Linear stability of natural symplectic maps

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Abstract

We calculate linear stability boundaries for natural symplectic maps, which are symplectic mappings derived from Lagrangian generating functions having positive definite kinetic energy. Simplified stability conditions are obtained in terms of the Hessian of the potential and applied to a four-dimensional pair of coupled standard maps.

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1 Introduction

Symplectic twist maps of the form [1, 2, 3]

$$T: \begin{cases} \mathbf{p}' &= \mathbf{p} - \nabla U \\ \mathbf{q}' &= \mathbf{q} + B^{-1}\mathbf{p}' \end{cases} \tag{1}$$

where typically $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ or $\mathbb{T}^n$, $U(\mathbf{q})$ is a smooth potential, and B is a positive definite constant matrix (the mass matrix), are frequently encountered in physical problems. For example, coupled standard maps [4, 5] are of this type. The mapping T is derivable from the Lagrangian generating function

$$F(\mathbf{q}, \mathbf{q}') = \frac{1}{2}(\mathbf{q}' - \mathbf{q})^T B(\mathbf{q}' - \mathbf{q}) - U(\mathbf{q}) \tag{2}$$

so that the kinetic part is positive definite in the "velocity," $\mathbf{q}' - \mathbf{q}$. If $U(\mathbf{q}) = U(-\mathbf{q})$ the mapping is reversible. In analogy with established nomenclature for Hamiltonian flows for "Natural Hamiltonian Systems" of the form [1]

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\mathbf{p}^T B^{-1}\mathbf{p} + U(\mathbf{q}) \tag{3}$$

where the matrix B is positive definite and U is a smooth potential, we term such mappings "Natural Symplectic Maps".

Stability of fixed points naturally plays a central role in understanding the dynamics of symplectic mappings, and considerable effort has been expended in developing useful stability criteria [1, 6]. The purpose of this paper is to derive simplified linear stability criteria for natural maps, entirely in terms of the invariants of the Hessian of U. We begin by recalling some basic notions of stability.

A fixed point $\mathbf{z_0} = (\mathbf{p_0}, \mathbf{q_0})$ of a mapping T is *Lyapunov stable* if for every neighborhood U of $\mathbf{z_0}$ there exists a subneighborhood $V \subset U$ such that $\mathbf{z} \in V \Rightarrow T^n(\mathbf{z}) \in U$ for all forward time n. It is *linearly stable* if all orbits of the tangent map $\mathcal{L} = DT(\mathbf{z_0})$ are bounded, and *spectrally stable* if all eigenvalues λ_i of $\mathcal{L}$ lie on the unit circle, S^1. A fixed point is linearly stable if it is spectrally stable and all Jordan blocks are trivial [14]. By the symplectic eigenvalue theorem [1, 7], the eigenvalues of a symplectic matrix occur in reciprocal pairs as well as complex conjugates.

In general a symplectic map can lose stability in just three ways:

Tangent Bifurcation. A pair of eigenvalues merge at $\lambda = 1$ and split off S^1 onto the positive real axis.

Period-Doubling Bifurcation. A pair of eigenvalues merge at $\lambda = -1$ and split off S^1 onto the negative real axis.

Krein Collision. Two pairs of eigenvalues merge at $\lambda, \bar{\lambda}$, $\lambda^2 \neq 1$, and split off S^1, forming a complex quadruplet. Whether a complex pair can actually leave S^1 depends on additional invariants, called the Krein signatures [8].

For a natural Hamiltonian *flow* the critical points of U give precisely the equilibria of H, for which we have [9, 10, 11]:

Theorem 0 *(Dirichlet) Let* $\mathbf{z_0} = (\mathbf{0}, \mathbf{q_0})$ *be an equilibrium of the natural Hamiltonian system (3), i.e.* $\nabla U(\mathbf{q}_0) = \mathbf{0}$. *Then* $\mathbf{z_0}$ *is Lyapunov stable if* $\mathbf{q}_0$ *is an isolated local minimum of* U.

For a natural Hamiltonian *map* the critical points of U give precisely the fixed points of T, for which we are going to prove the following

Theorem 1 *Let* $\mathbf{z_0} = (\mathbf{0}, \mathbf{q_0})$ *be a fixed point of the natural symplectic map (1), i.e.* $\nabla U(\mathbf{q_0}) = \mathbf{0}$. *Then* $\mathbf{z_0}$ *is linearly stable iff* $\mathbf{q_0}$ *is a quadratic local minimum of* U *and* $\mathbf{q_0}$ *is a quadratic local minimum of* $V = \frac{1}{2}\|\mathbf{q} - \mathbf{q_0}\|^2 - \frac{1}{4}U(\mathbf{q})$.

