Polynomials of equivariantly slice knots

Jonathan A. Hillman

School of Mathematics and Statistics, The University of Sydney,

Sydney, NSW 2006, AUSTRALIA

e-mail: jonh@maths.usyd.edu.au

ABSTRACT

We show that if K is a knot with semifree period q and which is equivariantly slice then the Murasugi polynomial $\Delta_{Z/qZ}(K)$ may be expressed as a quotient $a\bar{a}/q^{2n}$, for some $a \in \mathbb{Z}[t,t^{-1},u]/(u^q-1)$ and $n \geq 0$.

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If K is a knot invariant under a rotation h of order q about a disjoint axis A let $\Delta_{Z/qZ}(K)$ be the image of $\Delta_1(\overline{A} \cup \overline{K})$ in the ring $\mathbb{Z}[Z \times Z/qZ] = \mathbb{Z}[t,t^{-1},u]/(u^q-1)$, where $\overline{A} \cup \overline{K}$ is the orbit link in $S^3/\langle h \rangle$ and u,t are the meridians corresponding to $\overline{A}$ and $\overline{K}$, respectively. Davis and Naik used Milnor-Reidemeister torsion to show that there are $a = a(t,u)$ and $b = b(t,u)$ in $\mathbb{Z}[t,t^{-1},u]/(u^q-1)$ such that $a(1,u) = b(1,u) = 1$ and $\Delta_{Z/qZ}(K)b\overline{b} = a\overline{a}$. Moreover if $a(t,u) \in \mathbb{Z}[t,t^{-1},u]/(u^q-1)$ and $a(1,u) = 1$ then $a\bar{a} = \Delta_{Z/qZ}(K)$ for some equivariantly ribbon knot K with semifree period q [DN02]. This suggests a possible means of distinguishing between equivariantly ribbon and equivariantly slice knots (although there are as yet no candidates to test).

In Theorem 8.22 of [Hi] we attempted to recover the Davis-Naik result by means of Blanchfield duality with rational coefficients, and found instead that $\Delta_{Z/qZ}(K) = a\overline{a}/d^2$, for some $a = a(t,u) \in \mathbb{Z}[t,t^{-1},u]/(u^q-1)$ such that $a(1,1) = d \neq 0$. Here we shall show that we may always assume that the denominator d is a power of q. (It remains an open question whether in fact we may assume $d = 1$). Instead of using rational coefficients, we shall localize in two ways: (a) we invert q, so that $\mathbb{Z}[\frac{1}{q}][Z/qZ]$ is a direct sum of Dedekind domains; and (b) we invert $t-1$, so that we may reduce to modules which are finitely generated over $\mathbb{Z}[\frac{1}{q}]$. (A similar reduction is useful in relating Seifert forms to Blanchfield pairings).

Let $\Gamma = \mathbb{Z}[u]/(u^q-1) = \mathbb{Z}[Z/qZ]$, $\Lambda = \mathbb{Z}[t,t^{-1}]$, $\Lambda_2 = \mathbb{Z}[t,t^{-1},u,u^{-1}]$, and $\Psi = \Lambda\otimes_{\mathbb{Z}}\Gamma = \Lambda_2/(u^q-1) = \mathbb{Z}[Z\times Z/qZ]$. Let $\Gamma^+ = \Gamma[\frac{1}{q}] = \mathbb{Z}[\frac{1}{q},u]/(u^q-1) = \mathbb{Z}[\frac{1}{q}][Z/qZ]$, $\Lambda^+ = \Lambda[\frac{1}{q},(t-1)^{-1}]$ and $\Psi^+ = \Lambda^+\otimes_{\mathbb{Z}}\Gamma^+$. If M is a Γ-module let $M^+ = \Gamma^+\otimes_\Gamma M$. We shall use an overbar to denote both the canonical involution of the group ring Ψ, and also the conjugate module $\overline{M}$ derived from M via this involution. (In other respects our notation shall be based on that of Chapter 8 of [Hi]). Let $z = (t-1)^{-1}$. Then $t = z^{-1}(z-1)$ and $\Lambda^+ = \mathbb{Z}[\frac{1}{q},z,z^{-1},(z+1)^{-1}]$.

