Finitely Dominated Subnormal Covers of 4-manifolds

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ABSTRACT

Let M be a closed 4-manifold which has a finitely dominated covering space associated to a subnormal subgroup G of infinite index in $\pi = \pi_1(M)$. If G is FP_3, has finitely many ends and π is virtually torsion free then either M is aspherical or its universal covering space is homotopy equivalent to S^2 or S^3. In the aspherical case such a subnormal subgroup is usually Z, a surface group or a PD_3-group. (This is a revision of Research Report 1994-23).

AMS Subject Classification (1991): Primary 57N13. Secondary 20J05
Key words and phrases: finitely dominated, 4-manifold, Poincaré duality group, subnormal subgroup.

In this note we shall extend earlier work on 4-manifolds with a finitely dominated infinite covering space from the regular to the subnormal case. (See §2 of Chapter 3 of [9]). Finitely dominated covering spaces of aspherical manifolds correspond to FP subgroups of the fundamental group. In §1 we show that an FP_3 subnormal subgroup of a PD_4-group is usually a PD-group. However we have not been able to eliminate other possibilities completely. For instance, it is not known whether a Baumslag-Solitar group may be a subnormal subgroup of a PD_4-group. In §2 we assume that M is a closed 4-manifold and that $\pi = \pi_1(M)$ has an FP_3 subnormal subgroup G of infinite index such that the associated covering space is finitely dominated, and give homological conditions on π and G under which either M is aspherical or its universal covering space is homotopy equivalent to S^2 or S^3.

§1. Poincaré duality groups

The Hirsch-Plotkin radical $\sqrt{\pi}$ of a group π is the maximal locally nilpotent, normal subgroup of π. The Hirsch length $h(\nu)$ of a finitely generated nilpotent group ν is the number of infinite cyclic factors of a composition series for the group; $h(\sqrt{\pi})$ is the least upper bound of $h(\nu)$ as ν varies over finitely generated subgroups of $\sqrt{\pi}$. If G is a subgroup of π then $C_\pi(G)$ and $N_\pi(G)$ are the centralizer and normalizer of G in π, respectively. The centre of G is $\zeta G = G \cap C_\pi(G)$.

Theorem 1. *Let G be a nontrivial FP_3 normal subgroup of infinite index in a PD_4-group π. Then either*
(i) G is a PD_3-group and π/G has two ends;
(ii) G is a PD_2-group and π/G is virtually a PD_2-group; or
(iii) $G \cong Z$, $H^s(\pi/G; Z[\pi/G]) = 0$ for $s \leq 2$ and $H^3(\pi/G; Z[\pi/G]) \cong Z$.

Proof. The subgroup G is FP, since *c.d.*$G < 4$ [13], and hence so is π/G. The E_2 terms of the LHS spectral sequence with coefficients $Q[\pi]$ can then be expressed as tensor products $H^p(\pi/G; Q[\pi/G]) \otimes H^q(G; Q[G])$. If $H^j(\pi/G; Q[\pi/G])$ and $H^k(G; Q[G])$ are the first nonzero such cohomology groups then $H^j(\pi/G; Q[\pi/G]) \otimes H^k(G; Q[G])$ persists to E_∞ and hence $j+k = 4$ and this tensor product is Q. Hence $H^j(\pi/G; Q[\pi/G]) \cong H^{n-j}(G; Q[G]) \cong Q$. In particular, π/G has one or two ends and G is a PD_{4-j}-group over Q [6]. If π/G has two ends then it is virtually Z, and then G is a PD_3-group (over Z) by Theorem 9.11 of [1]. If $H^2(G; Q[G]) \cong H^2(\pi/G; Q[\pi/G]) \cong Q$ then G and π/G are virtually PD_2-groups [3]. Since G is torsion free it is then in fact a PD_2-group. The only remaining possibility is (*iii*). □

Is it sufficient that G be FP_2? Must the quotient π/G be virtually a PD-group in case (*iii*) also?

