Representations of twisted q-Yangians

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Abstract

The twisted q-Yangians are coideal subalgebras of the quantum affine algebra associated with $\mathfrak{gl}_N$. We prove a classification theorem for finite-dimensional irreducible representations of the twisted q-Yangians associated with the symplectic Lie algebras $\mathfrak{sp}_{2n}$. The representations are parameterized by their highest weights or by their Drinfeld polynomials. In the simplest case of $\mathfrak{sp}_2$ we give an explicit description of all the representations as tensor products of evaluation modules. We prove analogues of the Poincaré–Birkhoff–Witt theorem for the quantum affine algebra and for the twisted q-Yangians. We also reproduce a proof of the classification theorem for finite-dimensional irreducible representations of the quantum affine algebra by relying on its R-matrix presentation.

Contents

1 Introduction

The Yangian $\mathrm{Y}(\mathfrak{a})$ and quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{a}})$ associated with a simple Lie algebra $\mathfrak{a}$ are known as 'infinite-dimensional quantum groups'. They are deformations of the universal enveloping algebras $\mathrm{U}(\mathfrak{a}[z])$ and $\mathrm{U}(\widehat{\mathfrak{a}})$, respectively, in the class of Hopf algebras, and were introduced by Drinfeld [12] and Jimbo [20]. Here $\mathfrak{a}[z]$ denotes the Lie algebra of polynomials in a variable z with coefficients in $\mathfrak{a}$, while $\widehat{\mathfrak{a}}$ is the affine Kac–Moody algebra, i.e., a central extension of the Lie algebra $\mathfrak{a}[z, z^{-1}]$ of Laurent polynomials in z.

The case of $\mathfrak{a} = \mathfrak{sl}_N$ (the A type) is exceptional in the sense that only in this case do there exist epimorphisms $\mathrm{Y}(\mathfrak{a}) \to \mathrm{U}(\mathfrak{a})$ and $\mathrm{U}_q(\widehat{\mathfrak{a}}) \to \mathrm{U}_q(\mathfrak{a})$, called the evaluation homomorphisms, where $\mathrm{U}_q(\mathfrak{a})$ is the corresponding quantized enveloping algebra. These epimorphisms have important applications in the representation theory of both the finite- and infinite-dimensional quantum groups. For the classical Lie algebra $\mathfrak{a}$ (of type B, C or D) there are 'twisted' analogues of the Yangian and quantum affine algebra for which the corresponding epimorphisms do exist. Namely, the twisted Yangians $\mathrm{Y}'(\mathfrak{o}_N)$ and $\mathrm{Y}'(\mathfrak{sp}_{2n})$ associated with the orthogonal and symplectic Lie algebras were introduced by Olshanski [32], while their q-analogues $\mathrm{Y}'_q(\mathfrak{o}_N)$ and $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$, called the twisted q-Yangians, appeared in Molev, Ragoucy and Sorba [29]. These algebras do not possess natural Hopf algebra structures, but they are coideal subalgebras of the A type Yangian and quantum affine algebra, respectively. The evaluation homomorphisms have the form

$$\mathrm{Y}'(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N), \qquad \mathrm{Y}'_q(\mathfrak{g}_N) \to \mathrm{U}'_q(\mathfrak{g}_N),$$

where $\mathfrak{g}_N$ denotes either the orthogonal Lie algebra $\mathfrak{o}_N$ or the symplectic Lie algebra $\mathfrak{sp}_N$ (the latter with $N = 2n$) and $\mathrm{U}'_q(\mathfrak{g}_N)$ is the twisted (or nonstandard) quantized enveloping algebra associated with $\mathfrak{g}_N$ which was defined in [15], [30] and [31].

Finite-dimensional irreducible representations of the Yangians $\mathrm{Y}(\mathfrak{a})$ were classified by Drinfeld [13]. The particular case $\mathfrak{a} = \mathfrak{sl}_2$ plays a key role in the arguments and it was done earlier by Tarasov [36, 37]; see [28, Ch. 3] for a detailed exposition of these results. The classification theorem for the representations of the quantum affine algebras was proved by Chari and Pressley [7], [8, Ch. 12]. Again, the case of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_2)$ is crucial, and it is possible to prove the theorem here following Tarasov's arguments [36, 37]. The corresponding proof was also outlined in [28, Sec. 3.5] and we give more details below (Section 3), as the same approach will be used for the twisted q-Yangians.

A classification of finite-dimensional irreducible representations of the twisted Yangians $\mathrm{Y}(\mathfrak{o}_N)$ and $\mathrm{Y}(\mathfrak{sp}_{2n})$ was obtained in [26]; see also [28] for a detailed exposition, more references and applications to representation theory of the classical Lie algebras. Recent renewed interest in the representation theory of Yangians and twisted Yangians was caused by its surprising connection with the theory of finite W-algebras (see [4], [6], [33]) and by a generalized Howe duality (see [23], [24]). Note also the applications of the twisted Yangians and their q-analogues to the soliton spin chain models with special boundary conditions [1], [2].

In this paper we prove a classification theorem for finite-dimensional irreducible representations of the twisted q-Yangians associated with the symplectic Lie algebras $\mathfrak{sp}_{2n}$. The results and the arguments turn out to be parallel to both the twisted Yangians and quantum affine algebras; cf. [8, Ch. 12] and [28, Sec. 3.5 and 4.3]. First we prove that every finite-dimensional irreducible representation of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ is a highest weight representation. Then we give necessary and sufficient conditions on the highest weight representations to be finite-dimensional. These conditions involve a family of polynomials $P_1(u), \dots, P_n(u)$ in u (analogues of the Drinfeld polynomials) so that the finite-dimensional irreducible representations are essentially parameterized by n-tuples $\big(P_1(u), \dots, P_n(u)\big)$. In the case of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ we give an explicit construction of all finite-dimensional irreducible representations as tensor products of the evaluation modules over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$.

An important ingredient in our arguments is the Poincaré–Birkhoff–Witt theorem for the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$, where q is a fixed nonzero complex number; see Corollaries 2.12 and 2.13 below. This allows us to derive a new proof of this theorem for the twisted q-Yangians $\mathrm{Y}'_q(\mathfrak{o}_N)$ and $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$; cf. [29]. A version of the PBW theorem in terms of the 'new realization' of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{a}})$ over the field of rational functions in q was given by Beck [3], with the case of $\widehat{\mathfrak{sl}}_2$ previously done by Damiani [10]; see also Hernandez [17] for a weak version of this theorem for the quantum affinizations of symmetrized quantum Kac–Moody algebras, where q is regarded as a nonzero complex number, not a root of unity. Although it is believed that the PBW theorem (in the strong form) holds for the quantum affine algebras over $\mathbb{C}$ where q is considered as a fixed nonzero complex number (with some additional conditions of the form $q^{2d_i} \neq 1$), a proof of the theorem appears to be unavailable in the literature; cf. [8, Prop. 12.2.2]. The existence of PBW type bases follows also from the general results of Kharchenko [22]. Our proof of the PBW theorem for $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ applies to the RTT-presentation of this algebra.

The general approach of this paper developed for the C type twisted q-Yangians should be applicable to the B and D types as well, although some additional arguments will be needed in order to obtain analogous classification theorems for representations of the algebras $\mathrm{Y}'_q(\mathfrak{o}_N)$; cf. [19] and [27].

We are grateful to David Hernandez and Mikhail Kotchetov for discussions of the Poincaré–Birkhoff–Witt theorem for the quantum affine algebras.

2 Poincaré–Birkhoff–Witt theorem

We start by reviewing and proving analogues of the PBW theorem for some quantum algebras. In particular, we prove it for the RTT presentation of the quantum affine algebra

$\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ and then use it to get a new proof of the theorem for the twisted q-Yangians.

2.1 Quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_N)$ and its representations

Fix a nonzero complex number q. Following [21] and [34], consider the R-matrix presentation of the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_N)$. The R-matrix is given by

$$R = q \sum_{i} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji} \tag{2.1}$$

which is an element of $\operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N$, where the E_{ij} denote the standard matrix units and the indices run over the set $\{1, \dots, N\}$. The R-matrix satisfies the Yang–Baxter equation

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}, \tag{2.2}$$

where both sides take values in $\operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N$ and the subscripts indicate the copies of $\operatorname{End} \mathbb{C}^N$, e.g., $R_{12} = R \otimes 1$ etc.

The algebra $\mathrm{U}_q(\mathfrak{gl}_N)$ is generated by elements t_{ij} and $\bar{t}_{ij}$ with $1 \leqslant i, j \leqslant N$ subject to the relations

$$\begin{gathered}
t_{ij} = \bar{t}_{ji} = 0, \qquad 1 \leqslant i < j \leqslant N, \\
t_{ii}\, \bar{t}_{ii} = \bar{t}_{ii}\, t_{ii} = 1, \qquad 1 \leqslant i \leqslant N, \\
R\, T_1 T_2 = T_2 T_1 R, \qquad R\, \overline{T}_1 \overline{T}_2 = \overline{T}_2 \overline{T}_1 R, \qquad R\, \overline{T}_1 T_2 = T_2 \overline{T}_1 R.
\end{gathered} \tag{2.3}$$

Here T and $\overline{T}$ are the matrices

$$T = \sum_{i,j} t_{ij} \otimes E_{ij}, \qquad \overline{T} = \sum_{i,j} \bar{t}_{ij} \otimes E_{ij}, \tag{2.4}$$

which are regarded as elements of the algebra $\mathrm{U}_q(\mathfrak{gl}_N) \otimes \operatorname{End} \mathbb{C}^N$. Both sides of each of the R-matrix relations in (2.3) are elements of $\mathrm{U}_q(\mathfrak{gl}_N) \otimes \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N$ and the subscripts of T and $\overline{T}$ indicate the copies of $\operatorname{End} \mathbb{C}^N$ where T or $\overline{T}$ acts; e.g. $T_1 = T \otimes 1$. In terms of the generators the defining relations between the t_{ij} can be written as

$$q^{\delta_{ij}}\, t_{ia}\, t_{jb} - q^{\delta_{ab}}\, t_{jb}\, t_{ia} = (q - q^{-1})\, (\delta_{b<a} - \delta_{i<j})\, t_{ja}\, t_{ib} \tag{2.5}$$

where $\delta_{i<j}$ equals 1 if $i < j$ and 0 otherwise. The relations between the $\bar{t}_{ij}$ are obtained by replacing t_{ij} by $\bar{t}_{ij}$ everywhere in (2.5):

$$q^{\delta_{ij}}\, \bar{t}_{ia}\, \bar{t}_{jb} - q^{\delta_{ab}}\, \bar{t}_{jb}\, \bar{t}_{ia} = (q - q^{-1})\, (\delta_{b<a} - \delta_{i<j})\, \bar{t}_{ja}\, \bar{t}_{ib}, \tag{2.6}$$

while the relations involving both t_{ij} and $\bar{t}_{ij}$ have the form

$$q^{\delta_{ij}}\, \bar{t}_{ia}\, t_{jb} - q^{\delta_{ab}}\, t_{jb}\, \bar{t}_{ia} = (q - q^{-1})\, (\delta_{b<a}\, t_{ja}\, \bar{t}_{ib} - \delta_{i<j}\, \bar{t}_{ja}\, t_{ib}). \tag{2.7}$$

Note that for any nonzero complex number d the mapping

$$t_{ij} \mapsto d\, t_{ij}, \qquad \bar{t}_{ij} \mapsto d^{-1}\, \bar{t}_{ij} \tag{2.8}$$

defines an automorphism of the algebra $\mathrm{U}_q(\mathfrak{gl}_N)$.

Let z denote an indeterminate. Introduce the algebra $\mathrm{U}_z(\mathfrak{gl}_N)$ over $\mathbb{C}(z)$ with the generators t_{ij} and $\bar{t}_{ij}$ with $1 \leqslant i, j \leqslant N$ subject to the relations given in (2.3) with q replaced by z. Furthermore, we denote by $\mathrm{U}^{\circ}_z(\mathfrak{gl}_N)$ the algebra defined over the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$ with the same set of generators and relations. Then we have the isomorphism

$$\mathrm{U}^{\circ}_z(\mathfrak{gl}_N) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \mathrm{U}_q(\mathfrak{gl}_N), \tag{2.9}$$

where the $\mathbb{C}[z, z^{-1}]$-module $\mathbb{C}$ is defined via the evaluation of the Laurent polynomials at $z = q$.

The quantized enveloping algebras admit families of PBW bases depending on choices of reduced decompositions of the longest element of the Weyl group; see Lusztig [25]. In the A type such bases were previously constructed by Rosso [35] and Yamane [38]. These constructions use the Drinfeld–Jimbo presentation of the quantized enveloping algebras. In this presentation, the algebra $\mathrm{U}_z(\mathfrak{gl}_N)$ over $\mathbb{C}(z)$ is generated by the elements $t_1, \dots, t_N, t_1^{-1}, \dots, t_N^{-1}$, $e_1, \dots, e_{N-1}$ and $f_1, \dots, f_{N-1}$ subject to the defining relations

$$\begin{aligned}
t_i\, t_j = t_j\, t_i, &\qquad t_i\, t_i^{-1} = t_i^{-1}\, t_i = 1,\\
t_i\, e_j\, t_i^{-1} = e_j\, z^{\delta_{ij}-\delta_{i,j+1}}, &\qquad t_i\, f_j\, t_i^{-1} = f_j\, z^{-\delta_{ij}+\delta_{i,j+1}},\\
[e_i, f_j] = \delta_{ij}\, \frac{k_i - k_i^{-1}}{z - z^{-1}} &\qquad \text{with} \ \ k_i = t_i\, t_{i+1}^{-1},\\
[e_i, e_j] = [f_i, f_j] = 0 &\qquad \text{if} \ \ |i-j| > 1,\\
e_i^2\, e_j - (z + z^{-1})\, e_i\, e_j\, e_i + e_j\, e_i^2 = 0 &\qquad \text{if} \ \ |i-j| = 1,\\
f_i^2\, f_j - (z + z^{-1})\, f_i\, f_j\, f_i + f_j\, f_i^2 = 0 &\qquad \text{if} \ \ |i-j| = 1.
\end{aligned}$$

The root vectors can be defined inductively by

$$\begin{aligned}
e_{i,i+1} &= e_i, &\qquad e_{i+1,i} &= f_i,\\
e_{ij} &= e_{ip}\, e_{pj} - z\, e_{pj}\, e_{ip} &\qquad &\text{for} \ \ i < p < j,\\
e_{ij} &= e_{ip}\, e_{pj} - z^{-1}\, e_{pj}\, e_{ip} &\qquad &\text{for} \ \ i > p > j,
\end{aligned} \tag{2.10}$$

and the elements e_{ij} are independent of the choice of values of the index p.

An isomorphism between the two presentations of $\mathrm{U}_z(\mathfrak{gl}_N)$ is given by the formulas

$$t_{ii} \mapsto t_i, \qquad \bar{t}_{ii} \mapsto t_i^{-1}, \qquad \bar{t}_{ij} \mapsto -(z - z^{-1})\, e_{ij}\, t_i^{-1}, \qquad t_{ji} \mapsto (z - z^{-1})\, t_i\, e_{ji} \tag{2.11}$$

for $i < j$; see [11], [34]. We shall identify the corresponding elements of $\mathrm{U}_z(\mathfrak{gl}_N)$ via this isomorphism.

The quantized enveloping algebra $\mathrm{U}_z(\mathfrak{sl}_N)$ can be defined as the $\mathbb{C}(z)$-subalgebra of $\mathrm{U}_z(\mathfrak{gl}_N)$ generated by the elements k_i, k_i^{-1}, e_i, f_i for $i = 1, \dots, N-1$. Similarly, if q is a nonzero complex number such that $q^2 \neq 1$, then $\mathrm{U}_q(\mathfrak{sl}_N)$ can be defined as the subalgebra of $\mathrm{U}_q(\mathfrak{gl}_N)$ generated by the same elements.

We will be using the following form of the PBW theorem for the quantized enveloping algebra associated with $\mathfrak{gl}_N$.

Proposition 2.1. *The monomials*

$$t_{N,N-1}^{k_{N,N-1}}\, t_{N,N-2}^{k_{N,N-2}}\, t_{N-1,N-2}^{k_{N-1,N-2}} \dots t_{N2}^{k_{N2}} \dots t_{32}^{k_{32}}\, t_{N1}^{k_{N1}} \dots t_{21}^{k_{21}} \\ \times t_{11}^{l_1} \dots t_{NN}^{l_N}\, \bar{t}_{12}^{k_{12}} \dots \bar{t}_{1N}^{k_{1N}}\, \bar{t}_{23}^{k_{23}} \dots \bar{t}_{2N}^{k_{2N}} \dots \bar{t}_{N-1,N}^{k_{N-1,N}}, \tag{2.12}$$

where the k_{ij} run over non-negative integers and the l_i run over all integers, form a basis of the $\mathbb{C}[z, z^{-1}]$-algebra $\mathrm{U}_z^\circ(\mathfrak{gl}_N)$.

Proof. It follows easily from the defining relations of $\mathrm{U}_z^\circ(\mathfrak{gl}_N)$ that the monomials span the algebra over $\mathbb{C}[z, z^{-1}]$. Suppose now that there is a nontrivial linear combination of the monomials (2.12) with coefficients in $\mathbb{C}[z, z^{-1}]$ equal to zero. Applying the isomorphism (2.11) and the relations $t_i\, e_{jb} = z^{\delta_{ij}-\delta_{ib}}\, e_{jb}\, t_i$ we then obtain a nontrivial linear combination over $\mathbb{C}[z, z^{-1}]$ of the monomials

$$e_{N,N-1}^{k_{N,N-1}}\, e_{N,N-2}^{k_{N,N-2}}\, e_{N-1,N-2}^{k_{N-1,N-2}} \dots e_{N2}^{k_{N2}} \dots e_{32}^{k_{32}}\, e_{N1}^{k_{N1}} \dots e_{21}^{k_{21}} \\ \times t_1^{l_1} \dots t_N^{l_N}\, e_{12}^{k_{12}} \dots e_{1N}^{k_{1N}}\, e_{23}^{k_{23}} \dots e_{2N}^{k_{2N}} \dots e_{N-1,N}^{k_{N-1,N}} \tag{2.13}$$

equal to zero. Here the k_{ij} run over non-negative integers and the l_i run over all integers. However, by the PBW theorem for the Drinfeld–Jimbo presentation of the algebra $\mathrm{U}_z(\mathfrak{gl}_N)$ (see [25], [35], [38]), the monomials (2.13) form a basis of $\mathrm{U}_z(\mathfrak{gl}_N)$ over $\mathbb{C}(z)$. This makes a contradiction. □

The following corollary is immediate from the isomorphism (2.9).

Corollary 2.2. *Let q be a nonzero complex number. Then the monomials* (2.12) *form a basis of* $\mathrm{U}_q(\mathfrak{gl}_N)$ *over* $\mathbb{C}$. □

Note that in the particular case $q = 1$ the algebra $\mathrm{U}_1(\mathfrak{gl}_N)$ is commutative. Using Corollary 2.2 we will identify it with the algebra of polynomials $\mathcal{P}_N$ in the variables $\bar{x}_{ij}, x_{ji}$ with $1 \leqslant i < j \leqslant N$ and $x_{ii}, \bar{x}_{ii}$ with $i = 1, \dots, N$ subject to the relations $x_{ii}\bar{x}_{ii} = 1$ for all i. Thus, due to (2.9) we have the isomorphism

$$\mathrm{U}_z^\circ(\mathfrak{gl}_N) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \mathcal{P}_N, \tag{2.14}$$

where the $\mathbb{C}[z, z^{-1}]$-module $\mathbb{C}$ is defined via the evaluation of the Laurent polynomials at $z = 1$.

In the other degenerate case $q = -1$ the algebra $\mathrm{U}_{-1}(\mathfrak{gl}_N)$ is essentially a 'quasi-polynomial' algebra; see e.g. [9, Sec. 1.8]. It is well known that quasi-polynomial algebras admit PBW bases.

We will also use an extended version of the quantized enveloping algebra considered in [27]. Denote by $\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N)$ the algebra over $\mathbb{C}[z, z^{-1}]$ generated by elements t_{ij} and $\bar{t}_{ij}$ with $1 \leqslant i, j \leqslant N$ and elements t_{ii}^{-1} and $\bar{t}_{ii}^{-1}$ with $1 \leqslant i \leqslant N$ subject to the relations

$$
\begin{aligned}
&t_{ij} = \bar{t}_{ji} = 0, \qquad 1 \leqslant i < j \leqslant N, \\
&t_{ii}\,\bar{t}_{ii} = \bar{t}_{ii}\,t_{ii}, \qquad t_{ii}\,t_{ii}^{-1} = t_{ii}^{-1}\,t_{ii} = 1, \qquad \bar{t}_{ii}\,\bar{t}_{ii}^{-1} = \bar{t}_{ii}^{-1}\,\bar{t}_{ii} = 1, \qquad 1 \leqslant i \leqslant N, \\
&R\,T_1T_2 = T_2T_1R, \qquad R\,\overline{T}_1\overline{T}_2 = \overline{T}_2\overline{T}_1R, \qquad R\,\overline{T}_1T_2 = T_2\overline{T}_1R,
\end{aligned}
\tag{2.15}
$$

where we use the notation of (2.3) with q replaced by z in the definition of R. Although we use the same notation for the generators of the algebras $\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N)$ and $\mathrm{U}_z^{\circ}(\mathfrak{gl}_N)$, it should always be clear from the context which algebra is considered at any time. There is a natural epimorphism $\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N) \to \mathrm{U}_z^{\circ}(\mathfrak{gl}_N)$ which takes the generators t_{ij} and $\bar{t}_{ij}$ to the elements with the same name. The kernel K of this epimorphism is the two-sided ideal of the algebra $\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N)$ generated by the elements $t_{ii}\,\bar{t}_{ii} - 1$ for $i = 1, \dots, N$. All these elements are central in this algebra and we have the isomorphism $\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N)/K \cong \mathrm{U}_z^{\circ}(\mathfrak{gl}_N)$.

The following analogue of the PBW theorem is implied by Proposition 2.1; see also [27].

Proposition 2.3. *The monomials*

$$
\begin{aligned}
t_{N,N-1}^{k_{N,N-1}}\, t_{N,N-2}^{k_{N,N-2}}\, t_{N-1,N-2}^{k_{N-1,N-2}} \dots t_{N2}^{k_{N2}} \dots t_{32}^{k_{32}}\, t_{N1}^{k_{N1}} \dots t_{21}^{k_{21}}\, t_{11}^{l_1} \dots t_{NN}^{l_N} \\
\times\, \bar{t}_{11}^{m_1} \dots \bar{t}_{NN}^{m_N}\, \bar{t}_{12}^{k_{12}} \dots \bar{t}_{1N}^{k_{1N}}\, \bar{t}_{23}^{k_{23}} \dots \bar{t}_{2N}^{k_{2N}} \dots \bar{t}_{N-1,N}^{k_{N-1,N}},
\end{aligned}
\tag{2.16}
$$

where the k_{ij} run over non-negative integers and the l_i and m_i run over all integers, form a basis of the $\mathbb{C}[z, z^{-1}]$-algebra $\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N)$. □

By specializing z to a nonzero complex number q in the definition of $\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N)$ we obtain an algebra $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$ over $\mathbb{C}$ defined by the same set of relations (2.15). So we have the isomorphism

$$
\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N),
\tag{2.17}
$$

where the $\mathbb{C}[z, z^{-1}]$-module $\mathbb{C}$ is defined via the evaluation of the Laurent polynomials at $z = q$. The corresponding monomials (2.16) form a basis of $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$. In the particular case $q = 1$ the algebra $\mathrm{U}_1^{\mathrm{ext}}(\mathfrak{gl}_N)$ can be identified with the algebra of polynomials $\mathcal{P}_N^{\mathrm{ext}}$ in the variables $\bar{x}_{ij}, x_{ji}$ with $1 \leqslant i < j \leqslant N$ and $x_{ii}, x_{ii}^{-1}, \bar{x}_{ii}, \bar{x}_{ii}^{-1}$ with $i = 1, \dots, N$. Thus we have the isomorphism

$$
\mathrm{U}_z^{\mathrm{ext}}(\mathfrak{gl}_N) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \mathcal{P}_N^{\mathrm{ext}},
\tag{2.18}
$$

where the $\mathbb{C}[z,z^{-1}]$-module $\mathbb{C}$ is defined via the evaluation of the Laurent polynomials at $z=1$.

Suppose now that q is a nonzero complex number which is not a root of unity. A description of finite-dimensional irreducible representations of the algebra $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$ can be easily obtained from the corresponding results for the algebras $\mathrm{U}_q(\mathfrak{gl}_N)$ and $\mathrm{U}_q(\mathfrak{sl}_N)$; see e.g. [8, Ch. 10]. A representation L of $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$ is called a *highest weight representation* if L is generated by a nonzero vector ζ (the *highest vector*) such that

$$\begin{aligned} &\bar{t}_{ij}\,\zeta = 0 \qquad && \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}\\ &t_{ii}\,\zeta = \lambda_i\zeta, \qquad && \bar{t}_{ii}\,\zeta = \bar{\lambda}_i\zeta, \qquad \text{for} \quad 1 \leqslant i \leqslant N, \end{aligned}$$

for some nonzero complex numbers λ_i and $\bar{\lambda}_i$. The tuple $(\lambda_1,\dots,\lambda_N;\bar{\lambda}_1,\dots,\bar{\lambda}_N)$ is called the *highest weight* of L. Due to Proposition 2.3, for any N-tuples of nonzero complex numbers $\lambda=(\lambda_1,\dots,\lambda_N)$ and $\bar{\lambda}=(\bar{\lambda}_1,\dots,\bar{\lambda}_N)$, there exists an irreducible highest representation $L(\lambda;\bar{\lambda})$ with the highest weight $(\lambda;\bar{\lambda})$. This representation can be defined as a quotient of the corresponding Verma module in a standard way.