Note however that we can only prove linear stability, so that our treatment in fact parallels the theory of small oscillations for Hamiltonian flows, about which Dirichlet states [9] "... Theorie der kleinen Schwingungen, welche so viele interessante physikalische Anwendungen in sich begreift, und man muss sich in der That wundern, dass die Wahrheit desselben bisher nicht mit der nöthigen Strenge dargethan worden ist." [1] Having said this he proceeds to a proof demonstrating Lyapunov stability instead of just linear stability. Limiting ourselves to linear stability, however, has the advantage that we can obtain "if and only if" instead of just "if". We must therefore require a "quadratic local minimum", i.e. one with a positive definite Hessian, instead of just an isolated local minimum. The essential difference from Dirichlet's theorem is that stability can also be lost by making the neighborhood of the minimum very "steep" (or rigid in the language of the theory of small oscillations), thereby destroying the minimum in V.

The paper is organized as follows. In section 2 we employ the quadratic form $\xi^T J \mathcal{L} \xi$ to analyze linear stability. This "Krein form" is shown to lose definiteness only when $\lambda = \pm 1$, demonstrating that tangent and period-doubling bifurcations can occur, but not Krein collisions. Explicit conditions for stability boundaries are given in section 3

[1] free translation:... the theory of small oscillations, which encompasses so many interesting physical applications, and it is surprising that a sufficiently rigorous proof has not yet been carried out.

in terms of D^2U alone, including the Krein boundary (which cannot be crossed). The resulting stability boundaries are much simpler than those previously derived for a general symplectic map [6], and are worked out in detail for dimension 2, 4, and 6. Section 4 applies the results to the 4D Froeschlé map, for which a simple formula is found for the stability of all period-one fixed points. In addition an apparently previously unrecognized pitchfork bifurcation is discovered, giving birth to a pair of fixed points which can be stable for negative coupling.

2 Linear Stability and Quadratic Forms

In general, finding stability boundaries for an arbitrary symplectic map is complicated by the possibility of Krein collision of eigenvalues, requiring knowledge of the Krein signatures, as well as additional conditions to eliminate irrelevant merging of complex eigenvalues *off* the unit circle [6]. For natural maps, as for natural flows, the positivity of the kinetic energy precludes Krein collisions, making it possible to derive much simpler stability conditions.

For a Hamiltonian flow the preservation of energy is crucial to the proof of Dirichlet's theorem. For a $2n$-dimensional symplectic map there is no preserved energy. The closest analog is the preserved quadratic form in tangent space obtained from the symplectic two-form [1, 12, 13], which we dub the "Krein Form"

$$\Omega(\xi) = \xi^T J \xi' = \xi^T J \mathcal{L} \xi \tag{4}$$

where the tangent vector $\xi \in \mathbb{R}^{2n}$ and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{5}$$

with I the $n \times n$ unit matrix. The symplecticity of $\mathcal{L}$ guarantees that Ω is constant on orbits of $\mathcal{L}$. We call Ω the Krein form to emphasize that the signature of its restriction to an invariant subspace of $\mathcal{L}$ is exactly the Krein signature. (Note that (4) defines a quadratic form even though the matrix $J\mathcal{L}$ is not symmetric.) From (1) it follows that, in block form

$$\mathcal{L} = \begin{pmatrix} I & -D^2U \\ B^{-1} & I - B^{-1}D^2U \end{pmatrix}. \tag{6}$$

As in the theory of small oscillations the mass matrix B can be removed by a canonical transformation

$$M = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix} \tag{7}$$

where A is determined from $B = AA^T$, which is always possible for positive definite matrices. (In some cases it may be desirable to retain the mass matrix, for example to preserve the periodicity of U [15].) The tangent map now becomes simply

$$\hat{\mathcal{L}} = M^{-1}\mathcal{L}M = \begin{pmatrix} I & -\mathcal{H} \\ I & I - \mathcal{H} \end{pmatrix} \tag{8}$$

where

$$\mathcal{H} = A^{-1}D^2U(A^{-1})^T = \mathcal{H}^T \tag{9}$$

is the transformed Hessian of U. Again as in the theory of small oscillations a second symplectic transformation $M = \mathrm{diag}(R, R)$ with orthogonal R can be used to diagonalize $\mathcal{H}$. For calculating stability this step can be omitted because the stability criteria in Theorem 2 only involve the invariants of $\mathcal{H}$. But in the proofs we will assume a diagonal $\mathcal{H}$ whenever convenient.