Fix a primitive q^{th} root of unity ζ_q and let $\chi(m+qZ) = \zeta_q^m$ for $m \in Z$. Let $e_j = \frac{1}{q}\Sigma_{\{g\in Z/qZ\}}\chi(g)^j g$, for each $1 \le j \le q$. Then $\{e_j \mid 1 \le j \le q\}$ is the set of canonical idempotents in $\mathbb{C}[Z/qZ]$. For each divisor d of q let $\zeta_d = \zeta_q^{q/d}$ and $I(d) = \{i \mid 1 \le i \le n/d, (i,n/d) = 1\}$. Then $E_d = \Sigma_{i\in I(n/d)}e_{in/d}$ is the sum of the Galois conjugates of $e_{n/d}$, and $\{E_d\}_{d|q}$ is the set of canonical idempotents in Γ^+. Moreover $E_d\Gamma^+ \cong \Gamma^+/(\phi_d(u)) = \mathbb{Z}[\frac{1}{q},\zeta_d]$, where ϕ_d is the d^{th} cyclotomic polynomial. Hence $\Gamma^+ \cong \oplus_{d|q}E_d\Gamma^+$ is a direct sum of Dedekind domains. Note that the extension from Γ to Γ^+ is flat, whereas that from Γ to $\oplus_{d|q}\Gamma/(\phi_d(u))$ is not.

Lemma 1. *Let R be a ring and M a finitely generated $R[Z]$-module such that $\delta M = 0$ for some monic polynomial $\delta \in R[Z]$. Then M is finitely generated as an R-module.*

Proof. If $\{m_1, \ldots, m_r\}$ generates M over $R[Z]$ and δ has degree n then $\{Z^i m_j \mid 0 \leq i < n, 1 \leq j \leq r\}$ generates M over R. $\square$

Lemma 2. *Let P be a finitely generated Ψ-module such that P^+ is annihilated by a monic polynomial. Then $P^+ = \Psi^+ \otimes_{\mathbb{Z}[\frac{1}{q},z]} \tilde{P}$, where $\tilde{P}$ is an $\mathbb{Z}[\frac{1}{q}, z, u]/(u^q - 1)$-submodule which is finitely generated as a $\mathbb{Z}[\frac{1}{q}]$-module.*

Proof. We may take $\tilde{P}$ to be the $\mathbb{Z}[\frac{1}{q}, z, u]/(u^q - 1)$-submodule generated by a finite subset of P which generates P as a Ψ-module. $\square$

Lemma 3. *Let $\tilde{P}$ be a Γ^+-module which is finitely generated as a $\mathbb{Z}[\frac{1}{q}]$-module and $\mathbb{Z}$-torsion free. Then $\tilde{P}$ is projective as a Γ^+-module.*

Proof. The orthogonal idempotents $\{e_d\}_{d|q}$ determine a direct sum decomposition $\tilde{P} \cong \oplus e_d \tilde{P}$. As each summand $e_d\tilde{P}$ is finitely generated and $\mathbb{Z}$-torsion free, it is projective as a $\Gamma^+ e_d$-module and hence $\tilde{P}$ is projective as a Γ^+-module. $\square$

Theorem. *Let K be a 1-knot with semifree period q about an axis A and which is equivariantly slice. Then $|\mathrm{lk}(A, K)| = 1$ and we may normalize our choice of $\Delta_1(\overline{A} \cup \overline{K})$ so that $\Delta_{Z/qZ}(K) = a\overline{a}/q^{2m}$ for some $a = a(t, u) \in \mathbb{Z}[t, t^{-1}, u]/(u^q - 1)$ such that $a(1, u) = q^m$.*

Proof. The first assertion follows on considering the fixed point set of the restriction of the action to an invariant slice disc. Hence we may use the Torres conditions (Corollary 5.3.1 of [Hi]) to normalize $\Delta_{Z/qZ}(K)$ to be fixed under conjugation and augment to 1.

Let $X = X(K)$, $Y = X(A \cup K)$ and $\overline{Y} = X(\overline{A} \cup \overline{K})$ be the exteriors of the knot K and the links $A \cup K$ and $\overline{A} \cup \overline{K}$, and let X' and $Y' = \overline{Y}'$ be their universal covering spaces. Let Y^γ be the preimage of Y in X'. Then meridians for A lift to loops in Y^γ.

