ON HIGHER ORDER SUGAWARA OPERATORS

A. V. CHERVOV AND A. I. MOLEV

Abstract. The higher Sugawara operators acting on the Verma modules over the affine Kac–Moody algebra at the critical level are related to the higher Hamiltonians of the Gaudin model due to work of Feigin, Frenkel and Reshetikhin. An explicit construction of the higher Hamiltonians in the case of $\mathfrak{gl}_n$ was given recently by Talalaev. We propose a new approach to these results from the viewpoint of the vertex algebra theory by proving directly the formulas for the higher order Sugawara operators. The eigenvalues of the operators in the Wakimoto modules of critical level are also calculated.

Report ITEP-TH-36/08

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. The corresponding affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined as the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \qquad \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], \tag{1.1}$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t. Consider the Verma module $M(\lambda)$ over $\widehat{\mathfrak{g}}$ at the critical level, so that the central element K acts as multiplication by the dual Coxeter number $-h^\vee$. The Sugawara operators form a commuting family of $\widehat{\mathfrak{g}}$-endomorphisms of $M(\lambda)$. Such families of operators were first constructed by Goodman and Wallach [11] for the A series and independently by Hayashi [12] for the A, B, C series. These constructions were used in both papers for a derivation of the character formula for the irreducible quotient $L(\lambda)$ of $M(\lambda)$, under certain conditions on λ, thus proving the Kac–Kazhdan conjecture. The existence of families of Sugawara operators for any simple Lie algebra $\mathfrak{g}$ was established by Feigin and Frenkel [5] by providing a description of the center of the local completion $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ of the universal enveloping algebra of $\widehat{\mathfrak{g}}$ at the critical level. This work is based on the vertex algebra theory and makes use of the Wakimoto modules over $\widehat{\mathfrak{g}}$; see also [8].

On the other hand, due to work of Feigin, Frenkel and Reshetikhin [7] the central elements of the local completion $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ are related to the higher Hamiltonians of the Gaudin model describing quantum spin chain. In the case of $\mathfrak{g} = \mathfrak{gl}_n$, Talalaev [17] produced a remarkably simple explicit construction of the higher Hamiltonians. The relationship between the two problems was employed in [3] to produce central elements of $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ in explicit form. However, the proof given there is indirect and relies on [5], [16] and [17].

In this paper we derive this result directly with the use of the vertex algebra theory. We construct two complete sets of higher Sugawara operators for $\mathfrak{gl}_n$ which thus provides explicit formulas for the singular vectors of the Verma modules $M(\lambda)$ at the critical level. The first set is directly related to [17] and the construction recovers the corresponding results of [2], [3], as well as the main result of [17]. We also calculate the eigenvalues of the central elements in the Wakimoto modules.

2. Preliminaries on vertex algebras

2.1. **Universal affine vertex algebra.** A vertex algebra V is a vector space with the additional data $(Y, T, \mathbf{1})$, where the state-field correspondence Y is a map

$$Y : V \to \operatorname{End} V[[z, z^{-1}]],$$

the infinitesimal translation T is an operator $T : V \to V$, and $\mathbf{1}$ is a vacuum vector $\mathbf{1} \in V$. These data must satisfy certain axioms; see e.g., [4], [9], [14]. For $a \in V$ we write

$$Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \operatorname{End} V.$$

In particular, for all $a, b \in V$ we must have $a_{(n)} \cdot b = 0$ for $n \gg 0$. The span in $\operatorname{End} V$ of all *Fourier coefficients* $a_{(n)}$ of all vertex operators $Y(a, z)$ is a Lie algebra $\mathcal{U}$ with the bracket

$$[a_{(m)}, b_{(k)}] = \sum_{n\geqslant 0} \binom{m}{n} \big(a_{(n)} \cdot b\big)_{(m+k-n)}. \tag{2.1}$$

Note that (2.1) is equivalent to the formula for the commutator $[Y(a, z), Y(b, w)]$ implied by the operator product expansion formula; see e.g. [4], [9, Chap. 3]. The *center* of the vertex algebra V is its commutative vertex subalgebra spanned by all vectors $b \in V$ such that $a_{(n)} \cdot b = 0$ for all $a \in V$ and $n \geqslant 0$. The following observation will play an important role below. If b is an element of the center of V, then (2.1) implies $[Y(a, z), Y(b, w)] = 0$ for all $a \in V$. In other words, all Fourier coefficients $b_{(n)}$ belong to the center of the Lie algebra $\mathcal{U}$.

In this paper we are interested in the vertex algebras associated with affine Kac–Moody algebras $\widehat{\mathfrak{g}}$ defined in (1.1), where $\mathfrak{g}$ is a finite-dimensional simple (or reductive) Lie algebra over $\mathbb{C}$. Following [4], define the *universal affine* vertex algebra $V(\mathfrak{g})$ as the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$. The vacuum vector is 1 and the translation operator is determined by

$$T : 1 \mapsto 0, \quad \big[T, K\big] = 0 \quad \text{and} \quad \big[T, X[n]\big] = -n X[n-1], \qquad X \in \mathfrak{g}, \tag{2.2}$$

where we write $X[n] = X t^n$. The state-field correspondence Y is defined by setting $Y(1, z) = \mathrm{id}$,

$$Y(K, z) = K, \qquad Y(J^a[-1], z) = J^a(z) = \sum_{r\in\mathbb{Z}} J^a[r]\, z^{-r-1},$$

where $J^1, \dots, J^d$ is a basis of $\mathfrak{g}$, and then extending the map to the whole of $V(\mathfrak{g})$ with the use of *normal ordering*. Namely, the normally ordered product of fields

$$a(z) = \sum_{r\in\mathbb{Z}} a_{(r)} z^{-r-1} \qquad \text{and} \qquad b(w) = \sum_{r\in\mathbb{Z}} b_{(r)} w^{-r-1}$$

is the formal power series

$$:a(z)\, b(w): = a(z)_+\, b(w) + b(w)\, a(z)_-, \tag{2.3}$$

where

$$a(z)_+ = \sum_{r<0} a_{(r)} z^{-r-1} \qquad \text{and} \qquad a(z)_- = \sum_{r\geqslant 0} a_{(r)} z^{-r-1}.$$

This definition extends to an arbitrary number of fields with the convention that the normal ordering is read from right to left. Then

$$\begin{aligned} &Y(J^{a_1}[r_1] \dots J^{a_m}[r_m], z) \\ &\qquad = \frac{1}{(-r_1-1)! \dots (-r_m-1)!} :\partial_z^{-r_1-1} J^{a_1}(z) \dots \partial_z^{-r_m-1} J^{a_m}(z): \end{aligned}$$

with $r_1 \leqslant \dots \leqslant r_m < 0$, and $a_i \leqslant a_{i+1}$ if $r_i = r_{i+1}$.