In evaluating Ω it is convenient to take $\mathbf{z}_0 = (\mathbf{0}, \mathbf{0})$, so that $\xi = (\mathbf{p}, \mathbf{q})$. In the new coordinates, $\hat{\mathbf{p}} = A\mathbf{p}$, $\hat{\mathbf{q}} = (A^{-1})^T\mathbf{q}$, the quadratic form (4) becomes

$$\Omega(\xi) = \|\hat{\mathbf{p}}\|^2 - \hat{\mathbf{p}}^T\mathcal{H}\hat{\mathbf{q}} + \hat{\mathbf{q}}^T\mathcal{H}\hat{\mathbf{q}}. \tag{10}$$

From now on the transformed variables $\hat{\mathbf{z}} = \mathbf{Mz}$ will be always used, so that, e.g., $\mathcal{H} = D^2\hat{U}(\hat{\mathbf{q}})$. Completing the square and dropping the hats, we have

$$\Omega(\xi) = \|\mathbf{p} - \frac{1}{2}\mathcal{H}\mathbf{q}\|^2 + \mathbf{q}^T\mathcal{H}(I - \frac{1}{4}\mathcal{H})\mathbf{q}. \tag{11}$$

Thus, the definiteness of Ω hinges upon the positive definiteness of the $n \times n$ matrix

$$W = \mathcal{H}\,(I - \frac{1}{4}\mathcal{H}). \tag{12}$$

Recall that a symmetric matrix M is positive definite if the quadratic form $\mathbf{x}^T M \mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$.

Equation (12) suggests that linear stability might depend on $\mathcal{H}$ alone. The following Lemma provides the needed connection between the eigenvalues and eigenvectors of $\mathcal{H}$ and $\mathcal{L}$.

Lemma 1 *The eigenvalues ν_i of $\mathcal{H}$ are related to the eigenvalues λ_i of $\mathcal{L}$ via*

$$\nu_i = 2 - \rho_i = 2 - (\lambda_i + 1/\lambda_i) \tag{13}$$

where $\rho_i = \lambda_i + 1/\lambda_i$ is the stability index [6]. The eigenvectors of $\mathcal{L}$ are linearly independent iff $\lambda_i^2 \neq 1$.

Proof. In the basis $(p_1, q_1, ..., p_n, q_n)$ $\mathcal{L}$ takes the simple form

$$\mathcal{L} = \mathrm{diag}(B_1, B_2, ..., B_n) \tag{14}$$

where

$$B_i = \begin{pmatrix} 1 & -\nu_i \\ 1 & 1-\nu_i \end{pmatrix}. \tag{15}$$

The characteristic polynomial of B_i is then equivalent to (13), so that the eigenvectors corresponding to the $\lambda_i^\pm$ are $(\nu_i, 1-\lambda_i^\pm)$ or (0,1) for $\lambda_i = 1$. If all $\lambda_i^2 \neq 1$ each B_i is diagonalizable and the eigenvectors of $\mathcal{L}$ are linearly independent. If some $\lambda_i^2 = 1$ the corresponding B_i is not diagonalizable. For $\lambda_i = +1, B_i$ is a nontrivial Jordan block, while for $\lambda_i = -1, B_i$ is similar to a nontrivial Jordan block [14]. QED.

This takes us to our principal result:

Theorem 1 *Let T be a natural symplectic map with potential $U(\mathbf{q})$. Then a fixed point $\mathbf{z_0} = (\mathbf{0}, \mathbf{q_0})$ is linearly stable iff the Hessian $\mathcal{H}$ of U at $\mathbf{q_0}$ and the matrix $I - \frac{1}{4}\mathcal{H}$ are both positive definite.*

Proof. Assuming that $\mathcal{H}$ has been transformed to diagonal form, we immediately find

$$\omega_i = \nu_i(1 - \frac{1}{4}\nu_i), \tag{16}$$

where ω_i and ν_i are corresponding real eigenvalues of W and $\mathcal{H}$, respectively. Thus W and therefore Ω is positive definite iff $\mathcal{H}$ and $I - \frac{1}{4}\mathcal{H}$ are both positive definite. The orbits $\xi_i = L^i \xi_0$ of the tangent map are bounded because Ω is a conserved quantity for the tangent map, $\Omega(\xi_i) = \Omega(\xi_0)$, and if Ω is positive definite it can be bounded e.g. by the sphere containing the ellipsoid $\Omega(\xi) = \Omega(\xi_0)$. Conversely, suppose that $\mathbf{z_0}$ is linearly stable, i.e. $\mathcal{L}$ has a basis of $2n$ eigenvectors and all eigenvalues lie on S^1 [14]. By Lemma 1, this is equivalent to $0 < \nu_i < 4$. It follows from (16) that all the ω_i are positive and W is positive definite. Q.E.D.

