The $p$-canonical basis of the anti-spherical Hecke module

In this talk we provide a brief introduction to the p-canonical basis of the anti-spherical Hecke module. Moreover, we explicitly calculate it in type $\tilde{A}_1$; the only case where it is explicitly known.

The structure of this talk is as follows:

1. The Hecke algebra and its canonical bases;
2. The anti-spherical Hecke module;
3. Explicitly calculating the $p$-canonical basis in type $\tilde{A}_1$.

Notation

Throughout this talk we adopt the following notation:

- $k$: an algebraically closed field of characteristic $p > 0$;
- $G_k$: a simple, simply connected, algebraic group scheme over $k$;
- $(R, X, R^\vee, X^\vee)$: the root datum associated to $G_k$;
- $T_k \subset B_k \subset G_k$: a pinning of $G_k$;
- $W_f = N_G(T_k)/T$: the finite Weyl group;
- $W_p = W_f \rtimes p\mathbb{Z}$: the $p$-dilated affine Weyl group;
- $f_W$: the minimal left coset representatives;
- $w_f$: the longest element of $W_f$; and
- $h$: the Coxeter number.

The Hecke algebra and its canonical bases

Recall that the Hecke algebra $H$ associated to the Coxeter system $(W_p, S_p)$ is the associative, unital, $\mathbb{Z}[v^{\pm 1}]$-algebra generated by the symbols $\{\delta_w | w \in W_p\}$ subject to the relations:

$$\delta_w \delta_{w'} = \delta_{ww'} \quad \text{if} \quad \ell(w) + \ell(w') = \ell(ww')$$

$$(\delta_s + v)(\delta_s - v^{-1}) = 0 \quad \text{for all} \quad s \in S_p$$

Matsumoto’s lemma implies the symbols $\{\delta_w | w \in W_p\}$ are well defined. Moreover, $\{\delta_w | w \in W_p\}$ forms a basis of $H$ called the standard basis. Each standard basis element is invertible.

There exists an involution $\overline{\cdot}$ on $H$ given by:

$$\overline{\delta_w} = \delta_{w^{-1}}^{-1} \quad \quad \overline{v} = v^{-1}$$

By a theorem of Kazhdan and Lusztig, for each $w \in W_p$, there exists a unique element $b_w \in H$ such that $\overline{b_w} = b_w$ and $b_w = \delta_w + \sum_{u < w} \mathbb{Z}[v]\delta_u$. The set $\{b_w | w \in W_p\}$ form a basis of $H$ called the canonical basis.

To motivate the definition of the $p$-canonical basis, we have to first categorify the presentation of the Hecke algebra given by the canonical basis.

Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[\![t]\!]$. Consider the groups $G^\vee(\mathcal{O}) \subset G^\vee(\mathcal{K})$. There is a natural map $G^\vee(\mathcal{O}) \to G^\vee(\mathbb{C})$ induced by $t \mapsto 0$. The Iwahori subgroup $I \subset G^\vee(\mathcal{O})$ is defined to be the pre-image of $B$ under this map. The affine flag variety $\mathcal{F}I$ is the ind-projective ind-scheme whose $\mathbb{C}$-points may be identified with the space $G^\vee(\mathcal{K})/I$. It is a Kac-Moody flag variety, and thus admits a Bruhat decomposition:

$$\mathcal{F}I = \bigsqcup_{w \in W_p} \mathcal{F}b_w \quad \text{where} \quad \mathcal{F}b_w := IwI/I$$
The closure order is given by the Bruhat order on $W_p$. More precisely:

$$
\mathcal{F}_w = \bigcup_{u \leq w} \mathcal{F}_u
$$

Each $\mathcal{F}_w$ is called an affine Schubert variety.