The Lie algebra $\mathrm{U}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ spanned by the Fourier coefficients of the fields $Y(a, z)$ with $a \in V(\mathfrak{g})$ is known as the *local completion* of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ see [5], [9, Sec. 3.5]. More precisely, the Lie algebra $\mathrm{U}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ is embedded into the completion $\widetilde{\mathrm{U}}(\widehat{\mathfrak{g}})$ of $\mathrm{U}(\widehat{\mathfrak{g}})$ in the topology whose basis of the open neighborhoods of 0 is formed by the left ideals generated by $\mathfrak{g} \otimes t^N \mathbb{C}[t]$ with $N \geqslant 0$.

2.2. **Segal–Sugawara vectors.** As a vector space, $V(\mathfrak{g})$ can be identified with the space of polynomials in K with coefficients in the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_-)$, where $\mathfrak{g}_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$. An element $S \in \mathrm{U}(\mathfrak{g}_-) \subseteq V(\mathfrak{g})$ is called a *Segal–Sugawara vector* if

$$\mathfrak{g}[t]\, S \in (K + h^\vee)\, V(\mathfrak{g}), \tag{2.4}$$

where $h^\vee$ is the dual Coxeter number for $\mathfrak{g}$. A well-known example of such a vector is provided by the quadratic element

$$S = \frac{1}{2} \sum_{a=1}^{d} J^a[-1]\, J_a[-1], \tag{2.5}$$

where $J_1, \dots, J_d$ is a basis of $\mathfrak{g}$ dual to $J^1, \dots, J^d$ with respect to the suitably normalized invariant bilinear form; see e.g. [9, Chap. 2].

The Segal–Sugawara vectors form a vector subspace of $\mathrm{U}(\mathfrak{g}_-)$ which we will denote by $\mathfrak{z}(\widehat{\mathfrak{g}})$. Moreover, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is clearly closed under the multiplication in $\mathrm{U}(\mathfrak{g}_-)$. The structure of the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ was described in [5]; see also [8]. In order to formulate the result, for any element $S \in \mathrm{U}(\mathfrak{g}_-)$ denote by $\overline{S}$ its image in the associated graded algebra $\mathrm{gr}\,\mathrm{U}(\mathfrak{g}_-) \cong \mathrm{S}(\mathfrak{g}_-)$. We call the set of elements

$$S_1, \dots, S_n \in \mathrm{U}(\mathfrak{g}_-), \qquad n = \mathrm{rk}\,\mathfrak{g}$$

a *complete set of Segal–Sugawara vectors* if each S_l satisfies (2.4) and the corresponding elements $\overline{S}_1, \dots, \overline{S}_n$ coincide with the images of certain algebraically independent generators of the algebra of invariants $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ under the embedding $\mathrm{S}(\mathfrak{g}) \hookrightarrow \mathrm{S}(\mathfrak{g}_-)$ defined by the assignment $J^a \mapsto J^a[-1]$. In accordance to [5], $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials in infinitely many variables,

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n, \quad r \geqslant 0]. \tag{2.6}$$

A different proof of this result was given in a more recent work [8, Theorem 9.6].

Equivalently, the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ coincides with the center of the vertex algebra $V_{-h^\vee}(\mathfrak{g})$, where for each $\kappa \in \mathbb{C}$ the *universal affine* vertex algebra $V_\kappa(\mathfrak{g})$ *of level* κ is the quotient of $V(\mathfrak{g})$ by the ideal $(K-\kappa)V(\mathfrak{g})$. The space $V_\kappa(\mathfrak{g})$ is also called the *vacuum representation of* $\widehat{\mathfrak{g}}$ *of level* κ. The center of $V_\kappa(\mathfrak{g})$ is trivial (i.e., coincides with the multiples of the vacuum vector) for all values of κ except for the critical value $\kappa = -h^\vee$. If $\kappa \neq -h^\vee$ then the vertex algebra $V_\kappa(\mathfrak{g})$ has the standard conformal structure determined by the Segal–Sugawara vector (2.5). Due to the celebrated *Sugawara construction*, the Fourier coefficients of the field

$$\frac{1}{2(\kappa + h^\vee)} \sum_{a=1}^{d} : J^a(z)\, J_a(z) :$$

generate an action of the Virasoro algebra on the space $V_\kappa(\mathfrak{g})$. The existence of higher order Sugawara operators was established in [11] and [12], and certain complete (or fundamental) sets of such operators were produced for types A, B, C; see also [1].

Given a complete set of Segal–Sugawara vectors $S_l \in V(\mathfrak{g})$ with $l = 1, \dots, n$, the corresponding fields

$$S_l(z) = Y(S_l, z) = \sum_{r \in \mathbb{Z}} S_{l,(r)}\, z^{-r-1} \tag{2.7}$$

will form a *complete set of Sugawara fields*. This terminology can be shown to be consistent with [11] and [12]. The *Sugawara operators* are the Fourier coefficients $S_{l,(r)} \in \mathrm{U}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ of the fields (2.7). If the highest weight λ of the Verma module $M(\lambda)$ over $\widehat{\mathfrak{g}}$ is of critical level, then $K + h^\vee$ is the zero operator on $M(\lambda)$. Due to (2.1) and (2.4), the Sugawara operators form a commuting family of $\widehat{\mathfrak{g}}$-endomorphisms of $M(\lambda)$. Moreover, an explicit construction of a complete set of Sugawara fields (2.7) yields an explicit description of all singular vectors of the generic Verma modules $M(\lambda)$ as polynomials in the Sugawara operators $S_{l,(r)}$ with $r<0$ applied to the highest vector of $M(\lambda)$; see [12, Proposition 4.10] for a precise statement.

The local completion *at the critical level* is the quotient $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ of $\mathrm{U}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$ by the ideal generated by $K+h^\vee$. The Sugawara operators can be viewed as the elements of the center of $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})_{\mathrm{loc}}$. In accordance to the description of the center which was originally given in [5], all central elements are obtained from $\mathfrak{z}(\widehat{\mathfrak{g}})$ by the state-field correspondence.

3. Segal–Sugawara vectors for $\mathfrak{gl}_n$

We let e_{ij} denote the standard basis elements of the Lie algebra $\mathfrak{gl}_n$ which we equip with the invariant symmetric bilinear form

$$(X,\,Y)=\mathrm{tr}\,(XY)-\frac{1}{n}\,\mathrm{tr}\,X\,\mathrm{tr}\,Y,\qquad X,Y\in\mathfrak{gl}_n.$$