A meridian for A determines an element $m_A \in H_1(Y^\gamma;\mathbb{Z})$, such that $(t-1)m_A = (u-1)m_A = 0$, and $M = H_1(X;\Lambda) = H_1(Y^\gamma;\mathbb{Z})/\langle m_A\rangle$, by excision. Then M is a finitely generated Ψ-module, and $M = (t-1)M$ and is $\mathbb{Z}$-torsion free. In particular, $M^+ \cong \mathbb{Z}[\frac{1}{q}] \otimes M$. If we rewrite the Alexander polynomial $\Delta_K(t) = \Delta_0(M)$ in terms of $t = z^{-1}(z+1)$ and clear denominators the resulting polynomial in z is monic, and so Lemma 1 applies to M^+ and its submodules.

Let $H = H_1(\overline{Y};\Lambda_2) = H_1(Y';\mathbb{Z})$. Then H has a square presentation matrix over Λ_2 (see §7.1 of [Hi]), and the Wang sequence for the projection of Y' on Y^γ gives a short exact sequence

$$0 \to H/(u^q-1)H \to H_1(Y^\gamma;\mathbb{Z}) \to Z \to 0,$$

which is split by the inclusion of m_A, since $|\mathrm{lk}(A,L)| = 1$. Then $M^+ \cong \Psi^+ \otimes_{\Lambda_2} H$, and so M^+ has a square presentation matrix over Ψ^+. Hence $pd_{\Psi^+}M^+ \leq 1$ and $E_0(M^+) = (\Delta_{Z/qZ}(K))$ in Ψ^+.

The automorphism $u \in \Psi^+$ acts isometrically on the Blanchfield pairing $b : M^+ \times M^+ \to \mathbb{Q}(t)/\Lambda^+$ (i.e., $b(ug,uh) = b(g,h)$ for all $g,h \in M$).

Let $Z = D^4 - N(\mathcal{K})$ be the complement of an open regular neighbourhood of an equivariant null concordance $\mathcal{K}$ for K. Then $H_*(Z;\Lambda^+)$ and $H_*(Z,\partial Z;\Lambda^+)$ are finitely generated Λ^+-torsion modules. The inclusion of X into $\partial Z = X \cup (S^1 \times D^2)$ induces an isomorphism $M \cong H_1(\partial Z;\Lambda^+)$, and the Blanchfield pairings agree. Let P be the image of $H_2(Z,\partial Z;\Lambda^+)$ in $H_1(\partial Z;\Lambda^+)$. Then

P is invariant under the automorphism u, and so is a Ψ^+-submodule.

By Lemmas 2 and 3 we have $P^+ = \Psi^+ \otimes_{\mathbb{Z}[\frac{1}{q},z]} \tilde{P}$, where $\tilde{P}$ is finitely generated and projective as a Γ^+-module. Let $\hat{P}$ be a projective complement: $\tilde{P} \oplus \hat{P} \cong (\Gamma^+)^g$ for some $g \geq 0$. Let $P_1 = \mathbb{Z}[\frac{1}{q}, z] \otimes_{\mathbb{Z}} \tilde{P}$ and $P_2 = \mathbb{Z}[\frac{1}{q}, z] \otimes_{\mathbb{Z}} \hat{P}$, with the $\mathbb{Z}[\frac{1}{q}, z] \otimes_{\mathbb{Z}} \Gamma^+$-actions given by $z^m u^n (r \otimes p) = z^m r \otimes u^n p$, for all $r \in \mathbb{Z}[\frac{1}{q}, z]$ and $p \in \tilde{P}$ or $\hat{P}$, respectively. Then $P_1 \oplus P_2 \cong (\mathbb{Z}[\frac{1}{q}, z] \otimes_{\mathbb{Z}} \Gamma^+)^g$. Let $D(p_1, p_2) = ((z-1)p_1, p_2)$ for all $(p_1, p_2) \in P_1 \oplus P_2$. Then

$$0 \to (\mathbb{Z}[\frac{1}{q}, z] \otimes_{\mathbb{Z}} \Gamma^+)^g \xrightarrow{D} (\mathbb{Z}[\frac{1}{q}, z] \otimes_{\mathbb{Z}} \Gamma^+)^g \to \tilde{P} \to 0$$

is exact. Extending coefficients to Ψ^+ gives a square presentation matrix for P^+, and so $E_0(P^+) = (\alpha)$, where $\alpha = \det(D)$.