The category of $I$-equivariant perverse sheaves on the affine flag variety with coefficients in $\mathbb{C}$, $\text{Perv}_I(\mathcal{F}l, \mathbb{C})$, has simple objects $\{\text{IC}_w | w \in W_p\}$ called intersection cohomology sheaves. Each $\text{IC}_w$ is supported on the affine Schubert variety $\mathcal{F}_w$. We can also take the convolution of $I$-equivariant perverse sheaves

$$
* : \text{Perv}_I(\mathcal{F}l, \mathbb{C}) \times \text{Perv}_I(\mathcal{F}l, \mathbb{C}) \to \text{Perv}_I(\mathcal{F}l, \mathbb{C})
$$

which endows $\text{Perv}_I(\mathcal{F}l, \mathbb{C})$ with the structure of a monoidal category.

The split Grothendieck ring of $\text{Perv}_I(\mathcal{F}l, \mathbb{C})$, denoted $[\text{Perv}_I(\mathcal{F}l, \mathbb{C})]_{\oplus}$, can be endowed with the structure of a $\mathbb{Z}[v^\pm 1]$-algebra where $v[\mathcal{F}] = [\mathcal{F}[1]]$. We then have an isomorphism of $\mathbb{Z}[v^\pm 1]$-algebras:

$$
[\text{Perv}_I(\mathcal{F}l, \mathbb{C})]_{\oplus} \leftrightarrow H,
$$

$$
[\text{IC}_w] \mapsto b_w
$$

$$
[\mathcal{F}] \mapsto \sum_{w \in W} \sum_{i \in \mathbb{Z}} \dim H^{-i}(\mathcal{F}_w)v^{i-\ell(w)}\delta_u
$$

Thus the canonical basis can be interpreted geometrically as the characters of simple objects in the category of perverse sheaves on the affine flag variety.

The $p$-canonical basis is a generalisation of the canonical basis when the sheaf coefficients $\mathbb{C}$ are replaced by $k$, a field of positive characteristic.

The category of $I$-equivariant parity sheaves on the affine flag variety with coefficients in $k$, $\text{Parity}_I(\mathcal{F}l, k)$, has indecomposable objects $\{\mathcal{E}_w | w \in W_p\}$ called indecomposable parity sheaves. Each indecomposable parity sheaf $\mathcal{E}_w$ is the extension by zero of the constant sheaf $k_{\mathcal{F}_w}$. As before, there is a convolution of $I$-equivariant parity sheaves

$$
* : \text{Parity}_I(\mathcal{F}l, k) \times \text{Parity}_I(\mathcal{F}l, k) \to \text{Parity}_I(\mathcal{F}l, k)
$$

which endows $\text{Parity}_I(\mathcal{F}l, k)$ with the structure of a monoidal category.

The split Grothendieck ring of $\text{Parity}_I(\mathcal{F}l, k)$, denoted $[\text{Parity}_I(\mathcal{F}l, k)]_{\oplus}$, can be endowed with the structure of a $\mathbb{Z}[v^\pm 1]$-algebra where $v[\mathcal{F}] = [\mathcal{F}[1]]$. We then have an isomorphism of $\mathbb{Z}[v^\pm 1]$-algebras:

$$
[\text{Parity}_I(\mathcal{F}l, k)]_{\oplus} \leftrightarrow H,
$$

$$
[\mathcal{E}_w] \mapsto pb_w
$$

$$
[\mathcal{F}] \mapsto \sum_{w \in W_p} \sum_{i \in \mathbb{Z}} \dim H^{-i}(\mathcal{F}_w)v^{i-\ell(w)}\delta_u
$$

The $p$-canonical basis is defined to be the character of the indecomposable parity sheaves on the affine flag variety.

The $p$-canonical basis satisfies the following properties:
(1) \( p_{bw} = p_{bw} \);
(2) \( p_{bw} = bw + \sum_{u<w} p_{a_{u,w}b_u} \) with \( p_{a_{u,w}} \in \mathbb{Z}[v^{\pm 1}] \) and \( p_{a_{u,w}} = \overline{p_{a_{u,w}}} \);
(3) \( p_{bw} = bw \) for \( p \gg 0 \).