The irreducible highest weight representations $L(\mu)$, $\mu=(\mu_1,\dots,\mu_N)$, over the algebra $\mathrm{U}_q(\mathfrak{gl}_N)$ are defined in a similar way with the above conditions on the highest vector replaced by

$$\begin{aligned} &\bar{t}_{ij}\,\zeta = 0 \qquad && \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}\\ &t_{ii}\,\zeta = \mu_i\zeta, \qquad && \text{for} \quad 1 \leqslant i \leqslant N. \end{aligned}$$

The representation $L(\mu)$ is finite-dimensional if and only if there exist nonnegative integers m_i satisfying $m_1 \geqslant m_2 \geqslant \dots \geqslant m_N$, elements $\varepsilon_i \in \{-1,1\}$ for $i=1,\dots,N$, and a nonzero complex number d such that

$$\mu_i = d\,\varepsilon_i\,q^{m_i}, \qquad i=1,\dots,N.$$

Proposition 2.4. *Every finite-dimensional irreducible representation of* $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$ *is isomorphic to a highest weight representation* $L(\lambda;\bar{\lambda})$ *such that*

$$\lambda_i - a\,q^{2m_i}\bar{\lambda}_i = 0, \qquad i=1,\dots,N,$$

for some nonnegative integers m_i *satisfying* $m_1 \geqslant m_2 \geqslant \dots \geqslant m_N$ *and a nonzero complex number* a.

Proof. A standard argument shows that every finite-dimensional irreducible representation of $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$ is isomorphic to a highest weight representation; cf. [8, Ch. 10]. Hence we only need to determine when the representation $L(\lambda;\bar{\lambda})$ is finite-dimensional. Each central element $t_{ii}\bar{t}_{ii}$ of $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$ acts on $L(\lambda;\bar{\lambda})$ as multiplication by the scalar $\lambda_i\bar{\lambda}_i$. Fix constants $c_1,\dots,c_N$ such that

$$c_i^2 = \lambda_i\bar{\lambda}_i, \qquad i=1,\dots,N.$$

Then the mapping

$$t_{ij} \mapsto c_i\, t_{ij}, \qquad \bar{t}_{ij} \mapsto c_i\, \bar{t}_{ij}$$

defines an epimorphism $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N) \to \mathrm{U}_q(\mathfrak{gl}_N)$ whose kernel is generated by the elements

$$t_{ii}\bar{t}_{ii} - \lambda_i\bar{\lambda}_i, \qquad i = 1, \dots, N. \tag{2.19}$$

Hence, identifying $\mathrm{U}_q(\mathfrak{gl}_N)$ with the quotient of $\mathrm{U}_q^{\mathrm{ext}}(\mathfrak{gl}_N)$ by this kernel, we can equip $L(\lambda; \bar{\lambda})$ with the structure of an irreducible highest weight representation of $\mathrm{U}_q(\mathfrak{gl}_N)$. Its highest weight $(\mu_1, \dots, \mu_N)$ is given by

$$\mu_i = c_i^{-1}\, \lambda_i, \qquad i = 1, \dots, N.$$

This representation is finite-dimensional if and only if

$$c_i^{-1}\, \lambda_i = d\, \varepsilon_i\, q^{m_i}$$

for some nonnegative integers m_i satisfying $m_1 \geqslant m_2 \geqslant \dots \geqslant m_N$, a nonzero complex number d, and some elements $\varepsilon_i \in \{-1, 1\}$ for $i = 1, \dots, N$. By our choice of the constants c_i, this is equivalent to the relations $\lambda_i\, \bar{\lambda}_i^{-1} = a\, q^{2m_i}$ with $a = d^2$, as required. $\square$

2.2 Twisted quantized enveloping algebras $\mathrm{U}'_q(\mathfrak{o}_N)$ and $\mathrm{U}'_q(\mathfrak{sp}_{2n})$

The twisted quantized enveloping algebra $\mathrm{U}'_q(\mathfrak{o}_N)$ associated with the orthogonal Lie algebra $\mathfrak{o}_N$ was introduced independently in [15] and [30]. Its R-matrix presentation was given in [31]. We follow the notation of [29] and define $\mathrm{U}'_q(\mathfrak{o}_N)$ as the subalgebra of $\mathrm{U}_q(\mathfrak{gl}_N)$ generated by the matrix elements s_{ij} of the matrix $S = T\,\overline{T}^{\,t}$, where t denotes the usual matrix transposition. More explicitly, the elements s_{ij} are given by

$$s_{ij} = \sum_{a=1}^{N} t_{ia}\bar{t}_{ja}. \tag{2.20}$$

Hence, (2.3) implies

$$s_{ij} = 0, \qquad 1 \leqslant i < j \leqslant N, \tag{2.21}$$

$$s_{ii} = 1, \qquad 1 \leqslant i \leqslant N. \tag{2.22}$$

Furthermore, $\mathrm{U}'_q(\mathfrak{o}_N)$ is isomorphic to the algebra with (abstract) generators s_{ij} with the condition $i, j \in \{1, \dots, N\}$ subject to the defining relations (2.21), (2.22) and

$$R\, S_1 R^{t_1} S_2 = S_2 R^{t_1} S_1 R, \tag{2.23}$$

where

$$R^{t_1} = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ji} \otimes E_{ji}. \tag{2.24}$$

In terms of the generators, relation (2.23) takes the form

$$\begin{aligned} q^{\delta_{aj}+\delta_{ij}}\, s_{ia}\, s_{jb} - q^{\delta_{ab}+\delta_{ib}}\, s_{jb}\, s_{ia} &= (q-q^{-1})\, q^{\delta_{ai}}\, (\delta_{b<a} - \delta_{i<j})\, s_{ja}\, s_{ib} \\ &+ (q-q^{-1})\, \big(q^{\delta_{ab}}\, \delta_{b<i}\, s_{ji}\, s_{ba} - q^{\delta_{ij}}\, \delta_{a<j}\, s_{ij}\, s_{ab}\big) \\ &+ (q-q^{-1})^2\, (\delta_{b<a<i} - \delta_{a<i<j})\, s_{ji}\, s_{ab}, \end{aligned} \tag{2.25}$$

where $\delta_{i<j}$ or $\delta_{i<j<k}$ equals 1 if the subscript inequality is satisfied and 0 otherwise.

An analogue of the PBW theorem for the algebra $\mathrm{U}'_q(\mathfrak{o}_N)$ was proved in [18]; see also [27], [29]. Yet another proof is obtained from Proposition 2.1. We regard q as a nonzero complex number.

Proposition 2.5. *The monomials*

$$s_{21}^{k_{21}}\, s_{32}^{k_{32}}\, s_{31}^{k_{31}} \dots s_{N1}^{k_{N1}}\, s_{N2}^{k_{N2}} \dots s_{N,N-1}^{k_{N,N-1}}, \tag{2.26}$$

where the k_{ij} run over non-negative integers form a basis of the algebra $\mathrm{U}'_q(\mathfrak{o}_N)$.

Proof. Let us consider the $\mathbb{C}[z,z^{-1}]$-subalgebra $\mathrm{U}^\circ_z(\mathfrak{o}_N)$ of $\mathrm{U}^\circ_z(\mathfrak{gl}_N)$ generated by the elements s_{ij} defined by (2.20) and show that the monomials (2.26) form its basis. It follows easily from the defining relations that the monomials span the algebra; see [29, Lemma 2.1]. Suppose now that a nontrivial $\mathbb{C}[z,z^{-1}]$-linear combination of the monomials (2.26) is zero. By Proposition 2.1 we may suppose that at least one coefficient of the combination does not vanish at $z=1$. Using the isomorphism (2.14) we then get a nontrivial $\mathbb{C}$-linear combination of the corresponding monomials in the polynomial algebra $\mathcal{P}_N$. We will come to a contradiction if we show that the images σ_{ij} of the generators of s_{ij}, $i>j$, in $\mathcal{P}_N$ are algebraically independent.

We have

$$\sigma_{ij} = \sum_{a=1}^{N} x_{ia}\, \bar{x}_{ja}.$$

It suffices to verify that the differentials $d\sigma_{ij}$ are linearly independent. Calculate the differentials in terms of dx_{ia} and $d\bar{x}_{ia}$ and specialize the coefficient matrix by setting $x_{ij} = \bar{x}_{ij} = \delta_{ij}$. Then $d\sigma_{ij} = d\bar{x}_{ji} + dx_{ij}$ which implies that the differentials $d\sigma_{ij}$ are linearly independent even under the specialization.

This proves that the monomials (2.26) form a basis over $\mathbb{C}[z,z^{-1}]$ in the subalgebra $\mathrm{U}^\circ_z(\mathfrak{o}_N)$. The application of the isomorphism (2.9) shows that the monomials (2.26) form a basis over $\mathbb{C}$ in $\mathrm{U}'_q(\mathfrak{o}_N)$. $\square$

Finite-dimensional irreducible representations of the algebra $\mathrm{U}'_q(\mathfrak{o}_N)$ were classified in [19]. Moreover, that paper also contains explicit realization of the representations of 'classical type' via Gelfand–Tsetlin bases.

The twisted quantized enveloping algebra $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ associated with the symplectic Lie algebra $\mathfrak{sp}_{2n}$ was first introduced in [31]. In order to define it, consider the $2n \times 2n$ matrix G given by

$$G = q \sum_{k=1}^{n} E_{2k-1,2k} - \sum_{k=1}^{n} E_{2k,2k-1}, \tag{2.27}$$

that is,

$$G = \begin{bmatrix} 0 & q & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

We define $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ as the subalgebra of $\mathrm{U}^{\mathrm{ext}}_q(\mathfrak{gl}_{2n})$ generated by the matrix elements s_{ij} of the matrix $S = T\, G\, \overline{T}^{\, t}$ together with the elements

$$s^{-1}_{i,i+1} = q^{-1}\, t^{-1}_{ii}\, \bar{t}^{\,-1}_{i+1,i+1} \tag{2.28}$$

for $i = 1, 3, \ldots, 2n-1$. More explicitly,

$$s_{ij} = q \sum_{a=1}^{n} t_{i,2a-1} \bar{t}_{j,2a} - \sum_{a=1}^{n} t_{i,2a} \bar{t}_{j,2a-1}. \tag{2.29}$$

By (2.3) we have

$$s_{ij} = 0 \quad \text{for} \quad i < j \quad \text{unless} \quad j = i+1 \quad \text{with} \quad i \ \text{ odd}. \tag{2.30}$$

All matrix elements $\bar{s}_{ij}$ of the matrix $\overline{S} = \overline{T}\, G\, T^{\, t}$ also belong to the subalgebra. It was proved in [29] that $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ is isomorphic to the algebra with (abstract) generators s_{ij} with $i, j \in \{1, \ldots, 2n\}$ and $s^{-1}_{i,i+1}$ with $i = 1, 3, \ldots, 2n-1$, subject to the defining relations (2.23) (with $N = 2n$), (2.30) and

$$s_{i,i+1}\, s^{-1}_{i,i+1} = s^{-1}_{i,i+1}\, s_{i,i+1} = 1, \qquad i = 1, 3, \ldots, 2n-1. \tag{2.31}$$

Our definition of $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ follows closer the original paper [31], while a slightly different algebra $\mathrm{U}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$ was studied in [29]. The latter was defined as a subalgebra of $\mathrm{U}_q(\mathfrak{gl}_{2n})$ by the same formulas (2.28) and (2.29) which lead to extra relations for the generators: for any odd i

$$s_{i+1,i+1}\, s_{ii} - q^2\, s_{i+1,i}\, s_{i,i+1} = q^3. \tag{2.32}$$

They are implied by the relations $t_{ii}\, \bar{t}_{ii} = 1$ which hold in the algebra $\mathrm{U}_q(\mathfrak{gl}_{2n})$ but not in $\mathrm{U}^{\mathrm{ext}}_q(\mathfrak{gl}_{2n})$. Moreover, the elements $s_{i+1,i+1}\, s_{ii} - q^2\, s_{i+1,i}\, s_{i,i+1}$ are central in the algebra $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ and its quotient by the relations (2.32) is isomorphic to $\mathrm{U}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$; see also [27], where a slightly different notation was used. The latter algebra is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{sp}_{2n})$; see [29].

An analogue of the PBW theorem for the algebra $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ was proved in [27]. That paper and [29] also contain proofs of this theorem for the quotient algebra $\mathrm{U}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$. Here we give a more direct proof based on Proposition 2.3 for a slightly different order on the set of generators.

We define a function $\varsigma : \{1, 2, \dots, 2n\} \to \{\pm 1, \pm 3, \dots, \pm(2n-1)\}$ by

$$\varsigma(i) = \begin{cases} \;\; i & \text{if } \; i \; \text{ is odd,} \\ -i+1 & \text{if } \; i \; \text{ is even.} \end{cases} \tag{2.33}$$

We say $s_{ia} < s_{jb}$ if $\big(\varsigma(i) + \varsigma(a), \varsigma(i)\big) < \big(\varsigma(j) + \varsigma(b), \varsigma(j)\big)$ when ordered lexicographically.

In the next proposition we consider the corresponding ordered monomials in the generators s_{ij} with $i \geqslant j$ together with $s_{i,i+1}$ and $s^{-1}_{i,i+1}$ with odd i.

Proposition 2.6. *The ordered monomials*

$$s_{2n,2n}^{k_{2n,2n}}\, s_{2n,2n-2}^{k_{2n,2n-2}} \dots s_{2n,2n-1}^{k_{2n,2n-1}} \dots s_{21}^{k_{21}}\, s_{12}^{k_{12}} \dots s_{2n-1,2n}^{k_{2n-1,2n}} \dots s_{2n-1,2n-3}^{k_{2n-1,2n-3}}\, s_{2n-1,2n-1}^{k_{2n_1,2n-1}}, \tag{2.34}$$

where $k_{12}, k_{34}, \dots, k_{2n-1,2n}$ run over all integers and the remaining powers k_{ij} run over non-negative integers, form a basis of the algebra $\mathrm{U}'_q(\mathfrak{sp}_{2n})$.

Proof. Let us consider the $\mathbb{C}[z, z^{-1}]$-subalgebra $\mathrm{U}^{\circ}_z(\mathfrak{sp}_{2n})$ of $\mathrm{U}^{\mathrm{ext}}_z(\mathfrak{gl}_{2n})$ generated by the elements (2.28) and (2.29) with q replaced by z and show that the monomials (2.34) form its basis. The application of the isomorphism (2.17) will then imply that the monomials form a basis over $\mathbb{C}$ in $\mathrm{U}'_q(\mathfrak{sp}_{2n})$.

First we prove that an arbitrary monomial in the generators can be written as a linear combination of the ordered monomials; cf. [29, Lemma 2.1]. Due to the relations

$$s_{i,i+1}\, s_{kl} = z^{\delta_{ik} + \delta_{il} - \delta_{i+1,k} - \delta_{i+1,l}}\, s_{kl}\, s_{i,i+1}, \qquad i = 1, 3, \dots, 2n-1,$$

we can restrict our attention to those monomials where all generators occur in non-negative powers. We define the *degree* of a monomial $s_{i_1 a_1} \dots s_{i_p a_p}$, to be $d = i_1 + \dots + i_k$ and we argue by induction on the degree d. Modulo products of degree less than $i+j$, the relations (2.25) (with q replaced by z) imply:

$$\begin{aligned} & z^{\delta_{aj} + \delta_{ij}}\, s_{ia} s_{jb} - z^{\delta_{ab} + \delta_{ib}}\, s_{jb} s_{ia} \\ & \quad \equiv \;\; (z - z^{-1})\, z^{\delta_{ai}} (\delta_{b<a} - \delta_{i<j})\, s_{ja} s_{ib}. \end{aligned} \tag{2.35}$$

Swapping here i with j and a with b we can also write this in the form

$$\begin{aligned} & z^{\delta_{aj} + \delta_{ab}}\, s_{ia} s_{jb} - z^{\delta_{ij} + \delta_{ib}}\, s_{jb} s_{ia} \\ & \quad \equiv \;\; (z^{-1} - z)\, z^{\delta_{bj}} (\delta_{a<b} - \delta_{j<i})\, s_{ib} s_{ja}. \end{aligned} \tag{2.36}$$

Suppose $\varsigma(i)+\varsigma(a)>\varsigma(j)+\varsigma(b)$. Then if $\varsigma(i)+\varsigma(b)>\varsigma(j)+\varsigma(a)$, the equation (2.35) allows us to write $s_{ia}s_{jb}$ as a linear combination of ordered monomials and monomials of lower degree. On the other hand, if $\varsigma(i)+\varsigma(b)<\varsigma(j)+\varsigma(a)$ then the same outcome is achieved by using (2.36). In the case $\varsigma(i)+\varsigma(b)=\varsigma(j)+\varsigma(a)$ we have either $\varsigma(i)>\varsigma(j)$ or $\varsigma(i)<\varsigma(j)$ and we use (2.35) or (2.36), respectively; the equality $\varsigma(i)=\varsigma(j)$ is impossible as it would imply $\varsigma(i)+\varsigma(a)=\varsigma(j)+\varsigma(b)$.

Now suppose that we have a pair of generators s_{ia}, s_{jb} such that $\varsigma(i)+\varsigma(a)=\varsigma(j)+\varsigma(b)$, and that $\varsigma(i)>\varsigma(j)$. Then $\varsigma(a)<\varsigma(b)$, and so

$$\varsigma(i)+\varsigma(b)>\varsigma(j)+\varsigma(a).$$

This means that by applying (2.35), we can write $s_{ia}s_{jb}$ as a linear combination of ordered monomials. Thus, given an arbitrary monomial, we may rearrange each pair of generators in turn to write the monomial as a linear combination of ordered monomials and monomials of lower degree.

Suppose now that a nontrivial $\mathbb{C}[z,z^{-1}]$-linear combination of the monomials (2.34) is zero. By Proposition 2.3 we may suppose that at least one coefficient of the combination does not vanish at $z=1$. Using the isomorphism (2.18) we then get a nontrivial $\mathbb{C}$-linear combination of the corresponding monomials in the polynomial algebra $\mathcal{P}_{2n}^{\text{ext}}$.

Let σ_{ij} denote the image of s_{ij} in $\mathcal{P}_{2n}^{\text{ext}}$. Hence

$$\sigma_{ij}=\sum_{a=1}^{n}\big(x_{i,2a-1}\,\bar{x}_{j,2a}-x_{i,2a}\,\bar{x}_{j,2a-1}\big).$$

It suffices to verify that the polynomials σ_{ij} with $i\geqslant j$ and $\sigma_{i,i+1}$ with odd i are algebraically independent in $\mathcal{P}_{2n}^{\text{ext}}$. Calculate their differentials in terms of dx_{ia} and $d\bar{x}_{ia}$ and specialize the coefficient matrix by setting $x_{ij}=\bar{x}_{ij}=\delta_{ij}$. Then

$$d\sigma_{ij}=\begin{cases} d\bar{x}_{j,i+1}+dx_{i,j-1} & \text{if } \ i \ \text{ is odd,} \quad j \ \text{ is even,}\\ d\bar{x}_{j,i+1}-dx_{i,j+1} & \text{if } \ i \ \text{ is odd,} \quad j \ \text{ is odd,}\\ -d\bar{x}_{j,i-1}+dx_{i,j-1} & \text{if } \ i \ \text{ is even,} \quad j \ \text{ is even,}\\ -d\bar{x}_{j,i-1}-dx_{i,j+1} & \text{if } \ i \ \text{ is even,} \quad j \ \text{ is odd,}\end{cases}$$

so that the differentials are linearly independent. □

Finally, for the use in the next sections we reproduce the classification theorem for finite-dimensional irreducible representations of the algebra $\mathrm{U}'_q(\mathfrak{sp}_{2n})$. This theorem was proved in [27] for the quotient $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$ of this algebra by the relations (2.32), and it is not difficult to get the corresponding results for the algebra $\mathrm{U}'_q(\mathfrak{sp}_{2n})$. For the rest of this section we suppose that q is a nonzero complex number which is not a root of unity.

A representation V of $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ is called a *highest weight representation* if V is generated by a nonzero vector ξ (the *highest vector*) such that

$$\begin{aligned} s_{ij}\,\xi &= 0 \qquad &&\text{for}\quad i=1,3,\dots,2n-1,\quad j=1,2,\dots,i, \qquad \text{and}\\ s_{2i,2i-1}\,\xi &= \mu_i\,\xi, \qquad && s_{2i-1,2i}\,\xi = \mu_i'\,\xi, \qquad \text{for}\quad i=1,2,\dots,n, \end{aligned}$$

for some complex numbers μ_i and μ_i'. The numbers μ_i' have to be nonzero due to the relation (2.31). The tuple $(\mu_1,\dots,\mu_n;\mu_1',\dots,\mu_n')$ is called the *highest weight* of V.

Due to the PBW theorem for the algebra $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ (Proposition 2.6), given any two n-tuples of complex numbers $\mu=(\mu_1,\dots,\mu_n)$ and $\mu'=(\mu_1',\dots,\mu_n')$, where all μ_i' are nonzero, there exists an irreducible highest weight representation $V(\mu;\mu')$ with the highest weight $(\mu;\mu')$. It is defined as the unique irreducible quotient of the corresponding Verma module $M(\mu;\mu')$; cf. [27]. By definition, $M(\mu;\mu')$ is the quotient of $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ by the left ideal generated by the elements s_{ij} with $i=1,3,\dots,2n-1$, $j=1,2,\dots,i$, and by $s_{2i,2i-1}-\mu_i$, $s_{2i-1,2i}-\mu_i'$ with $i=1,\dots,n$.

Proposition 2.7. *Every finite-dimensional irreducible representation of* $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ *is isomorphic to a highest weight representation* $V(\mu;\mu')$ *such that*

$$\mu_i' + q^{2p_i+1}\mu_i = 0, \qquad i=1,\dots,n,$$

for some nonnegative integers p_i *satisfying* $p_1\leqslant p_2\leqslant\dots\leqslant p_n$.

Proof. A standard argument as in [27] shows that every finite-dimensional irreducible representation of $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ is isomorphic to $V(\mu;\mu')$ for certain μ and μ'. In order to find out when an irreducible highest weight representation $V(\mu;\mu')$ is finite-dimensional, consider first the case $n=1$. Let $M(\mu_1;\mu_1')$ be the Verma module over $\mathrm{U}_q'(\mathfrak{sp}_2)$ with the highest vector ξ. The vectors $s_{22}^k\xi$ with $k\geqslant 0$ form a basis of $M(\mu_1;\mu_1')$. The central element $s_{22}s_{11}-q^2s_{21}s_{12}$ acts on $M(\mu_1;\mu_1')$ as multiplication by the scalar $-q^2\mu_1\mu_1'$. Hence using the defining relations (2.25) we derive by induction on k that

$$s_{11}s_{22}^k\xi = \big(q^{-2k}-1\big)\big(q^2\mu_1\mu_1' + (\mu_1')^2q^{3-2k}\big)\, s_{22}^{k-1}\xi.$$

Since $\mu_1'\neq 0$, this implies that if $\mu_1=0$ then $M(\mu_1;\mu_1')$ is irreducible and so the representation $V(\mu_1;\mu_1')$ is infinite-dimensional.

By embedding $\mathrm{U}_q'(\mathfrak{sp}_2)$ into $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ as the subalgebra generated by the elements $s_{2i-1,2i}$, $s_{2i-1,2i}^{-1}$, $s_{2i,2i-1}$, $s_{2i-1,2i-1}$ and $s_{2i,2i}$ for $i\in\{1,\dots,n\}$, we can conclude that if the representation $V(\mu;\mu')$ of $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ is finite-dimensional, then all components μ_i must be nonzero. Furthermore, each central element $s_{2i,2i}\,s_{2i-1,2i-1}-q^2\,s_{2i,2i-1}\,s_{2i-1,2i}$ acts in $V(\mu;\mu')$ as multiplication by the nonzero scalar $-q^2\mu_i\mu_i'$.

