Melnikov's Method Applied to the Double Pendulum

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Abstract

Melnikov's method is applied to the planar double pendulum proving it to be a chaotic system. The parameter space of the double pendulum is discussed, and the integrable cases are identified. In the neighborhood of the integrable case of two uncoupled pendulums Melnikov's integral is evaluated using residue calculus. In the two limiting cases of one pendulum becoming a rotator or an oscillator, the parameter dependence of chaos, i. e. the width of the separatrix layer is analytically discussed. The results are compared with numerically computed Poincaré surfaces of section, and good agreement is found.

Keywords: Hamiltonian Systems, Melnikov's Method, Double Pendulum

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1 Introduction

Chaos as a generic phenomenon in Hamiltonian dynamics is by now well established. Nevertheless there are only few methods to actually prove a system to be chaotic. In this paper we want to use Melnikov's method as adapted to two degree of freedom Hamiltonians by Holmes and Marsden [1]. The idea is to use perturbation theoretical methods to calculate the splitting of separatrices as already described by Poincaré [2], and to prove the existence of homoclinic intersections and thus horseshoes. The extensive calculation does not only give the formal existence of homoclinic points but – more important from a physical point of view – also an estimate for the extent of chaos as measured by the size of the part of the energy surface where chaotic motion occurs.

In this paper we apply Melnikov's method to the planar double pendulum. In section two the mathematical double pendulum as defined, e.g., in Landau and Lifschitz [3] is generalized to be a physical double pendulum with rigid pendulum bodies. We analyze its parameter space and determine its integrable limiting cases. They serve as a starting point for the calculation of Melnikov's integral which is done in the next two sections. One pendulum is transformed to action-angle variables in order to describe the double pendulum as a system with one and a half degrees of freedom. Next the perturbation involving triple products of Jacobi elliptic functions is expressed as a Fourier series in the angle variables. The resulting Melnikov function has nondegenerate zeroes which demonstrates the double pendulum to be a chaotic system. In the final section the parameter dependence of the extent of chaos is estimated by the maximum of Melnikov's function, and compared to numerical results.

2 The Double Pendulum Hamiltonian

Consider the planar double pendulum shown in fig. 1. The first pendulum is suspended at point A_1 which is fixed in space. The second pendulum moves about point A_2 which is attached to the first. We assume that A_1, A_2, and the center of gravity S_1 of the first pendulum are collinear as in Richter and Scholz [4]. The configuration space is given by the two angles ϕ_1, ϕ_2, measuring the orientation of A_1A_2 and A_2S_2 with respect to the vertical, as in [3], § 5, exercise 1. fig.1

The physical system has seven parameters to start with: the masses m_1, m_2 of the two pendulums, and their moments of inertia Θ_1, Θ_2 with respect to the axes of rotation; the distance $a \geq 0$ between the two suspension points; the distances s_1 and s_2 between the respective suspension points and centers of gravity. We require that $\phi_1 = 0$, $\phi_2 = 0$ describes the stable equilibrium configuration. This implies $s_2 \geq 0$ and $m_1 s_1 + m_2 a \geq 0$ (allowing for negative values of s_1 which means that S_1 and A_2 are on different sides of A_1).

A specific double pendulum is thus described by the seven parameters a, m_i, s_i, and Θ_i. With the difference angle $\Delta := \phi_2 - \phi_1$ the Lagrangian is obtained as

$$\mathcal{L} \;=\; \frac{1}{2}(\Theta_1 + m_2 a^2)\dot{\phi_1}^2 + \frac{1}{2}\Theta_2\dot{\phi_2}^2 + m_2 s_2 a \dot{\phi_1}\dot{\phi_2}\cos\Delta \tag{1}$$

$$-g(m_1 s_1 + m_2 a)(1-\cos\phi_1) - g m_2 s_2 (1-\cos\phi_2)$$

$$=: \quad \sum \frac{1}{2} M_i \dot{\phi}_i^{\,2} + C\dot{\phi}_1\dot{\phi}_2 \cos\Delta + \sum V_i (1-\cos\phi_i) \tag{2}$$

introducing the parameter set (M_1, M_2, C, V_1, V_2). Since moments of inertia are always positive we have the constraint $M_1 M_2 > C^2$.

At this stage the Lagrangian (2) depends on only five relevant parameter combinations. In addition the total energy E of the system is a constant of motion and may be treated as another parameter, given by the initial conditions. Two of these six parameters can be removed by choosing appropriate scales for energy and time. We choose the second pendulum as a reference and turn it into a "unit mathematical pendulum", by scaling energy with $V_2 = g m_2 s_2$ and time with $\sqrt{M_2/V_2} = \sqrt{\Theta_2/(m_2 s_2 g)}$. Note that $\Theta_2/m_2 s_2$ is the reduced pendulum length. Dividing equation (2) by V_2 suggests that moments of inertia be measured in units of M_2. Thus the parameter set becomes $(M_1/M_2, 1, C/M_2, V_1/V_2, 1, E/V_2) =: (\alpha, 1, \varepsilon, \gamma, 1, h)$, i. e. the system is completely described by the four parameters

$$\alpha = \frac{M_1}{M_2} = \frac{\Theta_1 + m_2 a^2}{\Theta_2} \tag{3}$$

$$\varepsilon = \frac{C}{M_2} = \frac{a m_2 s_2}{\Theta_2} \tag{4}$$

$$\gamma = \frac{V_1}{V_2} = \frac{m_1 s_1 + m_2 a}{m_2 s_2} \tag{5}$$

$$h = \frac{E}{V_2} . \tag{6}$$

The constraint $M_1 M_2 > C^2$ becomes $\alpha > \varepsilon^2$ in scaled variables.

The range of the parameter values is given by

$$\mathcal{P} = \{(\alpha, \varepsilon, \gamma, h) \,|\, \varepsilon, \gamma, h \geq 0, \alpha > \varepsilon^2\} \tag{7}$$

where the last condition is the constraint $M_1 M_2 > C^2$ in scaled variables.