The element $e_{11}+\dots+e_{nn}$ spans the kernel of the form, therefore it defines a non-degenerate invariant symmetric bilinear form on the Lie algebra $\mathfrak{sl}_n$. The elements $e_{ij}t^r$ of $\mathfrak{gl}_n[t,t^{-1}]$ with $r\in\mathbb{Z}$ will be denoted by $e_{ij}[r]$. Then the affine Kac–Moody algebra $\widehat{\mathfrak{gl}}_n=\mathfrak{gl}_n[t,t^{-1}]\oplus\mathbb{C}K$ has the commutation relations

$$\big[e_{ij}[r],e_{kl}[s]\big]=\delta_{kj}\,e_{il}[r+s]-\delta_{il}\,e_{kj}[r+s]+K\Big(\delta_{kj}\,\delta_{il}-\frac{\delta_{ij}\,\delta_{kl}}{n}\Big)\,r\,\delta_{r,-s},\tag{3.1}$$

and the element K is central. The normalization of the form is chosen in such a way that the critical level $-n$ coincides with the negative of the dual Coxeter number for $\mathfrak{sl}_n$. We will also need the extended Lie algebra $\widehat{\mathfrak{gl}}_n\oplus\mathbb{C}\tau$, where for the element τ we have the relations

$$\big[\tau,e_{ij}[r]\big]=-r\,e_{ij}[r-1],\qquad\big[\tau,K\big]=0.\tag{3.2}$$

We identify the universal affine vertex algebra $V(\mathfrak{gl}_n)$ with the vector space of polynomials in K with coefficients in $\mathrm{U}(\mathfrak{g}_-)$ with $\mathfrak{g}_-=t^{-1}\mathfrak{gl}_n[t^{-1}]$. We will also consider the quotient of $\mathrm{U}(\widehat{\mathfrak{gl}}_n\oplus\mathbb{C}\tau)$ by the left ideal generated by $\mathfrak{gl}_n[t]$. As a vector space, this quotient will be identified with $\mathbb{C}[\tau]\otimes V(\mathfrak{gl}_n)$, which may be viewed as the space of polynomials in τ with coefficients in $V(\mathfrak{gl}_n)$.

For an arbitrary $n\times n$ matrix $A=[a_{ij}]$ with entries in a ring we define its *column-determinant* $\mathrm{cdet}\,A$ by the formula

$$\mathrm{cdet}\,A=\sum_{\sigma}\mathrm{sgn}\,\sigma\cdot a_{\sigma(1)1}\dots a_{\sigma(n)n},\tag{3.3}$$

summed over all permutations σ of the set $\{1,\dots,n\}$.

Consider the $n \times n$ matrix $\tau + E[-1]$ with entries in $\mathrm{U}(\mathfrak{g}_- \oplus \mathbb{C}\tau)$ given by

$$\tau + E[-1] = \begin{bmatrix} \tau + e_{11}[-1] & e_{12}[-1] & \dots & e_{1n}[-1] \\ e_{21}[-1] & \tau + e_{22}[-1] & \dots & e_{2n}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}[-1] & e_{n2}[-1] & \dots & \tau + e_{nn}[-1] \end{bmatrix}.$$

Its column-determinant $\mathrm{cdet}\big(\tau + E[-1]\big)$ is an element of $\mathrm{U}(\mathfrak{g}_- \oplus \mathbb{C}\tau)$ which we also regard as an element of $\mathbb{C}[\tau] \otimes V(\mathfrak{gl}_n)$.

Our main result is a direct proof of the following theorem.

Theorem 3.1. *The coefficients $S_1, \dots, S_n$ of the polynomial*

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^n + \tau^{n-1} S_1 + \dots + S_n \tag{3.4}$$

form a complete set of Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$. Hence, $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ is the algebra of polynomials,

$$\mathfrak{z}(\widehat{\mathfrak{gl}}_n) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n;\ r \geqslant 0].$$

We will prove the theorem in Section 4. Here we consider some examples and corollaries, and derive the second construction of a complete set of Segal–Sugawara vectors; see Theorem 3.5 below.

First we point out that regarding the Lie algebra $\mathfrak{sl}_n$ as the quotient of $\mathfrak{gl}_n$ by the relation $e_{11} + \dots + e_{nn} = 0$, we obtain the respective complete set of Segal–Sugawara vectors in $V(\mathfrak{sl}_n)$. In particular, the vector S_1 vanishes, while $-S_2$ coincides with the vector (2.5).

In what follows the usual vertical line notation for determinants will always be used for column-determinants.

Example 3.2. If $n = 2$ then

$$S_1 = e_{11}[-1] + e_{22}[-1],$$

$$S_2 = \begin{vmatrix} e_{11}[-1] & e_{12}[-1] \\ e_{21}[-1] & e_{22}[-1] \end{vmatrix} - e_{11}[-2] = e_{11}[-1]\, e_{22}[-1] - e_{21}[-1]\, e_{12}[-1] - e_{11}[-2].$$

If $n = 3$ then

$$S_1 = e_{11}[-1] + e_{22}[-1] + e_{33}[-1],$$

$$S_2 = \begin{vmatrix} e_{11}[-1] & e_{12}[-1] \\ e_{21}[-1] & e_{22}[-1] \end{vmatrix} + \begin{vmatrix} e_{11}[-1] & e_{13}[-1] \\ e_{31}[-1] & e_{33}[-1] \end{vmatrix} + \begin{vmatrix} e_{22}[-1] & e_{23}[-1] \\ e_{32}[-1] & e_{33}[-1] \end{vmatrix}$$
$$- 2\, e_{11}[-2] - e_{22}[-2]$$

and

$$S_3 = \begin{vmatrix} e_{11}[-1] & e_{12}[-1] & e_{13}[-1] \\ e_{21}[-1] & e_{22}[-1] & e_{23}[-1] \\ e_{31}[-1] & e_{32}[-1] & e_{33}[-1] \end{vmatrix} + 2\, e_{11}[-3]$$

$$- \begin{vmatrix} e_{11}[-2] & e_{12}[-1] \\ e_{21}[-2] & e_{22}[-1] \end{vmatrix} - \begin{vmatrix} e_{11}[-1] & e_{12}[-2] \\ e_{21}[-1] & e_{22}[-2] \end{vmatrix} - \begin{vmatrix} e_{11}[-2] & e_{13}[-1] \\ e_{31}[-2] & e_{33}[-1] \end{vmatrix}. \qquad \square$$

Note that using (3.2) we can write an alternative expansion

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^n + S_1'\,\tau^{n-1} + \dots + S_{n-1}'\,\tau + S_n'$$

and obtain another complete set of Segal–Sugawara vectors $S_1', \dots, S_n'$.

Applying Theorem 3.1, we can get a complete set of Sugawara fields in an explicit form. Consider the vertex algebra $V(\mathfrak{gl}_n)$ and set $e_{ij}(z) = Y(e_{ij}[-1], z)$ so that

$$e_{ij}(z) = \sum_{r\in\mathbb{Z}} e_{ij}[r]\, z^{-r-1}, \qquad i, j = 1, \dots, n.$$

Introduce the $n \times n$ matrix $\partial_z + E(z)$ by

$$\partial_z + E(z) = \begin{bmatrix} \partial_z + e_{11}(z) & e_{12}(z) & \dots & e_{1n}(z) \\ e_{21}(z) & \partial_z + e_{22}(z) & \dots & e_{2n}(z) \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}(z) & e_{n2}(z) & \dots & \partial_z + e_{nn}(z) \end{bmatrix}$$

and expand its normally ordered column-determinant

$$(3.5) \qquad :\mathrm{cdet}(\partial_z + E(z)): \; = \partial_z^n + \partial_z^{n-1}\, S_1(z) + \dots + \partial_z\, S_{n-1}(z) + S_n(z).$$

Equivalently, the fields $S_l(z)$ are given by $S_l(z) = Y(S_l, z)$, where the elements S_l are defined in (3.4).