On the other hand, the quotient of $\mathrm{U}_q'(\mathfrak{sp}_{2n})$ by the ideal generated by the elements

$$s_{2i,2i}\,s_{2i-1,2i-1}-q^2\,s_{2i,2i-1}\,s_{2i-1,2i}+q^2\mu_i\mu_i', \qquad i=1,\dots,n, \tag{2.37}$$

is isomorphic to the algebra $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$. Indeed, the mapping $s_{ij} \mapsto c_i c_j s_{ij}$ for nonzero scalars $c_1, \dots, c_{2n}$ such that

$$q^2 \mu_i \mu_i' = -q^3\, c_{2i-1}^2 c_{2i}^2, \qquad i = 1, \dots, n,$$

defines an epimorphism $\mathrm{U}_q'(\mathfrak{sp}_{2n}) \to \mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$ whose kernel is generated by the elements (2.37). Thus, $V(\mu;\mu')$ becomes an irreducible highest weight representation of the algebra $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$ whose highest weight $\lambda = (\lambda_1, \lambda_3, \dots, \lambda_{2n-1})$ in the notation of [27, Sec. 4] is found by

$$\lambda_{2i-1} = c_{2i-1}^{-1} c_{2i}^{-1}\, \mu_i', \qquad i = 1, \dots, n.$$

This implies $\lambda_{2i-1}^2 = -q\mu_i'\, \mu_i^{-1}$. By [27, Theorem 6.3] we must have

$$\lambda_{2i-1}^2 = q^{2m_i}, \qquad i = 1, \dots, n,$$

for some positive integers m_i satisfying $m_1 \leqslant m_2 \leqslant \dots \leqslant m_n$. This gives the desired conditions on the highest weight $(\mu;\mu')$. □

Remark 2.8. If $q^2 \neq 1$ then the algebra $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_2)$ is isomorphic to $\mathrm{U}_q(\mathfrak{sl}_2)$. An isomorphism can be given by

$$k \mapsto q^{-1} s_{12}, \qquad e \mapsto \frac{s_{11}}{q^3 - q}, \qquad f \mapsto \frac{s_{12}^{-1}\, s_{22}}{1 - q^2},$$

where e, f, k, k^{-1} are the standard generators of $\mathrm{U}_q(\mathfrak{sl}_2)$ satisfying the relations

$$ke = q^2 ek, \qquad kf = q^{-2} fk, \qquad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

This isomorphism can be used to get a description of finite-dimensional irreducible representations of $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_2)$; cf. [27]. □

2.3 Quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$

We start by recalling some well-known facts about the quantum affine algebra (or quantum loop algebra) associated with $\mathfrak{gl}_N$. We will keep the notation q for a fixed nonzero complex number. Consider the Lie algebra of Laurent polynomials $\mathfrak{gl}_N[\lambda, \lambda^{-1}]$ in an indeterminate λ. We denote it by $\widehat{\mathfrak{gl}}_N$ for brevity. The *quantum affine algebra* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ (with the trivial central charge) has countably many generators $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$ where $1 \leqslant i, j \leqslant N$ and r runs over nonnegative integers. They are combined into the matrices

$$T(u) = \sum_{i,j=1}^{N} t_{ij}(u) \otimes E_{ij}, \qquad \overline{T}(u) = \sum_{i,j=1}^{N} \bar{t}_{ij}(u) \otimes E_{ij}, \tag{2.38}$$

where $t_{ij}(u)$ and $\bar{t}_{ij}(u)$ are formal series in u^{-1} and u, respectively:

$$t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)}\, u^{-r}, \qquad \bar{t}_{ij}(u) = \sum_{r=0}^{\infty} \bar{t}_{ij}^{(r)}\, u^{r}. \tag{2.39}$$

The defining relations are

$$\begin{aligned}
t_{ij}^{(0)} &= \bar{t}_{ji}^{(0)} = 0, \qquad 1 \leqslant i < j \leqslant N,\\
t_{ii}^{(0)}\, \bar{t}_{ii}^{(0)} &= \bar{t}_{ii}^{(0)}\, t_{ii}^{(0)} = 1, \qquad 1 \leqslant i \leqslant N,\\
R(u,v)\, T_1(u) T_2(v) &= T_2(v) T_1(u) R(u,v),\\
R(u,v)\, \overline{T}_1(u) \overline{T}_2(v) &= \overline{T}_2(v) \overline{T}_1(u) R(u,v),\\
R(u,v)\, \overline{T}_1(u) T_2(v) &= T_2(v) \overline{T}_1(u) R(u,v),
\end{aligned} \tag{2.40}$$

where $R(u,v)$ is the trigonometric R-matrix given by

$$\begin{aligned}
R(u,v) = (u-v) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1}u - qv) \sum_{i} E_{ii} \otimes E_{ii}\\
+ (q^{-1} - q)\, u \sum_{i>j} E_{ij} \otimes E_{ji} + (q^{-1} - q)\, v \sum_{i<j} E_{ij} \otimes E_{ji}.
\end{aligned} \tag{2.41}$$

Both sides of each of the R-matrix relations are series with coefficients in the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N) \otimes \mathrm{End}\, \mathbb{C}^N \otimes \mathrm{End}\, \mathbb{C}^N$ and the subscripts of $T(u)$ and $\overline{T}(u)$ indicate the copies of $\mathrm{End}\, \mathbb{C}^N$; e.g. $T_1(u) = T(u) \otimes 1$. In terms of the generators these relations can be written more explicitly as

$$\begin{aligned}
&(q^{-\delta_{ij}} u - q^{\delta_{ij}} v)\, t_{ia}(u)\, t_{jb}(v) + (q^{-1} - q)\, (u\, \delta_{i>j} + v\, \delta_{i<j})\, t_{ja}(u)\, t_{ib}(v)\\
&= (q^{-\delta_{ab}} u - q^{\delta_{ab}} v)\, t_{jb}(v)\, t_{ia}(u) + (q^{-1} - q)\, (u\, \delta_{a<b} + v\, \delta_{a>b})\, t_{ja}(v)\, t_{ib}(u)
\end{aligned} \tag{2.42}$$

for the relations involving the $t_{ij}^{(r)}$,

$$\begin{aligned}
&(q^{-\delta_{ij}} u - q^{\delta_{ij}} v)\, \bar{t}_{ia}(u)\, \bar{t}_{jb}(v) + (q^{-1} - q)\, (u\, \delta_{i>j} + v\, \delta_{i<j})\, \bar{t}_{ja}(u)\, \bar{t}_{ib}(v)\\
&= (q^{-\delta_{ab}} u - q^{\delta_{ab}} v)\, \bar{t}_{jb}(v)\, \bar{t}_{ia}(u) + (q^{-1} - q)\, (u\, \delta_{a<b} + v\, \delta_{a>b})\, \bar{t}_{ja}(v)\, \bar{t}_{ib}(u)
\end{aligned} \tag{2.43}$$

for the relations involving the $\bar{t}_{ij}^{(r)}$ and

$$\begin{aligned}
&(q^{-\delta_{ij}} u - q^{\delta_{ij}} v)\, \bar{t}_{ia}(u)\, t_{jb}(v) + (q^{-1} - q)\, (u\, \delta_{i>j} + v\, \delta_{i<j})\, \bar{t}_{ja}(u)\, t_{ib}(v)\\
&= (q^{-\delta_{ab}} u - q^{\delta_{ab}} v)\, t_{jb}(v)\, \bar{t}_{ia}(u) + (q^{-1} - q)\, (u\, \delta_{a<b} + v\, \delta_{a>b})\, t_{ja}(v)\, \bar{t}_{ib}(u)
\end{aligned} \tag{2.44}$$

for the relations involving both $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$.

Note that the last relation in (2.40) can be equivalently written in the form

$$R(u,v)\, T_1(u)\, \overline{T}_2(v) = \overline{T}_2(v)\, T_1(u)\, R(u,v). \tag{2.45}$$

Indeed, we have the identity

$$R(u,v)\,R_{q^{-1}}(u,v)=(qu-q^{-1}v)(q^{-1}u-qv)\,1\otimes 1,$$

where $R_{q^{-1}}(u,v)$ is obtained from $R(u,v)$ by replacing q with q^{-1}. Therefore, the last relation in (2.40) can be written as

$$R_{q^{-1}}(u,v)\,T_2(v)\,\overline{T}_1(u)=\overline{T}_1(u)\;T_2(v)\,R_{q^{-1}}(u,v).$$

Now conjugate both sides by the permutation operator

$$P=\sum_{i,j=1}^{N}E_{ij}\otimes E_{ji}, \tag{2.46}$$

then swap u and v to get (2.45), as

$$R(u,v)=-P\,R_{q^{-1}}(v,u)\,P.$$

In terms of the generators the relation (2.45) takes the form

$$\begin{aligned}&(q^{-\delta_{ij}}u-q^{\delta_{ij}}v)\,t_{ia}(u)\,\bar t_{jb}(v)+(q^{-1}-q)\,(u\,\delta_{i>j}+v\,\delta_{i<j})\,t_{ja}(u)\,\bar t_{ib}(v)\\ &=(q^{-\delta_{ab}}u-q^{\delta_{ab}}v)\,\bar t_{jb}(v)\,t_{ia}(u)+(q^{-1}-q)\,(u\,\delta_{a<b}+v\,\delta_{a>b})\,\bar t_{ja}(v)\,t_{ib}(u).\end{aligned} \tag{2.47}$$

Let $f(u)$ and $\bar f(u)$ be formal power series in u^{-1} and u, respectively,

$$\begin{aligned}f(u)&=f_0+f_1\,u^{-1}+f_2\,u^{-2}+\dots,\\ \bar f(u)&=\bar f_0+\bar f_1\,u+\bar f_2\,u^2+\dots,\end{aligned}$$

such that $f_0\,\bar f_0=1$. Then it is immediate from the defining relations that the mapping

$$T(u)\mapsto f(u)\,T(u),\qquad \overline{T}(u)\mapsto \bar f(u)\,\overline{T}(u) \tag{2.48}$$

defines an automorphism of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$.

We will also use an involutive automorphism of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ given by

$$T(u)\mapsto \overline{T}(u^{-1})^t,\qquad \overline{T}(u)\mapsto T(u^{-1})^t, \tag{2.49}$$

where t denotes the matrix transposition. The first two sets of relations in (2.40) are obviously preserved by the map (2.49). In order to verify that the R-matrix relations are preserved as well, apply the transposition t_1 in the first copy of $\operatorname{End}\mathbb{C}^N$ to each of them, followed by the transposition t_2 in the second copy of $\operatorname{End}\mathbb{C}^N$. Then conjugate both sides by the permutation operator (2.46), replace u and v by v^{-1} and u^{-1} respectively, and observe that

$$uv\,PR^{t_1t_2}(v^{-1},u^{-1})\,P=R(u,v).$$

Another involutive automorphism is defined by the mapping

$$t_{ij}(u) \mapsto \varepsilon_i \, t_{ij}(u), \qquad \bar{t}_{ij}(u) \mapsto \varepsilon_i \, \bar{t}_{ij}(u), \tag{2.50}$$

where each ε_i equals 1 or -1.

It follows easily from the defining relations (2.40) that the mapping

$$t_{ij}(u) \mapsto t_{N-j+1,N-i+1}(u), \qquad \bar{t}_{ij}(u) \mapsto \bar{t}_{N-j+1,N-i+1}(u) \tag{2.51}$$

defines an involutive anti-automorphism of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$.

Ding and Frenkel [11] used the Gauss decompositions of the matrices $T(u)$ and $\overline{T}(u)$ to construct an isomorphism between the RTT-presentation (2.40) and Drinfeld's "new realization" of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$; see also [14]. However, the version of the PBW theorem given by Beck [3] for the new realization of the quantum affine algebras $\mathrm{U}_q(\widehat{\mathfrak{a}})$ over the field of rational functions in q does not immediately imply a PBW-type theorem for the RTT-presentation. Our next goal is to prove the PBW theorem for the RTT-presentation of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$, where q is an arbitrary fixed nonzero complex number.

As before, we let z denote an indeterminate. Introduce the algebra $\mathrm{U}^\circ_z(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}[z,z^{-1}]$ by the respective generators and relations given in (2.40) with q replaced by z. Then we have the isomorphism

$$\mathrm{U}^\circ_z(\widehat{\mathfrak{gl}}_N) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \mathrm{U}_q(\widehat{\mathfrak{gl}}_N), \tag{2.52}$$

where the $\mathbb{C}[z,z^{-1}]$-module $\mathbb{C}$ is defined via the evaluation of the Laurent polynomials at $z=q$. The next proposition takes care of the weak part of the PBW theorem. We use a particular total order on the generators of the algebra for which the argument appears to be the most straightforward. For the purposes of representation theory a different order is more useful and we will take care of that one in Corollary 2.13 below.

We associate the triple (i,a,r) to each nonzero generator $t^{(r)}_{ia}$ or $\bar{t}^{(r)}_{ia}$ of $\mathrm{U}^\circ_z(\widehat{\mathfrak{gl}}_N)$. If $(i,a,r) < (j,b,s)$ in the lexicographical order then we will say that each generator associated with (i,a,r) precedes each generator associated with (j,b,s). Moreover, we will suppose that $t^{(r)}_{ia}$ precedes $\bar{t}^{(r)}_{ia}$ for each triple (i,a,r) such that both generators are nonzero.

Proposition 2.9. *The ordered monomials in the generators span the algebra $\mathrm{U}^\circ_z(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}[z,z^{-1}]$.*

Proof. Let r and s be nonnegative integers. Multiply both sides of the relation (2.42) with q replaced with z by

$$\frac{1}{z^{-\delta_{ij}}u - z^{\delta_{ij}}v} = \sum_{k=1}^{\infty} z^{(2k-1)\,\delta_{ij}} u^{-k}\, v^{k-1}$$

and equate the coefficients of $u^{-r}v^{-s}$. This provides an expression for the product $t_{ia}^{(r)}t_{jb}^{(s)}$ with $i>j$ as a $\mathbb{C}[z,z^{-1}]$-linear combination of the elements of the form $t_{jc}^{(k)}t_{id}^{(l)}$. Furthermore, taking $i=j$ in (2.42) with q replaced with z we obtain

$$\begin{aligned}(z^{-ab}u-z^{ab}v)\,t_{ib}(v)\,t_{ia}(u)&\\ =(z^{-1}u-z\,v)\,t_{ia}(u)\,t_{ib}(v)&-(z^{-1}-z)\,(u\,\delta_{a<b}+v\,\delta_{a>b})\,t_{ia}(v)\,t_{ib}(u). \qquad (2.53)\end{aligned}$$

This allows us to express the product $t_{ib}^{(r)}t_{ia}^{(s)}$ with $b>a$ as a $\mathbb{C}[z,z^{-1}]$-linear combination of the elements of the form $t_{ia}^{(k)}t_{ib}^{(l)}$. Taking $a=b$ in (2.53), we find that the generators $t_{ia}^{(r)}$ and $t_{ia}^{(s)}$ commute for any r and s.

Applying similar arguments to the relations (2.43) and (2.44) with q replaced with z and using induction on the length of monomials we conclude that any monomial in the generators of $\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N)$ can be written as a $\mathbb{C}[z,z^{-1}]$-linear combination of the ordered monomials. □

Recall now that the algebra $\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N)$ possesses a Hopf algebra structure with the coproduct

$$\Delta:\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N)\to\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N)\otimes\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N),$$

where the tensor product is taken over $\mathbb{C}[z,z^{-1}]$, defined by

$$\Delta\big(t_{ij}(u)\big)=\sum_{k=1}^{N}t_{ik}(u)\otimes t_{kj}(u),\qquad \Delta\big(\bar{t}_{ij}(u)\big)=\sum_{k=1}^{N}\bar{t}_{ik}(u)\otimes\bar{t}_{kj}(u). \qquad (2.54)$$

The quantized enveloping algebra $\mathrm{U}_z^{\circ}(\mathfrak{gl}_N)$ is a Hopf subalgebra of $\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N)$ defined by the embedding

$$t_{ij}\mapsto t_{ij}^{(0)},\qquad \bar{t}_{ij}\mapsto\bar{t}_{ij}^{(0)}. \qquad (2.55)$$

Moreover, the mapping

$$\pi:T(u)\mapsto T+\overline{T}\,u^{-1},\qquad \overline{T}(u)\mapsto\overline{T}+T\,u \qquad (2.56)$$

defines a $\mathbb{C}[z,z^{-1}]$-algebra homomorphism $\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N)\to\mathrm{U}_z^{\circ}(\mathfrak{gl}_N)$ called the evaluation homomorphism.

In our proof of the PBW theorem for the quantum affine algebra we will follow the approach used in [5] to prove the corresponding theorem for the Yangian for $\mathfrak{gl}_N$; see also [16] for the super-version of the same approach.

We will need a simple lemma which is easily verified by induction. Let $x_1,\dots,x_l$ be indeterminates. For $r=0,\dots,l-1$ and $k=1,\dots,l$ consider the elementary symmetric polynomials in $l-1$ variables, where the variable x_k is skipped:

$$e_{rk}=e_r(x_1,\dots,\widehat{x}_k,\dots x_l)=\sum x_{i_1}\dots x_{i_r},$$

summed over indices $i_a\neq k$ with $1\leqslant i_1<\dots<i_r\leqslant l$.

Lemma 2.10. *We have*

$$\det \begin{bmatrix} e_{01} & e_{02} & \cdots & e_{0l} \\ e_{11} & e_{12} & \cdots & e_{1l} \\ \vdots & \vdots & \ddots & \vdots \\ e_{l-1,1} & e_{l-1,2} & \cdots & e_{l-1,l} \end{bmatrix} = \prod_{1 \leqslant i < j \leqslant l} (x_i - x_j). \tag{2.57}$$

In particular, the determinant is nonzero under any specialization of variables $x_i = a_i$, $i = 1, \dots, l$, where the a_i are distinct complex numbers. □

For each positive integer l introduce the $\mathbb{C}[z, z^{-1}]$-algebra homomorphism

$$\kappa_l : \mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N) \to \mathrm{U}_z^\circ(\mathfrak{gl}_N)^{\otimes l}$$

by setting

$$\kappa_l = \pi^{\otimes l} \circ \Delta^{(l-1)}, \tag{2.58}$$

where

$$\Delta^{(l-1)} : \mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N) \to \mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)^{\otimes l}$$

denotes the coproduct iterated $l - 1$ times. The explicit formulas for the images of the generators of $\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)$ under the homomorphism κ_l have the following form:

$$\begin{aligned} \kappa_l : t_{ij}^{(r)} &\mapsto \sum_{p_1 < \dots < p_r} \sum_{i_1, \dots, i_l} t_{ii_1} \otimes t_{i_1 i_2} \otimes \dots \otimes \bar{t}_{i_{p_1 - 1} i_{p_1}} \otimes \dots \otimes \bar{t}_{i_{p_r - 1} i_{p_r}} \otimes \dots \otimes t_{i_{l-1} j}, \\ \kappa_l : \bar{t}_{ij}^{(r)} &\mapsto \sum_{p_1 < \dots < p_r} \sum_{i_1, \dots, i_l} \bar{t}_{ii_1} \otimes \bar{t}_{i_1 i_2} \otimes \dots \otimes t_{i_{p_1 - 1} i_{p_1}} \otimes \dots \otimes t_{i_{p_r - 1} i_{p_r}} \otimes \dots \otimes \bar{t}_{i_{l-1} j}, \end{aligned} \tag{2.59}$$

where the indices $i_1, \dots, i_l$ in each formula run over the set $\{1, \dots, N\}$ and the indices $\{p_1, \dots, p_r\} \subset \{1, \dots, l\}$ indicate the places taken by the barred generators $\bar{t}_{kl}$ (resp. unbarred generators t_{kl}) in the first (resp. second) formula. The images of $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$ under the homomorphism κ_l are zero unless $l \geqslant r$.

With the order on the generators of $\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)$ introduced before Proposition 2.9 consider the corresponding ordered monomials. The zero generators $t_{ij}^{(0)}$ for $i < j$ and $\bar{t}_{ij}^{(0)}$ for $i > j$ will be excluded. Moreover, using the relation $t_{ii}^{(0)} \bar{t}_{ii}^{(0)} = 1$ we will suppose that for each $i = 1, \dots, N$ each monomial contains either a nonnegative power of $t_{ii}^{(0)}$ or a positive power of $\bar{t}_{ii}^{(0)}$. With these conventions we have the following version of the PBW theorem.

Theorem 2.11. *The ordered monomials in the generators $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$ form a basis of the algebra $\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}[z, z^{-1}]$.*

Proof. Due to Proposition 2.9, we only need to verify that the ordered monomials are linearly independent. We will argue by contradiction. Suppose that a nontrivial linear combination of the ordered monomials is zero. Let m be the minimum nonnegative integer such that for all generators $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$ occurring in the combination we have $0 \leqslant r \leqslant m$. Consider the homomorphism κ_l defined in (2.58) with $l = 2m + 1$ and apply it to the linear combination. We then get the respective nontrivial $\mathbb{C}[z, z^{-1}]$-linear combination of elements of the algebra $\mathrm{U}_z^{\circ}(\mathfrak{gl}_N)^{\otimes l}$ equal to zero. By Proposition 2.1 we may suppose that at least one coefficient of the combination does not vanish at $z = 1$.

On the other hand, due to (2.14) we have the isomorphism

$$\mathrm{U}_z^{\circ}(\mathfrak{gl}_N)^{\otimes l} \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \mathcal{P}_N^{\otimes l}.$$

Taking the image of the linear combination under this isomorphism we get a nontrivial $\mathbb{C}$-linear combination of elements of the polynomial algebra $\mathcal{P}_N^{\otimes l}$ equal to zero.

We will regard $\mathcal{P}_N^{\otimes l}$ as the algebra of polynomials in l sets of variables $\{x_{ij}^{[k]}, \bar{x}_{ij}^{[k]}\}$, where the parameter $k \in \{1, \dots, l\}$ indicates the k-th copy of $\mathcal{P}_N$ in the tensor product. Thus, the proof of the theorem is now reduced to verifying the following claim. Consider the elements $y_{ij}^{(r)}$ and $\bar{y}_{ij}^{(r)}$ of the algebra $\mathcal{P}_N^{\otimes l}$ defined by the relations

$$\begin{aligned} y_{ij}^{(r)} &= \sum_{p_1 < \dots < p_r} \sum_{i_1,\dots,i_l} x_{ii_1}^{[1]}\, x_{i_1 i_2}^{[2]} \dots \bar{x}_{i_{p_1-1} i_{p_1}}^{[p_1]} \dots \bar{x}_{i_{p_r-1} i_{p_r}}^{[p_r]} \dots x_{i_{l-1} j}^{[l]}, \\ \bar{y}_{ij}^{(r)} &= \sum_{p_1 < \dots < p_r} \sum_{i_1,\dots,i_l} \bar{x}_{ii_1}^{[1]}\, \bar{x}_{i_1 i_2}^{[2]} \dots x_{i_{p_1-1} i_{p_1}}^{[p_1]} \dots x_{i_{p_r-1} i_{p_r}}^{[p_r]} \dots \bar{x}_{i_{l-1} j}^{[l]}, \end{aligned}$$

with the same conditions on the summation indices as in (2.59) together with the relations $x_{ij}^{[s]} = \bar{x}_{ji}^{[s]} = 0$ for $i < j$. We need to verify that modulo the relations

$$\begin{aligned} y_{ij}^{(0)} = \bar{y}_{ji}^{(0)} &= 0, \qquad 1 \leqslant i < j \leqslant N, \\ y_{ii}^{(0)}\, \bar{y}_{ii}^{(0)} &= 1, \qquad 1 \leqslant i \leqslant N, \end{aligned}$$

the polynomials $y_{ij}^{(r)}, \bar{y}_{ij}^{(r)}$ with $1 \leqslant i, j \leqslant N$ and $0 \leqslant r \leqslant m$ are algebraically independent. It will be sufficient to show that the corresponding differentials $dy_{ij}^{(r)}, d\bar{y}_{ij}^{(r)}$ are linearly independent. In order to do this, we calculate the matrix of the map

$$\big(dx_{ij}^{[s]}, d\bar{x}_{ij}^{[s]}\big) \to \big(dy_{ij}^{(r)}, d\bar{y}_{ij}^{(r)}\big)$$

and show that its determinant is nonzero even when the variables are specialized to

$$x_{ij}^{[s]} = \delta_{ij} c_s, \qquad \bar{x}_{ij}^{[s]} = \delta_{ij} c_s^{-1}, \tag{2.60}$$

where $c_1, \dots, c_l$ are distinct nonzero complex numbers.

If $i > j$ then under the specialization (2.60) we have

$$dy_{ij}^{(r)} = c_1 \dots c_l \sum_{s=1}^{l} c_s^{-1} e_r(c_1^{-2}, \dots, \widehat{c_s^{-2}}, \dots, c_l^{-2})\, dx_{ij}^{[s]}$$

for $r=0,1,\dots,m$, and

$$d\bar{y}_{ij}^{(r)} = c_1 \dots c_l \sum_{s=1}^{l} c_s^{-1} e_{l-r}(c_1^{-2},\dots,\widehat{c_s^{-2}},\dots,c_l^{-2})\, dx_{ij}^{[s]}$$

for $r=1,\dots,m$. Similarly, for $i<j$ we have

$$d\bar{y}_{ij}^{(r)} = c_1^{-1} \dots c_l^{-1} \sum_{s=1}^{l} c_s\, e_r(c_1^{2},\dots,\widehat{c_s^{2}},\dots,c_l^{2})\, d\bar{x}_{ij}^{[s]}$$

for $r=0,1,\dots,m$, and

$$dy_{ij}^{(r)} = c_1^{-1} \dots c_l^{-1} \sum_{s=1}^{l} c_s\, e_{l-r}(c_1^{2},\dots,\widehat{c_s^{2}},\dots,c_l^{2})\, d\bar{x}_{ij}^{[s]}$$

for $r=1,\dots,m$. Note that since $x_{ii}^{[s]}\bar{x}_{ii}^{[s]}=1$, we have $d\bar{x}_{ii}^{[s]} = -\big(x_{ii}^{[s]}\big)^{-2}\, dx_{ii}^{[s]}$. Therefore, setting $e_{-1}=0$, for $i=j$ we obtain

$$dy_{ii}^{(r)} = c_1 \dots c_l \sum_{s=1}^{l} c_s^{-1}\Big(e_r(c_1^{-2},\dots,\widehat{c_s^{-2}},\dots,c_l^{-2}) - c_s^{-2}\, e_{r-1}(c_1^{-2},\dots,\widehat{c_s^{-2}},\dots,c_l^{-2})\Big)\, dx_{ii}^{[s]}$$

for $r=0,1,\dots,m$, and

$$d\bar{y}_{ii}^{(r)} = c_1 \dots c_l \sum_{s=1}^{l} c_s^{-1}\Big(e_{l-r}(c_1^{-2},\dots,\widehat{c_s^{-2}},\dots,c_l^{-2}) - c_s^{-2}\, e_{l-r-1}(c_1^{-2},\dots,\widehat{c_s^{-2}},\dots,c_l^{-2})\Big)\, dx_{ii}^{[s]}$$

for $r=1,\dots,m$.