There is one slight inconvenience with this choice of scales: the case $s_2 = 0$ is mapped to $\gamma = \infty$ and $h = \infty$. On the other hand, the Lagrangian (1) shows that this is the integrable case of uncoupled pendulum 1 and rotator 2. Clearly it would be preferable to treat this case by turning the *first* pendulum into a "unit mathematical pendulum", i. e. by scaling energy with $V_1 = g(m_1 s_1 + m_2 a)$ and time with $\sqrt{M_1/V_1}$. This corresponds to the parameter set $(1, M_2/M_1, C/M_1, 1, V_2/V_1, E/V_1) =:$ $(1, \alpha', \varepsilon', 1, \gamma', h')$. But notice that if we formally exchange the two pendulums, $\phi_1 \leftrightarrow \phi_2$, together with the parameter exchange

$$P : (\alpha, \varepsilon, \gamma, h) \mapsto (\alpha', \varepsilon', \gamma', h') = (1/\alpha, \varepsilon/\alpha, 1/\gamma, h/\gamma) \ , \tag{8}$$

then we are back to the set of pendulums described by the first scaling procedure. (The times in the two scalings are related by $t' = t\sqrt{\gamma/\alpha}$.) This symmetry of our Lagrangian with respect to exchange of pendulums and parameters shows that it is sufficient to consider only "half" of the parameter space (7). In order to identify this "half", we need to consider the mapping P in some detail.

It can be decomposed into two identical operations $P' : (\alpha, \varepsilon) \mapsto (\alpha', \varepsilon')$ and $P'' : (\gamma, h) \mapsto (\gamma', h')$. Consider only P'. It has a line of fixed points for $\alpha = 1$. Close to this line P' is almost an involution, since $P'^2 = id$ and $\det DP' = -1$ at $\alpha = 1$. Therefore P' can be considered as a generalized reflection about $\alpha = 1$ which is not area preserving. The diagonal $\alpha = \varepsilon$ is mapped to the horizontal line $\varepsilon = 1$ and vice versa. The boundaries of $\mathcal{P}$ are both invariant lines of P': The line $\varepsilon = 0$ is mapped onto itself by reflecting and expanding the unit interval to the interval $(1, \infty)$ and vice versa. The upper boundary of $\mathcal{P}$ as given by the constraint $\alpha = \varepsilon^2$ is reflected in a similar way around the point $(1, 1)$. Actually all curves $\alpha = (\varepsilon/\varepsilon_0)^2$ are invariant curves under P', ε_0 being the value of ε at the intersection with the fixed line $\alpha = 1$. Similarly, the curves $\gamma = (h/h_0)^2$ are reflected upon themselves under P''.

Now we have two choices for a parameter space that contains every double pendulum only once: Either restrict $\alpha \le 1$ or restrict $\gamma \le 1$. Since we are interested in the integrable case $s_2 = 0$ the latter choice is appropriate as $s_2 = 0$ is mapped to $\gamma' = 0$. Furthermore α cannot go to infinity (as opposed to γ) except for unphysical choices of masses and/or moments of inertia. Therefore our preferred parameter space that contains just one representation of every possible double pendulums is

$$\tilde{\mathcal{P}} = \{(\alpha, \varepsilon, \gamma, h) \,|\, \varepsilon \ge 0, \alpha > \varepsilon^2, 0 \le \gamma \le 1, h \ge 0\} \; . \tag{9}$$

The mathematical double pendulum studied in [4], with $m_1 = m_2$, $a = s_1 = s_2$, and $\Theta_1^c = \Theta_2^c = 0$, gives the parameter values $(\alpha, \varepsilon, \gamma) = (2, 1, 2) \notin \tilde{\mathcal{P}}$. Using P this is mapped to $(1/2, 1/2, 1/2) \in \tilde{\mathcal{P}}$. Such a double pendulum has the same dynamics as the mathematical double pendulum, except for the exchange $\phi_1 \leftrightarrow \phi_2$. A particularly simple representative of the parameter set $(1/2, 1/2, 1/2)$ is easily found to be $s_1 = 0$, $\Theta_2^c = 0$, $m_2 = 1$, $s_2 = 2a$ and $a^2 = \Theta_1^c$. The outer pendulum consists of a point mass at distance $2a$ from A_2. If the geometry of the inner pendulum body is specified to be a ring of radius a, then its mass must be $m_1 = 1$; if it is a disk of radius is a its mass must be $m_1 = 2$, and if it is a homogeneous rod with length $2a$ its mass is $m_1 = 3$.

There are of course lots of physical realizations of double pendulums corresponding to a given set of parameters $(\alpha, \varepsilon, \gamma)$; e. g. all parameter sets in the full seven dimensional parameter space for which the ratios l_1/l_2, m_1/m_2, s_1/s_2, a/s_2 and a/l_2 are constant, give the same set of $(\alpha, \varepsilon, \gamma)$. All these double pendulums are smoothly connected in full parameter space and are part of the equivalence class of double pendulums (a surface in full parameter space) corresponding to a fixed set of $(\alpha, \varepsilon, \gamma)$. The exchange map P typically leads to a different equivalence class so that the corresponding surfaces in full parameter space do not intersect. Generally speaking, a scaling of variables produces a continuous symmetry foliating the parameter space whereas the exchange map gives an additional discrete symmetry.

In order to apply Melnikov's method we need an integrable limiting case with a separatrix. The double pendulum has two kinds of integrable cases. Without gravity, i. e. for $g = 0$ (experimentally one would align the rotation axes of the pendulums with the direction of the gravity force), we have two dynamically coupled rotators with a stable orbit in the stretched configuration $\Delta = 0$, and an unstable orbit in the folded configuration $\Delta = \pi$. This case is very promising because of its interesting bifurcations depending on the ratio of the frequencies of the two rotators. Although the solutions are given in terms of hyperelliptic functions the separatrix will be – as usually – one order simpler, i. e. just an ordinary elliptic function. Therefore the analysis could be done in this case too. There is quite some evidence that a breathing chaos can be calculated in this case, using Melnikov's method (see below for the other breathing case(s)), because the resonant term $\cos(\varphi_1 + \cos\varphi_2)$ really is small, independently of the bifurcation parameter. This integrable limit can only be solved in terms of hyperelliptic functions [5]. We study the other integrable case of the double pendulum, namely $C = 0$ or $\varepsilon = 0$, where the system consists of two uncoupled pendulums. The nature of its motion depends on the value of γ; for $\gamma = 0$ pendulum 1 is a rotator; for $\gamma \to \infty$ pendulum 1 is an oscillator. In any case the other pendulum gives rise to the separatrix asymptotic to its unstable equilibrium point at $\phi_2 = \pi$.