Note that determining the definiteness of the matrix $I - \frac{1}{4}\mathcal{H}$ is tantamount to determining the types of the critical points of the "complementary potential"

$$V(\mathbf{q}) = \frac{1}{2}||\mathbf{q} - \mathbf{q_0}||^2 - \frac{1}{4}U(\mathbf{q}) \tag{17}$$

so that $W = D^2U\, D^2V$. Thus, we may paraphrase Theorem 1 as follows:

Theorem 1 *Let T be a natural symplectic map with potential U. Then a fixed point $\mathbf{z}_0 = (\mathbf{0},\ \mathbf{q_0})$ is linearly stable iff $\mathbf{q_0}$ is simultaneously a quadratic local minimum of U and the complementary potential V.*

Now suppose that U depends smoothly on parameters μ. Then the implicit function theorem guarantees the existence of a unique critical point $\mathbf{q}_0 = \mathbf{q}_0(\mu)$ provided that $\det \mathcal{H} \neq 0$. For natural flows this coincides with the condition for no tangent bifurcation. The essential difference in the case of natural symplectic maps is that stability might be lost without the Hessian becoming singular, but instead by $I - \frac{1}{4}\mathcal{H}$ having an eigenvalue pass through zero. The possible bifurcations are the content of

Corollary 1 *Let T be a natural symplectic map with potential U with a fixed point $(\mathbf{0}, \mathbf{q_0})$. The Hessian $\mathcal{H}$ of U losing positive definiteness corresponds to a tangent bifurcation and*

$I - \frac{1}{4}\mathcal{H}$ losing positive definiteness corresponds to a period doubling bifurcation of the fixed point. Krein collisions are not possible.

Proof. From Lemma 1, $\nu = 2 - \rho$ and $\rho = \lambda + 1/\lambda$. Therefore $\nu = 0$, ($\mathcal{H}$ losing positive definiteness) corresponds to $\lambda = 1$, and $\nu = 4$ ($I - \frac{1}{4}\mathcal{H}$ losing positive definiteness) corresponds to $\lambda = -1$. Since $\mathcal{H}$ is symmetric, its eigenvalues are all real, which implies that the stability index ρ is real, which in turn means that either $\lambda \in \mathbf{R}$ or $\lambda \in S^1$. Equivalently, the positivity of Ω implies that it is positive on any subspace, so that all Krein signatures are equal. It follows that complex bifurcations (Krein collisions) cannot occur. Q.E.D.

Nevertheless the Krein boundary plays an important role in delineating the stable region for natural maps. In general this boundary is readily calculated from the vanishing of the discriminant of $P_{2n}(\lambda) = \det(\mathcal{L} - \lambda I)$. As we shall see in Theorem 2, this is equivalent to the vanishing of the discriminant of the characteristic polynomial $Q_n(\nu)$ of the Hessian of U. On the Krein boundary multiple eigenvalues of $\mathcal{L}$ occur, so that there is a danger of $\mathcal{L}$ not being semisimple. However, Theorem 1 shows that $\mathcal{L}$ is linearly stable, even on the Krein boundary (excluding points where $\lambda^2 = 1$). Thus, linear stability prevails even in the case of multiple eigenvalues λ so long as $\lambda^2 \neq 1$. Besides the case of semisimple $\mathcal{L}$, in spite of multiple eigenvalues we also find the opposite situation, in which multiple eigenvalues lead to nontrivial Jordan blocks of $\mathcal{L}$. In the latter case one has spectral, but not linear stability, as shown by Lemma 1. For natural maps it corresponds exactly to the case of a degenerate minimum of U or V. By Lemma 1, for $\nu = 0$ or 4 there is an eigenvalue pair $\lambda = +1$ or -1, respectively. Since all eigenvalues are on S^1, we have spectral stability, but linear *instability* because of the nontrivial Jordan blocks.

It is of course possible for a (non-natural) symplectic map to be linearly stable in spite of the indefiniteness of Ω. For example, consider the 4D map analogous to a pair of counter-rotating oscillators:

$$M = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{18}$$

where

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_1 & 0 \\ 0 & -\beta_2 \end{pmatrix} \tag{19}$$

and $\alpha_1, \alpha_2, \beta_1$, and β_2 are real numbers with $\beta_i > 0$ and $\alpha_i^2 + \beta_i^2 = 1$. It follows that

$$\Omega = \beta_1(p_1^2 + q_1^2) - \beta_2(p_2^2 + q_2^2). \tag{20}$$

The eigenvalues of M are $\alpha_i \pm i\beta_i$, so the motion is linearly stable.