It follows from Lemma 2.10 that in each of the three cases, the determinant of the $l\times l$ matrix is nonzero. This proves that the differentials $dy_{ij}^{(r)}$ and $d\bar{y}_{ij}^{(r)}$ are linearly independent (excluding $dy_{ij}^{(0)}$ for $i<j$ and $d\bar{y}_{ij}^{(0)}$ for $i\geqslant j$), thus completing the proof. □

The following corollary is immediate from the isomorphism (2.52).

Corollary 2.12. *Let q be a nonzero complex number. With the same order on the generators as in Theorem 2.11, the ordered monomials in the generators $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$ form a basis of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}$.* □

Note that the proof of the linear independence of the ordered monomials in $\mathrm{U}_z^{\circ}(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}[z,z^{-1}]$ does not rely on the ordering used. Therefore, Theorem 2.11 holds in the same form for any other ordering, provided that the corresponding weak form of the PBW theorem holds; cf. Proposition 2.9. We will prove this weak form for another ordering which is useful for the description of representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$.

To each nonzero generator $t_{ia}^{(r)}$ and $\bar{t}_{ia}^{(r)}$ of $\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)$ we now associate the triple of integers $(a-i,i,r)$. The generators will now be ordered in accordance with the lexicographical order on the corresponding triples and we will also suppose that $t_{ia}^{(r)}$ precedes $\bar{t}_{ia}^{(r)}$ for each triple $(a-i,i,r)$ such that both generators are nonzero. We have the following version of the PBW theorem.

Corollary 2.13. *Let q be a nonzero complex number. With the order on the generators defined above, the ordered monomials in the generators $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$ form a basis of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}$.*

Proof. As we pointed out above, the linear independence of the corresponding monomials in the $\mathbb{C}[z,z^{-1}]$-algebra $\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)$ will follow by the argument used in the proof of Theorem 2.11. We only need to show that the ordered monomials span this algebra over $\mathbb{C}[z,z^{-1}]$. The corollary will then follow from the isomorphism (2.52).

Arguing as in the proof of Proposition 2.9, we derive from the relation (2.42) that

$$t_{ia}^{(r)}t_{jb}^{(s)} = \text{linear combination of} \quad t_{jb}^{(k)}t_{ia}^{(l)} \quad \text{and} \quad t_{ja}^{(m)}t_{ib}^{(p)} \tag{2.61}$$

for some k,l,m,p. Swapping i with j and a with b in (2.42) we also obtain

$$t_{ia}^{(r)}t_{jb}^{(s)} = \text{linear combination of} \quad t_{jb}^{(k)}t_{ia}^{(l)} \quad \text{and} \quad t_{ib}^{(m)}t_{ja}^{(p)}. \tag{2.62}$$

Suppose now that $a-i>b-j$. If $a-j\neq b-i$, then we use the formula (2.61) or (2.62) depending on whether $a-j<b-i$ or $b-i<a-j$ to write $t_{ia}^{(r)}t_{jb}^{(s)}$ as a linear combination of the ordered products of the generators. If $a-j=b-i$, then either $j<i$ or $i<j$; the equality $i=j$ is impossible due to the condition $a-i>b-j$. Again, the product $t_{ia}^{(r)}t_{jb}^{(s)}$ is then written as a linear combination of the ordered products of the generators by (2.61) or (2.62), respectively.

Further, suppose that $a-i=b-j$ and $i>j$. Then $a>b$ and $b-i<a-j$ so that (2.62) provides an expression of $t_{ia}^{(r)}t_{jb}^{(s)}$ as a linear combination of the ordered products of the generators.

The same arguments relying on (2.43) instead of (2.42) prove the corresponding statement for the products of the generators $\bar{t}_{ia}^{(r)}$.

Finally, relation (2.44) implies the following counterpart of (2.61):

$$\bar{t}_{ia}^{(r)}t_{jb}^{(s)} = \text{linear combination of} \quad t_{jb}^{(k)}\bar{t}_{ia}^{(l)}, \quad \bar{t}_{ja}^{(m)}t_{ib}^{(p)} \quad \text{and} \quad t_{ja}^{(h)}\bar{t}_{ib}^{(n)}. \tag{2.63}$$

The corresponding counterpart of (2.62) is obtained from (2.47) and it has the form

$$\bar{t}_{ia}^{(r)}t_{jb}^{(s)} = \text{linear combination of} \quad t_{jb}^{(k)}\bar{t}_{ia}^{(l)}, \quad t_{ib}^{(p)}\bar{t}_{ja}^{(m)} \quad \text{and} \quad \bar{t}_{ib}^{(n)}t_{ja}^{(h)}. \tag{2.64}$$

The above argument can now be applied to the products $\bar{t}_{ia}^{(r)}t_{jb}^{(s)}$ allowing one to write it as a linear combination of the ordered products of the generators.

Recalling that each of the generators $t_{ia}^{(r)}$ and $\bar{t}_{ia}^{(r)}$ commutes with each of $t_{ia}^{(s)}$ and $\bar{t}_{ia}^{(s)}$ for all r and s, we conclude by an easy induction that any monomial in $\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)$ can be written as a $\mathbb{C}[z, z^{-1}]$-linear combination of the ordered products of the generators. $\square$

As with the quantized enveloping algebra, we need to introduce an extended quantum affine algebra. We denote by $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ the algebra over $\mathbb{C}$ with countably many generators $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$, $1 \leqslant i, j \leqslant N$ and $r \geqslant 0$, together with $t_{ii}^{(0)-1}$ and $\bar{t}_{ii}^{(0)-1}$ with $1 \leqslant i \leqslant N$, subject to the defining relations (2.40), where the second set of relations is replaced with

$$t_{ii}^{(0)}\,\bar{t}_{ii}^{(0)} = \bar{t}_{ii}^{(0)}\,t_{ii}^{(0)}, \qquad t_{ii}^{(0)}\,t_{ii}^{(0)-1} = t_{ii}^{(0)-1}\,t_{ii}^{(0)} = 1, \qquad \bar{t}_{ii}^{(0)}\,\bar{t}_{ii}^{(0)-1} = \bar{t}_{ii}^{(0)-1}\,\bar{t}_{ii}^{(0)} = 1,$$

for $i = 1, \dots, N$. We have the natural epimorphism

$$\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N) \to \mathrm{U}_q(\widehat{\mathfrak{gl}}_N) \tag{2.65}$$

whose kernel is the ideal of $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ generated by the central elements $t_{ii}^{(0)}\,\bar{t}_{ii}^{(0)} - 1$ for $i = 1, \dots, N$. We also define the algebra $\mathrm{U}_z^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{C}[z, z^{-1}]$ with the same generators and relations, where q should be replaced with z.

It is straightforward to conclude that the PBW theorem for the algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ holds in the same form as in Corollaries 2.12 and 2.13, except for allowing the generators $t_{ii}^{(0)}$ and $\bar{t}_{ii}^{(0)}$ to occur simultaneously in the monomials and their powers can now run over the set of all integers.

Observe that given any tuple $(\phi_1, \dots, \phi_N)$ of nonzero complex numbers, the mapping

$$t_{ij}(u) \mapsto \phi_i\, t_{ij}(u), \qquad \bar{t}_{ij}(u) \mapsto \phi_i\, \bar{t}_{ij}(u), \tag{2.66}$$

defines an automorphism of the algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$.

2.4 Twisted q-Yangians $\mathrm{Y}'_q(\mathfrak{o}_N)$ and $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$

The *twisted q-Yangians* $\mathrm{Y}'_q(\mathfrak{o}_N)$ and $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ associated with the orthogonal Lie algebra $\mathfrak{o}_N$ and symplectic Lie algebra $\mathfrak{sp}_{2n}$ were introduced in [29]. By definition, $\mathrm{Y}'_q(\mathfrak{o}_N)$ is the subalgebra of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ generated by the coefficients $s_{ij}^{(r)}$, $r \geqslant 0$, of the series

$$s_{ij}(u) = \sum_{r=0}^{\infty} s_{ij}^{(r)}\, u^{-r}, \qquad 1 \leqslant i, j \leqslant N, \tag{2.67}$$

where

$$s_{ij}(u) = \sum_{a=1}^{N} t_{ia}(u)\, \bar{t}_{ja}(u^{-1}). \tag{2.68}$$

In the symplectic case, we define $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ as the subalgebra of $\mathrm{U}^{\mathrm{ext}}_q(\widehat{\mathfrak{gl}}_{2n})$ generated by the coefficients $s_{ij}^{(r)}, r \geqslant 0$, of the series

$$s_{ij}(u) = \sum_{r=0}^{\infty} s_{ij}^{(r)}\, u^{-r}, \qquad 1 \leqslant i,j \leqslant 2n, \tag{2.69}$$

where

$$s_{ij}(u) = q\sum_{a=1}^{n} t_{i,2a-1}(u)\, \bar{t}_{j,2a}(u^{-1}) - \sum_{a=1}^{n} t_{i,2a}(u)\, \bar{t}_{j,2a-1}(u^{-1}), \tag{2.70}$$

and by the elements $s_{i,i+1}^{(0)-1}$ with $i = 1,3,\dots,2n-1$.

Remark 2.14. The twisted q-Yangian in the symplectic case was defined in [29] by the above formulas as a subalgebra of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{2n})$ without using its extension. The generators of the corresponding algebra $\mathrm{Y}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$ satisfy some extra relations: for any odd i

$$s_{i+1,i+1}^{(0)}\, s_{ii}^{(0)} - q^2\, s_{i+1,i}^{(0)}\, s_{i,i+1}^{(0)} = q^3. \tag{2.71}$$

Moreover, the elements $s_{i+1,i+1}^{(0)}\, s_{ii}^{(0)} - q^2\, s_{i+1,i}^{(0)}\, s_{i,i+1}^{(0)}$ are central in $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ and its quotient by the relations (2.71) is isomorphic to $\mathrm{Y}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$. $\square$

Both in the orthogonal and symplectic cases, the twisted q-Yangians can be equivalently defined as abstract algebras with quadratic defining relations. Namely, consider the matrices $S(u) = T(u)\,\overline{T}(u^{-1})^t$ and $S(u) = T(u)\,G\,\overline{T}(u^{-1})^t$, where the matrix G is defined in (2.27). Then the matrix elements of $S(u)$ are the formal series $s_{ij}(u)$ given by (2.68) and (2.70), respectively. The coefficients $s_{ij}^{(r)}$ of these series then satisfy the relations

$$R(u,v)\, S_1(u)\, R^{t_1}(u^{-1},v)\, S_2(v) = S_2(v)\, R^{t_1}(u^{-1},v)\, S_1(u)\, R(u,v), \tag{2.72}$$

where the R-matrix $R(u,v)$ is defined in (2.41) and

$$\begin{aligned} R^{t_1}(u,v) = (u-v)\sum_{i\neq j} E_{ii}\otimes E_{jj} + (q^{-1}u - qv)\sum_{i} E_{ii}\otimes E_{ii} \\ + (q^{-1}-q)\,u\sum_{i>j} E_{ji}\otimes E_{ji} + (q^{-1}-q)\,v\sum_{i<j} E_{ji}\otimes E_{ji}. \end{aligned} \tag{2.73}$$

In terms of the generating series $s_{ij}(u)$ the relation (2.72) takes the form

$$\begin{aligned} (q^{-\delta_{ij}}u - q^{\delta_{ij}}v)\,\alpha_{ijab}(u,v) + (q^{-1}-q)(u\delta_{j<i} + v\delta_{i<j})\,\alpha_{jiab}(u,v) \\ = (q^{-\delta_{ab}}u - q^{\delta_{ab}}v)\,\alpha_{jiba}(v,u) + (q^{-1}-q)(u\delta_{a<b} + v\delta_{b<a})\,\alpha_{jiab}(v,u), \end{aligned} \tag{2.74}$$

where

$$\alpha_{ijab}(u,v) = (q^{-\delta_{aj}} - q^{\delta_{aj}}uv)\, s_{ia}(u)\, s_{jb}(v) + (q^{-1}-q)(\delta_{j<a} + uv\delta_{a<j})\, s_{ij}(u)\, s_{ab}(v).$$

All coefficients $\bar{s}_{ij}^{(r)}$ of the matrix elements $\bar{s}_{ij}(u)$ of the matrices

$$\overline{S}(u) = \overline{T}(u)\, T(u^{-1})^t \qquad \text{and} \qquad \overline{S}(u) = \overline{T}(u)\, G\, T(u^{-1})^t \tag{2.75}$$

belong to the subalgebras $\mathrm{Y}'_q(\mathfrak{o}_N) \subseteq \mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ and $\mathrm{Y}'_q(\mathfrak{sp}_{2n}) \subseteq \mathrm{U}^{\mathrm{ext}}_q(\widehat{\mathfrak{gl}}_{2n})$, respectively. Moreover, the relations between the elements $s_{ij}^{(r)}$ and $\bar{s}_{ij}^{(r)}$ can be derived from those of the algebras $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ and $\mathrm{U}^{\mathrm{ext}}_q(\widehat{\mathfrak{gl}}_{2n})$. They take the form

$$\begin{aligned}
R(u,v)\, \overline{S}_1(u)\, R^{t_1}(u^{-1},v)\, S_2(v) &= S_2(v)\, R^{t_1}(u^{-1},v)\, \overline{S}_1(u)\, R(u,v),\\
R(u,v)\, S_1(u)\, R^{t_1}(u^{-1},v)\, \overline{S}_2(v) &= \overline{S}_2(v)\, R^{t_1}(u^{-1},v)\, S_1(u)\, R(u,v),\\
R(u,v)\, \overline{S}_1(u)\, R^{t_1}(u^{-1},v)\, \overline{S}_2(v) &= \overline{S}_2(v)\, R^{t_1}(u^{-1},v)\, \overline{S}_1(u)\, R(u,v).
\end{aligned} \tag{2.76}$$

The proof of the equivalence of the two definitions of the twisted q-Yangians is based on analogues of the PBW theorem whose proofs were outlined in [29]. They use a specialization argument based on the fact the twisted q-Yangians are deformations of universal enveloping algebras. Here we give a different proof relying on Theorem 2.11.

In the orthogonal case we consider the same total order on the set of generators of $\mathrm{Y}'_q(\mathfrak{o}_N)$ as in [29]; we order the generators $s_{ia}^{(r)}$ in accordance with the lexicographical order on the corresponding triples (i, a, r).

In the symplectic case use the function $\varsigma : \{1, 2, \dots, 2n\} \to \{\pm 1, \pm 3, \dots, \pm(2n-1)\}$ defined in (2.33) and order the generators $s_{ia}^{(r)}$ of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ in accordance with the lexicographical order on the corresponding triples $(\varsigma(i) + \varsigma(a), \varsigma(i), r)$. Since for any odd i the generators $s_{i,i+1}^{(0)}$ and $s_{i,i+1}^{(0)-1}$ commute, it is unambiguous to associate each of them to the same triple $(0, i, 0)$.

By the definition (2.68) we have

$$s_{ij}^{(0)} = 0 \quad \text{for} \quad i < j \qquad \text{and} \qquad s_{ii}^{(0)} = 1 \quad \text{for all} \quad i \tag{2.77}$$

in the orthogonal case. Similarly, by (2.70) in the symplectic case we have

$$s_{ij}^{(0)} = 0 \quad \text{for} \quad i < j \quad \text{unless} \quad j = i+1 \quad \text{with} \quad i \ \text{odd}. \tag{2.78}$$

Consequently, the generators (2.77) and (2.78) will not occur in the ordered monomials.

Proposition 2.15. *Let q be a nonzero complex number. With the orders on the generators chosen as above, the ordered monomials in the generators form a basis of the respective algebra $\mathrm{Y}'_q(\mathfrak{o}_N)$ and $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$.*

Proof. The weak form of the PBW theorem was proved in [29, Lemma 3.2] for the order used in the orthogonal case. The proof for the order we chose in the symplectic case is obtained by obvious modifications of the same arguments; cf. the proof of Proposition 2.6. Thus, in both cases the ordered monomials span the algebra $\mathrm{Y}'_q(\mathfrak{o}_N)$ or $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$, respectively.

Furthermore, by Theorem 2.11, we have the isomorphism

$$\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \widehat{\mathcal{P}}_N, \tag{2.79}$$

where the $\mathbb{C}[z, z^{-1}]$-module $\mathbb{C}$ is defined via the evaluation of the Laurent polynomials at $z = 1$ and $\widehat{\mathcal{P}}_N$ is the algebra of polynomials in the variables $x_{ij}^{(r)}, \bar{x}_{ij}^{(r)}$ with $1 \leqslant i, j \leqslant N$ and $r \geqslant 0$ subject to the relations $x_{ij}^{(0)} = \bar{x}_{ji}^{(0)} = 0$ for $i < j$ and $x_{ii}^{(0)}\bar{x}_{ii}^{(0)} = 1$ for all i.

Define the algebra $\mathrm{Y}_z^\circ(\mathfrak{o}_N)$ over $\mathbb{C}[z, z^{-1}]$ as the $\mathbb{C}[z, z^{-1}]$-subalgebra of $\mathrm{U}_z^\circ(\widehat{\mathfrak{gl}}_N)$ generated by the elements $s_{ij}^{(r)}$ defined in (2.67) and (2.68). Suppose that a nontrivial $\mathbb{C}[z, z^{-1}]$-linear combination of the ordered monomials in the generators of $\mathrm{Y}_z^\circ(\mathfrak{o}_N)$ is zero. By Theorem 2.11 we may suppose that at least one coefficient of the combination does not vanish at $z = 1$. Using the isomorphism (2.79) we then get a nontrivial $\mathbb{C}$-linear combination of the ordered monomials in the images of the generators in the polynomial algebra $\widehat{\mathcal{P}}_N$. We will come to a contradiction if we show that the images of the generators of $\mathrm{Y}_z^\circ(\mathfrak{o}_N)$ in $\widehat{\mathcal{P}}_N$ are algebraically independent.

Let $\sigma_{ij}^{(r)}$ denote the image of $s_{ij}^{(r)}$ in $\widehat{\mathcal{P}}_N$. Then

$$\sigma_{ij}^{(r)} = \sum_{a=1}^{N} \sum_{k+l=r} x_{ia}^{(k)}\, \bar{x}_{ja}^{(l)}.$$

It suffices to verify that the differentials $d\sigma_{ij}^{(r)}$ are linearly independent. Calculate the differentials in terms of $dx_{ia}^{(k)}$ and $d\bar{x}_{ia}^{(k)}$ and specialize the coefficient matrix by setting

$$x_{ij}^{(k)} = \bar{x}_{ij}^{(k)} = \delta_{ij}\, \delta_{k0}. \tag{2.80}$$

Then

$$d\sigma_{ij}^{(r)} = d\bar{x}_{ji}^{(r)} + dx_{ij}^{(r)}$$

which implies that the differentials $d\sigma_{ij}^{(r)}$ are linearly independent even under the specialization (2.80). This completes the proof in the orthogonal case.

In the symplectic case define the algebra $\mathrm{Y}_z^\circ(\mathfrak{sp}_{2n})$ over $\mathbb{C}[z, z^{-1}]$ as the $\mathbb{C}[z, z^{-1}]$-subalgebra of $\mathrm{U}_z^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_{2n})$ generated by the elements $s_{ij}^{(r)}$ defined by (2.69) and (2.70) with q replaced by z and with the same additional generators $s_{i,i+1}^{(0)-1}$. Suppose that a nontrivial $\mathbb{C}[z, z^{-1}]$-linear combination of the ordered monomials in the generators of $\mathrm{Y}_z^\circ(\mathfrak{sp}_{2n})$ is zero. We may ignore the generators $s_{i,i+1}^{(0)-1}$ because they can be excluded from the linear combination by multiplying it by appropriate powers of the elements $s_{i,i+1}^{(0)}$ and using the following consequence of (2.74) (with q replaced by z):

$$s_{i,i+1}^{(0)}\, s_{kl}(u) = z^{\delta_{ik}+\delta_{il}-\delta_{i+1,k}-\delta_{i+1,l}}\, s_{kl}(u)\, s_{i,i+1}^{(0)}, \qquad i = 1, 3, \dots, 2n-1. \tag{2.81}$$

By the PBW theorem for the algebra $\mathrm{U}_z^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_{2n})$ (see Sec. 2.3) we have the isomorphism

$$\mathrm{U}_z^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_{2n}) \otimes_{\mathbb{C}[z,z^{-1}]} \mathbb{C} \cong \widehat{\mathcal{P}}_{2n}^{\mathrm{ext}}, \tag{2.82}$$

where the $\mathbb{C}[z, z^{-1}]$-module $\mathbb{C}$ is defined via the evaluation of the Laurent polynomials at $z = 1$ and $\widehat{\mathcal{P}}^{\text{ext}}_{2n}$ is the algebra of polynomials in the variables $x^{(r)}_{ij}, \bar{x}^{(r)}_{ij}$ with $1 \leqslant i, j \leqslant N$ and $r \geqslant 0$ together with $x^{(0)-1}_{ii}$ and $\bar{x}^{(0)-1}_{ii}$ subject to the relations $x^{(0)}_{ij} = \bar{x}^{(0)}_{ji} = 0$ for $i < j$ and

$$x^{(0)}_{ii} x^{(0)-1}_{ii} = 1, \qquad \bar{x}^{(0)}_{ii} \bar{x}^{(0)-1}_{ii} = 1$$

for all i. As in the orthogonal case, it suffices to verify that the images $\sigma^{(r)}_{ij}$ of the generators $s^{(r)}_{ij}$ in $\widehat{\mathcal{P}}^{\text{ext}}_{2n}$ are algebraically independent. Calculating the differentials and specializing the variables as in (2.80), we get

$$d\sigma^{(r)}_{ij} = \begin{cases} d\bar{x}^{(r)}_{j,i+1} + dx^{(r)}_{i,j-1} & \text{if} \quad i \quad \text{is odd,} \quad j \quad \text{is even,} \\ d\bar{x}^{(r)}_{j,i+1} - dx^{(r)}_{i,j+1} & \text{if} \quad i \quad \text{is odd,} \quad j \quad \text{is odd,} \\ d\bar{x}^{(r)}_{j,i-1} + dx^{(r)}_{i,j-1} & \text{if} \quad i \quad \text{is even,} \quad j \quad \text{is even,} \\ d\bar{x}^{(r)}_{j,i-1} - dx^{(r)}_{i,j+1} & \text{if} \quad i \quad \text{is even,} \quad j \quad \text{is odd,} \end{cases}$$

which shows that the differentials are linearly independent in this case as well. □

3 Representations of the quantum affine algebra

As in the Lie algebra representation theory, the representations of the quantum affine algebra associated with $\mathfrak{sl}_2$ plays a key role in the description of the representations of the quantum affine algebras $\mathrm{U}_q(\widehat{\mathfrak{a}})$; see [7], [8]. Finite-dimensional irreducible representations of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_2)$ were classified in [7]. In our proofs below the case of the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_2)$ will be similarly important for the general classification theorem. In order to make our arguments clearer, we first reproduce a proof of the classification theorem for the representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ following an approach used for the Yangian representations and which goes back to pioneering work of Tarasov [36, 37]. This approach is alternative to [7] and it also allows one to obtain a description of the finite-dimensional irreducible representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ as tensor products of the evaluation modules. The corresponding arguments were outlined in [28, Sec. 3.5].

Suppose that the complex number q is nonzero and not a root of unity. A representation L of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is called a *highest weight representation* if L is generated by a nonzero vector ζ (the *highest vector*) such that

$$\begin{aligned} &t_{ij}(u)\,\zeta = 0, && \bar{t}_{ij}(u)\,\zeta = 0 && \text{for} \quad 1 \leqslant i < j \leqslant N, \\ &t_{ii}(u)\,\zeta = \nu_i(u)\,\zeta, && \bar{t}_{ii}(u)\,\zeta = \bar{\nu}_i(u)\,\zeta && \text{for} \quad 1 \leqslant i \leqslant N, \end{aligned}$$

where $\nu(u) = (\nu_1(u), \dots, \nu_N(u))$ and $\bar\nu(u) = (\bar\nu_1(u), \dots, \bar\nu_N(u))$ are certain N-tuples of formal power series in u^{-1} and u, respectively:

$$\nu_i(u) = \sum_{r=0}^{\infty} \nu_i^{(r)} u^{-r}, \qquad \bar\nu_i(u) = \sum_{r=0}^{\infty} \bar\nu_i^{(r)} u^{r}. \tag{3.1}$$

We have $\nu_i^{(0)}\bar\nu_i^{(0)} = 1$ for each i due to the second set of relations in (2.40).

Note that this definition corresponds to *pseudo-highest weight representations* of the quantum loop algebras in the terminology of [8, Def. 12.2.4].