With scaled parameters the Lagrangian reads

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\dot{\boldsymbol{\phi}}^T \mathbf{M}\,\dot{\boldsymbol{\phi}} - V && (10)\\ V &= \gamma(1-\cos\phi_1) + 1 - \cos\phi_2 && (11)\\ \mathbf{M} &= \begin{pmatrix} \alpha & \varepsilon\cos\Delta \\ \varepsilon\cos\Delta & 1 \end{pmatrix}. && (12) \end{aligned}$$

Introducing the angular momenta $L_i = \partial\mathcal{L}/\partial\phi_i$ we obtain the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\mathbf{L}^T\mathbf{M}^{-1}\mathbf{L} + V && (13)\\ &= T_0 + V - \varepsilon\frac{\cos\Delta}{\alpha - \varepsilon^2\cos^2\Delta}(L_1L_2 - \varepsilon T_0\cos\Delta) \qquad \text{with} && (14)\\ T_0 &= \frac{1}{2}\mathbf{L}^T\mathbf{M}^{-1}\mathbf{L}\Big|_{\varepsilon=0} = \frac{1}{2\alpha}L_1^2 + \frac{1}{2}L_2^2 && (15) \end{aligned}$$

written in a form that is convenient for the following perturbation analysis. The perturbation can be Fourier expanded in Δ using integration by residues

$$\begin{aligned} H_1 &= \frac{1}{\sqrt{\alpha-\varepsilon^2}}\left[-T_0\varepsilon c + 2\sum_{j=1}^{\infty} c^j\cos j\Delta \begin{cases} L_1L_2 & j \text{ odd} \\ -\sqrt{\alpha}\,T_0 & j \text{ even} \end{cases}\right] && (16)\\ c &= \frac{\sqrt{\alpha}}{\varepsilon}(1 - \sqrt{1-\varepsilon^2/\alpha}) \approx \frac{\varepsilon}{2\sqrt{\alpha}}. && (17) \end{aligned}$$

Melnikov's method is applicable in near-integrable cases, therefore we assume $\varepsilon \ll 1$. Hence the above series can be approximated by the terms linear in ε. If in addition we assume $\varepsilon^2 \ll \alpha$ we obtain

$$\mathcal{H} = T_0 + V - \frac{\varepsilon}{\alpha}L_1L_2\cos\Delta + O(\frac{\varepsilon^2}{\alpha}). \qquad (18)$$

To keep the perturbation small we need to require $\varepsilon \ll \alpha$.

3 Melnikov's Function

We want to use Melnikov's method in order to show that the double pendulum has homoclinic intersections. The idea is [1, 6, 7] to integrate the perturbation along the explicitly given separatrix of the unperturbed system to get an approximation of the stable and unstable manifolds of the corresponding fixed point in the perturbed case. Initially the method was formulated for systems with one and a half degrees of freedom: two dimensional phase space plus time periodic perturbation. To apply it to a time independent Hamiltonian system with two degrees of freedom, one pair of canonical variables must first be transformed to action-angle variables (I, θ) of the unperturbed case. When the perturbation is turned on the action I is considered constant in first order, and the angle θ can be viewed as the new time variable. The resulting dynamics is analyzed in the Poincaré surface of section θ = const = 0. In the unperturbed case it shows the phase portrait of the remaining one degree of freedom system. The orbits of this planar system are used as a set of coordinate lines to measure the distance of the perturbed invariant manifolds from the unperturbed separatrix in the locally perpendicular direction. If this distance as given by Melnikov's function is zero the homoclinic point is located (to first order) at the corresponding point of the former separatrix.

For the double pendulum, we choose to transform ϕ_1, L_1 to action-angle variables (I, θ) of the integrable limit $\varepsilon = 0$. The Hamiltonian is then obtained in the form used by Holmes and Marsden [1] whose notation we adopt

$$H(\phi_2, L_2, \theta, I) = F(\phi_2, L_2) + G(I) + \varepsilon H^1(\phi_2, L_2, \theta, I). \tag{19}$$

The Melnikov function is then especially simple:

$$M(\theta^0) = \int_{-\infty}^{\infty} \{F, H^1\}\, dt. \tag{20}$$

The Poisson bracket is to be evaluated along the separatrix solution $\phi_2^0(t)$ and $L_2^0(t)$ of the unperturbed system F with energy $h^0 = F(\phi_2^0, L_2^0)$. The angle variable is $\theta = \omega t + \theta^0$ where the frequency ω is given by $\omega = \partial G/\partial I$, and the action I is determined by $I = G^{-1}(h - h^0)$ so that the total energy h is fixed. If $M(\theta_*^0) = 0$ the homoclinic intersection is located at $(\phi_2^0(\theta_*^0), L_2^0(\theta_*^0))$.

From now on we assume that ϕ_1 and L_1 are given as functions of I and θ. The transformation to action-angle variables which involves elliptic functions will be carried out later. Using (18) the Poisson bracket gives

$$\begin{aligned} \{F, H^1\} &= L_1(\sin\phi_2 \cos\Delta + L_2^2 \sin\Delta)/\alpha && (21)\\ &= L_1 \cos\phi_1(\sin\phi_2\cos\phi_2 + L_2^2 \sin\phi_2)/\alpha + \\ &\quad + L_1 \sin\phi_1(\sin^2\phi_2 - L_2^2\cos\phi_2)/\alpha\ , && (22) \end{aligned}$$

and a similar calculation can be done for every term in (16) to obtain higher order accuracy. Our strategy is to expand the terms involving L_1 and ϕ_1 as a Fourier series in the angle variable θ, as usual in canonical perturbation theory. After interchanging integration and summation the terms involving L_2 and ϕ_2 can be integrated term by term with $\sin j\theta$ and $\cos j\theta$. These integrals as well as the integrals giving the coefficients of the Fourier expansions, are solved by residues calculus.

Assume the Fourier series of the form

$$\begin{aligned} L_1 \sin\phi_1 &= \sum_{j=1}^{\infty} s_j \sin j\theta \qquad \text{and} \\ L_1 \cos\phi_1 &= \sum_{j=0}^{\infty} c_j \cos j\theta, \end{aligned} \tag{23}$$

with the coefficients s_j and c_j to be specified later as functions of I.