Corollary. *If K is FP_2 and is subnormal in N where N is an FP_3 normal subgroup of infinite index in the PD_4-group π then K is a PD_k-group for some $k < 4$.*

Proof. This follows immediately from Theorem 1 together with [2]. □

In [2] it was shown that if H is an FP_2 subnormal subgroup of a PD_3-group G then either H is an infinite cyclic normal subgroup or H is a surface group and $[G : N_G(H)] < \infty$ or G is virtually poly-Z. We shall consider next FP subnormal subgroups of PD_4-groups.

Theorem 2. *Let G be a nontrivial FP subnormal subgroup of infinite index in a PD_4-group π. Suppose that G has finitely many ends. Then either*

(i) G *is a* PD_3*-group,* $[\pi : N_\pi(G)] < \infty$ *and* $N_\pi(G)/G$ *has two ends; or*

(ii) $c.d.G = 3$ *and* $H^2(G; Z[G])$ *is not finitely generated; or*

(iii) G *is a* PD_2*-group,* $[\pi : N_\pi(G)] < \infty$ *and* π *is virtually the group of a surface bundle over a surface; or*

(iv) G *is a* PD_2*-group,* $\zeta G = 1$ *and* π *is virtually the group of the mapping torus of a self homeomorphism of a surface bundle over the circle; or*

(v) $c.d.G = 2$, $\chi(G) = 0$, $H^2(G; Z[G])$ *is not finitely generated and* $[\pi : N_\pi(G)] = \infty$*; or*

(vi) $G \cong Z$ *and* $G \leq \sqrt{\pi}$*, and either* $\sqrt{\pi}$ *is abelian of rank* ≤ 2 *or* π *is virtually poly-*Z*.*

Proof. Let $G = N_0 < N_1 < ... < N_r = \pi$ be a subnormal chain of minimal length. Let $j = \min\{i | [N_{i+1} : G] = \infty\}$. Then N_j is FP and is subnormal in π, and it is easily seen that the theorem holds for G if it holds for N_j. Thus we may assume that $[N_1 : G] = \infty$. Suppose first that G has one end. Then $c.d.G = 2$ or 3, since $[\pi : G] = \infty$. If $c.d.G = 3$ and $H^2(G; Z[G])$ is finitely generated then $H^s(G; Z[G]) = 0$ for $s \leq 2$, by [5]. It follows immediately from the LHS spectral sequence that $H^s(N_1; W) = 0$ for $s \leq 3$ and any free $Z[N_1]$-module W. Hence $c.d.N_1 = 4$ and so $[\pi : N_1] < \infty$, by [13]. Hence N_1 is a PD_4-group and (*i*) follows from Theorem 1. If $c.d.G = 3$ and $H^2(G; Z[G])$ is not finitely generated (*ii*) holds.

Suppose next that $c.d.G = 2$. If $G_1 < G_2$ are two such groups with G_1 normal in G_2 then $[G_2 : G_1]$ is finite, by Theorem 8.2 of [1], and $\chi(G_1) = [G_2 : G_1]\chi(G_2)$. Moreover if G_2 is normal in J then $[J : N_J(G_1)] < \infty$, since G_2 has only finitely many subgroups of index $[G_2 : G_1]$. Therefore if $\chi(G) \neq 0$ we may assume that G is maximal among normal subgroups of N_1 with cohomological dimension 2. Let n be an element of N_2 such that $nGn^{-1} \neq G$, and let $H = G.nGn^{-1}$. Then $G < H$ and H is normal in N_1 so $[H : G] = \infty$ and $c.d._Q H = 3$. Moreover H is FP and $H^s(H; Z[H]) = 0$ for $s \leq 2$, so either N_1/H is locally finite or $c.d_Q N_1 > c.d._Q H$, by Theorem 8.2 of [1]. If N_1/H is locally finite but not finite then we again have $c.d_Q N_1 > c.d._Q H$, by Theorem 3.3 of [8]. If $c.d._Q N_1 = 4$ then $[\pi : N_1] < \infty$, so N_1 is a PD_4-group and (*iii*) holds, by Theorem 1. Otherwise $[N_1 : H] < \infty$ and then $c.d.N_1 = 3$, N_1 is

FP and $H^s(N_1; Z[N_1]) = 0$ for $s \leq 2$. Hence N_1 is a PD_3-group by (*i*), and so (*iv*) holds.