A standard argument shows that any finite-dimensional irreducible representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is a highest weight representation; cf. [8, Prop. 12.2.3]. Furthermore, Corollary 2.13 implies that given any formal series of the form (3.1) with $\nu_i^{(0)}\bar\nu_i^{(0)} = 1$ for all i, there exists a nontrivial *Verma module* $M(\nu(u);\bar\nu(u))$ which is defined as the quotient of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$, $\bar t_{ij}(u)$ for $i<j$ and $t_{ii}(u)-\nu_i(u)$, $\bar t_{ii}(u)-\bar\nu_i(u)$ for all i. Moreover, $M(\nu(u);\bar\nu(u))$ has a unique irreducible quotient $L(\nu(u);\bar\nu(u))$. Therefore, in order to describe all finite-dimensional irreducible representations of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$, we need to determine for which highest weights $(\nu(u);\bar\nu(u))$ the representation $L(\nu(u);\bar\nu(u))$ is finite-dimensional. By considering 'simple root embeddings' $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2) \hookrightarrow \mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$, the problem is largely reduced to the particular case $N=2$.

3.1 Representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$

Consider an arbitrary irreducible highest weight representation $L\big(\nu_1(u),\nu_2(u);\bar\nu_1(u),\bar\nu_2(u)\big)$ of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$.

Proposition 3.1. *Suppose that* $\dim L\big(\nu_1(u),\nu_2(u);\bar\nu_1(u),\bar\nu_2(u)\big) < \infty$. *Then there exist polynomials $Q(u)$ and $R(u)$ in u of the same degree such that the product of the constant term and the leading coefficient of each polynomial is equal to* 1, *and*

$$\frac{\nu_1(u)}{\nu_2(u)} = \frac{Q(u)}{R(u)} = \frac{\bar\nu_1(u)}{\bar\nu_2(u)}, \tag{3.2}$$

where the first equality is understood in the sense that the ratio of polynomials has to be expanded as a power series in u^{-1}, while for the second equality the same ratio has to be expanded as a power series in u.

Proof. By twisting the representation $L\big(\nu_1(u),\nu_2(u);\bar\nu_1(u),\bar\nu_2(u)\big)$ with an appropriate automorphism of the form (2.48), we may assume without loss of generality that $\nu_2(u) = \bar\nu_2(u) = 1$. Consider the vector subspace L of $L\big(\nu_1(u),1;\bar\nu_1(u),1\big)$ spanned by all vectors $t_{21}^{(i)}\zeta$, $i \geqslant 0$ and $\bar t_{21}^{(j)}\zeta$, $j \geqslant 1$. Since $\dim L < \infty$, the space L is spanned by the vectors $t_{21}^{(i)}\zeta$

and $\bar{t}_{21}^{(j)}\zeta$, where i and j run over some finite sets of values. This implies that for sufficiently large n and m any vector $c_n\, t_{21}^{(n)}\zeta + d_m \bar{t}_{21}^{(m)}\zeta$ is a linear combination of the spanning vectors $t_{21}^{(i)}\zeta$ and $\bar{t}_{21}^{(j)}\zeta$. Hence, there exist integers $n \geqslant 0$, $m \geqslant 1$ and complex numbers c_i, d_j such that

$$\sum_{i=0}^{n} c_i\, t_{21}^{(i)}\zeta + \sum_{j=1}^{m} d_j\, \bar{t}_{21}^{(j)}\zeta = 0,$$

and $c_n, d_m \neq 0$. Denote the linear combination which occurs on the left-hand side by ξ. Then $t_{12}^{(r)}\,\xi = 0$ for all $r \geqslant 1$. On the other hand, by the defining relations (2.42) we have

$$(u-v)\,\big(t_{12}(u)\,t_{21}(v) - t_{21}(v)\,t_{12}(u)\big) = (q-q^{-1})\,v\,\big(t_{22}(u)\,t_{11}(v) - t_{22}(v)\,t_{11}(u)\big).$$

Now multiply both sides by

$$\frac{1}{u-v} = \sum_{k=1}^{\infty} u^{-k} v^{k-1}$$

and take the coefficients of $u^{-r}v^{-s}$ on both sides to get

$$t_{12}^{(r)}\,t_{21}^{(s)} - t_{21}^{(s)}\,t_{12}^{(r)} = (q-q^{-1}) \sum_{p=1}^{r} \big(t_{22}^{(r-p)}\,t_{11}^{(s+p)} - t_{22}^{(s+p)}\,t_{11}^{(r-p)}\big).$$

Similarly, using (2.47) we get

$$t_{12}^{(r)}\,\bar{t}_{21}^{(s)} - \bar{t}_{21}^{(s)}\,t_{12}^{(r)} = (q-q^{-1}) \sum_{p=1}^{\min\{r,s\}} \big(t_{22}^{(r-p)}\,\bar{t}_{11}^{(s-p)} - \bar{t}_{22}^{(s-p)}\,t_{11}^{(r-p)}\big).$$

Since $t_{22}^{(r)}\,\zeta = \bar{t}_{22}^{(r)}\,\zeta = 0$ for $r \geqslant 1$, we find that

$$t_{12}^{(r)}\,t_{21}^{(s)}\,\zeta = (q-q^{-1})\,\nu_1^{(r+s)}\,\zeta, \qquad t_{12}^{(r)}\,\bar{t}_{21}^{(s)}\,\zeta = (q-q^{-1})\,(\bar{\nu}_1^{(s-r)} - \nu_1^{(r-s)})\,\zeta,$$

where $s \geqslant 1$ in the second relation and we assume $\nu_1^{(s)} = \bar{\nu}_1^{(s)} = 0$ for $s < 0$. Hence, taking the coefficient of ζ in $t_{12}^{(r)}\,\xi = 0$ we get

$$\sum_{i=0}^{n} c_i\,\nu_1^{(r+i)} + \sum_{j=1}^{m} d_j\big(\bar{\nu}_1^{(j-r)} - \nu_1^{(r-j)}\big) = 0$$

for all $r \geqslant 1$. This is equivalent to the relation

$$\begin{aligned}\nu_1(u)\left(\sum_{i=0}^{n} c_i u^i - \sum_{j=1}^{m} d_j u^{-j}\right) &= \sum_{i=0}^{n} c_i u^i\,\big(\nu_1^{(0)} + \dots + \nu_1^{(i)} u^{-i}\big)\\ &\quad - \sum_{j=1}^{m} d_j u^{-j}\,\big(\bar{\nu}_1^{(0)} + \dots + \bar{\nu}_1^{(j-1)} u^{j-1}\big).\end{aligned}$$

Now use the relations $\bar{t}_{12}^{(r)}\,\xi = 0$, $r \geqslant 0$. The defining relations (2.43) give

$$(u-v)\,\big(\bar{t}_{12}(u)\,\bar{t}_{21}(v)-\bar{t}_{21}(v)\,\bar{t}_{12}(u)\big)=(q-q^{-1})\,v\,\big(\bar{t}_{22}(u)\,\bar{t}_{11}(v)-\bar{t}_{22}(v)\,\bar{t}_{11}(u)\big).$$

Divide both sides by $u-v$ and use the expansion

$$\frac{v}{u-v}=-\sum_{k=0}^{\infty}u^k v^{-k}.$$

Comparing the coefficients of $u^r v^s$ on both sides we get

$$\bar{t}_{12}^{(r)}\,\bar{t}_{21}^{(s)}-\bar{t}_{21}^{(s)}\,\bar{t}_{12}^{(r)}=(q-q^{-1})\sum_{p=0}^{r}\big(\bar{t}_{22}^{(s+p)}\,\bar{t}_{11}^{(r-p)}-\bar{t}_{22}^{(r-p)}\,\bar{t}_{11}^{(s+p)}\big).$$

Similarly, using (2.44) we get

$$\bar{t}_{12}^{(r)}\,t_{21}^{(s)}-t_{21}^{(s)}\,\bar{t}_{12}^{(r)}=(q-q^{-1})\sum_{p=0}^{\min\{r,s\}}\big(t_{22}^{(s-p)}\,\bar{t}_{11}^{(r-p)}-\bar{t}_{22}^{(r-p)}\,t_{11}^{(s-p)}\big).$$

Hence,

$$\bar{t}_{12}^{(r)}\,\bar{t}_{21}^{(s)}\,\zeta=(q^{-1}-q)\,\bar{\nu}_1^{(r+s)}\,\zeta,\qquad \bar{t}_{12}^{(r)}\,t_{21}^{(s)}\,\zeta=(q-q^{-1})\,(\bar{\nu}_1^{(r-s)}-\nu_1^{(s-r)})\,\zeta,$$

where $s \geqslant 1$ in the first relation. Taking the coefficient of ζ in $\bar{t}_{12}^{(r)}\,\xi=0$ we get

$$\sum_{i=0}^{n}c_i\,\big(\bar{\nu}_1^{(r-i)}-\nu_1^{(i-r)}\big)-\sum_{j=1}^{m}d_j\,\bar{\nu}_1^{(r+j)}=0$$

for all $r \geqslant 0$. This is equivalent to the relation

$$\begin{aligned}\bar{\nu}_1(u)\left(\sum_{i=0}^{n}c_iu^i-\sum_{j=1}^{m}d_ju^{-j}\right)&=\sum_{i=0}^{n}c_iu^i\,\big(\nu_1^{(0)}+\dots+\nu_1^{(i)}u^{-i}\big)\\&\quad-\sum_{j=1}^{m}d_ju^{-j}\,\big(\bar{\nu}_1^{(0)}+\dots+\bar{\nu}_1^{(j-1)}u^{j-1}\big).\end{aligned}$$

Thus, both series $\nu_1(u)$ and $\bar{\nu}_1(u)$ are expansions of the same rational function in u,

$$\nu_1(u)=\frac{Q(u)}{R(u)}=\bar{\nu}_1(u),$$

where the polynomials $Q(u)$ and $R(u)$ have the required properties. □

Write decompositions

$$Q(u) = (\alpha_1 u + \alpha_1^{-1}) \dots (\alpha_k u + \alpha_k^{-1}),$$
$$R(u) = (\beta_1 u + \beta_1^{-1}) \dots (\beta_k u + \beta_k^{-1}),$$

where α_i and β_i are nonzero complex numbers. By twisting the finite-dimensional representation $L\big(\nu_1(u), \nu_2(u); \bar{\nu}_1(u), \bar{\nu}_2(u)\big)$ by an appropriate automorphism of the form (2.48), we can get another finite-dimensional representation such that the components of the highest weight have the form:

$$\begin{aligned}
\nu_1(u) &= (\alpha_1 + \alpha_1^{-1} u^{-1}) \dots (\alpha_k + \alpha_k^{-1} u^{-1}),\\
\nu_2(u) &= (\beta_1 + \beta_1^{-1} u^{-1}) \dots (\beta_k + \beta_k^{-1} u^{-1}),\\
\bar{\nu}_1(u) &= (\alpha_1 u + \alpha_1^{-1}) \dots (\alpha_k u + \alpha_k^{-1}),\\
\bar{\nu}_2(u) &= (\beta_1 u + \beta_1^{-1}) \dots (\beta_k u + \beta_k^{-1}).
\end{aligned} \tag{3.3}$$

For any pair of nonzero complex numbers α and β consider the corresponding irreducible highest weight representation $L(\alpha, \beta)$ of $\mathrm{U}_q(\mathfrak{gl}_2)$. That is, $L(\alpha, \beta)$ is generated by a nonzero vector ζ such that

$$\bar{t}_{12}\, \zeta = 0, \qquad t_{11}\, \zeta = \alpha \zeta, \qquad t_{22}\, \zeta = \beta \zeta.$$

The representation $L(\alpha, \beta)$ is finite-dimensional if and only if $\alpha/\beta = \pm q^m$ for some nonnegative integer m. We make $L(\alpha, \beta)$ into a module over the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ via the evaluation homomorphism $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2) \to \mathrm{U}_q(\mathfrak{gl}_2)$ given by the formulas (2.56). This evaluation module is a highest weight representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ with the highest weight

$$\big(\alpha + \alpha^{-1} u^{-1}, \beta + \beta^{-1} u^{-1};\ \alpha\, u + \alpha^{-1}, \beta\, u + \beta^{-1}\big).$$

The comultiplication map (2.54) allows us to regard the tensor product

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \dots \otimes L(\alpha_k, \beta_k) \tag{3.4}$$

as a representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$. Moreover, it follows easily from (2.54) that the cyclic span $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)(\zeta_1 \otimes \dots \otimes \zeta_k)$ is a highest weight representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ with the highest weight given by the formulas (3.3); here ζ_i denotes the highest vector of $L(\alpha_i, \beta_i)$. Our next goal is to show that under some additional conditions on the parameters α_i and β_i the tensor product module (3.4) is irreducible and, hence, isomorphic to $L(\nu_1(u), \nu_2(u); \bar{\nu}_1(u), \bar{\nu}_2(u))$. Namely, we will suppose that for every $i = 1, \dots, k-1$ the following condition holds: if the multiset $\{\alpha_r/\beta_s \mid i \leqslant r, s \leqslant k\}$ contains numbers of the form $\pm q^m$ with nonnegative integers m, then $\alpha_i/\beta_i = \pm q^{m_0}$ and m_0 is minimal amongst these nonnegative integers. The following proposition goes back to [37].

Proposition 3.2. *If the above condition on the parameters α_i and β_i holds, then the representation* (3.4) *of* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ *is irreducible.*

Proof. We follow the corresponding argument used in the Yangian case; see e.g. [28, Prop. 3.3.2]. Denote the representation (3.4) by L. We start by proving the following claim: any vector $\xi \in L$ satisfying $t_{12}(u)\xi = 0$ is proportional to $\zeta = \zeta_1 \otimes \ldots \otimes \zeta_k$. We use the induction on k. The claim is obvious for $k = 1$ so suppose that $k \geqslant 2$. Write any such vector ξ, which is assumed to be nonzero, in the form

$$\xi = \sum_{r=0}^{p} (t_{21})^r \zeta_1 \otimes \xi_r, \qquad \text{where} \quad \xi_r \in L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k)$$

and p is some nonnegative integer. Moreover, if $\alpha_1/\beta_1 = \pm q^m$ for some nonnegative integer m, then we will assume that $p \leqslant m$. We will also assume that $\xi_p \neq 0$. Using the coproduct formulas (2.54), we get

$$\sum_{r=0}^{p} \Big(t_{11}(u)(t_{21})^r \zeta_1 \otimes t_{12}(u)\xi_r + t_{12}(u)(t_{21})^r \zeta_1 \otimes t_{22}(u)\xi_r \Big) = 0. \tag{3.5}$$

By (2.56), we obtain

$$t_{11}(u)(t_{21})^r \zeta_1 = (t_{11} + \bar{t}_{11} u^{-1})(t_{21})^r \zeta_1 = \big(q^{-r}\alpha_1 + q^r \alpha_1^{-1} u^{-1}\big)(t_{21})^r \zeta_1, \tag{3.6}$$

and

$$t_{12}(u)(t_{21})^r \zeta_1 = u^{-1} \bar{t}_{12} (t_{21})^r \zeta_1 = u^{-1} \left(q^r - q^{-r}\right) \big(q^{r-1}\beta_1/\alpha_1 - q^{-r+1}\alpha_1/\beta_1\big)(t_{21})^{r-1}\zeta_1. \tag{3.7}$$

Taking the coefficient of $(t_{21})^p \zeta_1$ in (3.5) we get $t_{12}(u)\xi_p = 0$. By the induction hypothesis, applied to the representation $L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_k, \beta_k)$, the vector ξ_p must be proportional to $\zeta_2 \otimes \ldots \otimes \zeta_k$. Therefore, using again (2.54) and (2.56), we obtain

$$t_{22}(u)\xi_p = (\beta_2 + \beta_2^{-1} u^{-1}) \ldots (\beta_k + \beta_k^{-1} u^{-1})\, \xi_p. \tag{3.8}$$

The proof of the claim will be completed if show that $p = 0$. Suppose on the contrary that $p \geqslant 1$. Then taking the coefficient of $(t_{21})^{p-1}\zeta_1$ in (3.5) we derive

$$\begin{aligned} &\big(q^{-p+1}\alpha_1 + q^{p-1}\alpha_1^{-1}u^{-1}\big)\, t_{12}(u)\xi_{p-1} \\ &\qquad\qquad + u^{-1} \left(q^p - q^{-p}\right)\big(q^{p-1}\beta_1/\alpha_1 - q^{-p+1}\alpha_1/\beta_1\big)\, t_{22}(u)\, \xi_p = 0. \end{aligned}$$

Note that $t_{12}(u)\xi_{p-1}$ is a polynomial in u^{-1}. Taking $u = -q^{2p-2}\alpha_1^{-2}$ and using (3.8) we obtain the relation

$$\big(q^{2p-2} - (\alpha_1/\beta_1)^2\big)\big(q^{2p-2} - (\alpha_1/\beta_2)^2\big) \ldots \big(q^{2p-2} - (\alpha_1/\beta_k)^2\big) = 0,$$

where we also used the assumption that q is not a root of unity. However, this is impossible due to the conditions on the parameters α_i and β_i. Thus, p must be zero and the claim follows.

Now suppose that M is a nonzero submodule of L. Then M must contain a nonzero vector ξ such that $t_{12}(u)\xi = 0$. This can be seen by considering $\mathrm{U}_q(\mathfrak{gl}_2)$-weights of M. If $\eta \in M$ is a vector of weight (μ_1, μ_2), i.e., $t_{11}\eta = \mu_1\eta$ and $t_{22}\eta = \mu_2\eta$, then $t_{12}(u)\eta$ has weight $(q\mu_1, q^{-1}\mu_2)$. So it suffices to observe that the set of $\mathrm{U}_q(\mathfrak{gl}_2)$-weights of L has a maximal element with respect to the natural ordering on the set of weights.

Due to the claim proved above, the highest vector ζ belongs to M. It remains to show that the vector ζ is cyclic in L, that is, the submodule $K = \mathrm{U}_q(\widehat{\mathfrak{gl}}_2)\,\zeta$ coincides with L.

Note that all $\mathrm{U}_q(\mathfrak{gl}_2)$-weight spaces of L are finite-dimensional. Denote by L^* the restricted dual vector space to L which is the direct sum of the dual vector spaces to the $\mathrm{U}_q(\mathfrak{gl}_2)$-weight spaces of L. We equip L^* with the $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$-module structure defined by

$$(y\,\omega)(\eta) = \omega(\varkappa(y)\,\eta) \quad \text{for} \quad y \in \mathrm{U}_q(\widehat{\mathfrak{gl}}_2) \qquad \text{and} \qquad \omega \in L^*,\ \eta \in L, \tag{3.9}$$

where $\varkappa$ is the involutive anti-automorphism of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$, defined by

$$\varkappa:\ t_{ij}(u) \mapsto \bar{t}_{3-i,3-j}(u^{-1}), \qquad \bar{t}_{ij}(u) \mapsto t_{3-i,3-j}(u^{-1}). \tag{3.10}$$

The latter is the composition of the automorphism (2.49) and the anti-automorphism (2.51). The anti-automorphism $\varkappa$ commutes with the comultiplication Δ in the sense that

$$\Delta \circ \varkappa = (\varkappa \otimes \varkappa) \circ \Delta.$$

This implies the isomorphism of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$-modules:

$$L^* \cong L(\beta_1^{-1}, \alpha_1^{-1}) \otimes \ldots \otimes L(\beta_k^{-1}, \alpha_k^{-1}). \tag{3.11}$$

Moreover, the highest vector ζ_i^* of the module $L(\beta_i^{-1}, \alpha_i^{-1}) \cong L(\alpha_i, \beta_i)^*$ can be identified with the element of $L(\alpha_i, \beta_i)^*$ such that $\zeta_i^*(\zeta_i) = 1$ and $\zeta_i^*(\eta_i) = 0$ for all $\mathrm{U}_q(\mathfrak{gl}_2)$-weight vectors $\eta_i \in L(\alpha_i, \beta_i)$ whose weights are different from the weight of ζ_i.

Now suppose on the contrary that the submodule $K = \mathrm{U}_q(\widehat{\mathfrak{gl}}_2)\,\zeta$ of L is proper. The annihilator of K defined by

$$\operatorname{Ann} K = \{\omega \in L^* \mid \omega(\eta) = 0 \quad \text{for all} \quad \eta \in K\} \tag{3.12}$$

is a submodule of the $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$-module L^*, which does not contain the vector $\zeta_1^* \otimes \ldots \otimes \zeta_k^*$. However, this contradicts the claim verified in the first part of the proof, because the condition on the parameters α_i and β_i remain satisfied after we replace each α_i by β_i^{-1} and each β_i by α_i^{-1}. $\square$

We can now describe the finite-dimensional irreducible representations of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$.

Theorem 3.3. *The irreducible highest weight representation* $L\big(\nu_1(u),\nu_2(u);\bar\nu_1(u),\bar\nu_2(u)\big)$ *of* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ *is finite-dimensional if and only if there exists a polynomial* $P(u)$ *in* u *with constant term* 1 *such that*

$$\frac{\varepsilon_1\,\nu_1(u)}{\varepsilon_2\,\nu_2(u)}=q^{-\deg P}\cdot\frac{P(uq^2)}{P(u)}=\frac{\varepsilon_1\,\bar\nu_1(u)}{\varepsilon_2\,\bar\nu_2(u)} \tag{3.13}$$

for some $\varepsilon_1,\varepsilon_2\in\{-1,1\}$. *In this case* $P(u)$ *is unique.*

Proof. Suppose that the representation $L\big(\nu_1(u),\nu_2(u);\bar\nu_1(u),\bar\nu_2(u)\big)$ is finite-dimensional. As was shown above, we may assume without loss of generality that the components of the highest weight have the form (3.3). Moreover, we may re-enumerate the parameters α_i and β_i to satisfy the conditions of Proposition 3.2. By that proposition, the representation $L\big(\nu_1(u),\nu_2(u);\bar\nu_1(u),\bar\nu_2(u)\big)$ is isomorphic to the tensor product (3.4). Therefore, all ratios α_i/β_i must have the form $\pm q^{m_i}$, where each m_i is a nonnegative integer. Then the polynomial

$$P(u)=\prod_{i=1}^{k}(1+\beta_i^2\,u)(1+\beta_i^2\,q^2\,u)\dots(1+\beta_i^2\,q^{2m_i-2}\,u) \tag{3.14}$$

satisfies (3.13) with an appropriate choice of the signs $\varepsilon_1,\varepsilon_2\in\{-1,1\}$.

Conversely, suppose (3.13) holds for a polynomial $P(u)=(1+\gamma_1u)\dots(1+\gamma_pu)$ and some $\varepsilon_1,\varepsilon_2\in\{-1,1\}$. Choose square roots β_i so that $\beta_i^2=\gamma_i$ for $i=1,\dots,p$ and set

$$\begin{aligned}
\mu_1(u)&=(\beta_1q+\beta_1^{-1}q^{-1}u^{-1})\dots(\beta_pq+\beta_p^{-1}q^{-1}u^{-1}),\\
\mu_2(u)&=(\beta_1+\beta_1^{-1}u^{-1})\dots(\beta_p+\beta_p^{-1}u^{-1}),\\
\bar\mu_1(u)&=(\beta_1q\,u+\beta_1^{-1}q^{-1})\dots(\beta_pq\,u+\beta_p^{-1}q^{-1}),\\
\bar\mu_2(u)&=(\beta_1u+\beta_1^{-1})\dots(\beta_pu+\beta_p^{-1}).
\end{aligned}$$

Consider the tensor product module

$$L(\beta_1\,q,\beta_1)\otimes L(\beta_2\,q,\beta_2)\otimes\dots\otimes L(\beta_p\,q,\beta_p)$$

of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$. This module is finite-dimensional and the cyclic $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$-span of the tensor product of the highest vectors of $L(\beta_i\,q,\beta_i)$ is a highest weight module with the highest weight $(\mu_1(u),\mu_2(u);\bar\mu_1(u),\bar\mu_2(u))$. Hence, the irreducible highest weight module $L(\mu_1(u),\mu_2(u);\bar\mu_1(u),\bar\mu_2(u))$ is finite-dimensional. Since

$$\frac{\mu_1(u)}{\mu_2(u)}=\frac{\varepsilon_1\,\nu_1(u)}{\varepsilon_2\,\nu_2(u)},\qquad\frac{\bar\mu_1(u)}{\bar\mu_2(u)}=\frac{\varepsilon_1\,\bar\nu_1(u)}{\varepsilon_2\,\bar\nu_2(u)}$$

there exist automorphisms of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ of the form (2.48) and (2.50) such that their composition with the representation $L(\mu_1(u),\mu_2(u);\bar\mu_1(u),\bar\mu_2(u))$ is isomorphic to the irreducible highest weight representation $L\big(\nu_1(u),\nu_2(u);\bar\nu_1(u),\bar\nu_2(u)\big)$. Thus, the latter is also finite-dimensional.

The uniqueness of $P(u)$ is easily verified. □

The above arguments imply that, up to twisting with an automorphism of the form (2.48), every finite-dimensional irreducible representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ is isomorphic to a tensor product representation of the form (3.4). We will now establish a criterion of irreducibility of such representations which we will use in Sec. 4.2 below. It is essentially a version of the well-known results; see [8, Ch. 12], [37].

We will define a *q-string* to be any subset of $\mathbb{C}$ of the form $\{\beta, \beta q, \dots, \beta q^p\}$, where $\beta \neq 0$ and p is a nonnegative integer. Since q is not a root of unity, the q-strings $\{\beta, \beta q, \dots, \beta q^p\}$ and $\{-\beta, -\beta q, \dots, -\beta q^p\}$ have no common elements. Their union will be called a *q-spiral*. Two q-spirals S_1 and S_2 are *in general position* if either

(i) $S_1 \cup S_2$ is not a q-spiral; or

(ii) $S_1 \subset S_2$, or $S_2 \subset S_1$.

