endofuctors $\mathcal{F}unct(\mathcal{C}, \mathcal{C})$, possesses a Yang–Baxter operator c, which is (as a morphism in the functor category $\mathcal{F}unct(\mathcal{C}, \mathcal{C})$) the natural transformation $O_{c_{X,X}}$:

$$O \circ O(Y) = Y \otimes X^{\otimes 2} \xrightarrow{1_Y \otimes c_{X,X}} Y \otimes X^{\otimes 2} = O \circ O(Y)\,.$$

Define an endomorphism $x : O \to O$ (a natural transformation):

$$O(Y) = Y \otimes X \xrightarrow{\lambda^{-1} h_{Y,X}} Y \otimes X = O(Y).$$

Proposition 8.8. *Let $\mathcal{C}$ be a chorded monoidal category and let X be a degenerate Hecke object of $\mathcal{C}$. Then the triple (O, x, c) defines a monoidal functor $CAS_X : \mathcal{L} \to \mathcal{F}unct(\mathcal{C}, \mathcal{C})$.*

Proof. The condition (6.1) follows from the chording axiom. Indeed, the natural transformation $(x \otimes 1)\, c - c\,(1 \otimes x) \in \mathrm{End}\big(O(X) \circ O(X)\big)$ evaluated at $Y \in \mathcal{C}$ has the form

$$\begin{aligned}
&\lambda^{-1}(h_{Y,X} \otimes 1_X)(1_Y \otimes c_{X,X}) - \lambda^{-1}(1_Y \otimes c_{X,X}) h_{Y\otimes X,X} \\
&\qquad = \lambda^{-1}(h_{Y,X} \otimes 1_X)(1_Y \otimes c_{X,X}) - \lambda^{-1}(1_Y \otimes c_{X,X}) \\
&\qquad\qquad \times \Big(1_Y \otimes h_{X,X} + (1_Y \otimes c_{X,X})(h_{Y,X} \otimes 1_X)(1_Y \otimes c_{X,X})\Big) \\
&\qquad\qquad\qquad = \lambda^{-1}(1_Y \otimes c_{X,X})(1_Y \otimes h_{X,X}) = 1_{Y\otimes X\otimes X},
\end{aligned}$$

as required. $\square$

We call the functor CAS_X the *Cherednik–Arakawa–Suzuki* functor corresponding to $X \in \mathcal{C}$; see the following examples.

Example 8.9. *Representations of* $\mathfrak{gl}_N$. Let V be the N-dimensional vector representation of the Lie algebra $\mathfrak{gl}_N$. Consider the chorded structure on the category $\mathcal{C} = \mathcal{R}ep(\mathfrak{gl}_N)$ of representations of $\mathfrak{gl}_N$ given by the standard Casimir element $C \in \mathfrak{gl}_N \otimes \mathfrak{gl}_N$. Then V is a degenerate Hecke object of $\mathcal{C}$. This gives a functor $CAS_V : \mathcal{L} \to \mathcal{F}unct(\mathcal{C}, \mathcal{C})$ studied in some form in [2] and [6]; see also [7, Lemma 7.21]. $\square$

Example 8.10. *The category $\mathcal{S}$.* The category $\mathcal{S}$ has a chorded structure uniquely defined by $h_{X,X} = c_{X,X}$. In particular, the generator X is a degenerate Hecke object of $\mathcal{S}$. Thus we get a monoidal functor $CAS_X : \mathcal{L} \to \mathcal{F}unct(\mathcal{S}, \mathcal{S})$. □

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