Suppose that $\chi(G) = 0$ and that G is a PD_2-group. Then $G \cong Z^2$ or $Z \times_{-1} Z$, so $h(\sqrt{\pi}) \geq 2$ and $\chi(\pi) = 0$. We may assume that π is orientable, so $Hom(\pi, Z) \neq 0$. If $h(\sqrt{\pi}) > 2$ then π is virtually poly-Z, by Theorem 8.1 of [9]. Therefore we may also assume that $h(\sqrt{\pi}) = 2$. In this case $\sqrt{\pi} \cong Z^2$ and π is virtually the group of a torus bundle over a surface, by Theorem 9.2 of [9]. Since $[\sqrt{\pi} : G] < \infty$ it follows also that $[\pi : N_\pi(G)] < \infty$ and so (*iii*) holds. If G has one end and $c.d.G = 2$ but G is not a PD_2-group then $H^2(G; Z[G])$ is not finitely generated [6] and $[\pi : N_\pi(G)] = \infty$, and so (*v*) covers the remaining possibilities.

Finally, if G has two ends then $G \cong Z$, so $G \leq \sqrt{\pi}$. If $h = h(\sqrt{\pi}) > 2$ then π is virtually poly-Z. If $h \leq 2$ then $\sqrt{\pi}$ is abelian of rank h. $\square$

To what extent can the hypotheses be relaxed? Are all FP subnormal subgroups PD-groups? If so then cases (*ii*) and (*vi*) cannot arise. (This is certainly so if there is a subnormal chain consisting of FP subgroups). If G is FP and $c.d.G = 3$ then G has one end (cf [2]). Can a finitely generated noncyclic free group be a subnormal subgroup of a PD_4-group?

Examples. 1. Let π be the semidirect product of $S = \langle a, b, c, d \mid [a, b][c, d] = 1\rangle$ (the genus 2 surface group) with the rank 2 free abelian normal subgroup G generated by x and y, with the action of S on G given by $axa^{-1} = xy^2, cyc^{-1} = x^2y$, b, c, d commute with x and a, b, d commute with y. Then $\sqrt{\pi} = G$ and $C_\pi(G) \cong Z^2 \times F(\infty)$. In particular, $C_\pi(\sqrt{\pi})$ need not be finitely generated.

2. Let G be a PD_2-group such that $\zeta G = 1$. Let $\theta : G \to G$ have infinite order in $Out(G)$, and let $\lambda : G \to Z$ be an epimorphism. Let $\pi = (G \times Z) \times_\phi Z$ where $\phi(g, n) = (\theta(g), \lambda(g) + n)$ for all $g \in G$ and $n \in Z$. Then G is subnormal in π but this group is not virtually the group of a surface bundle over a surface.

3. Any group with a finite 2-dimensional Eilenberg - Mac Lane complex is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to D^4. On applying the orbifold hyperbolization technique of Gromov, Davis and Januszkiewicz to the boundary

we see that each such group embeds in a PD_4-group. (See [10]). (Conjecturally such groups are exactly the finitely presentable groups of cohomological dimension 2). The simplest such groups G with $\chi(G) = 0$ which are not PD_2-groups are the Baumslag-Solitar 1-relator groups $G_{p,q} = \langle a, t \mid ta^p t^{-1} = a^q \rangle$ with $|pq| > 1$. Can they be realised as *subnormal* subgroups of PD_4-groups?

§2. Closed 4-manifolds

In this section we shall investigate closed 4-manifolds whose fundamental groups have subnormal subgroups and whose homology is constrained in other ways.

Theorem 3. *Let M be a closed 4-manifold with fundamental group π and let $p : \hat{M} \to M$ be a covering projection such that $\hat{M}$ is finitely dominated and such that $G = \pi_1(\hat{M})$ is a nontrivial subnormal subgroup of infinite index in π. Suppose also that G is FP_3. Then*

(i) if G is finite then the universal covering space $\tilde{M}$ is homotopy equivalent to S^2 or S^3 and $[\pi : N_\pi(G)]$ is finite;

(ii) if G has one end then M is aspherical;

(iii) if G has two ends then either M is aspherical or it is finitely covered by $S^2 \times S^1 \times S^1$ or $h(\sqrt{\pi}) = 1$ and $H^2(\pi; Z[\pi])$ is not finitely generated;

(iv) if G has infinitely many ends and is subnormal in N where N is an FP_2 normal subgroup of infinite index in π then either M has a finite covering space which is homotopy equivalent to the mapping torus of a self homotopy equivalence of a PD_3-complex and $[\pi : N_\pi(G)]$ is finite or M is aspherical and N is not FP_3.