Corollary 3.3. *The fields $S_1(z), \dots, S_n(z)$ form a complete set of Sugawara fields for $\widehat{\mathfrak{gl}}_n$.* $\square$

Example 3.4. For $n = 2$ we have

$$:\mathrm{cdet} \begin{bmatrix} \partial_z + e_{11}(z) & e_{12}(z) \\ e_{21}(z) & \partial_z + e_{22}(z) \end{bmatrix} : \; = \; :\big(\partial_z + e_{11}(z)\big)\big(\partial_z + e_{22}(z)\big) - e_{21}(z)\, e_{12}(z):$$

so that

$$S_1(z) = e_{11}(z) + e_{22}(z), \qquad S_2(z) = \; :e_{11}(z)\, e_{22}(z) - e_{21}(z)\, e_{12}(z): \; - e_{11}'(z). \qquad \square$$

Now we use Theorem 3.1 to obtain the second complete set of Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$.

Theorem 3.5. *For any $k \geqslant 0$ all coefficients T_{kl} in the expansion*

$$\mathrm{tr}(\tau + E[-1])^k = \tau^k\, T_{k0} + \tau^{k-1}\, T_{k1} + \cdots + T_{kk}$$

are Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$. Moreover, the elements $T_{11}, \dots, T_{nn}$ form a complete set of Segal–Sugawara vectors. Hence, $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ is the algebra of polynomials,

$$\mathfrak{z}(\widehat{\mathfrak{gl}}_n) = \mathbb{C}[T^r T_{ll} \mid l = 1, \dots, n;\ r \geqslant 0].$$

Proof. Observe that if we replace τ by $u + \tau$, where u is a complex variable, then relations (3.2) will still hold. Therefore, the coefficients in the expansion of the polynomial $\mathrm{cdet}(u+\tau+E[-1])$ in the powers of u and τ are Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$. Hence, the first part of the theorem is implied by the identity

$$\partial_u\, \mathrm{cdet}(u+\tau+E[-1]) = \mathrm{cdet}(u+\tau+E[-1]) \sum_{k=0}^{\infty} (-1)^k\, u^{-k-1}\, \mathrm{tr}(\tau+E[-1])^k, \tag{3.6}$$

since it provides an expression for the vectors T_{kl} in terms of the S_l regarded as elements of $\mathrm{U}(\mathfrak{g}_-)$. The identity (3.6) can be viewed as a noncommutative analogue of the Liouville formula (as well as the Newton identities). It follows from [2, Theorem 4], due to Lemma 4.3 below.[1]

In order to prove the second part, note that the elements $\overline{T}_{11}, \dots, \overline{T}_{nn}$ coincide with the images of the traces of powers $\mathrm{tr}\, E^k$ of the matrix $E = [e_{ij}]$ with $k = 1, \dots, n$ under the embedding $\mathrm{S}(\mathfrak{gl}_n) \hookrightarrow \mathrm{S}(\mathfrak{g}_-)$ defined by $e_{ij} \mapsto e_{ij}[-1]$. However, the elements $\mathrm{tr}\, E, \dots, \mathrm{tr}\, E^n$ are algebraically independent and generate the algebra of invariants $\mathrm{S}(\mathfrak{gl}_n)^{\mathfrak{gl}_n}$. □

Example 3.6. We have

$$\begin{aligned} &T_{10} = n, && T_{11} = \mathrm{tr}\, E[-1] && \\ &T_{20} = n, && T_{21} = 2\,\mathrm{tr}\, E[-1], && T_{22} = \mathrm{tr}\, E[-1]^2 - \mathrm{tr}\, E[-2], \\ &T_{30} = n, && T_{31} = 3\,\mathrm{tr}\, E[-1], && T_{32} = 3\,\mathrm{tr}\, E[-1]^2 - 3\,\mathrm{tr}\, E[-2] \end{aligned}$$

and

$$T_{33} = \mathrm{tr}\, E[-1]^3 - \mathrm{tr}\, E[-1]\, E[-2] - 2\,\mathrm{tr}\, E[-2]\, E[-1] + 2\,\mathrm{tr}\, E[-3]. \qquad \square$$

Introduce the fields $T_{kl}(z) = Y(T_{kl}, z)$ corresponding to the Segal–Sugawara vectors T_{kl}. More explicitly, they can be defined by the expansion of the normally ordered trace

$$:\mathrm{tr}\big(\partial_z + E(z)\big)^k: \ = \partial_z^k\, T_{k0}(z) + \partial_z^{k-1}\, T_{k1}(z) + \cdots + T_{kk}(z), \qquad k \geqslant 0. \tag{3.7}$$

Theorem 3.5 implies the following.

[1] We thank Pavel Pyatov for pointing out that Theorem 4 in [2] can also be proved with the use of R-matrix arguments; cf. [13].

Corollary 3.7. *The fields $T_{11}(z), \dots, T_{nn}(z)$ form a complete set of Sugawara fields for $\widehat{\mathfrak{gl}}_n$.* $\square$

Remark 3.8. The Sugawara operators associated with the vectors T_{kl} exhibit some similarity with the families of operators constructed in [11] and [12], although the exact relation with those families is unclear.

Some other families of Segal–Sugawara vectors can be constructed by using properties of Manin matrices. In particular, the quantum MacMahon master theorem proved in [10] leads to a construction of permanent-type vectors; see also [2]. $\square$

4. Proof of Theorem 3.1

We start by establishing some general properties of column-determinants. First we note that the column-determinant $\operatorname{cdet} A$ of a matrix A over an arbitrary ring changes sign if two rows of the matrix A are swapped. In particular, $\operatorname{cdet} A = 0$ if A has two identical rows.

The next lemma is immediate from the definition of the column-determinant.

Lemma 4.1. *Let $A = [a_{ij}]$ be an arbitrary $n \times n$ matrix with entries in a ring and b an element of the ring. Then*

$$\big[b, \operatorname{cdet} A\big] = \sum_{i=1}^{n} \begin{vmatrix} a_{11} & \cdots & [b, a_{1i}] & \cdots & a_{1n} \\ a_{21} & \cdots & [b, a_{2i}] & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & [b, a_{ni}] & \cdots & a_{nn} \end{vmatrix}.$$

Lemma 4.1 implies one more property of column-determinants.

