Syntactic and semantic reasoning in mathematics teaching and learning

David Easdown*

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

Abstract

This paper discusses a variety of examples in errors in mathematical reasoning, the source of which is due to the tension between syntax (form of mathematical expression) and semantics (underlying ideas or meaning). The paper suggests that heightened awareness of syntactic and semantic reasoning, and consequent resolution of the tension and errors in particular cases, may lead to enhanced mathematics learning outcomes, robustness and creativity.

Keywords: mathematical reasoning, syntax, semantics

1. Introduction

Students who choose to major in pure mathematics often comment that they enjoy the emphasis on proof and rigour, coupled with the sense that they can see why something is true for themselves rather than rely on the spoken or written word of others. There are many more students, however, that have drifted away or even been repelled from the discipline because they dislike proofs and rigour, especially when the level of abstraction or detail becomes tedious or obscures the main ideas or applications. At the heart of this dichotomy of attraction and repulsion is the *activity of reasoning*, about which we appear to know very little, though its prominence in western education traditions goes back at least to Socrates, and in eastern traditions probably much earlier (see, for example, discussion of extracts from the Upanishads in Mason and Johnston-Wilder [1]). The author [2] has argued that art-of-proof is one of the most important threshold concepts in teaching and learning mathematics, in the sense of Meyer and Land [3,4], and it is worthwhile to analyse and try to understand the dynamics and processes involved, especially when sustained mathematical reasoning is encountered for the first time by naive students.

Kant [5] is often quoted (see, for example, Polya [6, p. 99] and [1, p. 46]) for his seminal remark that

> all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas.

But, despite the prominence of intuition, Mason and Johnston-Wilder [1] paraphrase Kant as cautioning that 'a succession of perceptions does not necessarily add up to a perception of succession', and themselves go on to warn that

> a learner can go through a succession of activities but not end up with any sense of it as anything more than a succession, even a smorgasbord, of activities.

The whole may not be more than the sum of its parts. The parts may not add up to anything tangible for the learner. It is extremely common to follow a proof or technique, for example, by means of verifying small steps, but nevertheless remain afterwards far removed from progress towards genuine understanding. A student who is aware of this can feel a deep sense of dissatisfaction or frustration and not know what to do about it.

*Email: de@maths.usyd.edu.au

Why is verification of a proof without promoting full or even partial understanding any more valuable than just accepting on faith a statement made by a reliable expert? The latter is certainly less time-consuming. Even worse: simple but detailed verification is subject to error as much as any other human activity. One can be lulled into a false sense of security or even fool oneself that one 'understands' a result by this process. Almost every research mathematician has had the experience of false elation at having 'proved' and subsequently 'understood' a difficult result, only to find his or her hopes dashed later when an error is uncovered. Pathways can be rocky that deliver one from the unistructural and multistructural phases of learning to the relational and extended abstract phases of the SOLO taxonomy (Biggs and Collis [7]). Indeed an overemphasis on the qualitative phases of SOLO has come under some criticism: students may think they are making relational or extended abstract constructs, which are in fact wrong, and may be 'pursuing wild geese' because they have not spent enough time practising at the lower levels (Atherton [8]). Balance in phases of learning is always important.

This paper examines a number of contrasting examples of erroneous reasoning. Errors are very interesting and revealing, not just of a student's current state of knowledge or understanding, but of the process of thinking itself, and strategies for tackling difficult or sophisticated mathematical problems. This paper proposes that there are at least two types of mathematical reasoning that tend to lie at opposite ends of a spectrum. They may, if one is aware of them, interact, balance and reinforce each other, to further learning and deepen an appreciation of the particular phenomenon under investigation. On the one hand we have *syntactic reasoning*, which broadly speaking one associates with the superficial end of the learning spectrum, and relies on simple or naive, incremental rules, searching or pattern matching. It can, for example, involve very literal interpretations and superficial relationships, pasting together ideas, almost like a collage. It is extremely common, especially when students are under pressure in examinations, or with a cascade of deadlines, and must respond quickly without thinking too carefully about their answers. *Semantic reasoning*, on the other hand, is more naturally associated with the deeper end of the learning spectrum, and relies on solid intuition, insight or experience. These can relate to physical, visual and other sorts of models that become ingrained in our minds, and may be the result of practice over many years or some inherent instinct (such as a naturally evolved facility for thinking in three dimensions to go 'hunting' in an environment fraught with difficulties and obstacles, to second-guess the prey, or even avoid becoming prey).