The separatrix solution for the pendulum F is

$$(\phi_2^0(t), L_2^0(t)) = (\pm 2 \arctan(\sinh t), \pm 2 \cosh^{-1} t) \tag{24}$$

with energy $h^0 = 2$, the "turn over energy" of the pendulum. Note that the point $t = 0$ on the separatrix is $(\phi_2^0, L_2^0) = (0, \pm 2)$: the place where one would naturally expect the homoclinic intersection to appear. Inserting the separatrix solution into the Poisson bracket (21) gives

$$\{F, H^1\}|_{\text{sep.}} = \frac{2L_1}{\alpha \cosh^4 t}[\sin \phi_1 (4 \cosh^2 t - 6) - \cos \phi_1 \sinh t (\cosh^2 t - 6)]. \tag{25}$$

Now we substitute this and the series (23) into (20). θ is replaced by $\omega t + \theta^0$ and only the even parts with respect to t contribute to Melnikov's integral. Four types of integrals with poles of second and fourth order remain. They can be solved by residue integration, see e.g. [8] and the appendix:

$$\int_{-\infty}^{\infty} \frac{\cos \omega t}{\cosh^2 t} dt = \pi \frac{\omega}{\sinh \omega \pi / 2} \tag{26}$$

$$\int_{-\infty}^{\infty} \frac{\cos \omega t}{\cosh^4 t} dt = \frac{\pi}{6} \frac{\omega (4 + \omega^2)}{\sinh \omega \pi / 2} \tag{27}$$

$$\int_{-\infty}^{\infty} \frac{\sin \omega t \sinh t}{\cosh^2 t} dt = \pi \frac{\omega}{\cosh \omega \pi / 2} \tag{28}$$

$$\int_{-\infty}^{\infty} \frac{\sin \omega t \sinh t}{\cosh^4 t} dt = \frac{\pi}{6} \frac{\omega (1 + \omega^2)}{\cosh \omega \pi / 2}. \tag{29}$$

Taking the terms of next order from (16) into account, already leads to poles of sixth order. Omitting these corrections we obtain

$$M(\theta^0) = -\frac{2\pi}{\alpha} \omega^3 \sum_{j=1}^{\infty} j^3 \sin j\theta^0 \left(\frac{s_j}{\sinh j\omega\pi/2} + \frac{c_j}{\cosh j\omega\pi/2} \right) \tag{30}$$

4 Angle Fourier Series

In the integrable limit $\varepsilon = 0$ of the double pendulum, the system to be transformed to action-angle variables is of course the single pendulum 1. The calculations are easy if the pendulum can be approximated by a rotator or by an oscillator. This is only justified for special values of the parameters α and γ. The general case requires elliptic functions and integrals. There are four cases depending on the value of k, the modulus of the elliptic functions, which turns out to be

$$k^2 = \frac{\delta}{2\gamma}, \qquad \text{with} \tag{31}$$

$$\delta = h - h^0 . \tag{32}$$

The four cases describe four different types of motion of pendulum 1:

- small oscillations for $k \ll 1$, i. e. $\gamma \to \infty$,

- librations for $k < 1$,
- rotations for $k > 1$,
- pure rotations for $k \gg 1$, i. e. $\gamma \to 0$.

We treat these cases one by one.

$\boldsymbol{k \gg 1}$ We start with the pure rotation case since it is the easiest. L_1 and ϕ_1 are already action-angle variables: $G(L_1) = L_1^2/2\alpha$. Therefore $L_1 = \sqrt{2\alpha\delta}$, and the frequency of rotation is

$$\omega_\infty^2 = 2\delta/\alpha. \tag{33}$$

The only nonzero coefficients in (23) are $s_1 = c_1 = L_1$, and substituting into (30) gives

$$M_\infty(\theta^0) = -2\pi\omega_\infty^4 \sin\theta^0 \left(\frac{1}{\sinh \omega_\infty \pi/2} + \frac{1}{\cosh \omega_\infty \pi/2} \right). \tag{34}$$

This function obviously has nondegenerate zeroes: this proves the existence of chaos in the double pendulum. One zero is at $\theta_0 = 0$ which corresponds to the first intersection of the invariant manifolds at the expected point $(0, \pm 2)$.

$\boldsymbol{k \ll 1}$ If γ is large enough we can approximate the first pendulum by a harmonic oscillator with frequency

$$\omega_0^2 = \gamma/\alpha, \tag{35}$$

and $G(I) = \omega_0 I$, so we can express L_1 and ϕ_1 in action-angle variables:

$$(\phi_1, L_1) = (\sqrt{\frac{2I}{\omega_0\alpha}} \sin\theta, \sqrt{2\omega_0\alpha I} \cos\theta). \tag{36}$$

With $I = G^{-1}(h - h^0) = \delta/\omega_0$ we obtain

$$\begin{aligned} L_1 \sin\phi_1 &= \sqrt{2\delta\alpha} \cos\theta \sin(2k \sin\theta) && (37) \\ &= 2\alpha\omega_0 \sum_{j \text{ even}>0}^{\infty} j \mathrm{J}_j(2k) \sin j\theta && (38) \\ L_1 \cos\phi_1 &= \sqrt{2\delta\alpha} \cos\theta \cos(2k \sin\theta) && (39) \\ &= 2\alpha\omega_0 \sum_{j \text{ odd}>0}^{\infty} j \mathrm{J}_j(2k) \cos j\theta && (40) \end{aligned}$$

where J_j is the j-th Bessel function of integer order. Putting it all together gives

$$M_0(\theta^0) = -4\pi\omega_0^4 \sum_{j=1}^{\infty} j^4 \mathrm{J}_j(2k) \sin j\theta^0 \left\{ \begin{array}{ll} \sinh^{-1} j\omega_0\pi/2 & j \text{ even} \\ \cosh^{-1} j\omega_0\pi/2 & j \text{ odd} \end{array} \right. . \tag{41}$$

Again this function has nondegenerate zeroes, and hence we have proven the existence of chaos in this case too. Since k is assumed to be small we might just as well approximate $\mathrm{J}_j(2k)$ by $k^j/j!$.

The two remaining intermediate cases involve the general action-angle variables for the pendulum, and thus incomplete elliptic integrals $\mathrm{F}(\phi, k)$, complete elliptic integrals $\mathrm{K}(k) = \mathrm{F}(\pi/2, k)$ and $\mathrm{K}'(k) = \mathrm{K}(\sqrt{1-k^2})$, and the Jacobian elliptic functions sn$(u|k)$, cn, dn, and am, see e.g. [9, 10, 11]. If the modulus k_m of the elliptic functions and integrals is not explicitly given in the following, then $k_m = k$ for $k < 1$ and $k_m = 1/k$ for $k > 1$ is assumed.