The coefficients of the \( p \)-canonical basis have the following representation theoretic interpretation:

\[
p_{bw_{\lambda,\mu}}(1) = \dim T_{\lambda,\mu}
\]

where \( T_{\lambda,\mu} \) is \( \mu \)-weight space of the the indecomposable tilting module with highest \( \lambda \), \( T_{\lambda} \), \( w_{\lambda}(0) = \lambda \), and \( w_{\mu}(0) = \mu \).

Remarks.

(1) The character of the indecomposable parity sheaf depends only on the characteristic of \( k \), not on the field \( k \) itself.
(2) Whilst the canonical basis may be defined relative only to the Hecke algebra \( H \), the \( p \)-canonical basis requires the addition data of a root system associated to \( H \). For example, Jensen and Williamson show that the 2-canonical bases for the Hecke algebras of types \( \tilde{C}_2 \) and \( \tilde{B}_2 \) differ.
(3) The \( p \)-canonical basis is typically calculated using intersection forms and Elias-Williamson-Khovanov diagrammatics. When the associated Schubert variety is relatively nice (i.e. smooth/rationally smooth/low dimensional) the \( p \)-canonical basis can be determined using geometric techniques. It may also be calculated using the Braden-Macpherson algorithm.

The anti-spherical Hecke module

The quadratic relation \((\delta_s + v)(\delta_s - v^{-1}) = 0\) gives a morphism of \( \mathbb{Z}[v^{\pm 1}] \)-algebras

\[
\begin{align*}
H & \longrightarrow \mathbb{Z}[v^{\pm}] \\
\delta_s & \longmapsto -v.
\end{align*}
\]

For any parabolic subset of \( S_p \) containing \( s \). In particular, if we take \( S_f \subset S_p \) as the parabolic subset then the resulting \( H_f \)-module is denoted \( \text{sign}_v \).

Inducing \( \text{sign}_v \) to a representation of \( H \) produces the anti-spherical Hecke module \( N \). Explicitly:

\[
N = \text{sign}_v \otimes_{H_f} H
\]

It is a free \( \mathbb{Z}[v^{\pm 1}] \)-module with basis \( \{ \nu_w := 1 \otimes \delta_w | w \in fW_p \} \) called the standard basis of \( N \).

The Kazhdan-Lusztig involution extends to an involution of \( N \) in the following way:

\[
\nu_w = 1 \otimes \overline{\delta_w}.
\]

We analogously define the canonical basis of \( N \) to be the elements \( \{ d_w | w \in W_a \} \) such that \( d_w = d_w \) and \( d_w \in \nu_w + \sum_{u<w} v\mathbb{Z}[v]\nu_u v \).

The \( p \)-canonical basis of the anti-spherical Hecke module is then defined as:

\[
p_{d_w} := 1 \otimes p_{bw}
\]

for any \( w \in fW_a \).

Remarks.
Analogous to the universal enveloping algebra $U$, then $(G)$ modules (where $A$ result of Carell and Petersen implies the affine Schubert variety of the Bruhat interval $[id, w]$. Moreover the Bruhat order is particularly simple. [INSERT PICTURE].

Calculations for $SL_2$

Recall that for $SL_2$ we have

- $W_f \cong \langle s | s^2 = id \rangle$;
- $W_p \cong \langle s, t | s^2 = t^2 = id \rangle$; and
- $f W_p = \{ w_l \in W_p | \ell(w_l) = l \text{ and } sw_l > w_l \}$. 

Moreover the Bruhat order is particularly simple. [INSERT PICTURE].

The canonical basis is particularly simple for $SL_2$. For any $w \in W_p$ the Poincare polynomial of the Bruhat interval $[id, w]$ is $1 + 2r + 2r^2 + \cdots + 2r^{\ell(w)} - 1 + r^{\ell(w)}$. In particular it is palindromic. A result of Carell and Petersen implies the affine Schubert variety $\mathcal{F}_w$ is rationally smooth. Thus the canonical basis is

$$b_w = \sum_{u \leq w} v^{\ell(w) - \ell(u)} \delta_u.$$

It is then immediate that the canonical basis of the anti-spherical module is

$$d_{w_n} = v_{w_n} + v v_{w_{n-1}}$$

where $w_{n-1}$ is taken to be 0 if $n = 0$.