Given a pair of nonzero complex numbers (α, β) with $\alpha/\beta = \pm q^m$ and $m \in \mathbb{Z}_+$ the corresponding q-spiral is defined as

$$S_q(\alpha, \beta) = \{\beta, \beta q, \dots, \beta q^{m-1}\} \cup \{-\beta, -\beta q, \dots, -\beta q^{m-1}\}.$$

If $\alpha = \beta$, then the set $S_q(\alpha, \beta)$ is regarded to be empty. Note that changing sign of α or β does not affect the q-spiral.

Denote by L the tensor product (3.4), where for all ratios we have $\alpha_i/\beta_i = \pm q^{m_i}$ for some nonnegative integers m_i.

Corollary 3.4. *The representation L of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ is irreducible if and only if the q-spirals $S_q(\alpha_1, \beta_1), \dots, S_q(\alpha_k, \beta_k)$ are pairwise in general position.*

Proof. Suppose that the q-spirals are pairwise in general position and assume first that the nonnegative integers m_i satisfy the inequalities $m_1 \leqslant \dots \leqslant m_k$. This implies that the condition of Proposition 3.2 on the parameters α_i and β_i holds. Indeed, if this is not the case, then $\alpha_r/\beta_s = \pm q^p$ for some $i \leqslant r, s \leqslant k$ and a nonnegative integer p with $p < m_i$. By our assumption, $r \neq s$. Suppose that $r > s$. Then $m_i \leqslant m_s$ so we may assume that $s = i$. The condition $\alpha_r = \pm\beta_i q^p$ means that α_r belongs to the q-spiral $S_q(\alpha_i, \beta_i)$. Hence, the q-spirals $S_q(\alpha_i, \beta_i)$ and $S_q(\alpha_r, \beta_r)$ are not in general position, a contradiction. The opposite inequality $r < s$ leads to a similar contradiction.

Thus, L is irreducible by Proposition 3.2. It is easy to verify (cf. [28, Prop. 3.2.10]) that any permutation of the tensor factors yields an isomorphic irreducible representation.

Conversely, let $k = 2$ and let $L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2)$ be irreducible. Suppose that the q-spirals $S_q(\alpha_1, \beta_1)$ and $S_q(\alpha_2, \beta_2)$ are not in general position. Then the q-spirals $S_q(\alpha_1, \beta_2)$ and $S_q(\alpha_2, \beta_1)$ are in general position. Hence, the representation $L(\alpha_1, \beta_2) \otimes L(\alpha_2, \beta_1)$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ is irreducible due to the first part of the proof. Comparing the dimension of this representation with the dimension of $L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2)$ we come to a contradiction.

The case of general $k \geqslant 3$ is reduced to $k = 2$ by permuting the tensor factors in (3.4), if necessary. Indeed, if L is irreducible, but a pair of q-spirals $S_q(\alpha_i, \beta_i)$ and $S_q(\alpha_j, \beta_j)$ is not in general position, then we may assume that i and j are adjacent. However, the representation $L(\alpha_i, \beta_i) \otimes L(\alpha_j, \beta_j)$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ is reducible as shown above. This implies that L is reducible, a contradiction. $\square$

Remark 3.5. An isomorphism between the RTT-presentation of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ and its new realization is provided in [11] by using the Gauss decomposition of the matrices $T(u)$ and $\overline{T}(u)$; cf. [5], [14]. Thus, Theorem 3.3 provides a description of finite-dimensional irreducible representations of the algebras $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ and $\mathrm{U}_q(\widehat{\mathfrak{sl}}_2)$ in terms of the new realization via this isomorphism. This argument is alternative to [7] and it is straightforward to apply this description to prove the classification theorem for finite-dimensional irreducible representations of an arbitrary quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{a}})$; cf. the A type case considered below. $\square$

3.2 Representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$

The evaluation homomorphism $\pi : \mathrm{U}_q(\widehat{\mathfrak{gl}}_N) \to \mathrm{U}_q(\mathfrak{gl}_N)$ defined in (2.56) allows us to regard any $\mathrm{U}_q(\mathfrak{gl}_N)$-module as a $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$-module. In particular, we thus obtain the *evaluation modules* $L(\mu)$ over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$; see Sec. 2.1.

As we pointed out in the beginning of Sec. 3, in order to describe all finite-dimensional irreducible representations of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$, we need to determine for which highest weights $(\nu(u); \bar{\nu}(u))$ the representation $L(\nu(u); \bar{\nu}(u))$ is finite-dimensional. These conditions are provided by the following theorem which is essentially equivalent to [8, Theorem 12.2.6].

Theorem 3.6. *The irreducible highest weight representation $L\big(\nu(u); \bar{\nu}(u)\big)$ of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is finite-dimensional if and only if there exist polynomials $P_1(u), \dots, P_{N-1}(u)$ in u, all with constant term* 1, *such that*

$$\frac{\varepsilon_i\, \nu_i(u)}{\varepsilon_{i+1}\, \nu_{i+1}(u)} = q^{-\deg P_i} \cdot \frac{P_i(uq^2)}{P_i(u)} = \frac{\varepsilon_i\, \bar{\nu}_i(u)}{\varepsilon_{i+1}\, \bar{\nu}_{i+1}(u)} \tag{3.15}$$

for $i = 1, \dots, N-1$ and some $\varepsilon_j \in \{-1, 1\}$. The polynomials $P_1(u), \dots, P_{N-1}(u)$ are determined uniquely while the tuple $(\varepsilon_1, \dots, \varepsilon_N)$ is determined uniquely, up to the simultaneous change of sign of the ε_i.

Proof. Suppose that the representation $L\big(\nu(u); \bar{\nu}(u)\big)$ is finite-dimensional. Let us fix $k \in \{0, \dots, N-2\}$ and let $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ act on $L\big(\nu(u); \bar{\nu}(u)\big)$ via the homomorphism $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2) \to \mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ which sends $t_{ij}(u)$ and $\bar{t}_{ij}(u)$ to $t_{k+i,k+j}(u)$ and $\bar{t}_{k+i,k+j}(u)$, respectively, for any

$i, j \in \{1, 2\}$. The cyclic $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$-span of the highest vector of $L\big(\nu(u); \bar{\nu}(u)\big)$ is a highest weight representation of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ with the highest weight

$$\big(\nu_{k+1}(u), \nu_{k+2}(u); \bar{\nu}_{k+1}(u), \bar{\nu}_{k+2}(u)\big).$$

Its irreducible quotient is finite-dimensional, and so the required conditions follow from Theorem 3.3.

Conversely, taking into account the automorphisms (2.48) and (2.50), it is enough to show that given any set of polynomials $P_1(u), \dots, P_{N-1}(u)$ in u with constant terms equal to 1, there exists an irreducible finite-dimensional representation whose highest weight satisfies (3.15). Such a representation can be constructed by using the following inductive procedure. Consider the irreducible highest weight representation $L(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_N)$ with the highest weight

$$\lambda = (d q^{m_1}, \dots, d q^{m_N}),$$

where d is a nonzero complex number and the integers m_i satisfy $m_1 \geqslant \dots \geqslant m_N$. This representation is finite-dimensional and we regard it as the evaluation module over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ by using the homomorphism (2.56). By the first part of the proof we can associate a family of polynomials $P_1(u), \dots, P_{N-1}(u)$ to any finite-dimensional representation $L(\nu(u); \bar{\nu}(u))$. Let ζ and ξ be the highest vectors of the representations $L(\lambda)$ and $L(\nu(u); \bar{\nu}(u))$, respectively, and equip $L(\lambda) \otimes L(\nu(u); \bar{\nu}(u))$ with the $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$-module structure by using the coproduct (2.54). It is easily verified that the cyclic span $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)(\zeta \otimes \xi)$ is a highest weight representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ such that

$$t_{ii}(u)(\zeta \otimes \xi) = \big(d\, q^{m_i} + d^{-1} q^{-m_i} u^{-1}\big)\, \nu_i(u)\, (\zeta \otimes \xi),$$
$$\bar{t}_{ii}(u)(\zeta \otimes \xi) = \big(d^{-1} q^{-m_i} + d\, q^{m_i} u\big)\, \bar{\nu}_i(u)\, (\zeta \otimes \xi).$$

Hence, the irreducible quotient of this representation corresponds to the family of polynomials $Q_1(u)P_1(u), \dots, Q_{N-1}(u)P_{N-1}(u)$, where

$$Q_i(u) = (1 + d^2 q^{2m_{i+1}} u)(1 + d^2 q^{2m_{i+1}+2} u) \dots (1 + d^2 q^{2m_i - 2} u), \qquad i = 1, \dots, N-1.$$

Starting from the trivial representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ and choosing appropriate parameters d and m_i we will be able to produce a finite-dimensional highest weight representation associated with an arbitrary family of polynomials $P_1(u), \dots, P_{N-1}(u)$ by iterating this construction. The last statement of the theorem is easily verified. $\square$

The polynomials $P_1(u), \dots, P_{N-1}(u)$ introduced in Theorem 3.6 are called the *Drinfeld polynomials* of the representation $L\big(\nu(u); \bar{\nu}(u)\big)$. Moreover, any finite-dimensional irreducible representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ is obtained from a representation $L\big(\nu(u); \bar{\nu}(u)\big)$ associated with the tuple $(\varepsilon_1, \dots, \varepsilon_N) = (1, \dots, 1)$ by twisting with an automorphism of the form (2.50). Note also that the evaluation module $L(\mu)$ with

$$\mu_i = q^{m_i}, \qquad i = 1, \dots, N,$$

where $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_N$ are arbitrary integers, is a representation associated with the tuple $(1, \dots, 1)$. Its Drinfeld polynomials are given by

$$P_i(u) = (1 + q^{2m_{i+1}} u)(1 + q^{2m_{i+1}+2} u) \dots (1 + q^{2m_i - 2} u),$$

for $i = 1, \dots, N-1$.

A description of finite-dimensional irreducible representations of the extended algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ can be easily obtained from that of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$. Namely, every finite-dimensional irreducible representation of $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ is isomorphic to the highest weight representation $L(\nu(u); \bar{\nu}(u))$. The latter is defined in the same way as for the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ except that the relations $\nu_i^{(0)} \bar{\nu}_i^{(0)} = 1$ for the series (3.1) are replaced by the condition that all constants $\nu_i^{(0)}$ and $\bar{\nu}_i^{(0)}$ are nonzero. We have the following corollary of Theorem 3.6.

Corollary 3.7. *The irreducible highest weight representation $L\big(\nu(u); \bar{\nu}(u)\big)$ of the algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ is finite-dimensional if and only if there exist polynomials $P_1(u), \dots, P_{N-1}(u)$ in u, all with constant term 1, and nonzero constants $\phi_1, \dots, \phi_N$ such that*

$$\frac{\phi_i\, \nu_i(u)}{\phi_{i+1}\, \nu_{i+1}(u)} = q^{-\deg P_i} \cdot \frac{P_i(uq^2)}{P_i(u)} = \frac{\phi_i\, \bar{\nu}_i(u)}{\phi_{i+1}\, \bar{\nu}_{i+1}(u)} \tag{3.16}$$

for $i = 1, \dots, N-1$. The polynomials $P_1(u), \dots, P_{N-1}(u)$ are determined uniquely while the tuple $(\phi_1, \dots, \phi_N)$ is determined uniquely, up to a common factor.

Proof. By twisting the representation $L\big(\nu(u); \bar{\nu}(u)\big)$ by an appropriate automorphism of the algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_N)$ of the form (2.66), we can get the representation where all central elements $t_{ii}^{(0)}\, \bar{t}_{ii}^{(0)}$, $i = 1, \dots, N$, act as the identity operators. Therefore, due to (2.65), we get the irreducible highest weight representation of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$, such that the components of the highest weight have the form $\phi_i \nu_i(u)$ and $\phi_i \bar{\nu}_i(u)$. Now all statements follow from Theorem 3.6. □

4 Representations of the twisted q-Yangians

We will combine the approaches of Sec. 3 and [28, Ch. 4] to classify the finite-dimensional irreducible representations of the twisted q-Yangians $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$. As with the quantum affine algebras, the case $n = 1$ will play a key role. We start by proving some general results about highest weight representation of the twisted q-Yangians.

4.1 Highest weight representations

As we recalled in Sec. 2.4, the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ can be defined as the algebra generated by the elements $s_{ij}^{(r)}$ with $r \geqslant 0$ and $1 \leqslant i, j \leqslant 2n$ and by the elements $s_{i,i+1}^{(0)-1}$

with $i = 1, 3, \dots, 2n-1$. The defining relations are written in terms of the generating series (2.69) and they take the form (2.74) together with the relations

$$s_{ij}^{(0)} = 0 \quad \text{for} \quad i < j \quad \text{unless} \quad j = i+1 \quad \text{with} \quad i \ \text{ odd}$$

and

$$s_{i,i+1}^{(0)} s_{i,i+1}^{(0)-1} = s_{i,i+1}^{(0)-1} s_{i,i+1}^{(0)} = 1, \qquad i = 1, 3, \dots, 2n-1. \tag{4.1}$$

Observe that given any formal series $g(u)$ in u^{-1} of the form

$$g(u) = g_0 + g_1 u^{-1} + g_2 u^{-2} + \dots, \qquad g_0 \neq 0,$$

the mapping

$$s_{ij}(u) \mapsto g(u)\, s_{ij}(u) \tag{4.2}$$

defines an automorphism of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$.

Given any tuple $(\psi_1, \dots, \psi_{2n})$ of nonzero complex numbers, the mapping

$$s_{ij}(u) \mapsto \psi_i\, \psi_j\, s_{ij}(u) \tag{4.3}$$

defines another automorphism of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$.

Furthermore, the mapping

$$\varkappa : s_{ij}(u) \mapsto s_{2n-j+1, 2n-i+1}(u) \tag{4.4}$$

defines an involutive anti-automorphism of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$. This can be verified directly from the defining relations. Alternatively, one can show that the mappings (2.49) and (2.51) respectively define an automorphism and anti-automorphism of the extended algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_{2n})$, and their composition $\varkappa$ preserves the subalgebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$.

We will use the elements $\bar{s}_{ij}^{(r)}$ of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ which are defined as the coefficients of the power series $\bar{s}_{ij}(u)$ in u; see (2.75). The relationship between the elements is given by the formulas[1]

$$\begin{aligned}(u^{-1}q - uq^{-1})\, \bar{s}_{ij}(u) &= \\ &(uq^{\delta_{ij}} - u^{-1}q^{-\delta_{ij}})\, s_{ji}(u^{-1}) + (q - q^{-1})(u^{-1}\delta_{i<j} + u\,\delta_{j<i})\, s_{ij}(u^{-1}).\end{aligned} \tag{4.5}$$

For the rest of this section we will suppose that the complex number q is nonzero and not a root of unity. Recall the function $\varsigma : \{1, 2, \dots, 2n\} \to \{\pm 1, \pm 3, \dots, \pm(2n-1)\}$ defined in (2.33).

[1]The corresponding relation (3.62) in [29] should be corrected by swapping $\delta_{i<j}$ and $\delta_{j<i}$.

Definition 4.1. A representation V of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ is called a *highest weight representation* if V is generated by a nonzero vector ξ (the *highest vector*) such that

$$
\begin{aligned}
s_{kl}(u)\,\xi &= 0, && \text{for} \quad \varsigma(k)+\varsigma(l)>0,\\
s_{2i,2i-1}(u)\,\xi &= \mu_i(u)\,\xi, && \text{for} \quad 1\leqslant i\leqslant n,\\
\bar{s}_{2i,2i-1}(u)\,\xi &= \bar{\mu}_i(u)\,\xi, && \text{for} \quad 1\leqslant i\leqslant n,
\end{aligned}
\tag{4.6}
$$

where $\mu(u)=(\mu_1(u),\dots,\mu_n(u))$ and $\bar{\mu}(u)=(\bar{\mu}_1(u),\dots,\bar{\mu}_n(u))$ are certain n-tuples of formal power series in u^{-1} and u, respectively:

$$
\mu_i(u)=\sum_{r=0}^{\infty}\mu_i^{(r)}\,u^{-r}, \qquad \bar{\mu}_i(u)=\sum_{r=0}^{\infty}\bar{\mu}_i^{(r)}\,u^{r}. \tag{4.7}
$$

Due to (4.5), the first relation in (4.6) is equivalent to

$$
s_{ij}(u)\,\xi=\bar{s}_{ij}(u)\,\xi=0 \tag{4.8}
$$

for $j=1,3,\dots,2n-1$ and $i=1,2,\dots,j$. □

The definition of the highest weight representation is consistent with a particular choice of the positive root system of type C_n. Namely, the root system Φ is the subset of vectors in $\mathbb{R}^n$ of the form

$$
\pm\,2\,\varepsilon_i \quad \text{with} \quad 1\leqslant i\leqslant n \qquad \text{and} \qquad \pm\,\varepsilon_i\pm\varepsilon_j \quad \text{with} \quad 1\leqslant i<j\leqslant n,
$$

where ε_i denotes the n-tuple which has 1 on the i-th position and zeros elsewhere. Partition this set into positive and negative roots $\Phi=\Phi^+\cup(-\Phi^+)$, where the set of positive roots Φ^+ consists of the vectors

$$
2\,\varepsilon_i \quad \text{with} \quad 1\leqslant i\leqslant n \qquad \text{and} \qquad \varepsilon_i+\varepsilon_j, \quad -\,\varepsilon_i+\varepsilon_j \quad \text{with} \quad 1\leqslant i<j\leqslant n. \tag{4.9}
$$

We will regard $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ as a subalgebra of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ defined via the embedding

$$
s_{ij}\mapsto s_{ij}^{(0)}. \tag{4.10}
$$

By (2.81) we have

$$
s_{i,i+1}\,s_{kl}(u)=q^{\delta_{ik}+\delta_{il}-\delta_{i+1,k}-\delta_{i+1,l}}\,s_{kl}(u)\,s_{i,i+1}, \tag{4.11}
$$

for any $i=1,3,\dots,2n-1$. Hence, the generating series $s_{kl}(u)$ with $\varsigma(k)+\varsigma(l)>0$ can be associated with the elements of Φ^+ listed in (4.9) as follows:

$$
\begin{aligned}
2\,\varepsilon_i\longleftrightarrow s_{2i-1,2i-1}(u), \qquad \varepsilon_i+\varepsilon_j &\longleftrightarrow \{s_{2i-1,2j-1}(u),s_{2j-1,2i-1}(u)\},\\
-\varepsilon_i+\varepsilon_j &\longleftrightarrow \{s_{2i,2j-1}(u),s_{2j-1,2i}(u)\}.
\end{aligned}
\tag{4.12}
$$

Here the commutative subalgebra of $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ generated by the elements $s_{i,i+1}$ with $i=1,3,\dots,2n-1$ plays the role of a Cartan subalgebra; cf. [27].

Theorem 4.2. *Any finite-dimensional irreducible representation V of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ is a highest weight representation. Moreover, V contains a unique, up to a constant factor, highest vector.*

Proof. We use a standard argument with some necessary modifications; cf. [8, Sec. 12.2] and [28, Sec. 4.2]. Set

$$\begin{aligned} V^0 = \{\eta \in V \mid s_{ij}(u)\,\eta = \bar{s}_{ij}(u)\,\eta = 0,& \\ j = 1, 3, \dots, 2n-1 \qquad \text{and} \qquad i = 1, 2, \dots, j\}.& \end{aligned} \tag{4.13}$$

Equivalently, V^0 is spanned by the vectors annihilated by all operators $s_{kl}(u)$ such that $\varsigma(k) + \varsigma(l) > 0$. Let us show that V^0 is nonzero. The operators $s_{2i-1,2i} = s^{(0)}_{2i-1,2i}$ with $i = 1, \dots, n$ pairwise commute on V and so V contains a common eigenvector θ for these operators:

$$s_{2i-1,2i}\,\theta = \rho_i\,\theta, \qquad i = 1, \dots, n.$$

By (4.11), every coefficient of the series $s_{kl}(u)\,\theta$ with $\varsigma(k) + \varsigma(l) > 0$ is again a common eigenvector for the operators $s_{2i-1,2i}$, whose eigenvalues have the form $\rho_i\, q^{\alpha_i}$, $i = 1, \dots, n$, where $\alpha = \alpha_1\varepsilon_1 + \dots + \alpha_n\varepsilon_n$ is the element of Φ^+ associated with $s_{kl}(u)$ by (4.12). Since the sets of eigenvalues obtained in this way are distinct and $\dim V < \infty$, we can conclude that there exists a nonzero element of V annihilated by all operators $s_{kl}(u)$ with $\varsigma(k)+\varsigma(l) > 0$. Thus, $V^0 \neq \{0\}$.

Next, we show that the subspace V^0 is invariant with respect to the action of all operators $s_{b+1,b}(v)$ and $\bar{s}_{b+1,b}(v)$ with odd b. Let us show first that if $\eta \in V^0$, then for any odd a and $i \leqslant a$ we have

$$s_{ia}(u)\, s_{b+1,b}(v)\,\eta = 0. \tag{4.14}$$

If $i \leqslant b$, then this follows by (2.74) with $j = b + 1$. If $i > b$, then $b + 1 \leqslant i \leqslant a$ and (4.14) follows by the application of (2.74), where i, j, a, b are respectively replaced with $b + 1, i, b, a$, and u is swapped with v.

Furthermore, three more relations of the form (4.14) where $s_{ia}(u)$ is replaced with $\bar{s}_{ia}(u)$ or $s_{b+1,b}(v)$ is replaced with $\bar{s}_{b+1,b}(v)$, are verified by exactly the same argument with the use of the corresponding relation in (2.76) instead of (2.72).

A similar argument shows that all operators $s^{(r)}_{a+1,a}$ and $\bar{s}^{(r)}_{a+1,a}$ on the space V^0 with odd a and $r \geqslant 0$ pairwise commute. Indeed, suppose that both a and b are odd and $a \leqslant b$. The application of (2.74) with $i = a + 1$ and $j = b + 1$ proves that all operators $s^{(r)}_{a+1,a}$ pairwise commute. The remaining commutativity relations are verified in the same way with the use of (2.76).

Thus, the operators $s^{(r)}_{a+1,a}$ and $\bar{s}^{(r)}_{a+1,a}$ on the space V^0 with odd a and $r \geqslant 0$ are simultaneously diagonalizable. We let ξ be a simultaneous eigenvector for all these operators. Then $V = \mathrm{Y}'_q(\mathfrak{sp}_{2n})\xi$ by the irreducibility of V, and ξ is a highest weight vector. By considering the $\mathrm{U}'_q(\mathfrak{sp}_{2n})$-weights of V and using (4.11) we may also conclude that ξ is determined uniquely, up to a constant factor. $\square$

Consider now the tuples $\mu(u) = (\mu_1(u), \dots, \mu_n(u))$ and $\bar{\mu}(u) = (\bar{\mu}_1(u), \dots, \bar{\mu}_n(u))$ of arbitrary formal power series of the form (4.7).

The *Verma module* $M(\mu(u); \bar{\mu}(u))$ over the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ is the quotient of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ by the left ideal generated by all coefficients of the series $s_{kl}(u)$ for $\varsigma(k)+\varsigma(l) > 0$, $s_{2i,2i-1}(u) - \mu_i(u)$ and $\bar{s}_{2i,2i-1}(u) - \bar{\mu}_i(u)$ for $i = 1, \dots, n$.

Clearly, the Verma module $M(\mu(u); \bar{\mu}(u))$ is a highest weight representation of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ with the highest weight $(\mu(u); \bar{\mu}(u))$. Moreover, any highest weight representation with the same highest weight is isomorphic to a quotient of $M(\mu(u); \bar{\mu}(u))$.

The Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ (Proposition 2.15) implies that the ordered monomials of the form

$$s^{(r_1)}_{i_1 j_1} \dots s^{(r_m)}_{i_m j_m} 1, \qquad m \geqslant 0, \qquad \varsigma(i_a) + \varsigma(j_a) < 0, \tag{4.15}$$

form a basis of $M(\mu(u); \bar{\mu}(u))$. Moreover, using (4.11) and considering the weights of $M(\mu(u); \bar{\mu}(u))$ with respect to the operators $s_{i,i+1}$ with odd i, we derive that the Verma module $M(\mu(u); \bar{\mu}(u))$ possesses a unique maximal proper submodule K.

The *irreducible highest weight representation* $V(\mu(u); \bar{\mu}(u))$ of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ with the highest weight $(\mu(u); \bar{\mu}(u))$ is defined as the quotient of the Verma module $M(\mu(u); \bar{\mu}(u))$ by the submodule K.

Due to Theorem 4.2, all finite-dimensional irreducible representations of the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ have the form $V(\mu(u); \bar{\mu}(u))$ for a certain highest weight $(\mu(u); \bar{\mu}(u))$. Hence, in order to classify such representations it remains to describe the set of highest weights $(\mu(u); \bar{\mu}(u))$ such that $V(\mu(u); \bar{\mu}(u))$ is finite-dimensional. As with the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ (see Sec. 3), a key role will be played by the particular case $\mathrm{Y}'_q(\mathfrak{sp}_2)$ which we consider in the next section.