Let $\chi : \Psi^+ \to \Lambda^+$ be the Λ^+-linear function defined by $\chi(\Sigma r_j u^j) = r_0$. If N is a Ψ^+-module let $N|_{\Lambda^+}$ denote the underlying Λ^+-module. Composition with χ induces natural isomorphisms $Hom_{\Psi^+}(N, \Psi^+) \cong Hom_{\Lambda^+}(N|_{\Lambda^+}, \Lambda^+)$ for all finitely generated Ψ^+-modules. These are Ψ^+-linear if we set $u\phi(n) = \phi(un)$ for all $\phi : N \to \Lambda^+$ and $n \in N$. Since restriction is exact and takes projectives to projectives these lead to natural isomorphisms $Ext^i_{\Psi^+}(N, \Psi^+) \cong Ext^i_{\Lambda^+}(N|_{\Lambda^+}, \Lambda^+)$ for all $i \geq 0$. In particular, if $e(P^+) = Ext^1_{\Psi^+}(P^+, \Psi^+)$ then $e(P^+) \cong Hom_{\Lambda^+}(P^+|_{\Lambda^+}, \mathbb{Q}(t)/\Lambda^+)$. Moreover this is an isomorphism of Ψ^+-modules if we set $u\phi(p) = \phi(up)$ for all $p \in P^+$ and $\phi : P^+ \to \mathbb{Q}(t)/\Lambda^+$. (This is the functorial Γ-action via P). Since the transpose D^{tr} is a (square) presentation matrix for $e(P^+)$ we have $E_0(e(P^+)) = (det(D^{tr})) = E_0(P^+)$.

The isomorphism $M/P \cong \overline{e(P)}$ implicit in Theorem 2.3 of [Hi] is Ψ-linear. The module M^+ has a presentation matrix which is block-triangular, with diagonal blocks D and $\overline{D^{tr}}$. Hence $(\Delta_{Z/qZ}(K)) = (\Delta_0(M)) = (\Delta_0(M^+)) = (det(D)\overline{det(D)}) = (\alpha\overline{\alpha})$, and so $\Delta_{Z/qZ}(K) = vt^n\alpha\overline{\alpha}$ for some $v \in (\Gamma^+)^\times$

and $n \in \mathbb{Z}$ (since $(\Psi^+)^\times = (\Gamma^+)^\times\langle t\rangle$). Since $\Delta_{Z/qZ}(K)$ has been normalized $1 = \Delta_1(\overline{A}) = v\alpha(1,u)\alpha(1,u^{-1})$ and $n = 0$, by Corollary 5.3.1 of [Hi]. Hence $\alpha(1,u) \in \Gamma^\times$, and so $\Delta_{Z/qZ}(K) = \beta\overline{\beta}$, where $\beta = \beta(t,u) = \alpha(t,u)/\alpha(1,u)$. We may clearly write $\beta = a(t,u)/q^m$ for some $a = a(t,u) \in \mathbb{Z}[t,t^{-1},u]/(u^q-1)$ such that $a(1,u) = q^m$. $\square$

In particular, $\Delta_1(K)(t) = \Pi_{\zeta^q=1} a(t,\zeta)a(t^{-1},\zeta^{-1})$. Does P have a square presentation matrix over Ψ (i.e., before localization)? Does this theorem together with the original Davis-Naik result imply that $\Delta_{Z/qZ}(K) = a\overline{a}$ for some $a = a(t,u) \in \Psi$ such that $a(1,u) = 1$?

The theorem extends easily to equivariantly slice knots with period p in p-homology 3-spheres. This special case may be useful in an inductive argument; it is also of interest algebraically, as we may then view Ψ as a pullback of Λ and $\Lambda[\zeta_p]$ over $\mathbb{F}_p\Lambda$, by Rim's Theorem (cf [DL90]).

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