Proof. Let $G = N_0 < N_1 < ... < N_r = \pi$ be a subnormal chain. Suppose first that G is finite. Then $\tilde{M}$ is also finitely dominated. Since π has nontrivial torsion M cannot be aspherical, so is homotopy equivalent to S^2 or S^3, by Theorem 3.9 of [9]. If $\tilde{M} \simeq S^2$ then the kernel of the natural homomorphism from π to $Aut(\pi_2(M))$ is torsion free. Hence $G = Z/2Z$ and so G is central in N_1. Moreover as it is the torsion subgroup of ζN_1 it is characteristic in N_1, and hence normal in N_2. A finite induction now shows that G is normal in π. If $\tilde{M} \simeq S^3$ then π has two ends, and so $[\pi : N_\pi(G)]$ is finite.

If G is infinite then a finite induction using the LHS spectral sequence shows that π has one end, and that if moreover G has one end then $H^2(\pi; Z[\pi]) =$

0. Since G is FP_3 and $\hat{M}$ is finitely dominated $\pi_2(M) = \pi_2(\hat{M})$ is finitely generated as a $Z[G]$-module, and so $Hom_\pi(\pi_2(M), Z[\pi]) = 0$. Therefore $\pi_2(M) \cong \overline{H^2(\pi; Z[\pi])}$, by Lemma 3.3 of [9], and so M is aspherical if and only if $H^2(\pi; Z[\pi]) = 0$. In particular, M is aspherical if G has one end.

If G has two ends then it has an infinite cyclic normal subgroup of finite index, and so we may assume without loss of generality that $G \cong Z$. A finite induction then shows that $G \leq \sqrt{\pi}$. If $h(\sqrt{\pi}) > 2$ then $H^2(\pi; Z[\pi]) = 0$, by Theorem 1.16 of [9], and so M is aspherical. (In fact M is then homeomorphic to an infrasolvmanifold, by Theorem 8.1 of [9]). If $h(\sqrt{\pi}) = 2$ and $\sqrt{\pi}$ has infinite index in π then we again have $H^2(\pi; Z[\pi]) = 0$ and so M is aspherical. (If $\sqrt{\pi}$ is finitely generated it is nilpotent, hence FP, and the vanishing of $H^2(\pi; Z[\pi])$ follows immediately from an LHS spectral sequence argument. If $\sqrt{\pi}$ is not finitely generated then it is the increasing union of finitely generated subgroups of Hirsch rank 2, and we may apply Theorem 3.3 of [7] to conclude that $H^s(\sqrt{\pi}; Z[\pi]) = 0$ for $s \leq 2$). If $h(\sqrt{\pi}) = 2$ and $\sqrt{\pi}$ has finite index in π then π is virtually Z^2. We may then assume that $\pi \cong Z^2$ and $\pi/G \cong Z$. Since $H_*(\hat{M}; Q)$ is finitely generated it follows from the Wang sequence for the projection of $\hat{M}$ onto M that $\chi(M) = 0$. Hence M is finitely covered by $S^2 \times S^1 \times S^1$, by Theorem 10.10 of [9].

Suppose that $h(\sqrt{\pi}) = 1$ and let $\sqrt{M}$ be the associated covering space. Since $h(G) = h(\sqrt{\pi})$ the stages of a subnormal chain between G and $\sqrt{\pi}$ are locally finite, and so the rational homology spectral sequences between the corresponding covering spaces collapse, to show that $H_*(\sqrt{M}; Q)$ is finitely generated and $\chi(\sqrt{M}) = \chi(\hat{M})$. In particular, $\pi/\sqrt{\pi}$ has finitely many ends, since $H_3(\sqrt{M}; Q)$ is finite dimensional.