$\boldsymbol{k<1}$ For librations, i. e. motion of pendulum 1 inside its separatrix, we have (see e.g. [12])

$$\theta = \frac{\pi}{2\mathrm{K}(k)}\mathrm{F}(\eta, k) \qquad \text{with} \tag{42}$$

$$k\sin\eta = \sin\frac{\phi_1}{2} \tag{43}$$

$$\omega = \omega_0\frac{\pi}{2\mathrm{K}(k)}. \tag{44}$$

We do not need the explicit equation for I because we will express everything in terms of k instead. We obtain

$$L_1 = 2k\sqrt{\gamma\alpha}\,\mathrm{cn}(2\theta\mathrm{K}(k)/\pi) = 2k\sqrt{\gamma\alpha}\,\mathrm{cn}(\theta\omega_0/\omega) \tag{45}$$

$$\sin\phi_1 = 2k\,\mathrm{sn}(2\theta\mathrm{K}(k)/\pi)\,\mathrm{dn}(2\theta\mathrm{K}(k)/\pi) \tag{46}$$

$$\cos\phi_1 = 2\,\mathrm{dn}^2(2\theta\mathrm{K}(k)/\pi) - 1 = 1 - 2k^2\,\mathrm{sn}^2(2\theta\mathrm{K}(k)/\pi) \tag{47}$$

and a calculation given in the appendix shows (omitting the arguments of elliptic functions)

$$L_1\sin\phi_1 = 4k^2\sqrt{\gamma\alpha}\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn} \tag{48}$$

$$= 4\sqrt{\gamma\alpha}\left(\frac{\pi}{2\mathrm{K}}\right)^3\sum_{j\ \mathrm{even}>0}\frac{j^2\sin j\theta}{\sinh j\xi} \tag{49}$$

$$L_1\cos\phi_1 = 2k\sqrt{\gamma\alpha}\,\mathrm{cn}(2\,\mathrm{dn}^2-1) \tag{50}$$

$$= 4\sqrt{\gamma\alpha}\left(\frac{\pi}{2\mathrm{K}}\right)^3\sum_{j\ \mathrm{odd}>0}\frac{j^2\cos j\theta}{\cosh j\xi} \qquad \text{with} \tag{51}$$

$$\xi = \frac{\pi}{2}\frac{\mathrm{K}'}{\mathrm{K}}. \tag{52}$$

Using these results to identify the coefficients s_j and c_j in (30), we obtain the Melnikov function for $k < 1$,

$$M_<(\theta^0) = -\frac{\pi}{8}alpha\omega_0^4\left(\frac{\pi}{\mathrm{K}}\right)^6\sum_{j=1}^{\infty}j^5\sin j\theta^0\begin{cases}\sinh^{-1}j\xi\sinh^{-1}(j\omega_0\pi^2/4\mathrm{K}) & j\ \text{even}\\ \cosh^{-1}j\xi\cosh^{-1}(j\omega_0\pi^2/4\mathrm{K}) & j\ \text{odd}\end{cases}$$

$$= -8\pi\frac{\omega^6}{\omega_0^2}\sum_{j=1}^{\infty}j^5\sin j\theta^0\begin{cases}\sinh^{-1}j\xi\sinh^{-1}(j\omega\pi/2) & j\ \text{even}\\ \cosh^{-1}j\xi\cosh^{-1}(j\omega\pi/2) & j\ \text{odd}\end{cases} \tag{53}$$

$\boldsymbol{k>1}$ For rotations, i. e. motion of pendulum 1 outside its separatrix, we have

$$\theta = \frac{\pi}{\mathrm{K}(k_m)}\mathrm{F}(\frac{\phi_1}{2}, k_m) \qquad \text{and} \tag{54}$$

$$\omega = \omega_0\frac{\pi k}{\mathrm{K}(k_m)} = \omega_\infty\frac{\pi}{2\mathrm{K}(k_m)} \tag{55}$$

and similarly as in the case of libration, but with a factor of 2 missing in the arguments,

$$
\begin{aligned}
L_1 &= 2k\sqrt{\gamma\alpha}\,\mathrm{dn}(\theta \mathrm{K}(k_m)/\pi) = 2k\sqrt{\gamma\alpha}\,\mathrm{dn}(\theta k\omega_0/\omega) && (56)\\
\sin\phi_1 &= 2\,\mathrm{sn}(\theta \mathrm{K}(k_m)/\pi)\,\mathrm{cn}(\theta \mathrm{K}(k_m)/\pi) && (57)\\
\cos\phi_1 &= 1 - 2\,\mathrm{sn}^2(\theta \mathrm{K}(k_m)/\pi). && (58)
\end{aligned}
$$

Omitting the arguments of elliptic functions (remember that now they are corresponding to modulus $k_m = 1/k$ since $k > 1$ and there is a factor of 2 missing compared to the libration case, see appendix) we find

$$
\begin{aligned}
L_1\sin\phi_1 &= 4k\sqrt{\gamma\alpha}\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn} && (59)\\
&= \sqrt{\gamma\alpha}\frac{1}{2}\left(\frac{k\pi}{\mathrm{K}}\right)^3 \sum_{j\ \text{even}>0} \frac{j^2 \sin j\theta/2}{\sinh j\xi} && (60)\\
&= 2\sqrt{\gamma\alpha}\left(\frac{\pi k}{\mathrm{K}}\right)^3 \sum_{j=1}^{\infty} \frac{j^2\sin j\theta}{\sinh 2j\xi} && (61)\\
L_1\cos\phi_1 &= 2k\sqrt{\gamma\alpha}\,\mathrm{dn}(1-2\,\mathrm{sn}^2) && (62)\\
&= 2\sqrt{\gamma\alpha}\left(\frac{\pi k}{\mathrm{K}}\right)^3 \sum_{j=1}^{\infty} \frac{j^2\cos j\theta}{\cosh 2j\xi} && (63)
\end{aligned}
$$

Again reading the coefficients s_j and c_j off these expressions and using eqn.(30), we obtain the Melnikov function for $k > 1$:

$$
\begin{aligned}
M_>(\theta^0) &= -2\pi\omega_0^4\left(\frac{\pi k}{\mathrm{K}}\right)^6 \sum_{j=1}^{\infty} j^5 \sin j\theta^0 && (64)\\
&\quad \left\{\sinh^{-1} 2j\xi \sinh^{-1}(j\omega_\infty\pi^2/4\mathrm{K}) + \cosh^{-1} 2j\xi \cosh^{-1}(j\omega_\infty\pi^2/4\mathrm{K})\right\}\\
&= -16\pi k^2 \frac{\omega^6}{\omega_\infty^2}\sum_{j=1}^{\infty} j^5\sin j\theta^0\\
&\quad \left\{\sinh^{-1} 2j\xi \sinh^{-1} j\omega\pi/2) + \cosh^{-1} 2j\xi \cosh^{-1}(j\omega\pi/2)\right\} && (65)\\
&= -4\pi\frac{\omega^6}{\omega_0^2}\sum_{j=1}^{\infty} j^5 \sin j\theta^0\ \times\\
&\quad \left(\frac{1}{\sinh 2j\xi\ \sinh j\omega\pi/2} + \frac{1}{\cosh 2j\xi\ \cosh j\omega\pi/2}\right) && (66)
\end{aligned}
$$

The convergence of the Fourier series (53) and (66) is good: The coefficients become exponentially small for sufficiently large values of j. However for $k \to 1$ the complete elliptic integral K goes to infinity logarithmically, and therefore ξ and ω approach zero. The number of terms contributing to the sum goes to infinity in this case. Thus the convergence could be a problem for $k \approx 1$. Yet numerically the convergence has not been a problem even very close to $k = 1$. In the critical case $k = 1$ pendulum 1 is on the separatrix between librational and rotational motion. This means that the perturbation of pendulum 2 is no longer periodic in θ, and Melnikov's method cannot be expected to be valid in this singular limit. Excluding this critical case we have a non degenerate zero in Melnikov's function at $\theta^0 = 0$, and thereby the existence of chaos in the double pendulum is proven.