Since Lemma 1 connects the eigenvalues of the Hessian $\mathcal{H}$ with the eigenvalues of $\mathcal{L}$ it is possible to obtain the type (a mixture of elliptic $\mathcal{E}$, hyperbolic $\mathcal{H}$, and inverse hyperbolic $\mathcal{I}$) of a fixed point from the index [7] of the critical points of U and V. By the index $\mathrm{ind}(\mathrm{U}, \mathbf{q}_0)$ we mean the number of negative eigenvalues of the Hessian of U at the critical point $\mathbf{q}_0$, similarly for V. With this notation we have the important

Corollary 2 *Let* $(\mathbf{0}, \mathbf{q_0})$ *be a fixed point of a natural symplectic map* T. *The linearized map* $\mathcal{L}$ *has* $\mathrm{ind}(\mathrm{U}, \mathbf{q}_0)$ *hyperbolic pairs of eigenvalues and* $\mathrm{ind}(\mathrm{V}, \mathbf{q}_0)$ *inverse hyperbolic pairs of eigenvalues. If* $\mathbf{q}_0$ *is a nondegenerate critical point of* U *and* V *then the remaining eigenvalues are elliptic.*

Proof. If $\nu_i < 0$, then $\rho_i = 2 - \nu_i > 2$, and since $\rho = \lambda + 1/\lambda$ this implies $\lambda > 0$. Thus $\mathrm{ind}(\mathrm{U}, \mathbf{q}_0)$ gives the number of pairs of positive eigenvalues. The eigenvalues η of $D^2V = I - \frac{1}{4}\mathcal{H}$ are related to the those of $\mathcal{H}$ by $\eta = 1 - \nu/4$ since $\mathcal{H}$ can be assumed diagonal. Thus, if $\eta_i < 0$, then $\rho_i = 2 - 4(1 - \eta_i) < -2$, and this implies $\lambda_i^{\pm} < 0$. Hence $\mathrm{ind}(\mathrm{V}, \mathbf{q}_0)$ gives the number of pairs of negative eigenvalues. Since there are no complex quadruplets the remaining eigenvalues are elliptic, because nondegeneracy excludes $\lambda^2 = 1$. Q.E.D.

3 Stability Criteria

While Theorem 1 yields complete necessary and sufficient conditions for the stability of natural maps in arbitrary dimension, they can be cumbersome to implement for dimension greater than two. For example one could utilize the well-known fact that a matrix is positive definite iff its principal minors are all positive [16]. Fortunately, the results of Lemma 1 suggest an easier way. In fact the stability properties of T are encapsulated in the characteristic polynomial

$$Q_n(\nu) = \nu^n - A_1\nu^{n-1} + A_2\nu^{n-2} + \cdots + (-)^n A_n \tag{21}$$

of D^2U, which is half the degree of $P_{2n}(\lambda)$. Similarly we denote the characteristic polynomial of D^2V by $\tilde{Q}_n(\eta)$. Analysis of Q_n yields a set of stability conditions similar to, but much easier to calculate, than those derived in [6], which we shall refer to as Ref I. The treatment here roughly follows that of Ref I, to which the interested reader is referred for proofs and other details. From Corollary 1 it is obvious that the the symplectic matrix $\mathcal{L}$ is linearly stable iff all the eigenvalues ν lie in $(0,4)$. Thus, it can be shown that the stable region is simply connected.

Now let Π_n be the set of polynomials Q_n satisfying

i. $(-1)^n Q_n(0) = \det D^2U > 0$ ($= 0$ means Tangent Bifurcation)

ii. $(-4)^n \tilde{Q}_n(0) = Q_n(4) = 4^n \det D^2V > 0$ ($= 0$ means Period-Doubling)

By Theorem 1 linear stability of $\mathcal{L}$ implies positivity of W, hence $\det D^2U$ and $\det D^2V$ are both positive, so that $Q_n \in \Pi_n$. Conditions (i) and (ii) define hyperplanes in the space of coefficients of Q_n, which, together with the discriminantal surface of Q_n, form the boundary of the stable region. The latter is part of the linearly stable region, while the former are excluded. Even for $n = 2$ (see Fig. 1) these conditions are not sufficient and have to be supplemented by the absolute bounds on the A_k. The absolute bounds on the polynomial coefficients are obtained by setting all ν_i either to zero or to 4, yielding

$0 \le A_k \le 4^k \begin{pmatrix} n \\ k \end{pmatrix}$. However, for $n \ge 3$ the conditions to be in Π_n, even together with the absolute bounds are not sufficient. For example, in 6D there is an additional "wedge region" which would not be excluded.

A set of sufficient conditions could be obtained by checking that all principal minors of the two factors of W are positive. Alternatively, one could use Sturm's theorem to derive a set of necessary and sufficient conditions that the zeros of Q_n lie in the interval $(0, 4)$. But this method is equally unwieldy for $n > 3$. Once more there is an easier way. We have seen that a fixed point is stable iff the matrices $\mathcal{H}$ and $I - \frac{1}{4}\mathcal{H}$ are both positive definite. Since complex eigenvalues are impossible, this is true only if all eigenvalues of D^2U and D^2V are positive. Now write

$$\tilde{Q}_n(\eta) = \eta^n - \tilde{A}_1\eta^{n-1} + \cdots + (-)^n \tilde{A}_n, \tag{22}$$

and by direct computation we find, using $\eta = 1 - \nu/4$ in (21), that

$$\tilde{A}_q = \sum_{p=0}^{q} \begin{pmatrix} n-p \\ n-q \end{pmatrix} \frac{A_p}{(-4)^p} \tag{23}$$

where by definition $A_0 = 1$. This enables us to formulate simple stability conditions for natural symplectic maps in