If $[\pi : \sqrt{\pi}]$ is finite then $\sqrt{\pi}$ is finitely generated. But then $[\sqrt{\pi} : G] < \infty$ and so $[\pi : G] < \infty$, contrary to hypothesis.

If $\pi/\sqrt{\pi}$ has two ends then we may assume that $\pi/\sqrt{\pi} \cong Z$. But then π is an ascending HNN construction over a finitely generated base, and so the torsion subgroup T of $\sqrt{\pi}$ is finite, while $\sqrt{\pi}/T$ is abelian. Therefore $\sqrt{\pi}$ has a finitely generated infinite normal subgroup and so $H^2(\pi; Z[\pi])$ is free abelian [11]. Since $H_*(\sqrt{M}; Q)$ is finitely generated $\sqrt{M}$ satisfies Poincaré duality with simple coefficients Q and formal dimension 3 [12] and so $\chi(\sqrt{M}) = 0$. Hence $\chi(\hat{M}) = 0$. This in turn implies that $\pi_2(\hat{M})$ is a torsion $Z[G]$-

module. Since $Z[G] \cong Z[t,t^{-1}]$ and $\pi_2(\hat{M}) = \pi_2(M) \cong H^2(\pi;Z[\pi])$ is free abelian it must be finitely generated. Since π has elements of infinite order $H^2(\pi;Z[\pi])$ must therefore be 0 or Z, by Corollary 5.2 of [5]. But M cannot be aspherical as $c.d._Q(\pi) \leq c.d._Q\sqrt{\pi} + c.d._Q Z = 2$. Therefore $\tilde{M} \simeq S^2$. As π is elementary amenable it must be virtually Z^2, by Theorem 10.10 of [9]. But this contradicts the assumption that $h(\sqrt{\pi}) = 1$. Therefore $\pi/\sqrt{\pi}$ has one end. As we may again exclude the possibility that $H^2(\pi;Z[\pi]) \cong Z$, either M is aspherical or $H^2(\pi;Z[\pi])$ is not finitely generated.

Suppose that G has infinitely many ends and is subnormal in N where N is an FP_2 normal subgroup of infinite index in π. If $[N:G]$ is finite then N has infinitely many ends and the regular covering space associated to N is finitely dominated, so π/N has two ends and the covering space associated to N is a PD_3-complex, by Theorem 3.9 of [9]. If $[N:G] = \infty$ then $H^s(\pi;Z[\pi]) = 0$ for $s \leq 2$ and so M is aspherical, as before. This cannot happen if N is FP_3, by the corollary to Theorem 2. □

What happens if we drop the subnormality hypothesis? It can be shown that a closed 3-manifold has a finitely dominated infinite covering space if and only if its fundamental group has one or two ends. Does this condition remain necessary in dimension 4? The hypothesis that G be FP_3 is automatic if π is finite or has two ends. It is used only to ensure that $Hom_\pi(\pi_2(M), Z[\pi]) = 0$. Can it be relaxed to FP_2 in general? The final possibility in case (*iii*) surely never occurs. Mapping tori of self homeomorphisms of 3-manifolds whose fundamental group is a nontrivial free product give examples in which G is FP_3, has infinitely many ends and is normal in π.

Corollary. *If π is virtually torsion free and G has finitely many ends then either M is aspherical or its universal covering space is homotopy equivalent to S^2 or S^3.*

Proof. It is sufficient to note that if $\sqrt{\pi}$ is torsion free and $h(\sqrt{\pi}) = 1$ then $\sqrt{\pi}$ is abelian and has a finitely generated infinite normal subgroup. Hence $H^2(\pi;Z[\pi]) = 0$, by [7] and [11], and so M is aspherical. □

Conversely, if $\tilde{M}$ is finitely dominated then π is virtually torsion free, by Theorems 3.9, 10.1 and 11.1 of [9]. This also holds if M is finitely covered by the mapping torus of a self homotopy equivalence of a PD_3-complex [4].

If M is a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ has a subnormal subgroup G of infinite index which is a PD_2-group then M is aspherical and either has a finite regular covering space which is homotopy equivalent to the total space of a torus bundle over an aspherical closed surface or has a finite covering space which is homotopy equivalent to a mapping torus. See Chapters 4 and 5 of [9].

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