This dichotomy between syntax (form) and semantics (meaning) is ancient and goes back to Euclid's *Elements*, the first published attempt to create an axiomatic deductive system in mathematics, providing a paradigm for the formal development of any area of mathematics, and even mathematics itself. This led to the question whether mathematics could in some precise sense be reduced to syntax, through the formal and mechanical manipulation of axioms. This idea, for example, is behind Hilbert's Tenth Problem on the solvability of Diophantine equations, resolved in the negative only as recently as 1970 (Matiyasevich [9]), the method of which implies Gödel's celebrated Incompleteness Theorem (Gödel [10]). The latter implies that any axiomatic system including number theory is inherently 'incomplete' in the sense that there will be true statements that can't be proved 'syntactically' using the axioms of the system. The main idea also relates to the unsolvability of the Halting Problem: roughly speaking this says that mechanisation itself can't be mechanised (Turing [11], related also to Hilbert's Second Problem). Therefore we should expect (and possibly celebrate) an unresovable tension between syntax and semantics. Rather than regarding this as a nuisance or source of frustration in teaching mathematics, one can exploit the differences between syntactic and semantic reasoning to create opportunities to enhance learning and expose weaknesses or gaps in understanding. Almost always, in the author's experience, errors in reasoning tell us more than we imagined and their resolution make us more robust and creative in the long run.

2. Examples

A number of specimens are discussed in detail, most of which are extracts taken from the author's students' work, some very recent, or feature in the author's teaching. The first two are adapted from Miller [12] and Halmos [13] respectively, and the last is beautiful and succinct but probably apocryphal.

The first specimen involves averages and an irresistible urge to apply the usual template of adding up two numbers and dividing by 2:

> Specimen 1:
>
> **Question (Julius Sumner-Miller):** You travel from A to B at a speed of 20 miles per hour, and immediately return from B to A at a speed of 30 miles per hour. What is your average speed for the return journey?
>
> **Syntactic response:** Average speed is $\dfrac{20+30}{2} = 25$ m.p.h.

This is a superficial response that pulls out of the problem sufficient syntax to create a neat formulaic answer. But this ignores meaning: the problem isn't asking for the average of two numbers. It is asking for an average speed over a journey, and to successfully answer this question one must think about semantics: we require a fraction whose numerator is the total distance travelled and denominator the overall accumulated time. (The correct average is 24 m.p.h.)

The next specimen is intriguing, and the author is yet to find a person whose immediate response is semantic (and he has tested it now on thousands of students):

> Specimen 2:
>
> **Question (Paul Halmos):** You build a straight railway track 20 km long on a flat horizontal plane, fixed at one end, but realise when you get to the other end that you have built it one metre too long. You push the metre in, absorbing it evenly into the 20 km to make a gentle arc over the plane. How high above the plane does the track arch in the middle?
>
> **Syntactic response:** Absorbing one metre in 20 km means the height above the plane must be negligibly small.

It is a typical experience or pattern in life that small perturbations have small effects. This creates a natural 'template' for slipping almost uncontrollably into the above syntactic response. In actual fact, the 20 km has a magnifying effect because the perturbation (1 metre along the horizontal railway track) is orthogonal to the final effect (vertical arching above the plane). If one appoximates half of the arch by the hypotenuse of a right-angled triangle, then one quickly discovers, using the Theorem of Pythagoras, that the height above the plane is approximately 100 metres! At first this is a shock, but really it accords with natural semantics if one thinks about the correct analogical experience. That the height should be large rather than small is intuitively clear, for example, to anyone who has gently squeezed the ends of a metal ruler, or to anyone who has observed the buckle that results from trying to plug a gap in a wooden floor using a board that is slightly too long. This example of Halmos is connected to the sensitivity of matrix multiplication to small perturbations with consequences for dynamical systems and ergodic theory. Rolf Harris' wonderful wobble board musical instrument is a two-dimensional version of this, and a brilliant sound is obtained not by shaking but by gentle squeezing/flicking and letting Pythagoras take effect.