Theorem 2 *A fixed point of T is linearly stable iff all coefficients $A_k, \tilde{A}_k$ of $Q_n(\nu)$ and $\tilde{Q}_n(\eta)$ are positive.*

Proof. Recalling that the A_k and $\tilde{A}_k$ are just the symmetric functions of the ν_i and η_i, direct application of Descartes' rule of signs shows that the ν_i are all positive iff the A_k are all positive; the η_i are all positive iff the $\tilde{A}_k$ are all positive. Descartes' rule gives the exact number of positive roots because we know that there are no complex roots because D^2U and D^2V are symmetric. But the positivity of ν_i and η_i together guarantee that $0 < \nu_i < 4$, so that Ω is positive definite and $\mathcal{L}$ is linearly stable. Q.E.D.

This yields a set of $2n$ inequalities which are much simpler to calculate and evaluate than the equivalent number found from Sturm's theorem. Moreover the conditions are *linear* in the A_k and the highly nonlinear discriminant of Q need not be calculated. Let us now work out the explicit stability boundaries for natural maps of dimension two, four, and six.

Two-Dimensional Maps (n = 1)

Here $\mathcal{H} = U_{qq}$ and (11) becomes

$$\Omega(\xi) = (p - \frac{1}{2}U_{qq}q)^2 + U_{qq}(1 - \frac{1}{4}U_{qq})q^2 \tag{24}$$

which is clearly definite only in the range $0 < U_{qq} < 4$. For the standard map $U(q) = -K\cos q$ and we recover the familiar result that the central fixed point at $q_0 = p_0 = 0$ is stable for $0 < K < 4$.

Four-Dimensional Maps (n=2)

In this case the fixed point is stable if the 2×2 matrices $\mathcal{H}$ and $I - \frac{1}{4}\mathcal{H}$ are positive definite, which requires that the determinant and trace of each be positive, that is $\det\mathcal{H} > 0$, $\det(I - \frac{1}{4}\mathcal{H}) > 0$, and $0 < \mathrm{tr}\,\mathcal{H} < 8$. Setting $\tau = \mathrm{tr}\,\mathcal{H}$ and $\Delta = \det\mathcal{H}$, these conditions reduce to

$$0 < \tau < 8, \qquad \Delta > 4(\tau - 4), \qquad \Delta > 0, \tag{25}$$

which are equivalent to $A_i > 0$ and $\tilde{A}_i > 0$. Fig. 1 shows the stable region in the $\tau - \Delta$ plane, which is to be compared with Fig. 2 of Ref. I.

Alternatively, these conditions may be obtained by setting $Q_2(0) > 0$, $Q_2(4) > 0$ in the reduced characteristic polynomial

$$Q_2(\nu) = \nu^2 - \tau\nu + \Delta \tag{26}$$

together with the absolute bounds. The boundary for Krein collisions is given by $\mathrm{disc}(Q_2) = 0$, i.e. $\Delta = \frac{1}{4}\tau^2$. Since $\tau^2 - 4\Delta = (U_{xx} - U_{yy})^2 + U_{xy}^2 \geq 0$, we confirm that a stable fixed point cannot cross the Krein boundary. The next section applies these results to the 4D Froeschlé map.

Six-Dimensional Maps (n=3)

For dimension greater than four it is advisable to directly use the conditions on A_k and $\tilde{A}_k$. Denote the characteristic polynomial of $\mathcal{H}$ by

$$Q_3(\nu) = \nu^3 - \tau\nu^2 + \sigma\nu - \Delta \tag{27}$$

where $2\sigma = (\mathrm{tr}\,\mathcal{H})^2 - \mathrm{tr}(\mathcal{H}^2)$. Working out the $\tilde{A}_k$ and noting that $A_k = \tau,\ \sigma,\ \Delta > 0$, we find

$$\begin{aligned}
\tilde{A}_1 &= 3 - \frac{1}{4}\tau > 0 \quad &&\Rightarrow \quad 0 < \tau < 12 \\
\tilde{A}_2 &= 3 - \frac{1}{2}\tau + \frac{1}{16}\sigma > 0 \quad &&\Rightarrow \quad 0 < \sigma > 8(\tau - 6) \\
\tilde{A}_3 &= 1 - \frac{1}{4}\tau + \frac{1}{16}\sigma - \frac{1}{64}\Delta > 0 \quad &&\Rightarrow \quad 0 < \Delta < 4(\sigma - 4\tau + 16).
\end{aligned} \tag{28}$$