4.2 Representations of $\mathrm{Y}'_q(\mathfrak{sp}_2)$

The irreducible highest weight representations $V(\mu(u); \bar{\mu}(u))$ of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ are parameterized by formal series of the form

$$\begin{aligned} \mu(u) &= \mu^{(0)} + \mu^{(1)} u^{-1} + \mu^{(2)} u^{-2} + \dots, \\ \bar{\mu}(u) &= \bar{\mu}^{(0)} + \bar{\mu}^{(1)} u + \bar{\mu}^{(2)} u^2 + \dots, \qquad \mu^{(r)}, \bar{\mu}^{(r)} \in \mathbb{C}. \end{aligned} \tag{4.16}$$

The highest vector ξ of $V(\mu(u); \bar{\mu}(u))$ satisfies the conditions

$$s_{11}(u)\, \xi = 0, \qquad s_{21}(u)\, \xi = \mu(u)\, \xi, \qquad \bar{s}_{21}(u)\, \xi = \bar{\mu}(u)\, \xi.$$

Note that due to the relations (4.5), the vector ξ is also an eigenvector for the operators $s_{12}(u)$ and $\bar{s}_{12}(u)$,

$$
\begin{aligned}
s_{12}(u)\,\xi &= \mu'(u)\,\xi, \qquad \mu'(u) = \frac{(q^2-1)\,\mu(u) + (1-u^2q^2)\,\bar{\mu}(u^{-1})}{q(u^2-1)},\\
\bar{s}_{12}(u)\,\xi &= \bar{\mu}'(u)\,\xi, \qquad \bar{\mu}'(u) = \frac{(q^2-1)\,\bar{\mu}(u) + (1-u^2q^2)\,\mu(u^{-1})}{q(u^2-1)},
\end{aligned}
\tag{4.17}
$$

where $\mu'(u)$ and $\bar{\mu}'(u)$ are regarded as formal series in u^{-1} and u, respectively.

Proposition 4.3. *If the representation $V(\mu(u);\bar{\mu}(u))$ of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ is finite-dimensional, then both coefficients $\mu^{(0)}$ and $\bar{\mu}^{(0)}$ in (4.16) are nonzero.*

Proof. The constant term $\mu'^{(0)}$ of the series $\mu'(u)$ is nonzero due to the relation (4.1). By (4.17) we have $\mu'^{(0)} = -q\bar{\mu}^{(0)}$ and so, $\bar{\mu}^{(0)} \neq 0$. Furthermore, consider the restriction of $V(\mu(u);\bar{\mu}(u))$ to the subalgebra $\mathrm{U}'_q(\mathfrak{sp}_2)$ defined by the embedding (4.10). The cyclic span $\mathrm{U}'_q(\mathfrak{sp}_2)\,\xi$ of the highest vector is a finite-dimensional representation of the subalgebra with the highest weight $(\mu^{(0)};\mu'^{(0)})$. However, as was pointed out in the proof of Proposition 2.7, this implies that $\mu^{(0)} \neq 0$. □

The following is an analogue of Proposition 3.1 and its proof follows a similar approach.

Proposition 4.4. *Suppose that $\dim V(\mu(u);\bar{\mu}(u)) < \infty$. Then there exists a polynomial $Q(u)$ in u of even degree with the constant term equal to 1 such that*

$$
\frac{\bar{\mu}(u^{-1})}{\mu(u)} = \frac{u^{\deg Q}\,Q(u^{-1})}{Q(u)}. \tag{4.18}
$$

Proof. By twisting the representation $V(\mu(u);\bar{\mu}(u))$ with an automorphism of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ of the form (4.2) we get a representation isomorphic to $V(g(u)\mu(u);g(u^{-1})\bar{\mu}(u))$. Since $\mu'^{(0)} \neq 0$, we may consider such an automorphism with

$$
g(u) = \big(\mu'(u) + q^{-1}u^{-2}\mu(u)\big)^{-1}.
$$

Hence, we may assume without loss of generality that the highest weight of the representation $V(\mu(u);\bar{\mu}(u))$ satisfies the condition $\mu'(u) + q^{-1}u^{-2}\mu(u) = 1$.

As in the proof of Proposition 3.1, the assumption $\dim V(\mu(u);\bar{\mu}(u)) < \infty$ implies that

$$
\sum_{l=0}^{k} c_l\, s_{22}^{(l)}\,\xi = 0 \tag{4.19}
$$

for some $k \geqslant 0$ and some $c_l \in \mathbb{C}$ with $c_k \neq 0$.

On the other hand, the defining relations (2.74) imply

$$\begin{aligned}&(u^{-1}-v^{-1})(1-u^{-1}v^{-1})\, s_{11}(u)\, s_{22}(v)\, \xi\\ &= (q^{-1}-q)\Big(u^{-1}\big(\mu'(u)+q\,\mu(u)\big)\big(\mu'(v)+q^{-1}v^{-2}\mu(v)\big)\\ &\qquad - v^{-1}\big(\mu'(v)+q\,\mu(v)\big)\big(\mu'(u)+q^{-1}u^{-2}\mu(u)\big)\Big)\xi.\end{aligned}$$

Taking into account the assumption $\mu'(u)+q^{-1}u^{-2}\mu(u)=1$, we can write these relations as

$$(1-u^{-1}v^{-1})\, s_{11}(u)\, s_{22}(v)\, \xi = (q^{-1}-q)\,\frac{u^{-1}\rho(u)-v^{-1}\rho(v)}{u^{-1}-v^{-1}}\,\xi, \tag{4.20}$$

where

$$\rho(u)=\mu'(u)+q\,\mu(u).$$

Write

$$\rho(u)=\sum_{r=0}^{\infty}\rho^{(r)}\,u^{-r}. \tag{4.21}$$

Divide both sides of (4.20) by $1-u^{-1}v^{-1}$ and compare the coefficients of $u^{-m}v^{-l}$. This gives

$$s_{11}^{(m)}\, s_{22}^{(l)}\,\xi = (q^{-1}-q)\,\widetilde{\rho}^{(m,l)}\,\xi,$$

where

$$\widetilde{\rho}^{(m,l)}=\sum_{i=0}^{\min\{m,l\}}\rho^{(m+l-2i)}.$$

Hence, applying the operator $s_{11}^{(m)}$ to the vector (4.19) and taking the coefficient of ξ we get

$$\sum_{l=0}^{k} c_l\,\widetilde{\rho}^{(m,l)}=0 \tag{4.22}$$

for all $m\geqslant 0$. Our next step is to demonstrate that this set of relations for the coefficients of the series (4.21) implies that $\rho(u)$ is the expansion of a rational function in u with the property

$$\rho(u)=u^2\rho(u^{-1}). \tag{4.23}$$

To this end, introduce the coefficients $d_{-k},d_{-k+1},\dots,d_k$ by the formulas

$$d_r=d_{-r}=\begin{cases} c_r+c_{r+2}+\dots+c_k & \text{if } k-r \text{ is even,}\\ c_r+c_{r+2}+\dots+c_{k-1} & \text{if } k-r \text{ is odd,}\end{cases} \tag{4.24}$$

where $r=0,1,\dots,k$. For any $m\geqslant k$ the relation (4.22) takes the form

$$\sum_{l=0}^{k} c_l\left(\rho^{(m+l)}+\rho^{(m+l-2)}+\dots+\rho^{(m-l)}\right)=0,$$

which can written as

$$\sum_{r=-k}^{k} d_r\, \rho^{(m+r)} = 0.$$

Therefore, we have

$$\begin{aligned}\big(d_k u^k + d_{k-1}u^{k-1} + \dots + d_{-k}u^{-k}\big)\, \rho(u)&\\ = \sum_{r=-k+1}^{k} \big(d_k \rho^{(k-r)} + d_{k-1}\rho^{(k-r-1)} + \dots + d_r \rho^{(0)}\big)\, u^r.&\end{aligned} \tag{4.25}$$

This shows that $\rho(u)$ is a rational function in u. The property (4.23) is equivalent to the relations

$$d_k \rho^{(k-r)} + d_{k-1}\rho^{(k-r-1)} + \dots + d_r\rho^{(0)} = d_k\rho^{(k+r-2)} + d_{k-1}\rho^{(k+r-3)} + \dots + d_{-r+2}\rho^{(0)}$$

for $r = 2, 3, \dots, k+1$, where we assume that $\rho^{(i)}$ and d_i with out-of-range indices are zero. However, the relations are easily verified by using (4.24): after writing them in terms of the coefficients c_l they take the form of (4.22) with $m = r - 2$.

The argument is now completed by noting that the series $\mu(u)$ and $\mu'(u)$ are expressed in terms of $\rho(u)$ as

$$\mu(u) = \frac{q\,(1-\rho(u))}{u^{-2} - q^2}, \qquad \mu'(u) = \frac{\rho(u)u^{-2} - q^2}{u^{-2} - q^2}.$$

Hence, using (4.17), we obtain

$$\frac{\bar{\mu}(u^{-1})}{\mu(u)} = \frac{1 - u^{-2}\rho(u)}{1-\rho(u)}. \tag{4.26}$$

Now write (4.25) as $D(u)\rho(u) = F(u)$ so that $F(u)$ and $D(u)$ are Laurent polynomials in u with $D(u^{-1}) = D(u)$. Moreover, $u^2F(u^{-1}) = F(u)$ due to (4.23). Hence, (4.26) implies

$$\frac{\bar{\mu}(u^{-1})}{\mu(u)} = \frac{D(u) - u^{-2}F(u)}{D(u) - F(u)}.$$

Recalling that $d_k = d_{-k} = c_k \neq 0$ set

$$Q(u) = d_k^{-1} u^k \big(D(u) - F(u)\big) = (1-\rho^{(0)})u^{2k} + \dots + 1.$$

The coefficient $1-\rho^{(0)}$ is nonzero by (4.26) and Proposition 4.3. Hence $Q(u)$ is a polynomial in u of degree $2k$ with the constant term equal to 1 and $Q(u)$ satisfies (4.18). $\square$

Proposition 4.4 implies that if $\dim V(\mu(u);\bar{\mu}(u)) < \infty$ then there exist nonzero constants $\gamma_1,\dots,\gamma_{2k}$ such that

$$\frac{\bar{\mu}(u^{-1})}{\mu(u)} = \frac{(u+\gamma_1)\dots(u+\gamma_{2k})}{(1+\gamma_1 u)\dots(1+\gamma_{2k}u)}. \tag{4.27}$$

Therefore, in order to determine which representations $V(\mu(u);\bar{\mu}(u))$ are finite-dimensional, we may restrict our attention to those whose highest weights satisfy (4.27). We now aim to prove a tensor product decomposition for such representations analogous to Proposition 3.2; cf. [28, Prop. 4.3.2].

We will re-enumerate the numbers γ_i, if necessary, so that for each $i = 1,\dots,k$ the following condition holds: if the multiset $\{\gamma_r\gamma_s \mid 2i-1 \leqslant r < s \leqslant 2k\}$ contains numbers of the form q^{-2m} with nonnegative integers m, then $\gamma_{2i-1}\gamma_{2i} = q^{-2m_0}$ and m_0 is minimal amongst these nonnegative integers.

Assuming that the γ_i satisfy these conditions, let us choose square roots α_i and β_i so that

$$\alpha_i^2 = \gamma_{2i-1}^{-1}, \qquad \beta_i^2 = \gamma_{2i}, \qquad i = 1,2,\dots,k. \tag{4.28}$$

Recall the evaluation modules $L(\alpha,\beta)$ over the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ defined in Sec. 3.1. Each of them may also be regarded as a module over the extended algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_2)$ via the epimorphism (2.65) so that the elements $t_{11}^{(0)}\,\bar{t}_{11}^{(0)}$ and $t_{22}^{(0)}\,\bar{t}_{22}^{(0)}$ act as the identity operators. More generally, the tensor product

$$L(\alpha_1,\beta_1)\otimes\dots\otimes L(\alpha_k,\beta_k) \tag{4.29}$$

can be regarded as a representation of the algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_2)$ and hence as a representation over its subalgebra $\mathrm{Y}'_q(\mathfrak{sp}_2)$. In other words, as far as the action of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ on the space (4.29) is concerned, the operators $s_{ij}(u)$ are related with the action of the generators of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ by the formulas (2.70).

Proposition 4.5. *If the above condition on the parameters γ_i holds, then there exists an automorphism of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_2)$ of the form* (4.2) *such that its composition with the representation $V(\mu(u);\bar{\mu}(u))$ is isomorphic to the representation* (4.29) *of $\mathrm{Y}'_q(\mathfrak{sp}_2)$.*

Proof. Due to (2.70) the generators of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ act on the tensor product module (4.29) by the formulas

$$\begin{aligned} s_{11}(u) &= q\,t_{11}(u)\,\bar{t}_{12}(u^{-1}) - t_{12}(u)\,\bar{t}_{11}(u^{-1}),\\ s_{21}(u) &= q\,t_{21}(u)\,\bar{t}_{12}(u^{-1}) - t_{22}(u)\,\bar{t}_{11}(u^{-1}),\\ \bar{s}_{21}(u) &= q\,\bar{t}_{21}(u)\,t_{12}(u^{-1}) - \bar{t}_{22}(u)\,t_{11}(u^{-1}). \end{aligned} \tag{4.30}$$

Consider the tensor product $\zeta = \zeta_1\otimes\dots\otimes\zeta_k$ of the highest vectors of the representations $L(\alpha_i,\beta_i)$. As we pointed out in Sec. 3.1, this vector generates a highest weight submodule of the tensor product module (4.29) over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$. Therefore, the formulas (3.3) and (4.30) imply that

$$\begin{aligned} s_{11}(u)\,\zeta &= 0,\\ s_{21}(u)\,\zeta &= -\prod_{i=1}^{k}(\alpha_i^{-1}+\alpha_i u^{-1})(\beta_i+\beta_i^{-1}u^{-1})\,\zeta,\\ \bar{s}_{21}(u^{-1})\,\zeta &= -\prod_{i=1}^{k}(\alpha_i+\alpha_i^{-1}u^{-1})(\beta_i^{-1}+\beta_i u^{-1})\,\zeta. \end{aligned} \tag{4.31}$$

Hence the ratio of the eigenvalues of $\bar{s}_{21}(u^{-1})$ and $s_{21}(u)$ equals

$$\prod_{i=1}^{k}\frac{(\alpha_i+\alpha_i^{-1}u^{-1})(\beta_i^{-1}+\beta_i u^{-1})}{(\alpha_i^{-1}+\alpha_i u^{-1})(\beta_i+\beta_i^{-1}u^{-1})}=\prod_{i=1}^{k}\frac{(u+\alpha_i^{-2})(u+\beta_i^{2})}{(1+\alpha_i^{-2}u)(1+\beta_i^{2}u)}=\prod_{i=1}^{2k}\frac{u+\gamma_i}{1+\gamma_i u},$$

which coincides with $\bar{\mu}(u^{-1})/\mu(u)$ by (4.27). We may conclude that there exists an automorphism of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_2)$ of the form (4.2) such that its composition with the representation $V(\mu(u);\bar{\mu}(u))$ is isomorphic to the irreducible quotient of the cyclic span $\mathrm{Y}'_q(\mathfrak{sp}_2)\,\zeta$. In order to complete the argument, we will now be proving that $\mathrm{Y}'_q(\mathfrak{sp}_2)$-module (4.29) is irreducible and so it coincides with the cyclic span of ζ.

Denote the representation (4.29) of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ by L. We first prove the following claim: any vector $\xi\in L$ satisfying $s_{11}(u)\xi=0$ is proportional to ζ. We use the induction on k and suppose that $k\geqslant 1$. Write

$$\xi=\sum_{r=0}^{p}(t_{21})^r\zeta_1\otimes\xi_r,\qquad \text{where}\quad \xi_r\in L(\alpha_2,\beta_2)\otimes\ldots\otimes L(\alpha_k,\beta_k)$$

and p is some nonnegative integer. Moreover, if $\alpha_1/\beta_1=\pm q^m$ for some nonnegative integer m, then we will assume that $p\leqslant m$. We will also assume that $\xi_p\neq 0$. Using (2.70) and the coproduct formulas (2.54), we get

$$\begin{aligned} s_{11}(u)\big((t_{21})^r\zeta_1\otimes\xi_r\big) &= t_{11}(u)\,\bar{t}_{11}(u^{-1})(t_{21})^r\zeta_1\otimes s_{11}(u)\,\xi_r\\ &+t_{11}(u)\,\bar{t}_{12}(u^{-1})(t_{21})^r\zeta_1\otimes s_{12}(u)\,\xi_r\\ &+t_{12}(u)\,\bar{t}_{11}(u^{-1})(t_{21})^r\zeta_1\otimes s_{21}(u)\,\xi_r\\ &+t_{12}(u)\,\bar{t}_{12}(u^{-1})(t_{21})^r\zeta_1\otimes s_{22}(u)\,\xi_r. \end{aligned}$$

Now use relations (3.6) and (3.7) together with the following formulas which are implied by (2.56):

$$\bar{t}_{11}(u^{-1})(t_{21})^r\zeta_1=(\bar{t}_{11}+t_{11}u^{-1})(t_{21})^r\zeta_1=\big(q^r\alpha_1^{-1}+q^{-r}\alpha_1u^{-1}\big)(t_{21})^r\zeta_1,$$

and

$$\bar{t}_{12}(u^{-1})(t_{21})^r\zeta_1=\bar{t}_{12}\,(t_{21})^r\zeta_1=(q^r-q^{-r})\big(q^{r-1}\beta_1/\alpha_1-q^{-r+1}\alpha_1/\beta_1\big)(t_{21})^{r-1}\zeta_1.$$

Taking the coefficient of $(t_{21})^p\zeta_1$ in the expansion of $s_{11}(u)\xi$ we get $s_{11}(u)\xi_p = 0$. By the induction hypothesis, applied to the representation $L(\alpha_2,\beta_2)\otimes\ldots\otimes L(\alpha_k,\beta_k)$, the vector ξ_p must be proportional to $\zeta_2\otimes\ldots\otimes\zeta_k$. As we observed in Sec. 3.1, the cyclic $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$-span of the vector ξ_p is a highest weight representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$ whose highest weight is found by formulas (3.3). Hence, using (2.70), we find that

$$s_{21}(u)\,\xi_p = \nu(u)\,\xi_p, \qquad \bar{s}_{21}(u)\,\xi_p = \bar{\nu}(u)\,\xi_p, \qquad s_{12}(u)\,\xi_p = \nu'(u)\,\xi_p,$$

where

$$\nu(u) = -\prod_{i=2}^{k}(\alpha_i^{-1}+\alpha_i u^{-1})(\beta_i+\beta_i^{-1}u^{-1}),$$
$$\bar{\nu}(u^{-1}) = -\prod_{i=2}^{k}(\alpha_i+\alpha_i^{-1}u^{-1})(\beta_i^{-1}+\beta_i u^{-1}),$$

and

$$\nu'(u) = \frac{(q^2-1)\,\nu(u)+(1-u^2q^2)\,\bar{\nu}(u^{-1})}{q(u^2-1)}.$$

Note that these are polynomials in u^{-1}. To complete the proof of the claim, we need to show that $p=0$. Suppose on the contrary that $p\geqslant 1$. Then taking the coefficient of $(t_{21})^{p-1}\zeta_1$ in the expansion of $s_{11}(u)\xi$ we get

$$\begin{aligned}&\big(q^{-p+1}\alpha_1+q^{p-1}\alpha_1^{-1}u^{-1}\big)\big(q^{p-1}\alpha_1^{-1}+q^{-p+1}\alpha_1u^{-1}\big)\,s_{11}(u)\,\xi_{p-1}\\ &+(q^p-q^{-p})\big(q^{p-1}\beta_1/\alpha_1-q^{-p+1}\alpha_1/\beta_1\big)\\ &\times\Big(\big(q^{-p+1}\alpha_1+q^{p-1}\alpha_1^{-1}u^{-1}\big)\,\nu'(u)+u^{-1}\big(q^p\alpha_1^{-1}+q^{-p}\alpha_1u^{-1}\big)\,\nu(u)\Big)\,\xi_p=0.\end{aligned}\tag{4.32}$$

By the definition of the action of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_2)$ on the vector space (4.29), the expression $s_{11}(u)\xi_{p-1}$ is a polynomial in u^{-1}. Now we consider two cases. Suppose first that the expression $q^p\alpha_1^{-1}+q^{-p}\alpha_1u^{-1}$ does not vanish at $u=-q^{2p-2}\alpha_1^{-2}$. Then putting this value of u into (4.32) and recalling the notation (4.28) we get the relation

$$\big(\gamma_1\gamma_2-q^{-2p+2}\big)\big(\gamma_1\gamma_3-q^{-2p+2}\big)\ldots\big(\gamma_1\gamma_{2k}-q^{-2p+2}\big)=0.$$

However, this is impossible due to the conditions on the parameters γ_i. Thus, in the case under consideration, p must be zero.

Now suppose that the expression $q^p\alpha_1^{-1}+q^{-p}\alpha_1u^{-1}$ vanishes at $u=-q^{2p-2}\alpha_1^{-2}$ so that $\gamma_1=\alpha_1^{-2}=\varepsilon\,q^{-2p+1}$ for some $\varepsilon\in\{-1,1\}$. In this case we may simplify (4.32) by canceling the common factor $q^{-p+1}\alpha_1+q^{p-1}\alpha_1^{-1}u^{-1}$. Then setting $u=-\varepsilon\,q$ we obtain

$$\nu'(-\varepsilon\,q)-q^{-1}\nu(-\varepsilon\,q)=-(q+q^{-1})\,\bar{\nu}(-\varepsilon\,q^{-1})=0$$

which gives the relation

$$\big(\gamma_3-\varepsilon\,q\big)\big(\gamma_4-\varepsilon\,q\big)\ldots\big(\gamma_{2k}-\varepsilon\,q\big)=0.$$

Hence, $\gamma_1\gamma_j = q^{-2p+2}$ for some $j \in \{3, \dots, 2k\}$ which contradicts the condition of the γ_i. Thus, p must be zero is this case as well, and the claim is proved.

Suppose that M is a nonzero submodule of L. Then M must contain a nonzero vector ξ such that $s_{11}(u)\xi = 0$. By the claim proved above, ξ is proportional to the highest vector ζ, and so ζ belongs to M. It remains to show that the vector ζ is cyclic in L, that is, the submodule $K = \mathrm{Y}'_q(\mathfrak{sp}_2)\zeta$ coincides with L. We will do this by employing the dual space L^* introduced in the proof of Proposition 3.2. We equip L^* with a $\mathrm{Y}'_q(\mathfrak{sp}_2)$-module structure by using the anti-automorphism (4.4). Namely, we set

$$(y\,\omega)(\eta) = \omega(\varkappa(y)\,\eta) \quad \text{for} \quad y \in \mathrm{Y}'_q(\mathfrak{sp}_2) \qquad \text{and} \qquad \omega \in L^*,\ \eta \in L. \tag{4.33}$$

Since $\varkappa$ is obtained as the restriction of the anti-automorphism (3.10), we conclude that (3.11) is a $\mathrm{Y}'_q(\mathfrak{sp}_2)$-module isomorphism. Arguing as in the proof of Proposition 3.2, suppose now that the submodule $K = \mathrm{Y}'_q(\mathfrak{sp}_2)\zeta$ of L is proper. The annihilator

$$\operatorname{Ann} K = \{\omega \in L^* \mid \omega(\eta) = 0 \quad \text{for all} \quad \eta \in K\}$$

is a submodule of the $\mathrm{Y}'_q(\mathfrak{sp}_2)$-module L^*, which does not contain the vector $\zeta_1^* \otimes \dots \otimes \zeta_k^*$. However, this contradicts the claim verified in the first part of the proof, because the tensor product in (3.11) is associated with the set of parameters obtained by swapping γ_{2i-1} and γ_{2i} for each $i = 1, \dots, k$ so that the condition on the parameters remain satisfied after this swap. □

Proposition 3.2 allows us to describe the finite-dimensional irreducible representations of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_2)$.

Theorem 4.6. *The irreducible highest weight representation $V\big(\mu(u);\bar\mu(u)\big)$ of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ is finite-dimensional if and only if there exists a polynomial $P(u)$ in u of even degree with constant term 1 such that $u^{\deg P}\, P(u^{-1}) = q^{-\deg P}\, P(uq^2)$ and*

$$\frac{\bar\mu(u^{-1})}{\mu(u)} = q^{-\deg P} \cdot \frac{P(uq^2)}{P(u)}. \tag{4.34}$$

In this case $P(u)$ is unique.

Proof. Suppose that the representation $V\big(\mu(u);\bar\mu(u)\big)$ is finite-dimensional. By Proposition 4.4 there exist constants γ_i such that (4.27) holds. Re-enumerate these constants to satisfy the assumptions of Proposition 4.5. This proposition implies that each representation $L(\alpha_i, \beta_i)$ occurring in (4.29) is finite-dimensional. Therefore, all ratios α_i/β_i must have the form $\pm q^{m_i}$, where each m_i is a nonnegative integer. Then the polynomial

$$\begin{aligned} P(u) = \prod_{i=1}^{k} &(1+\beta_i^2 u)(1+\beta_i^2 q^2 u)\dots(1+\beta_i^2 q^{2m_i-2} u) \\ \times \prod_{i=1}^{k} &(1+\alpha_i^{-2} u)(1+\alpha_i^{-2} q^2 u)\dots(1+\alpha_i^{-2} q^{2m_i-2} u) \end{aligned}$$

has the property $u^{\deg P}\,P(u^{-1}) = q^{-\deg P}\,P(uq^2)$ and satisfies (4.34).