In introducing the Cayley-Hamilton Theorem in teaching linear algebra, the author has given the following well-known single line 'proof' many times in a traditional lecture setting, and has *never* fielded an objection from within a student audience:

> Specimen 3:
>
> **Cayley-Hamilton Theorem:** If M is a matrix and $p(\lambda) = \det(M - \lambda I)$ is its characteristic polynomial, then $p(M)$ is the zero matrix.
>
> **Proof:** $p(M) = \det(M{-}MI) = \det(M{-}M) = \det(0) = 0$.

It may be that there are students who see the fallacy straight away and prefer to say nothing, but the overwhelming impression is one of universal, uncritical acceptance. The brevity is appealing and the chain of equalities looks impregnable. Indeed the last three equalities are absolutely correct, and the first equality looks so plausible that it seems churlish to question it. But of course it is a syntactic illusion and the proof cannot be correct. Thinking about the semantics reveals that $p(M)$ is a matrix and 0 at the end is just a number. (The 0 inside $\det(0)$ is in fact a matrix, so there is the added abuse of notation, but that is harmless and not a fault in the argument.) The fallacy is in the substitution at the first step. The matrix $p(M)$ is formed by taking the characteristic polynomial $p(\lambda)$ and replacing λ by M throughout and the constant term $p(0)$ by $p(0)I$ and then evaluating to the zero matrix using matrix arithmetic. The expression $\det(M - MI)$ at the first step is trying to circumvent this process quickly by performing a matrix substitution before taking a determinant!

In the language of algebra, the two *operations*, of matrix substitution and of taking determinants, *do not commute.* Operations A and B in mathematics often *do* commute, expressed by the simple formula $AB = BA$. This expresses a common pattern and following it superficially without regard to meaning is one of the most common sources of syntactic errors in reasoning. This mirrors 'formulaic' behaviour, which one tends to identify with 'mindlessness'. A famous example of this is *The Freshman's Dream*, where the naive student conveniently, but unfortunately incorrectly, assumes operations of addition and taking powers commute:

> Specimen 4:
>
> **Freshman's Dream:** $(x + y)^n = x^n + y^n$

The next is an extract from an examination script by an apparently talented student taking a first year course in calculus given by the author:

Specimen 5:

$$\int \frac{dx}{1+e^x} \;=\; \int (1+e^x)^{-1}\,dx \;=\; \int 1^{-1} + e^{-x}\,dx \;=\; x - e^{-x} + C$$

Evidently, by the high quality of the rest of the examination script, the student was in a rush and not able to check his or her answer or consider its meaning. (Differentiating the answer would reveal the error immediately.) It is interesting to speculate about the student's line of thought. The integration is tricky. Possibly the presence of the integral sign and differential obscured the logic, and added to the desperation, because it is hard to believe this student would ordinarily assume the operations of inversion and addition commute. This was a final exam, so the student has no opportunity for feedback. He or she used the Freshman's Dream, which is in fact true if n is a prime number and the arithmetic happens to be a finite field of characteristic n. Unfortunately for the student, the field here has characteristic 0 and -1 isn't a prime number! Nevertheless, it is a learning opportunity, and if there had been feedback or discussion, one could speculate as a follow up exercise whether inversion and addition can commute in any reasonable arithmetic. (They commute in the field with three elements when both are defined.)

The next pair of examples are responses received when the author set quizzes in second year linear algebra classes, when asking students to complete a list of axioms for a vector space. The first example involves an error that is common even amongst the best students, who have not had much practice with mixing quantifiers. The error is purely syntactic, either because the student is reproducing an axiom from memory without regard to meaning, or the student is thinking about the meaning but has not followed the usual conventions about interpreting quantifiers in the order in which they are listed.

Specimen 6:

Vector space axiom: $(\forall \mathbf{v} \in V)(\exists \mathbf{0} \in V)\;\; \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$

For a thoughtful student, this answer is correct from a semantic point of view, just as the meaning of the following statement in normal English is obvious:

> A person dies every five minutes in Australia (on average).

Here the quantifiers are in the wrong order, but the reader is almost certainly oblivious to the semantics dictated by a ridiculous literal meaning:

> There exists a person who dies, then is resurrected, and dies again, and is resurrected, and so on, ad infinitum, every five minutes (on average).

The next answer is an interesting blend of psychology and syntax, and offered by a surprising number of good students. (For weak students, it is just syntax, and extremely common.)