5 Parameter Dependence of Chaos

Before evaluating Melnikov's function let us recall the restrictions on the parameters. Firstly, the perturbation parameter ε, i. e. the coupling of the two pendulums, must be small to be close to the integrable case. According to equation 4 this either means $a \to 0$ (suspension points of the pendulums are close to each other) or $s_2 \to 0$ (the outer pendulum is a rotator). In the latter case pendulum 2 has no separatrix but pendulum 1 has; thus, interchanging the indices 1 and 2, we are back to the situation where $F(\phi_2, L_2)$ has a separatrix. The approximation of the Hamiltonian by (18) is valid for $\varepsilon \ll \alpha$ (the perturbation is small compared to the moment of inertia of pendulum 1).

The second restriction is that the total energy h has to be close to $h_0 = 2$ so that the motion stays close to the separatrix of pendulum 2; this requires $\delta = h - h_0$ to be small. The parameter γ can be varied independently from δ. If $\gamma \gg \delta$, pendulum 1 is heavy and its motion is restricted to harmonic oscillations. If $\gamma \ll \delta$, pendulum 1 is almost a free rotor. In the following discussion we take ε and δ as fixed. The modulus k_m can then be adjusted to any value by changing γ. It describes the nature of the motion of pendulum 1. At given γ, we may still vary the frequencies ω_0 or ω_∞ of pendulum 1 by changing α (see eqs. (33) and (35)); but note that they may not be too large because of $\varepsilon \ll \alpha$. We expect that the splitting of the separatrix of pendulum 2 depends on both k and ω. fig. 2

The Melnikov function can be interpreted as a deviation in energy from the value on the unperturbed separatrix. The maximum of M measures how far the perturbation can drive the energy of pendulum 2 from the value $h^0 = 2$, at the point of the largest splitting of the invariant manifolds. We therefore denote this maximum by ΔE and take it as an estimate for the size of the area in the Poincaré surface of section where chaotic motion takes place. We thus take ΔE as an estimate for the extent of chaos in the system.

ΔE is obtained by numerically searching for the maximum of the expressions (53) and (66). The results are shown in fig. 2. To present all parameter choices in one diagram it is divided into four regions.

1. Lower left: librations of pendulum 1 ($k < 1$) with low frequencies, $\omega_0 < 1$.
2. Upper left: librations with high frequencies. Instead of ω_0 its inverse is shown along the vertical axis.
3. Lower right: rotations ($k > 1$) with low frequencies, $\omega_\infty < 2$.
4. Upper right: rotations with high frequencies. The vertical axis shows $2/\omega_\infty$.

These are the natural parameters on which the solution depends in an essential way. As discussed above, the allowed values of α (at given ε) restrict the values of ω_0 respectively ω_∞. Therefore, depending on the strength of the perturbation ε, the results near the top of fig. 2 become unreliable. Nevertheless they seem to be qualitatively correct as a comparison with a series of Poincaré surfaces of section on that same parameter plane demonstrates. Figure 3 shows such a series for $\varepsilon = 0.02$ fig. 3
and $\delta = 0.3$. The sections are done with the condition $\phi_1 = 0$, and projected onto

the (ϕ_2, L_2)-plane. They illustrate the splitting of the separatrices of pendulum 2 as a result of the coupling to pendulum 1. In each individual picture, a single orbit is shown with initial point very close to the unstable fixed point, and 10000 iterations of the full equations of motion. The main features of the analytical and the numerical results agree quite well. Even for values of ε up to 0.1 and δ up to 1.0, the qualitative picture does not change. This is remarkable considering 1) that the Hamiltonian and the Melnikov function were only taken to first order in the perturbation ε, 2) that the method cannot really be applied on the line $k = 1$, and 3) that the upper part of the figure, corresponding to high frequencies of pendulum 1, is outside the range of validity of $\varepsilon \ll \alpha$.

The lower left corner of fig. 2 corresponds to small amplitude and slow frequency oscillations of the perturbing pendulum. The extent of chaos is quite small in this region. The "iso-chaos" lines are approximately hyperbolas $\omega_0 k = 2\delta/\alpha = \omega_\infty =$ const. In the lower right quarter, the lines of constant chaos follow approximately constant ω_∞, especially at small ω_∞ where the perturbation is generated by a slowly rotating pendulum 1. In the upper left quarter, we have fast oscillations, and the extent of chaos is mainly determined by the value of k, i. e. their amplitude. If the frequency is too fast to disturb the system, the extent of chaos becomes small again. Finally the upper right quarter corresponds to fast rotations of the perturbing pendulum, and the decay of chaos in the direction of decreasing k_m takes place earlier than in the case of oscillations. But the largest extent of chaos, according to the results of the Melnikov method, occurs in this region, at intermediate values of the frequency ω_∞. This prediction is, however, not quite supported by the numerical studies in fig. 3; they show maximal chaotic motion in the oscillatory regime of pendulum 1, though also at intermediate frequencies.

The three physical parameters α, γ and δ (ε enters only as a linear factor) are mapped to the natural parameters k, ω_0 respectively ω_∞ by (31), (33) and (35). Clearly only the ratios of the physical parameters matter. This means that the extent of chaos for a given parameter set (α, γ, δ) should be the same as for the set $(s\alpha, s\gamma, s\delta)$ obtained by scaling with a number $s > 0$. Of course, the constraints for the validity of the solution as discussed above, have to be fulfilled. This limits the range of this scaling law: for small s, the value $s\alpha$ becomes too small, and for large s, the value of $s\delta$ too large. Moreover, this scaling is only expected to hold in first order perturbation theory. The detailed features of the numerically calculated pictures in fig. 3 show that there is a lot more structure which could only be explained by higher order calculations.