Figure 2 depicts the stable region in the space of invariants τ, σ, and Δ, bounded by the tangent bifurcation plane Σ_{TB} ($A_3 = \Delta = 0$), the period-doubling plane Σ_{PD} ($\tilde{A}_3 = 0$), and the two-sheeted, quartic discriminantal surface Σ_0, given by

$$(9\Delta - \tau\sigma)^2 = 4(\tau^2 - 3\sigma)(\sigma^2 - 3\tau\Delta). \tag{29}$$

Again note that although (29) is useful for delimiting the stable region, it is not really needed to determine stability. The effect of the plane $\sigma = 8(\tau - 6)$ is to eliminate the unstable wedge region between Σ_0 and Σ_{PD}, which intersect tangentially along the line $(\tau, \sigma, \Delta) = (\tau, 8(\tau - 6), 16(\tau - 8))$. It is easily seen that $\nu_1 = \nu_2 = \nu_3$ along the cusped ridge joining the two sheets of Σ_0.

In the general case of a non-natural map a full set of necessary and sufficient stability conditions can be found by applying Sturm's theorem to the characteristic equation. Unfortunately, the 6D stability conditions given in Ref. I are incomplete, although Fig.

3 of that paper is correct. The missing stability condition is $|B+1| > A + C/2$, where A, B, and C are coefficients of the characteristic equation.

Finally we briefly remark on the question of codimension and unfolding about degenerate fixed points. As explained in Ref. I, for a general symplectic map there may be "avoided collisions." For natural maps, while Krein collisions are forbidden, double eigenvalues can nevertheless occur on S^1 away from ± 1 as well as at ± 1. But for natural maps both configurations are codimension one and therefore generic. This is in contrast to the general case where a collision of eigenvalues having the same Krein signature is codimension 3. The same configuration is codimension 1 for natural maps because the signatures are automatically definite.

4 Application to the Froeschlé Map

Various forms of coupled standard maps have been devised to study the transition to global chaos in higher dimensions [4, 17, 5, 18]. We shall adopt the version

$$T: \begin{cases} p_x' &= p_x - K_1 \sin x - h \sin(x+y) \\ p_y' &= p_y - K_2 \sin y - h \sin(x+y) \\ x' &= x + p_x' \\ y' &= y + p_y' \end{cases} \tag{30}$$

which has the underlying averaged potential

$$U(x,y) = -K_1 \cos x - K_2 \cos y - h \cos(x+y), \tag{31}$$

where K_1 and K_2 are real amplitudes and h is the (positive or negative) coupling constant. Since U is even, T is reversible. Without loss of generality we may take $K_1, K_2 > 0$, since shifting x, y by π changes the signs of K_1, K_2; by exchanging x and y we can always achieve $K_1 \geq K_2$. Moreover, shifting both x and y by π and changing h to $-h$ transforms U to $-U$. Note however, that this only applies to U, but not to the complementary potential V, so that the stability of T behaves in a more complicated way upon changing the sign of h.

The fixed points of T are $\mathbf{p}_0 = 0$ and the critical points $\mathbf{q}_0 = (x_0, y_0)$ of U, i.e. the simultaneous solutions of

$$\begin{aligned} K_1 \sin x_0 + h \sin(x_0 + y_0) &= 0 \\ K_2 \sin y_0 + h \sin(x_0 + y_0) &= 0. \end{aligned} \tag{32}$$

There are two families of period-one fixed points, the four primary fixed points at $\{x_0, y_0\} = \{0, \pi\}$, and pairs of secondary fixed points which bifurcate from these at certain values of the coupling constant h. Let us now trace the metamorphosis of the level sets of U, which yields the tangent bifurcations of the fixed points of T. The reader may find it helpful to consult the suite of contour plots in Fig. 3, generated for the parameters $K_1 = 0.5,\ K_2 = 0.3$. The type of each critical point is given by the the signs of

$$\begin{aligned} \Delta = \det D^2 U &= K_1 K_2 \cos x_0 \cos y_0 + h \cos(x_0 + y_0)(K_1 \cos x_0 + K_2 \cos y_0) \\ \tau = \mathrm{tr}\ D^2 U &= K_1 \cos x_0 + K_2 \cos y_0 + 2h \cos(x_0 + y_0). \end{aligned} \tag{33}$$

For $\Delta < 0$, U has a saddle; for $\Delta > 0$ there is a minimum if $\tau > 0$ and a maximum if $\tau < 0$.