Conversely, suppose (4.34) holds for a polynomial $P(u) = (1+\gamma_1 u)\dots(1+\gamma_{2k}u)$ with the property $u^{\deg P}\,P(u^{-1}) = q^{-\deg P}\,P(uq^2)$. This property implies that the multiset of parameters γ_i can be written in the form

$$\{\gamma_1,\dots,\gamma_{2k}\} = \{\alpha_1,\dots,\alpha_k,\alpha_1^{-1}q^{-2},\dots,\alpha_k^{-1}q^{-2}\}.$$

Consider the irreducible highest weight representation $L(\nu_1(u),\nu_2(u);\bar{\nu}_1(u),\bar{\nu}_2(u))$ of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$, where the components of the highest weight are given by

$$\begin{aligned}
\nu_1(u) &= (\alpha_1 q + q^{-1}u^{-1})\dots(\alpha_k q + q^{-1}u^{-1}),\\
\nu_2(u) &= (\alpha_1 + u^{-1})\dots(\alpha_k + u^{-1}),\\
\bar{\nu}_1(u) &= (q^{-1} + \alpha_1 q\, u)\dots(q^{-1} + \alpha_k q\, u),\\
\bar{\nu}_2(u) &= (1+\alpha_1 u)\dots(1+\alpha_k u).
\end{aligned}$$

By Theorem 3.3, the representation $L(\nu_1(u),\nu_2(u);\bar{\nu}_1(u),\bar{\nu}_2(u))$ is finite-dimensional as

$$\frac{\nu_1(u)}{\nu_2(u)} = q^{-\deg Q}\cdot\frac{Q(uq^2)}{Q(u)} = \frac{\bar{\nu}_1(u)}{\bar{\nu}_2(u)}$$

with $Q(u) = (1+\alpha_1 u)\dots(1+\alpha_k u)$. We will regard $L(\nu_1(u),\nu_2(u);\bar{\nu}_1(u),\bar{\nu}_2(u))$ as a representation of the extended algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_2)$ via the epimorphism (2.65). The formulas (4.30) imply that the cyclic span $\mathrm{Y}'_q(\mathfrak{sp}_2)\,\zeta$ of the highest vector ζ of $L(\nu_1(u),\nu_2(u);\bar{\nu}_1(u),\bar{\nu}_2(u))$ is a highest weight representation of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ with the highest weight $(\lambda(u);\bar{\lambda}(u))$, where

$$\lambda(u) = -\nu_2(u)\,\bar{\nu}_1(u^{-1}),\qquad \bar{\lambda}(u) = -\bar{\nu}_2(u)\,\nu_1(u^{-1}).$$

Therefore, the irreducible highest weight representation $V(\lambda(u);\bar{\lambda}(u))$ is finite-dimensional and

$$\frac{\bar{\lambda}(u^{-1})}{\lambda(u)} = \frac{\nu_1(u)}{\nu_2(u)}\cdot\frac{\bar{\nu}_2(u^{-1})}{\bar{\nu}_1(u^{-1})} = \frac{Q(uq^2)\,Q(u^{-1})}{Q(u)\,Q(u^{-1}q^2)} = q^{-\deg P}\cdot\frac{P(uq^2)}{P(u)}.$$

Due to (4.34) there exists an automorphism of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_2)$ of the form (4.2) such that the composition of the representation $V(\lambda(u);\bar{\lambda}(u))$ with this automorphism is isomorphic to $V(\mu(u);\bar{\mu}(u))$. Hence the latter is also finite-dimensional.

The uniqueness of $P(u)$ is easily verified. □

In the following corollary we use the q-spirals introduced in Sec. 3.1. Denote by L the tensor product (4.29), where for all $i = 1,\dots,k$ we have $\alpha_i/\beta_i = \pm q^{m_i}$ for some nonnegative integers m_i.

Corollary 4.7. *The representation L of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ is irreducible if and only if each pair of the q-spirals*

$$S_q(\alpha_i,\beta_i),\quad S_q(\alpha_j,\beta_j)\qquad \textit{and}\qquad S_q(\beta_i^{-1},\alpha_i^{-1}),\quad S_q(\alpha_j,\beta_j)$$

is in general position for all $1\leqslant i<j\leqslant k$.

Proof. Suppose that the condition on the q-spirals is satisfied. Then Corollary 3.4 implies that L is irreducible as a representation of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$. Moreover, as we pointed out in the proof of that corollary, any permutation of the tensor factors in (4.29) yields an isomorphic representation. Hence we may assume that the nonnegative integers m_i satisfy the inequalities $m_1 \leqslant \cdots \leqslant m_k$. Let us verify that in this case the condition of Proposition 4.5 is satisfied. Indeed, if this is not the case, then $\gamma_r \gamma_s = q^{-2p}$ for some $2i-1 \leqslant r < s \leqslant 2k$ and a nonnegative integer p such that $p < m_i$. Suppose first that r and s are both odd. Then we may assume that $r = 2i-1$ and $s = 2j-1$ for some $j > i$. By (4.28) we have $\gamma_{2i-1} = \alpha_i^{-2}$ and $\gamma_{2j-1} = \alpha_j^{-2}$. Hence, $\alpha_j = \pm \alpha_i^{-1} q^p$ which means that α_j belong to the q-spiral $S_q(\beta_i^{-1}, \alpha_i^{-1})$. However, the condition $m_i \leqslant m_j$ then implies that the q-spirals $S_q(\beta_i^{-1}, \alpha_i^{-1})$ and $S_q(\alpha_j, \beta_j)$ are not in general position. This contradicts the assumptions of the proposition. The remaining cases, where r or s is even lead to similar contradictions. Thus, Proposition 4.5 allows us to conclude that L is irreducible as a representation of $\mathrm{Y}'_q(\mathfrak{sp}_2)$.

Conversely, suppose that the representation L of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ is irreducible. Then L is irreducible as a representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$. By Corollary 3.4 the q-spirals $S_q(\alpha_i, \beta_i)$ and $S_q(\alpha_j, \beta_j)$ are in general position for all $i < j$. Now fix an index $i \in \{1, \ldots, k\}$ and consider the $\mathrm{Y}'_q(\mathfrak{sp}_2)$-module L' obtained by replacement of the tensor factor $L(\alpha_i, \beta_i)$ by $L(\beta_i^{-1}, \alpha_i^{-1})$. We claim that L' is isomorphic to L. Indeed, the formulas (4.31) show that the highest weight of the cyclic $\mathrm{Y}'_q(\mathfrak{sp}_2)$-span of the tensor product of the highest vectors of the tensor factors occurring in L' is unchanged under the replacement $\alpha_i \mapsto \beta_i^{-1}$, $\beta_i \mapsto \alpha_i^{-1}$. This implies that the module L is isomorphic to the irreducible quotient of this span. Since $\dim L = \dim L'$, the claim follows.

Thus, L' is irreducible as a $\mathrm{Y}'_q(\mathfrak{sp}_2)$-module and, hence, as a $\mathrm{U}_q(\widehat{\mathfrak{gl}}_2)$-module. By Corollary 3.4, the q-spiral $S_q(\beta_i^{-1}, \alpha_i^{-1})$ is in general position with any q-spiral $S_q(\alpha_j, \beta_j)$ for $i \neq j$. This gives the required condition on the q-spirals. □

4.3 Classification theorem

We can now prove the classification theorem for finite-dimensional irreducible representations of the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ for arbitrary $n \geqslant 1$. By Theorem 4.2, all finite-dimensional irreducible representations of the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ have the form $V(\mu(u); \bar{\mu}(u))$ for a certain highest weight $(\mu(u); \bar{\mu}(u))$.

Theorem 4.8. *The irreducible highest weight representation $V(\mu(u); \bar{\mu}(u))$ of the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ is finite-dimensional if and only if there exist polynomials $P_1(u), \ldots, P_n(u)$ in u, all with constant term 1, where $P_1(u)$ is of even degree and satisfies $u^{\deg P_1} P_1(u^{-1}) = q^{-\deg P_1} P_1(uq^2)$, and nonzero constants $\phi_1, \ldots, \phi_n$ such that*

$$\frac{\phi_{i-1}\mu_{i-1}(u)}{\phi_i\, \mu_i(u)} = q^{-\deg P_i} \cdot \frac{P_i(uq^2)}{P_i(u)} = \frac{\phi_{i-1}\bar{\mu}_{i-1}(u)}{\phi_i\, \bar{\mu}_i(u)} \tag{4.35}$$

for $i = 2, \dots, n$ *and*

$$\frac{\bar{\mu}_1(u^{-1})}{\mu_1(u)} = q^{-\deg P_1} \cdot \frac{P_1(uq^2)}{P_1(u)}. \tag{4.36}$$

The polynomials $P_1(u), \dots, P_n(u)$ *are determined uniquely, while the tuple* $(\phi_1, \dots, \phi_n)$ *is determined uniquely, up to a common factor.*

Proof. Suppose first that $\dim V(\mu(u); \bar{\mu}(u)) < \infty$. Let J be the left ideal of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ generated by all coefficients of the series $s_{ij}(u)$ with $i, j = 1, 3, \dots, 2n-1$. Due to (4.5), all coefficients of the series $\bar{s}_{ij}(u)$ with $i, j = 1, 3, \dots, 2n-1$ also belong to J. Consider the subspace V^J of $V(\mu(u); \bar{\mu}(u))$ defined by

$$V^J = \{\eta \in V(\mu(u); \bar{\mu}(u)) \mid s_{ij}(u)\, \eta = 0 \quad \text{for all} \quad i, j = 1, 3, \dots, 2n-1\}.$$

Note that the highest vector ξ of $V(\mu(u); \bar{\mu}(u))$ belongs to V^J. The defining relations (2.74) together with (2.76) imply that if the indices i, a, b are odd and j is even, then

$$s_{ia}(u)\, s_{jb}(v), \quad s_{ia}(u)\, \bar{s}_{jb}(v) \in J.$$

Therefore the subspace V^J is stable under the action of the operators $s_{2i,2a-1}(u)$ and $\bar{s}_{2i,2a-1}(u)$. Moreover, regarding the relations (2.74) and (2.76) modulo the left ideal J, we find that the mapping

$$t_{ia}(u) \mapsto s_{2i,2a-1}(u), \qquad \bar{t}_{ia}(u) \mapsto \bar{s}_{2i,2a-1}(u), \qquad i, a = 1, \dots, n, \tag{4.37}$$

defines an action of the algebra $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_n)$ on the space V^J. The cyclic span $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_n)\,\xi$ is a finite-dimensional highest weight representation of $\mathrm{U}_q^{\mathrm{ext}}(\widehat{\mathfrak{gl}}_n)$ with the highest weight $(\mu(u); \bar{\mu}(u))$. It follows from Corollary 3.7 that the highest weight satisfies the conditions (4.35) for appropriate nonzero constants ϕ_i.

Furthermore, the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_2)$ act on $V(\mu(u); \bar{\mu}(u))$ via the homomorphism $\mathrm{Y}'_q(\mathfrak{sp}_2) \to \mathrm{Y}'_q(\mathfrak{sp}_{2n})$ which sends $s_{ij}(u)$ to the series with the same name in $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$. The cyclic span $\mathrm{Y}'_q(\mathfrak{sp}_2)\xi$ is a highest weight representation of $\mathrm{Y}'_q(\mathfrak{sp}_2)$ with the highest weight $(\mu_1(u); \bar{\mu}_1(u))$. Its irreducible quotient is finite-dimensional, and so (4.36) follows from Theorem 4.6.

In order to prove the converse statement, note that given two irreducible highest weight representations $V(\mu(u); \bar{\mu}(u))$ and $V(\lambda(u); \bar{\lambda}(u))$ such that the components of the highest weights satisfy the conditions (4.35) and (4.36) with the same set of polynomials $P_1(u), \dots, P_n(u)$, there exist automorphisms of the form (4.2) and (4.3) such that the composition of the representation $V(\mu(u); \bar{\mu}(u))$ with these automorphisms is isomorphic to $V(\lambda(u); \bar{\lambda}(u))$. Hence, it suffices to show that given any set of polynomials $P_1(u), \dots, P_n(u)$ of the form described in the formulation of the theorem, there exists a finite-dimensional representation $V(\mu(u); \bar{\mu}(u))$ whose highest weight satisfies (4.35) and (4.36) with $\phi_i = 1$

for all i. We will use a result from [27, Sec. 6] concerning a particular irreducible highest weight representation $L(\nu)$ of the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_{2n})$; see Sec. 2.1 above for the definition. The highest weight ν has the form

$$\nu = (q^{r_n}, \dots, q^{r_1}, 1, \dots, 1), \qquad r_n \geqslant \dots \geqslant r_1 \geqslant 0,$$

where the parameters r_i are integers. The representation $L(\nu)$ is finite-dimensional and it admits a basis parameterized by the Gelfand–Tsetlin patterns associated with ν. As in [27] consider the pattern Ω^0 such that for each $k = 1, 2, \dots, n$ its row $2k-1$ counted from the bottom is $(r_k, r_{k-1}, \dots, r_1, 0, \dots, 0)$ with $k-1$ zeros, while the row $2k$ from the bottom is $(r_k, r_{k-1}, \dots, r_1, 0, \dots, 0)$ with k zeros. Then the corresponding basis vector ζ_{Ω^0} has the properties

$$\begin{aligned} &\bar{t}_{ij}\, \zeta_{\Omega^0} = 0 \qquad \text{if } j \text{ is even and } i < j, \\ &t_{ij}\, \zeta_{\Omega^0} = 0 \qquad \text{if } i \text{ is odd and } i > j. \end{aligned} \tag{4.38}$$

We denote by $L(d\,\nu)$ the composition of the representation $L(\nu)$ with the automorphism of $\mathrm{U}_q(\mathfrak{gl}_{2n})$ given in (2.8). We will consider $L(d\,\nu)$ as an evaluation module over $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{2n})$ by using the homomorphism (2.56).

Suppose now that $V(\mu(u); \bar{\mu}(u))$ is a finite-dimensional highest weight representation of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ with the highest vector ξ. By the first part of the proof we can associate a family of polynomials $P_1(u), \dots, P_n(u)$ to $V(\mu(u); \bar{\mu}(u))$. The coproduct structure on $\mathrm{U}_q(\widehat{\mathfrak{gl}}_{2n})$ given by (2.54) allows us to equip the vector space $L(d\,\nu) \otimes V(\mu(u); \bar{\mu}(u))$ with a structure of a $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$-module so that for the action of the generators we have

$$s_{ij}(u)(\eta \otimes \theta) = \sum_{k,l=1}^{2n} t_{ik}(u) \bar{t}_{jl}(u^{-1})\, \eta \otimes s_{kl}(u)\, \theta, \quad \eta \in L(d\,\nu), \quad \theta \in V(\mu(u); \bar{\mu}(u)). \tag{4.39}$$

Let us verify that $\zeta_{\Omega^0} \otimes \xi$ is the highest vector of the $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$-module $\mathrm{Y}'_q(\mathfrak{sp}_{2n})(\zeta_{\Omega^0} \otimes \xi)$. Take $\eta = \zeta_{\Omega^0}$ and $\zeta = \xi$ in (4.39) and suppose that j is odd and $i \leqslant j$. Using (2.56) and (4.38), we find that

$$\bar{t}_{jl}(u^{-1})\, \zeta_{\Omega^0} = u^{-1} t_{jl}\, \zeta_{\Omega^0} = 0$$

for $j > l$. If l is even and $j < l$, then by (4.38)

$$\bar{t}_{jl}(u^{-1})\, \zeta_{\Omega^0} = \bar{t}_{jl}\, \zeta_{\Omega^0} = 0.$$

Hence, we may assume that $l = j + 2p$ for a nonnegative integer p; in particular, l is odd. Then the index k in (4.39) may be assumed to be even as otherwise $s_{kl}(u)\, \xi = 0$. Furthermore, if $k \leqslant i$ then $k \leqslant j + 2p = l$ so that $s_{kl}(u)\, \xi = 0$ in this case too. Therefore, we may assume that $k > i$. In this case we have

$$t_{ik}(u) \bar{t}_{jl}(u^{-1})\, \zeta_{\Omega^0} = u^{-1} \bar{t}_{ik}\, (\bar{t}_{jl} + u^{-1} \delta_{jl}\, t_{jl})\, \zeta_{\Omega^0}. \tag{4.40}$$

By the defining relations (2.6) we have

$$\bar{t}_{ik}\,\bar{t}_{jl} = q^{\delta_{ij}}\,\bar{t}_{jl}\,\bar{t}_{ik} - (q-q^{-1})\,(\delta_{k<l}-\delta_{j<i})\,\bar{t}_{il}\,\bar{t}_{jk}.$$

Now, if $k<l$ then $s_{kl}(u)\,\xi=0$. Otherwise, $\delta_{k<l}-\delta_{j<i}\neq 0$ only if $j<i$. But in this case $j<k$ and $\bar{t}_{jk}\,\zeta_{\Omega^0}=0$. Thus, in all cases (4.40) is zero due to (4.38). A similar calculation shows that $\bar{s}_{ij}(u)(\zeta_{\Omega^0}\otimes\xi)=0$ for odd j and $i\leqslant j$. By (4.5) this proves $s_{ij}(u)(\zeta_{\Omega^0}\otimes\xi)=0$ if $\varsigma(i)+\varsigma(j)>0$; see Definition 4.1.

Let us now calculate the eigenvalues of $\zeta_{\Omega^0}\otimes\xi$ with respect to the operators $s_{2i,2i-1}(u)$ and $\bar{s}_{2i,2i-1}(u)$. The above arguments show that (4.39) with $\eta=\zeta_{\Omega^0}$ and $\zeta=\xi$ simplifies to

$$\begin{aligned} s_{2i,2i-1}(u)(\zeta_{\Omega^0}\otimes\xi) &= t_{2i,2i}(u)\,\bar{t}_{2i-1,2i-1}(u^{-1})\,\zeta_{\Omega^0}\otimes s_{2i,2i-1}(u)\,\xi\\ &= \big(d+d^{-1}u^{-1}\big)\big(d^{-1}q^{-r_i}+dq^{r_i}u^{-1}\big)\,\mu_i(u)(\zeta_{\Omega^0}\otimes\xi). \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{s}_{2i,2i-1}(u)(\zeta_{\Omega^0}\otimes\xi) &= \bar{t}_{2i,2i}(u)\,t_{2i-1,2i-1}(u^{-1})\,\zeta_{\Omega^0}\otimes\bar{s}_{2i,2i-1}(u)\,\xi\\ &= \big(d^{-1}+du\big)\big(dq^{r_i}+d^{-1}q^{-r_i}\big)\,\bar{\mu}_i(u)(\zeta_{\Omega^0}\otimes\xi). \end{aligned}$$

The cyclic span $\mathrm{Y}'_q(\mathfrak{sp}_{2n})(\zeta_{\Omega^0}\otimes\xi)$ is finite-dimensional. By the above formulas, the irreducible quotient of this representation of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ corresponds to the family of polynomials $Q_1(u)P_1(u),\dots,Q_n(u)P_n(u)$, where

$$Q_i(u)=(1+d^{-2}q^{-2r_i}u)(1+d^{-2}q^{-2r_i+2}u)\dots(1+d^{-2}q^{-2r_{i-1}-2}u),\qquad i=2,\dots,n,$$

and

$$\begin{aligned} Q_1(u) &= (1+d^2u)(1+d^2q^2u)\dots(1+d^2q^{2r_1-2}u)\\ &\quad\times(1+d^{-2}q^{-2r_1}u)(1+d^{-2}q^{-2r_1+2}u)\dots(1+d^{-2}q^{-2}u). \end{aligned}$$

Thus, starting from the trivial representation $V(\mu(u);\bar{\mu}(u))$ and choosing appropriate parameters d and r_i we will be able to produce a finite-dimensional highest weight representation of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ associated with an arbitrary family of polynomials $P_1(u),\dots,P_n(u)$ by iterating this construction; cf. the proof of Theorem 3.6. The last statement of the theorem is easily verified. □

We will call $P_1(u),\dots,P_n(u)$ the *Drinfeld polynomials* of the finite-dimensional representation $V(\mu(u);\bar{\mu}(u))$.

We will now use Theorem 4.8 to describe finite-dimensional irreducible representations of the quotient algebra $\mathrm{Y}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$ of $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ by the relations (2.71); see Remark 2.14. Every finite-dimensional irreducible representation of the algebra $\mathrm{Y}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$ is isomorphic to the highest weight representation $V(\mu(u);\bar{\mu}(u))$ which is defined in the same way as for the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$; see Definition 4.1. This time the constant terms of the series (4.7) should satisfy the conditions $\mu_i^{(0)}\bar{\mu}_i^{(0)}=1$ for $i=1,\dots,n$.

Corollary 4.9. *The irreducible highest weight representation $V(\mu(u);\bar{\mu}(u))$ of the algebra $\mathrm{Y}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$ is finite-dimensional if and only if there exist polynomials $P_1(u),\dots,P_n(u)$ in u, all with constant term 1, where $P_1(u)$ is of even degree and satisfies $u^{\deg P_1}\,P_1(u^{-1})=q^{-\deg P_1}\,P_1(uq^2)$ such that*

$$\frac{\varepsilon_{i-1}\mu_{i-1}(u)}{\varepsilon_i\,\mu_i(u)}=q^{-\deg P_i}\cdot\frac{P_i(uq^2)}{P_i(u)}=\frac{\varepsilon_{i-1}\bar{\mu}_{i-1}(u)}{\varepsilon_i\,\bar{\mu}_i(u)} \tag{4.41}$$

for $i=2,\dots,n$ and

$$\frac{\bar{\mu}_1(u^{-1})}{\mu_1(u)}=q^{-\deg P_1}\cdot\frac{P_1(uq^2)}{P_1(u)} \tag{4.42}$$

for some $\varepsilon_i\in\{-1,1\}$. The polynomials $P_1(u),\dots,P_n(u)$ are determined uniquely, while the tuple $(\varepsilon_1,\dots,\varepsilon_n)$ is determined uniquely, up to a simultaneous change of sign.

Proof. Suppose that $\dim V(\mu(u);\bar{\mu}(u))<\infty$. We argue as in the proof of Theorem 4.8. The first part of that proof is now modified so that the mapping (4.37) defines an action of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ on the corresponding space V^J. The necessary conditions on the components of the highest weight come from the application of Theorem 3.6.

Conversely, suppose that conditions (4.41) and (4.42) hold. Using the natural epimorphism $\mathrm{Y}'_q(\mathfrak{sp}_{2n})\to\mathrm{Y}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$ we may regard $V(\mu(u);\bar{\mu}(u))$ as a $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$-module. This module is finite-dimensional by Theorem 4.8. □

We conclude with a discussion of a particular class of representations of the twisted q-Yangians associated with the evaluation homomorphisms. By [29, Theorem 3.15] there exists a homomorphism $\mathrm{Y}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})\to\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$ which is identical on the subalgebra $\mathrm{U}_q^{\mathrm{tw}}(\mathfrak{sp}_{2n})$. The arguments used for the proof of that theorem apply to the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ without any changes so that we have the homomorphism $\mathrm{Y}'_q(\mathfrak{sp}_{2n})\to\mathrm{U}'_q(\mathfrak{sp}_{2n})$ given by

$$S(u)\mapsto S+q\,u^{-1}\overline{S}. \tag{4.43}$$

It allows one to extend any representation of $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ to the twisted q-Yangian $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$. Consider the highest weight representations $V(\mu;\mu')$ defined in Sec. 2.2. The $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$-modules $V(\mu;\mu')$ will be called the *evaluation modules.*

Suppose that this representation is finite-dimensional with the parameters p_i as defined in Proposition 2.7.

Proposition 4.10. *The Drinfeld polynomials of the evaluation module $V(\mu;\mu')$ over the algebra $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$ are given by*

$$P_1(u)=(1+qu)(1+q^3u)\dots(1+q^{2p_1-1}u)(1+q^{-2p_1-1}u)(1+q^{-2p_1+1}u)\dots(1+q^{-3}u)$$

and

$$P_i(u)=(1+q^{-2p_i-1}u)(1+q^{-2p_i+1}u)\dots(1+q^{-2p_{i-1}-3}u)$$

for $i = 2, \dots, n$. The parameters ϕ_i are found by

$$\phi_i = \mu_i^{-1} q^{-p_i}, \qquad i = 1, \dots, n.$$

Proof. The highest vector of the representation $V(\mu;\mu')$ of $\mathrm{U}'_q(\mathfrak{sp}_{2n})$ is also the highest vector of the evaluation module over $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$. The claims are now verified by calculating the highest weight of the $\mathrm{Y}'_q(\mathfrak{sp}_{2n})$-module $V(\mu;\mu')$ with the use of (4.43) and the formulas relating the matrix elements of the matrices S and $\overline{S}$; cf. [29, (2.52)]. The components of the highest weight are found by

$$\mu_i(u) = \mu_i - u^{-1}\mu'_i, \qquad \bar{\mu}_i(u) = \frac{1+qu}{u+q}\,(u\mu_i - \mu'_i), \qquad i = 1, \dots, n.$$

Together with (4.35) and (4.36) this implies all the statements. □

As we pointed out in the proof of Proposition 2.7, if the highest weight $(\mu;\mu')$ satisfies the additional conditions $\mu_i\,\mu'_i = -q$ for $i = 1, \dots, n$, then $V(\mu;\mu')$ can be regarded as a representation of the quotient algebra $\mathrm{Y}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$. In this case we have $\mu_i = \varepsilon_i\, q^{-p_i}$ for all i and some $\varepsilon_i \in \{-1, 1\}$. The corresponding evaluation module over $\mathrm{Y}^{\mathrm{tw}}_q(\mathfrak{sp}_{2n})$ has the same Drinfeld polynomials as given in Proposition 4.10, while $\phi_i = \varepsilon_i$ for all i.

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