Lemma 4.2. *Let $A = [a_{ij}]$ be an arbitrary $n \times n$ matrix with entries in a ring and b an element of the ring. Replace column j of A by the column whose all entries are zero except for the i-th entry equal to b. Then the column-determinant of this matrix can be written as*

$$\begin{vmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & b & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} = (-1)^{n-j} \begin{vmatrix} a_{11} & \cdots & \cdots & a_{1n} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & \cdots & a_{in} & b \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & \cdots & a_{nn} & 0 \end{vmatrix}$$

$$+ (-1)^{i+j} \sum_{k=j+1}^{n} \begin{vmatrix} a_{11} & \cdots & [b, a_{1k}] & \cdots & a_{1n} \\ a_{21} & \cdots & [b, a_{2k}] & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & [b, a_{nk}] & \cdots & a_{nn} \end{vmatrix},$$

where the first determinant on the right hand side is obtained by moving column j to the last position, while row i and column j in the determinants occurring in the sum are deleted and the commutators $[b, a_{ik}]$ occur in column $k-1$. $\square$

Now we recall some properties of a class of matrices introduced by Manin [15]. Following [2] we call a matrix $A = [a_{ij}]$ with entries in a ring a *Manin matrix* if

$$a_{ij}\, a_{kl} - a_{kl}\, a_{ij} = a_{kj}\, a_{il} - a_{il}\, a_{kj} \qquad \text{for all possible} \quad i, j, k, l.$$

Such matrices are also known in the literature as *right-quantum matrices* (with $q = 1$); see [10]. It is straightforward to verify that the column-determinant of a (square) Manin matrix will change sign if two columns are swapped. In particular, if a Manin matrix has two identical columns, then its column-determinant is zero. The next observation will be useful in the calculations below.

Lemma 4.3. *The matrix* $\tau + E[-1]$ *with entries in* $\mathrm{U}(\mathfrak{g}_- \oplus \mathbb{C}\tau)$ *is a Manin matrix.* □

We begin proving Theorem 3.1 by verifying that the elements $S_1, \dots, S_n$ satisfy (2.4). Since

$$\big[e_{ij}[0], \tau\big] = 0 \qquad \text{and} \qquad \big[e_{nn}[1], \tau^k\big] = k\, \tau^{k-1} e_{nn}[0],$$

it will be sufficient to check that for all i, j

$$e_{ij}[0]\, \mathrm{cdet}\big(\tau + E[-1]\big) = 0 \quad \text{and} \quad e_{nn}[1]\, \mathrm{cdet}\big(\tau + E[-1]\big) \in (K+n)\, V(\mathfrak{gl}_n) \tag{4.1}$$

in the $\widehat{\mathfrak{gl}}_n$-module $\mathbb{C}[\tau] \otimes V(\mathfrak{gl}_n)$. The first relation in (4.1) is analogous to the well-known property of the Capelli determinant $\mathrm{cdet}[\delta_{ij}(u-i+1)+e_{ij}]$; this is a polynomial in u with coefficients in the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. This property can be verified by a direct argument, and we argue in a similar way to prove the relation. By Lemma 4.1, the polynomial $e_{ij}[0]\, \mathrm{cdet}\big(\tau + E[-1]\big)$ equals

$$\begin{vmatrix} \cdots & -e_{1j}[-1] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{ii}[-1] - e_{jj}[-1] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & -e_{nj}[-1] & \cdots \end{vmatrix} + \sum_{k \neq i} \begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{ik}[-1] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots \end{vmatrix}, \tag{4.2}$$

where the dots indicate the same entries as in the column-determinant of the matrix $\tau + E[-1]$, except for column i in the first determinant and column k in the k-th term in the sum; the entries shown in the middle belong to row j of each determinant. The first determinant can be written as the difference of the determinants

$$\begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \tau + e_{ii}[-1] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots \end{vmatrix} - \begin{vmatrix} \cdots & e_{1j}[-1] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \tau + e_{jj}[-1] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nj}[-1] & \cdots \end{vmatrix}. \tag{4.3}$$

Now, if $i = j$ then the second determinant here equals $\mathrm{cdet}(\tau + E[-1])$, while the first determinant together with the sum over k in (4.2) equals $\mathrm{cdet}(\tau + E[-1])$ by the analogue of the j-th row expansion formula for column-determinants. Hence the first relation in (4.1) holds in this case.

If $i \neq j$, then the second determinant in (4.3) is obtained from $\mathrm{cdet}(\tau + E[-1])$ by replacing column i with column j. Similarly, by the row expansion formula for column-determinants, the first determinant in (4.3) together with the sum over k in (4.2) is obtained from $\mathrm{cdet}(\tau + E[-1])$ by replacing row j with row i. Both determinants are zero due to Lemma 4.3 and the properties of column-determinants.

Now we turn to the second relation in (4.1). By (3.1) and (3.2) we have

$$\big[e_{nn}[1], \tau + e_{ii}[-1]\,\big] = e_{nn}[0] - K' \qquad \text{and} \qquad \big[e_{nn}[1], e_{ni}[-1]\,\big] = e_{ni}[0],$$

where $i \neq n$ and we have put $K' = K/n$, while

$$\big[e_{nn}[1], e_{in}[-1]\,\big] = -e_{in}[0] \qquad \text{and} \qquad \big[e_{nn}[1], \tau + e_{nn}[-1]\,\big] = (n-1)\,K' + e_{nn}[0].$$

Therefore, by Lemma 4.1, the polynomial $e_{nn}[1]\,\mathrm{cdet}\big(\tau + E[-1]\big)$ equals

$$\sum_{i=1}^{n-1} \begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nn}[0] - K' & \cdots \\ \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{ni}[0] & \cdots \end{vmatrix} + \begin{vmatrix} \cdots & -e_{1n}[0] \\ \cdots & \cdots \\ \cdots & -e_{in}[0] \\ \cdots & \cdots \\ \cdots & -e_{n-1,n}[0] \\ \cdots & (n-1)\,K' + e_{nn}[0] \end{vmatrix}, \tag{4.4}$$

where the dots replace the entries of the determinant $\mathrm{cdet}(\tau + E[-1])$, except for column i in the i-th summand and the last column in the last summand. Since each element $e_{in}[0]$ annihilates the vacuum vector of $V(\mathfrak{gl}_n)$, the last summand can be written as $(n-1)\,K' \,\big|\tau + E[-1]\big|_{nn}$, where by $\big|\tau + E[-1]\big|_{ii}$ we denote the column-determinant of the matrix obtained from $\tau + E[-1]$ by deleting row and column i. Using this notation, we can represent (4.4) in the form

$$\sum_{i=1}^{n-1} \begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nn}[0] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{ni}[0] & \cdots \end{vmatrix} - K' \sum_{i=1}^{n-1} \big|\tau + E[-1]\big|_{ii} + (n-1)\,K' \,\big|\tau + E[-1]\big|_{nn}. \tag{4.5}$$

Our next step is to prove that the following key relation holds for each value of $i = 1, \dots, n-1$:

$$\begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nn}[0] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{ni}[0] & \cdots \end{vmatrix} = \big|\tau + E[-1]\big|_{nn} - \big|\tau + E[-1]\big|_{ii}. \tag{4.6}$$