Good analytical approximations for ΔE can be given for small and large values of k. An easy calculation shows that the maximum amplitude in the Fourier series occurs approximately at

$$j^* = \frac{5}{r\xi + \omega\pi/2} \tag{67}$$

where $r = 1$ for librations and $r = 2$ for rotations. If $j^* < 1$ (which is always the case except for $k \approx 1$ and $\alpha \gg 1$) it is safe to approximate the Fourier series in Melnikov's function by its first term. We then find for oscillations with small k

$$\Delta E_< \quad = \quad |\varepsilon M^*_<| \tag{68}$$

$$\approx \quad 8\pi\varepsilon\frac{\omega^6}{\omega_0^2}\frac{1}{\cosh\xi\cosh\omega\pi/2}\ , \tag{69}$$

and for the case of rotations with large k,

$$\Delta E_> \quad \approx \quad 4\pi\varepsilon\frac{\omega^6}{\omega_0^2}\left(\frac{1}{\sinh 2\xi\sinh\omega\pi/2}+\frac{1}{\cosh 2\xi\cosh\omega\pi/2}\right) \tag{70}$$

$$\approx \quad 2\pi\varepsilon\omega_\infty^4\left(\frac{1}{\sinh\omega\pi/2}+\frac{1}{\cosh\omega\pi/2}\right) \tag{71}$$

where the last approximation is valid for $k \ll 1$, compare to (34). $\Delta E_>$ for large k as a function of ω_∞ has a maximum at $\omega_\infty = 2.546$: the widest separatrix layer occurs at $3.23\,\alpha = \delta$. But to fulfill $\alpha \gg \varepsilon$ in this case we need $\delta \gg \varepsilon$, i. e. we approach the boundary of the range of validity of Melnikov's function. fig.4

On closer inspection, the numerical experiments show that there is a lot more structure than the first order Melnikov method can predict, especially for larger perturbations. We demonstrate this for the case $\gamma = 0$, or $k = \infty$ (i. e. pure rotations of the inner pendulum) in fig. 4. This series of pictures reveals the scenario of breathing chaos (as described in [13]), i. e. an oscillation of the extent of chaos as the parameter α increases. This phenomenon is connected with a series of bifurcations of the central fixed point for values of α given by $\alpha n^2 = 2(2+\delta)$, for any integer n. Physically this means that one pendulum rotates n times while the other one oscillates once. For fixed energy we can make the rotator lighter and lighter and thereby make it rotate faster and faster. Upon variation of α, resonances emanate from the central fixed point and get swallowed by the chaotic region. This seems to be a common global feature of chaotic systems, but it is beyond explanation by simple versions of Melnikov's method. Higher order perturbation theory is required to deal with this fine structure.

6 Conclusion

Melnikov's method can serve as a tool not only to prove the existence of homoclinic points and thus chaos, but beyond that to study the parameter dependence of chaos. Being an analytic perturbation method, it can naturally give results close to an integrable case only. For the double pendulum these results remain valid for surprisingly large perturbation parameters. However, finer structures in the parameter dependence of the extent of chaos such as breathing require higher orders of the analysis. In principle this can be done using the methods presented here, but the computational effort will be considerably larger.

7 Acknowledgment

The author would like to thank P.H. Richter and A. Paul for illuminating discussions and B. Bruhn for a motivating introduction to Melnikov's method.

The cases of rotation and libration differ basically only by the period of the function to be Fourier expanded. Therefore we introduce $r = 1$ for librations and $r = 2$ for rotations and then calculate the Fourier coefficients of

$$
\begin{aligned}
f_1(u) &= 2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u && (72)\\
f_2(u) &= \mathrm{cn}\,u(2\,\mathrm{dn}^2\,u-1) && (73)\\
f_3(u) &= \mathrm{dn}\,u(1-2\,\mathrm{sn}^2\,u) && (74)\\
u &= \theta\frac{2K}{\pi r}. && (75)
\end{aligned}
$$

which are all $2r\pi$ periodic. We seek a Fourier series expansion of the form

$$
f(u)=\sum_j d_j e^{iju/r}. \qquad (76)
$$

The coefficients are given by

$$
\begin{aligned}
d_j &= \frac{1}{2\pi r}\int_{-r\pi}^{r\pi} f(\theta\frac{2K}{\pi r})e^{-ij\theta/r}\,d\theta && (77)\\
&= \frac{1}{4K}\int_{-2K}^{2K} f(u)\exp(-iju\frac{\pi}{2K})\,du. && (78)
\end{aligned}
$$

Denote by Γ the contour $(-2K,0)-(2K,0)-(2K,-2K')-(-2K,-2K')-(-2K,0)$ where poles on the left boundary of the contour are excluded and the ones on the right are included. Using the periodicity properties of the Jacobi elliptic functions we find that the result of the integration along $\Im u=-2K'$ can be expressed as a multiple of the integration along $\Im u=0$:

$$
\begin{aligned}
\oint_\Gamma f(u)\exp(-ij\frac{\pi}{2K}u)\,du &= \int_{-2K}^{2K} f(u)\exp(-ij\frac{\pi}{2K}u)+ && (79)\\
&\quad+\int_{-2K-i2K'}^{-2K-i2K'} f(u)\exp(-ij\frac{\pi}{2K}u) && (80)\\
&= (1\pm\exp(-2j\xi))\int_{-2K}^{2K} f(u)\exp(-ij\frac{\pi}{2K}u). && (81)
\end{aligned}
$$

Solving the contour integral using residue calculus and using (78) gives

$$
d_j(1\pm\exp(-2j\xi)) = i\frac{\pi}{2K}\sum_{n=0}^{1}\operatorname*{Res}_{2nK-iK'} f(u)\exp(-ij\frac{\pi}{2K}u) \qquad (82)
$$

where the minus is taken for f_1 while the plus for f_2 and f_3. The residues are evaluated by using the periodicity properties of the Jacobi elliptic functions to move the pole to $z_0=0$.

$$
\begin{aligned}
f_1(z_0-iK'+2nK) &= 2\frac{\mathrm{dn}\,z_0\,\mathrm{cn}\,z_0}{k_m^2\,\mathrm{sn}^3\,z_0} =: f_1^*(z_0) && (83)\\
f_2(z_0-iK'+2nK) &= (-1)^n i\frac{\mathrm{dn}\,z_0(1+\mathrm{cn}^2\,z_0)}{k_m\,\mathrm{sn}^3\,z_0} =: (-1)^n f_2^*(z_0) && (84)\\
f_3(z_0-iK'+2nK) &= i\frac{\mathrm{cn}\,z_0(1+\mathrm{dn}^2\,z_0)}{k_m^2\,\mathrm{sn}^3\,z_0} =: f_3^*(z_0). && (85)
\end{aligned}
$$

OLD:

The coefficients are given by

$$
\begin{aligned}
c_l &= \frac{1}{2\pi r}\int_{-r\pi}^{r\pi} f(\theta\frac{2K}{\pi r})e^{-il\theta/r}\,d\theta && (86)\\
&= \frac{1}{4K}\int_{-2K}^{2K} f(u)\exp(-ilu\frac{\pi}{2K})\,du && (87)\\
&= \frac{i\pi}{2K}\sum_{n=0}^{1}\sum_{j=0}^{\infty}\operatorname*{Res}_{z_0=0} f(z_0-i(2j+1)K'+2nK)\\
&\qquad \exp(-il\frac{\pi}{2K}(z_0-i(2j+1)K'+2nK)) && (88)
\end{aligned}
$$

where the contour $(-2K,0)-(2K,0)-(2K,-\infty)-(-2K,-\infty)-(-2K,0)$ is used for the residue integration. The poles on the left boundary of the contour are excluded and the ones on the right are included. Using the periodicity properties of the Jacobi elliptic functions we find

$$
\begin{aligned}
f_1(z_0-i(2j+1)K'+2nK) &= 2\frac{\operatorname{dn} z_0 \operatorname{cn} z_0}{k^2\operatorname{sn}^3 z_0} =: f_1^*(z_0) && (89)\\
f_2(z_0-i(2j+1)K'+2nK) &= -1^{n+j} i\frac{\operatorname{dn} z_0(1+\operatorname{cn}^2 z_0)}{k\operatorname{sn}^3 z_0} =: -1^{n+j} f_2^*(z_0) && (90)\\
f_3(z_0-i(2j+1)K'+2nK) &= -1^{j} i\frac{\operatorname{cn} z_0(1+\operatorname{dn}^2 z_0)}{k^2\operatorname{sn}^3 z_0} =: -1^{j} f_3^*(z_0). && (91)
\end{aligned}
$$

If we take the alternating signs in these expressions into account we obtain

$$
\begin{aligned}
c_l &= \frac{\pi}{2K} i\sum_{j=0;n=0,1}^{\infty} e^{-l(2j+1)\xi}e^{-iln\pi}(\pm 1)\operatorname*{Res}_{z_0=0} f_m^*\exp(-il\frac{\pi}{2K}z_0) && (92)\\
c_l &= \frac{\pi}{2K} i\operatorname*{Res}_{z_0=0} f_m^*\exp(-il\frac{\pi}{2K}z_0)(1+(2\delta_{m,2}-1)e^{-il\pi})e^{-l\xi}\sum_{j=0}^{\infty}(\pm e)^{-j2l\xi} && (93)
\end{aligned}
$$

where the plus sign is needed for $m=1$ and the minus sign otherwise, and therefore the geometric series gives $2/\sinh l\xi$ respectively $2/\cosh l\xi$. The term in front of the sum picks out odd l's for $m=2$, and even l's otherwise.

:OLD

The residues have poles of third order and can be evaluated using the general formula

$$
\operatorname*{Res}_{z=z_0}\frac{g(z)}{h(z)^3} = \left.\frac{h'^2 g'' - 3g'h'h'' + 3gh''^2 - gh'h'''}{2h'^5}\right|_{z_0} \tag{94}
$$

obtained by the following little Mathematica [14] program which calculates the formula for the residue of an n-th order pole of the form $g(z)/h(z)^n$, $h'(z_0)\neq 0$:

```
cbmat[n_] := Table[ If[ j <= i,
                        If[ j==n,b[n-1],c[n + i - j]],
                        If[ j==n,b[i-1],0]
                      ], {i,n}, {j,n}];
ccoef[n_,i_] := Expand[ D[h[x]^n,{x,i}]/i! /. h[x] -> 0 ];
ResOrdN[n_] := Det[cbmat[[n]]/c[n]^n
```

```
                 /. Table[b[i] -> D[g[x],{x,i}]/i!,{i,0,n-1}]
                 /. Table[c[i] -> ccoef[n,i],{i,n,2 n - 1}];
ResOrdN[3]
```

All the residues give basically the same value

$$\operatorname*{Res}_{z_0=0} f_l^*(z_0)\exp(-ij\frac{\pi}{2K}z_0) = -j^2\left(\frac{\pi}{2k_m K}\right)^2 \begin{cases} 1 & l=1 \\ ik_m & l=2 \\ i & l=3 \end{cases} \tag{95}$$

The term -1^n in (84) picks out odd j's for f_2 while coefficients with even j's contribute to f_1 and f_2 only, so the final result is

$$d_j = \left(\frac{\pi}{2K}\right)^3 \frac{j^2}{k_m^2}\delta_{j-l \bmod 2,1} \begin{cases} -i/\sinh j\xi & l=1 \\ k_m/\cosh j\xi & l=2 \\ 1/\cosh j\xi & l=3 \end{cases} . \tag{96}$$

and

$$f_1(\theta) = \tag{97}$$

$$f_2(\theta) = \tag{98}$$

$$f_3(\theta) = \tag{99}$$

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Figure 1: The physical double pendulum with suspension points A_i, center of mass S_i, distance s_i from A_i to S_i and the distance a of the two suspension points.

Figure 2: Lines of constant ΔE in the planes of (k, ω_0) (lower left quarter), $(k, 1/\omega_0)$ (upper left quarter), $(1/k, \omega_\infty/2)$ (lower right), and $(1/k, 2/\omega_\infty)$ (upper right). The graph was obtained by a numerical maximum search from eqs. (53) and (66).

Figure 3: On the same parameter plane as in fig. 2, a set of 10×10 Poincaré surfaces of section $\phi_1 = 0$ is shown, in projection onto the (ϕ_2, L_2)-plane. Parameters are $\varepsilon = 0.02$, $\delta = 0.3$. Labelling the pictures by integers n_x and n_y, starting with 0 in the lower left corner, the values $x = (0.5 + n_x)/10$, $y = (0.5 + n_y)/10$ determine which parameter is used where the quadrants are interpreted like in fig. 2.

Figure 4: A series of Poincaré surfaces of section similar to those in fig. 3, but for the case of pure rotation of pendulum 1, $\gamma = 0$ or $k = \infty$, with $\varepsilon = 0.05$ and $\delta = 0.5$. Labeling the pictures by n, starting with 0 in the upper left corner increasing from left to right and from top to bottom, the parameter value is $\omega_\infty = \sqrt{\delta/(2+\delta)}(n+4)/9$, so that in the fifth column always the resonance condition is met. In essence this means to look at the rightmost column of fig. 3 in greater detail (and at higher value of ε). There is clear evidence of "breathing chaos", i. e. oscillatory increase and decrease in the extent of chaos in connection with resonances of the central fixed point.