Tilting modules for $SL_2$ can all be explicitly described. This allows an explicit description of the $p$-canonical bases of the Hecke algebra and the anti-spherical module of the Hecke algebra.

First, observe that $T_\lambda \cong \Delta_\lambda$ for $0 \leq \lambda \leq p - 1$ by the linkage principle.

The $T_\lambda$ where $p \leq \lambda \leq 2p - 2$ are known to be the projective covers of the simple $G_1$-modules (where $G_1$ denotes the first Frobenius kernel of $G = SL_2$). The category $Rep G_1$ is equivalent to $Rep U_p(\mathfrak{sl}_2)$ where $U_p(\mathfrak{sl}_2)$ is the restricted Lie algebra of $\mathfrak{sl}_2$. Recall the restricted Lie algebra $\mathfrak{sl}_2$ is the Lie algebra $\mathfrak{sl}_2$ over a field $k$ of characteristic $p$, endowed with a $p$-operation $(\cdot)^{[p]} : \mathfrak{sl}_2 \to \mathfrak{sl}_2$. If we realise $\mathfrak{sl}_2 \subset \mathfrak{gl}_2$ as the matrices

$$f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then $(\cdot)^{[p]}$ can be realised as the $p$-th power of each matrix:

$$f^{[p]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0, \quad h^{[p]} = \begin{bmatrix} 1^p & 0 \\ 0 & (-1)^p \end{bmatrix} = h, \quad e^{[p]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Analogous to the universal enveloping algebra $U(\mathfrak{sl}_2)$ we have the restricted universal enveloping algebra $U_p(\mathfrak{sl}_2)$ which is the quotient $U(\mathfrak{sl}_2)/(x^p - x^{[p]})$. The Poincare-Birkhoff-Witt basis of $U(\mathfrak{sl}_2)$ descends to a basis of $U_p(\mathfrak{sl}_2)$ given by $\{ [i^j] e^k \}_{0 \leq i, j, k < p}$. It can be shown that

$$[T_\lambda] = [\Delta_\lambda] + [\Delta_{\ell, \lambda}]$$
when \( p \leq \lambda \leq 2p - 2 \).

Finally Donkin’s tensor product theorem (for \( SL_2 \)) states that if we write \( \lambda = \lambda_0 + p\lambda_1 \) where \( p - 1 \leq \lambda_0 \leq 2p - 2 \) and \( \lambda_1 \in \) then:

\[
T_\lambda = T_{\lambda_0} \otimes T^{(1)}_{\lambda_1}
\]

where \((-)^{(1)}: Rep G \to Rep G \) denotes the Frobenius twist.

For \( SL_2 \) these suffice to inductively prove that for any fixed \( \lambda = \sum_{i \geq 0} \lambda ip^i \in X_+ \) where \( 0 \leq \lambda_i \leq p - 1 \). Set \( \lambda_{(k)} = \sum_{i \geq k} \lambda_i p^i \). Then

\[
[T_\lambda] = \left( \prod_{k \geq 1} (s_{\alpha, \lambda_{(k)}} + 1) \right) \cdot [\Delta_\lambda]
\]

where the product acts on the left (i.e. \( \prod_{k \geq 1} x_i = \ldots x_3 x_2 x_1 \)), and the action of \( W_\alpha \) on \( Rep G \) is given by \( w \cdot \Delta_\lambda = \Delta_{w \cdot \lambda} \).

Using explicit knowledge of the Weyl modules for \( SL_2 \) and the characterisation of the \( p \)-canonical basis of \( H \) in terms of tilting modules we find

\[
pb_{w\lambda} = \left( \prod_{k \geq 1} (s_{\alpha, \lambda_{(k)}} + 1) \right) \cdot bw_\lambda
\]

and consequently

\[
pd_{w\lambda} = \left( \prod_{k \geq 1} (s_{\alpha, \lambda_{(k)}} + 1) \right) \cdot dw_\lambda
\]