The determinant on the left hand side equals the sum of two determinants obtained by replacing $e_{ni}[0]$ or $e_{nn}[0]$ by 0, respectively. Now we apply Lemma 4.2 to each of these determinants. For the first one we have

$$
\begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nn}[0] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots \end{vmatrix} = (-1)^{n-i} \begin{vmatrix} \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & e_{nn}[0] \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 \end{vmatrix}
$$
$$
+ \sum_{k=i+1}^{n-1} \begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nk}[-1] & \cdots \end{vmatrix} + \begin{vmatrix} \cdots & \cdots & -e_{1n}[-1] \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & -e_{n-1,n}[-1] \\ \cdots & \cdots & 0 \end{vmatrix}, \tag{4.7}
$$

where the determinants in the second line are of the size $(n-1)\times(n-1)$ with row and column i deleted. Now add and subtract the following $(n-1)\times(n-1)$ determinant with row and column i deleted

$$
\begin{vmatrix} \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & -\tau - e_{nn}[-1] \end{vmatrix} = \big|\tau + E[-1]\big|_{ii,nn} \cdot \big(-\tau - e_{nn}[-1]\big),
$$

where $\big|\tau + E[-1]\big|_{ii,nn}$ denotes the column-determinant of the matrix obtained from $\tau + E[-1]$ by deleting rows and columns i and n. Since the element $e_{nn}[0]$ annihilates the vacuum vector of $V(\mathfrak{gl}_n)$, the expression for the determinant in (4.7) simplifies to

$$
\sum_{k=i+1}^{n-1} \begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nk}[-1] & \cdots \end{vmatrix} - \big|\tau + E[-1]\big|_{ii} + \big|\tau + E[-1]\big|_{ii,nn} \big(-\tau - e_{nn}[-1]\big).
$$

Applying now Lemma 4.2 in a similar way to the determinant in (4.6) with $e_{nn}[0]$ replaced with 0, we get

$$
\begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{ni}[0] & \cdots \end{vmatrix} = (-1)^{n-i} \begin{vmatrix} \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & e_{ni}[0] \end{vmatrix}
$$

$$
+ (-1)^{n-i} \sum_{k=i+1}^{n-1} \begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nk}[-1] & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots \end{vmatrix} + (-1)^{n-i} \begin{vmatrix} \cdots & -e_{1i}[-1] \\ \cdots & \cdots \\ \cdots & e_{nn}[-1] - e_{ii}[-1] \\ \cdots & \cdots \\ \cdots & -e_{n-1\,i}[-1] \end{vmatrix}, \tag{4.8}
$$

where, as before, the dots indicate the same entries as in the column-determinant $\operatorname{cdet}\big(\tau + E[-1]\big)$, the determinants in the second line are of the size $(n-1)\times(n-1)$ with row n and column i deleted, and the entry $e_{nk}[-1]$ in the summation term occur in row i and column $k-1$. Since the element $e_{ni}[0]$ annihilates the vacuum vector of $V(\mathfrak{gl}_n)$, the first determinant on the right hand side of (4.8) vanishes. In order to transform the remaining combination, add and subtract the $(n-1)\times(n-1)$ column-determinant

$$
(-1)^{n-i} \begin{vmatrix} \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & -\tau \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 \end{vmatrix},
$$

obtained from $\operatorname{cdet}(\tau + E[-1])$ by deleting row n and column i and replacing the last column as indicated, with $-\tau$ in the row i. After combining this difference with the last term in the expansion (4.8) we obtain the difference of two $(n-1)\times(n-1)$ column-determinants multiplied by $(-1)^{n-i}$, where one of them is obtained from $\operatorname{cdet}(\tau + E[-1])$ by deleting row n and moving column i to replace the last column. By Lemma 4.3, the corresponding $(n-1)\times(n-1)$ matrix is a Manin matrix. Hence, taking the signs into account we conclude that the determinant in question equals the minor $\big|\tau+E[-1]\big|_{nn}$. For the remaining determinants we use the general property of column-determinants allowing us to permute rows. Moving row i in each of the determinants down to the last position and taking signs into account, we find that the determinant in (4.8) equals

$$
-\sum_{k=i+1}^{n-1} \begin{vmatrix} \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & e_{nk}[-1] & \cdots \end{vmatrix} + \big|\tau + E[-1]\big|_{nn} - \big|\tau + E[-1]\big|_{ii,nn}\big(-\tau - e_{nn}[-1]\big).
$$

Combining this with the expression for the determinant (4.7) derived above, we complete the verification of (4.6).

Recalling now the expression (4.5), we arrive at the relation

$$e_{nn}[1]\,\mathrm{cdet}\big(\tau+E[-1]\big)=\frac{K+n}{n}\,\Big((n-1)\,\big|\tau+E[-1]\big|_{nn}-\sum_{i=1}^{n-1}\big|\tau+E[-1]\big|_{ii}\Big)$$

thus completing the proof of (4.1).

Finally, under the embedding $\mathrm{S}(\mathfrak{gl}_n)\hookrightarrow\mathrm{S}(\mathfrak{g}_-)$ defined by $e_{ij}\mapsto e_{ij}[-1]$, the elements $\overline{S}_1,\dots,\overline{S}_n$ coincide with the images of the respective coefficients of the characteristic polynomial $\det[\delta_{ij}\,u+e_{ij}]$ which are algebraically independent and generate the algebra of invariants $\mathrm{S}(\mathfrak{gl}_n)^{\mathfrak{gl}_n}$. Therefore, $S_1,\dots,S_n$ form a complete set of Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$.

5. Center of the local completion and commutative subalgebras

Recall that $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ is the algebra of Segal–Sugawara vectors in $V(\mathfrak{gl}_n)$ which can also be regarded as the center of the vertex algebra $V_{-n}(\mathfrak{gl}_n)$. The center of the local completion $\mathrm{U}_{-n}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ is the vector subspace $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ which consists of the elements commuting with the action of $\widehat{\mathfrak{gl}}_n$. It was proved in [5] that any element of $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ is a Fourier component of a field corresponding to an element of $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$. By Theorems 3.1 and 3.5, the fields $Y(S,z)$ with $S\in\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ can be interpreted as differential polynomials either in the fields $S_1(z),\dots,S_n(z)$ or in the fields $T_{11}(z),\dots,T_{nn}(z)$ with normally ordered products; see (3.5) and (3.7). Hence, we obtain two explicit descriptions of $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ formulated below. The first of them was originally given in [3], and the arguments there rely on the results of [5], [16] and [17].