Specimen 7:

Vector space axiom: $(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V)\;\; (\mathbf{u}\mathbf{v})\mathbf{w} \;=\; \mathbf{u}(\mathbf{v}\mathbf{w})$

A capable student who makes this mistake, for example, will correctly complete the list of axioms for a vector space, and then add the associativity of multiplication for good measure.

(Associativity is yet another syntactic template.) A conscientious student does not want to miss out on getting full marks if possible and prefers to provide the most thorough list of axioms imaginable! Unfortunately this gives the game away and reveals a lapse (hopefully momentary) in understanding, undermining an otherwise faultless response to the question. Once again, with feedback, this is a learning opportunity. There is no multiplication of vectors in the axioms of a vector space, but there is multiplication in a ring. In more advanced mathematics, one studies arithmetics that are rings and vector spaces simultaneously, and then Specimen 7 becomes a valid *algebra* axiom. The student will never forget his or her error, and may be pleased to think of stumbling unwittingly on something potentially interesting.

The next two examples are incorrect attempts at an assignment question in a fourth year course on commutative algebra taught by the author. The context for the question was discussion in lectures about unique factorisation domains, where every nonzero nonunit can be expressed as a product of irreducibles, unique up to order and associates. It was clear from the semantics, though never explicitly stated, that the expression 'product of irreducibles' included the trivial case of a single irreducible. Without that interpretation the concept of a unique factorisation domain would not make any sense at all. The question was then posed in lectures, whether it is possible to have a ring in which factorisation into products of irreducibles failed. Such an example, namely $A = \mathbb{Z} + x\mathbb{Q}[x]$, was provided in the assignment exercise, and students were asked to verify that the element x finishes off the job. Both of the following attempts are beautifully crafted collages of syntactic reasoning, by talented students, and wrong for different (though closely related) reasons.

Specimen 8:

> **Exercise:** Prove that the element x cannot be expressed as a product of irreducibles in the ring $A = \mathbb{Z} + x\mathbb{Q}[x]$.
>
> **Solution:** Since A is an integral domain it suffices to prove that x is prime. Suppose x divides the product fg, so $fg = xq$ for some $q \in A$. Note $xq \in x\mathbb{Q}[x]$ has constant term equal to zero, so f or g must also. Without loss of generality f has constant term equal to zero so x divides f. Thus x is prime and hence irreducible, q.e.d.

Specimen 9:

> **Exercise:** Prove that the element x cannot be expressed as a product of irreducibles in the ring $A = \mathbb{Z} + x\mathbb{Q}[x]$.
>
> **Solution:** The quotient $A/xA = A/x\mathbb{Q}[x] \cong \mathbb{Z}$ is an integral domain, so the principal ideal xA is prime, whence x is a prime element. But A is a subring of $\mathbb{Q}[x]$, which is also an integral domain, and we know prime elements are irreducible in an integral domain. Hence x is irreducible, so by definition cannot be expressed as a product of irreducibles.

Both students interpreted 'product of irreducible' literally in the sense of 'product of *two* (or more) irreducibles'. By proving that x is irreducible then, by definition, x cannot be a product or two (or more) irreducibles, and they are done. How then to prove x is irreducible?

Another assignment question asked them to verify that primes are irreducible in an integral domain, so both students quoted this result and set about proving x is prime. The first made a direct attempt based on a straightforward divisibility argument that shows x is prime in the full polynomial ring $\mathbb{Q}[x]$. Lifting an argument in one context to another related context is a paradigm of syntactic reasoning. Unfortunately, in this case, divisibility in the subring $\mathbb{Z} + x\mathbb{Q}[x]$ is not inherited from the overring $\mathbb{Q}[x]$ and the student falls into error.

In the second case, the argument is an elaborate pastiche and the error becomes even more difficult to trace. In that example, the student is relying on two more facts, that an element is prime if and only if the principal ideal it generates is prime, and that an ideal is prime if and only if the quotient of the ring by this ideal is an integral domain. It is extremely tempting, from the syntax of the expressions, to jump to the conclusion that multiples of x form precisely everything in sight that involves x as a factor, namely $x\mathbb{Q}[x]$ and deduce immediately that this equals xA, whence the quotient should be what is left, namely $\mathbb{Z}$, and the rest follows. Unfortunately $x\mathbb{Q}[x]$ is strictly bigger than xA, and the quotient A/xA is a more complicated ring than $\mathbb{Z}$, and certainly not an integral domain.