Corollary 5.1. *The center $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ of the local completion $\mathrm{U}_{-n}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ consists of the Fourier coefficients of all differential polynomials in either family of the fields $S_1(z),\dots,S_n(z)$ or $T_{11}(z),\dots,T_{nn}(z)$.* □

By the vacuum axiom of a vertex algebra, the application of the fields $S_l(z)$ and $T_{kl}(z)$ to the vacuum vector 1 of $V_{-n}(\mathfrak{gl}_n)$ yields power series in z which we denote respectively by

$$S_l(z)_+=\sum_{r<0}S^+_{l,(r)}z^{-r-1}\qquad\text{and}\qquad T_{kl}(z)_+=\sum_{r<0}T^+_{kl,(r)}z^{-r-1}. \tag{5.1}$$

Since the Fourier coefficients of the fields $S_l(z)$ and $T_{kl}(z)$ belong to $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$, all coefficients of the series (5.1) belong to the center $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ of the vertex algebra $V_{-n}(\mathfrak{gl}_n)$. By (3.5) and (3.7), the series (5.1) can be written in explicit form with the use of the matrix $E(z)_+=[e_{ij}(z)_+]$. We have

$$\mathrm{cdet}(\partial_z+E(z)_+)=\partial_z^n+\partial_z^{n-1}\,S_1(z)_++\dots+\partial_z\,S_{n-1}(z)_++S_n(z)_+$$

and

$$\operatorname{tr}\big(\partial_z + E(z)_+\big)^k = \partial_z^k\, T_{k0}(z)_+ + \partial_z^{k-1}\, T_{k1}(z)_+ + \dots + T_{kk}(z)_+.$$

We arrive at the following result, whose first part dealing with the commuting family of the elements $S^+_{l,(r)}$ goes back to the original work [17]; see also [2], [3].

Corollary 5.2. *The elements of each of the families*

$$S^+_{l,(r)} \quad \textit{with} \quad l = 1, \dots, n \quad \textit{and} \quad r < 0, \qquad T^+_{kl,(r)} \quad \textit{with} \quad 0 \leqslant l \leqslant k \quad \textit{and} \quad r < 0,$$

belong to $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$. *Moreover,* $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ *is the algebra of polynomials*

$$\mathfrak{z}(\widehat{\mathfrak{gl}}_n) = \mathbb{C}[S^+_{l,(r)} \mid l = 1, \dots, n, \quad r < 0] = \mathbb{C}[T^+_{ll,(r)} \mid l = 1, \dots, n; \ r < 0].$$

Proof. We only need to prove the algebraic independence of the families of the elements $S^+_{l,(r)}$ and $T^+_{ll,(r)}$. This follows by comparing their highest degree components in the graded algebra $\operatorname{gr} \mathrm{U}(\mathfrak{g}_-) \cong \mathrm{S}(\mathfrak{g}_-)$ with those of the elements $T^r S_l$ and $T^r T_{ll}$, respectively; see Theorems 3.1 and 3.5. In the general case such relationship between two families of generators of $\mathrm{S}(\mathfrak{g}_-)$ was pointed out in [16]. □

The higher Gaudin Hamiltonians can be obtained by taking the images of the elements of the commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ of $\mathrm{U}(\mathfrak{g}_-)$ in the algebras $\mathrm{U}(\mathfrak{gl}_n)^{\otimes m}$ under certain evaluation homomorphisms; see e.g. [7], [17] for details.

6. Eigenvalues in the Wakimoto modules

As another application of the explicit formulas of Theorems 3.1 and 3.5, we recover a description of the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{gl}_n)$ which allows one to find the eigenvalues of the elements of the center $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ in the Wakimoto modules of critical level. We start by recalling the main steps in the construction of these modules following [8].

Consider the *Weyl algebra* $\mathcal{A}(\mathfrak{gl}_n)$ generated by the elements $a_{ij}[r]$ with $r \in \mathbb{Z}$, $i, j = 1, \dots, n$ and $i \neq j$ and the defining relations

$$\big[a_{ij}[r], a_{kl}[s]\big] = \delta_{kj}\,\delta_{il}\,\delta_{r,-s} \qquad \text{for} \quad i < j,$$

whereas all other pairs of the generators commute. The Fock representation $M(\mathfrak{gl}_n)$ of $\mathcal{A}(\mathfrak{gl}_n)$ is generated by a vector $|0\rangle$ such that for $i < j$ we have

$$a_{ij}[r]|0\rangle = 0, \quad r \geqslant 0 \qquad \text{and} \qquad a_{ji}[r]|0\rangle = 0, \quad r > 0.$$

The vector space $M(\mathfrak{gl}_n)$ carries a vertex algebra structure. In particular, $|0\rangle$ is the vacuum vector, and for $i < j$ we have

$$Y(a_{ij}[-1]\,|0\rangle, z) = \sum_{r \in \mathbb{Z}} a_{ij}[r]\, z^{-r-1} \qquad \text{and} \qquad Y(a_{ji}[0]\,|0\rangle, z) = \sum_{r \in \mathbb{Z}} a_{ji}[r]\, z^{-r}.$$

We will denote these series by $a_{ij}(z)$ and $a_{ji}(z)$, respectively. Furthermore, consider the algebra of polynomials

$$\pi_0 = \mathbb{C}[b_i[r] \mid i = 1, \dots, n;\ r < 0]$$

in the variables $b_i[r]$ as a commutative vertex algebra and set

$$b_i(z) = \sum_{r \in \mathbb{Z}} b_i[r]\, z^{-r-1}.$$

The translation operator on π_0 is defined by

$$T\,1 = 0, \qquad \big[T, b_i[r]\big] = -r\, b_i[r-1].$$

The key fact leading to the construction of Wakimoto modules is the existence of the vertex algebra homomorphism

$$\rho : V_{-n}(\mathfrak{gl}_n) \to M(\mathfrak{gl}_n) \otimes \pi_0, \tag{6.1}$$

given in terms of fields as follows:

$$\begin{aligned} e_{i,i+1}(z) &\mapsto a_{i,i+1}(z) + \sum_{k<l} :P^i_{kl}\, a_{kl}(z): \\ e_{ii}(z) &\mapsto \sum_{k<l} d^i_{kl}\ :a_{lk}(z) a_{kl}(z): \ + b_i(z) \\ e_{i+1,i}(z) &\mapsto \sum_{k<l} :Q^i_{kl}\, a_{kl}(z): + c_i\, \partial_z\, a_{i+1,i}(z) + b_i(z)\, a_{i+1,i}(z), \end{aligned}$$

where P^i_{kl} and Q^i_{kl} are certain polynomials in the series $a_{pq}(z)$ with $p > q$, while d^i_{kl} and c_i are certain coefficients; see [8, Theorem 4.7] for the precise formulas. Take an n-tuple

$$\chi(t) = \big(\chi_1(t), \dots, \chi_n(t)\big), \qquad \chi_i(t) = \sum_{r \in \mathbb{Z}} \chi_i[r]\, t^{-r-1} \in \mathbb{C}((t)),$$

where $\mathbb{C}((t))$ denotes the algebra of formal Laurent series in t containing only a finite number of negative powers of t. Consider the one-dimensional π_0-module $\mathbb{C}_{\chi(t)}$ on which $b_i[r]$ acts as multiplication by $\chi_i[r]$. Taking the composition of this module with the homomorphism (6.1) we get a $\widehat{\mathfrak{gl}}_n$-module structure on $M(\mathfrak{gl}_n)$. These are the *Wakimoto modules of critical level*, they are denoted by $W_{\chi(t)}$.

The image of the center $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ of the vertex algebra $V_{-n}(\mathfrak{gl}_n)$ under the homomorphism ρ given in (6.1) is contained in $\pi_0 \cong 1 \otimes \pi_0$; see [8, Lemma 9.1]. This image is known as the *classical $\mathcal{W}$-algebra* $\mathcal{W}(\mathfrak{gl}_n)$. Applying Theorems 3.1 and 3.5, we recover its explicit description; cf. [6, Sec. 2.4.11]. In order to formulate the result, consider the extended algebra, which is isomorphic to $\mathbb{C}[\tau] \otimes \pi_0$, as a vector space, with the relations

$$\big[\tau, b_i[r]\big] = -r\, b_i[r-1].$$

Corollary 6.1. *The image of the column-determinant* $\mathrm{cdet}(\tau + E[-1])$ *under the homomorphism* (6.1) *is given by*

$$\rho : \mathrm{cdet}(\tau + E[-1]) \mapsto (\tau + b_1[-1]) \cdots (\tau + b_n[-1]).$$

Hence, if the elements B_i *are defined by*

$$(\tau + b_1[-1]) \cdots (\tau + b_n[-1]) = \tau^n + \tau^{n-1} B_1 + \cdots + B_n,$$

then $\mathcal{W}(\mathfrak{gl}_n)$ *is the algebra of polynomials in the variables* $T^r B_i$ *with* $i = 1, \dots, n$ *and* $r \geqslant 0$.

Proof. The images of the elements $e_{ij}[-1]$ with $i < j$ under the homomorphism (6.1) have trivial intersection with the algebra π_0. Therefore the image of the expansion of $\mathrm{cdet}(\tau + E[-1])$ will only contain diagonal terms arising from the $e_{ii}[-1]$. □

Applying the identity (3.6) and Corollary 6.1, we can also calculate the images of the Segal–Sugawara vectors T_{kl}.

Corollary 6.2. *The images of the elements* $\mathrm{tr}(\tau + E[-1])^k$ *under the homomorphism* (6.1) *are found from the formula*

$$\rho : \sum_{k=0}^{\infty} t^k \, \mathrm{tr}(\tau + E[-1])^k \mapsto \sum_{i=1}^{n} \Big(1 - t\big(\tau + b_n[-1]\big)\Big)^{-1} \cdots \Big(1 - t\big(\tau + b_i[-1]\big)\Big)^{-1}$$
$$\times \Big(1 - t\big(\tau + b_{i+1}[-1]\big)\Big) \cdots \Big(1 - t\big(\tau + b_n[-1]\big)\Big),$$

where t *is a complex variable.*

Proof. The formula follows from (3.6) by calculating the derivative on the left hand side and then replacing u by $-t^{-1}$. □

The elements of the center $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ of $\mathrm{U}_{-n}(\widehat{\mathfrak{gl}}_n)_{\mathrm{loc}}$ act on the Wakimoto modules $W_{\chi(t)}$ as multiplications by scalars which can be calculated by using Corollaries 6.1 and 6.2. Recall the description of $\mathfrak{Z}(\widehat{\mathfrak{gl}}_n)$ provided by Corollary 5.1.

Corollary 6.3. *The coefficients of* $:\mathrm{cdet}(\partial_z + E(z)):$ *and* $:\mathrm{tr}\big(\partial_z + E(z)\big)^k:$ *act on* $W_{\chi(t)}$ *as multiplications by scalars found from the respective formulas*

$$:\mathrm{cdet}(\partial_z + E(z)): \ \mapsto \big(\partial_z + \chi_1(z)\big) \dots \big(\partial_z + \chi_n(z)\big)$$

and

$$\sum_{k=0}^{\infty} t^k \ :\mathrm{tr}\big(\partial_z + E(z)\big)^k: \ \mapsto \sum_{i=1}^{n} \Big(1 - t\big(\partial_z + \chi_n(z)\big)\Big)^{-1} \cdots \Big(1 - t\big(\partial_z + \chi_i(z)\big)\Big)^{-1}$$
$$\times \Big(1 - t\big(\partial_z + \chi_{i+1}(z)\big)\Big) \cdots \Big(1 - t\big(\partial_z + \chi_n(z)\big)\Big).$$

Acknowledgments

We acknowledge the support of the Australian Research Council. The first author is grateful to the University of Sydney for support and hospitality during his visit. His research is also supported by the grant of Support for the Scientific Schools 8004.2006.2, RFBR grant 08-02-00287a, the ANR grant GIMP (Geometry and Integrability in Mathematics and Physics) and the grant MK-5056.2007.1.

References

[1] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, *Extensions of the Virasoro algebra constructed from Kac–Moody algebras using higher order Casimir invariants*, Nucl. Phys. B**304** (1988), 348–370.

[2] A. Chervov and G. Falqui, *Manin matrices and Talalaev's formula*, J. Phys. A: Math. Theor. **41** (2008), 194006 (28pp).

[3] A. Chervov and D. Talalaev, *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, preprint `arXiv:hep-th/0604128`.

[4] A. De Sole and V. G. Kac, *Finite vs affine W-algebras*, Japan. J. Math., **1** (2006), 137-261.

[5] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras*, Int. J. Mod. Phys. A**7**, Suppl. 1A (1992), 197–215.

[6] B. Feigin and E. Frenkel, *Integrals of motion and quantum groups*, in "Integrable systems and quantum groups" (Montecatini Terme, 1993), 349–418, Lecture Notes in Math., 1620, Springer, Berlin, 1996.

[7] B. Feigin, E. Frenkel and N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys. **166** (1994), 27–62.

[8] E. Frenkel, *Wakimoto modules, opers and the center at the critical level*, Adv. Math. **195** (2005), 297–404.

[9] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, 88. American Mathematical Society, Providence, RI, 2001.

[10] S. Garoufalidis, Thang T. Q. Lê and D. Zeilberger, *The quantum MacMahon master theorem*, Proc. Nat. Acad. Sci. USA **103** (2006), 13928–13931.

[11] R. Goodman and N. Wallach, *Higher-order Sugawara operators for affine Lie algebras*, Trans. Amer. Math. Soc. **315** (1989), 1–55.

[12] T. Hayashi, *Sugawara operators and Kac–Kazhdan conjecture*, Invent. Math. **94** (1988), 13–52.

[13] A. Isaev, O. Ogievetsky and P. Pyatov, *On quantum matrix algebras satisfying the Cayley-Hamilton-Newton identities*, J. Phys. A **32** (1999), L115–L121.

[14] V. Kac, *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, Providence, RI, 1997.

[15] Yu. I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montreal, QC, 1988.

[16] L. G. Rybnikov, *Uniqueness of higher Gaudin hamiltonians*, preprint `arXiv:math/0608588`.

[17] D. V. Talalaev, *The quantum Gaudin system*, Funct. Anal. Appl. **40** (2006), 73–77.

Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail address: chervov@itep.ru

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: alexm@maths.usyd.edu.au