All of these considerations enhance the original problem and lead these students, in correcting their errors, to a far greater understanding and appreciation of the associated mathematics. In a certain sense these students are better off in the long run than the student who finds the very brief correct solution (that any decomposition of x must comprise constants together with just one scalar multiple of x, and the latter cannot be irreducible because $\lambda x = 2\dfrac{\lambda x}{2}$ for any λ) and doesn't get embroiled in interesting questions, such as 'What really is the quotient ring A/xA?'

The preceding errors in Specimens 8 and 9 were by very talented students, and it should come as no surprise that no one is immune to the vagaries of syntactic reasoning. There is a famous story about Grothendieck (described in Jackson [14]), being asked to illustrate some concept using prime numbers to which he replied:

> Consider the prime number 57.

His off-the-cuff response was almost certainly pure and simple syntax (perhaps since only a low proportion of composite numbers finish with the digit 7)!

The next and final example was passed on to the author by the late Douglas Munn, who had heard it from someone else and unsure whether it actually occurred or was just an amusing joke. Even if apocryphal, it illustrates the idea that a student attempts to salvage some credit in an assessment, knowing full well that he or she has no idea at all what is going on in the mathematics, and is a combination *par excellence* of syntactic reasoning with the psychology of native cunning:

Specimen 10:

> When asked in an algebra exam whether groups G and H were isomorphic, a student wrote: "G is isomorphic but H isn't."

In the opinion and experience of the author, developing and exploiting the nexus, between syntax of symbolic representation and the underlying semantics, is perhaps the most important feature of successful mathematics teaching and learning. Resolution of the interplay between syntax and semantics can be extremely interesting and also leads to novel teaching principles, introduced in another paper of the author [15], such as the Principle of Reflected Blindness and the Principle of Trivial Complexity, which inhibit learning, and the Halmos Principle, which enhances learning.

References

[1] J. Mason, and S. Johnston-Wilder, *Fundamental contructs in mathematics education*, The Open University, RoutledgeFalmer, Cornwall, 2004.

[2] D. Easdown, *The role of proof in mathematics teaching and the Plateau Principle*, Proc. Science Teaching and Learning Research Symposium, UniServe Science, University of Sydney, 2007, pp. 28–33.

[3] J.H.F. Meyer, and R. Land, *Threshold concepts and troublesome knowledge: linkages to ways of thinking and practising within the discipline*, in C. Rust (ed.), Improving Student Learning: Improving Student Learning Theory and Practice – Ten Years On, Oxford Centre for Staff and Learning Development, Oxford, 2003.

[4] J.H.F. Meyer, and R. Land, *Threshold concepts and troublesome knowledge (2): epistemological considerations and a conceptual framework for teaching and learning*, High. Educ. 49 (2005), pp. 373–388.

[5] I. Kant, *Critique of Pure Reason*, MacMillan, London, 1781.

[6] G. Polya, *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, Wiley, New York, 1962.

[7] J.B. Biggs, and K.F. Collis, *Evaluating the Quality of Learning: the SOLO Taxonomy*, Academic Press, New York, 1982.

[8] J.S. Atherton, *Learning and Teaching: SOLO taxonomy*, 2005, Available at: http://www.learningandteaching.info/learning/solo.htm

[9] Y.V. Matiyasevich, *Hilbert's Tenth Problem*, MIT Press, Cambridge, Massachusetts, 1993.

[10] K. Gödel, *Über formal unentscheidbare Sätze der Prinicipia Mathematica und verwandter Systeme*, Monatsh. Math. Phys. 38 (1931), pp. 173-198.

[11] A. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proc. London Math. Soc. 42 (1936), pp. 230-265.

[12] J.S. Miller, *Why It Is So?*, ABC books, Sydney, 1971.

[13] P.R. Halmos, *Problems for Mathematicians, Young and Old*, Mathematical Association of America, Washington, 1991.

[14] A. Jackson, *Comme Appelé du Néant – As If Summoned From the Void: The Life of Alexandre Grothendieck*, Notices Amer. Math. Soc. 51 (2004), pp. 1196-1212.

[15] D. Easdown, *Teaching mathematics: the gulf between semantics (meaning) and syntax (form)*, Proceedings of the 3rd International Conference on the Teaching of Mathematics at the Undergraduate Level, Turkish Mathematical Society, Istanbul, 2006.