Primary Fixed Points

As argued above, we can restrict attention to positive h and positive $K_1 \geq K_2$ for the discussion of the critical points of U. Introducing $s_1 = \cos(x_0)$ and $s_2 = \cos(y_0)$, which take only the values ± 1 on the primary fixed points, we find

$$\begin{aligned} \Delta &= h(s_2 K_1 + s_1 K_2) + s_1 s_2 K_1 K_2 \\ \tau &= 2 s_1 s_2 h + s_1 K_1 + s_2 K_2. \end{aligned} \tag{34}$$

$\mathbf{q}_0 = (0, 0)$. Since $\Delta > 0$ and $\tau > 0$ the origin always is a minimum of U. The stability of the corresponding fixed point depends on V.

$\mathbf{q}_0 = (0, \pi)$. Since $\Delta < 0$, $\mathbf{q}_0$ is a saddle. The corresponding fixed point is always unstable.
$\mathbf{q}_0 = (\pi, 0)$. Here $\Delta = -K_1K_2 + h(K_1 - K_2)$ and $\tau = -(K_1 - K_2) - 2h < 0$, so that $\mathbf{q}_0$ changes from a saddle to a maximum upon increasing h through the critical value $h_- = K_1K_2/(K_1 - K_2)$. The secondary fixed points are destroyed in this tangent bifurcation. This primary fixed point is always unstable.
$\mathbf{q}_0 = (\pi, \pi)$. Here $\Delta = K_1K_2 - h(K_1 + K_2)$ and $\tau = -K_1 - K_2 + 2h$, so that $\mathbf{q}_0$ changes from a maximum to a saddle upon increasing h through the critical value $h_+ = K_1K_2/(K_1 + K_2)$. At $h = h_+$ we find $\tau = -(K_1^2 + K_2^2)/(K_1 + K_2) < 0$ always. The secondary fixed points are created in this tangent bifurcation. This primary fixed point is always unstable.

The stability of the primary fixed points is mostly determined by U, with the exception of the origin, where it depends on V having a minimum, which is the case if $\Delta > 4(\tau - 4)$ and $\tau < 8$, yielding

$$(K_1 + h - 4)(K_2 + h - 4) > h^2 \quad \text{and} \quad 2h + K_1 + K_2 < 8. \tag{35}$$

Thus, the origin is stable with zero coupling only if it is stable in both decoupled maps ($K_i < 4$). Stability can be lost in a period-doubling bifurcation upon increasing h through the critical value

$$\frac{1}{h_p} = \frac{1}{4 - K_1} + \frac{1}{4 - K_2}. \tag{36}$$

When $K_1 = K_2 = K < 4$, we find stability for $0 < h < (4 - K)/2$. At h_p a transition from $\mathcal{EE}$ to $\mathcal{EI}$ takes place. If both decoupled maps are unstable ($K_i > 4$), the origin is $\mathcal{II}$ for small h and changes to $\mathcal{EI}$ at $h = h_p$. If one of the decoupled maps has a stable origin, and the other one an unstable one, the origin of the coupled map is $\mathcal{EI}$ for small h and can turn into $\mathcal{II}$ at h_p, provided that $\tau > 8$.

Secondary Fixed Points

Next we consider the stability of the bifurcated pair of fixed points which migrate from (π,π), to $(\pi,0)$ as h increases from h_+ to h_-. From (32) it follows that

$$\cos x_0 = \frac{1}{2}\left(\frac{hK_2}{K_1^2} - \frac{K_2}{h} - \frac{h}{K_2}\right), \quad \cos y_0 = \frac{1}{2}\left(\frac{hK_1}{K_2^2} - \frac{K_1}{h} - \frac{h}{K_1}\right) \tag{37}$$

and $\sin y_0 = (K_1/K_2)\sin x_0$. The second primary critical point of U is given by $\mathbf{q}_0 = (-x_0, -y_0)$. It can be shown that for both points

$$\begin{aligned} \Delta &= -\frac{1}{4K_1^2K_2^2h^2}\prod_{s_i=\pm}\left(h(s_2K_1+s_1K_2)+s_1s_2K_1K_2\right) \\ \tau &= -h\left(\frac{1}{K_1}+\frac{1}{K_2}\right), \end{aligned} \tag{38}$$

so that, for $h>0$ the secondary fixed points are unstable because $\tau<0$. The product runs over all four sign combinations. In summary, for $K_1>K_2>0$ the Froeschlé potential undergoes the following metamorphosis as h increases from zero, as shown in Fig. 3. When h increases through h_+ a pitchfork bifurcation (in U) takes place, the maximum at (π,π) becoming a saddle and giving birth to two new maxima. When $h=K_2$ and K_1 two global bifurcations occur (reconnections) in which the critical points of U do not change type. When h increases through h_- an inverse pitchfork bifurcation takes place at $(\pi,0)$. For positive h this scenario does not greatly affect the dynamics of the mapping itself, as the bifurcated maximum points of U correspond to *unstable* fixed points of T. Of the four principal fixed points, only that at the origin is ever stable, when eq. (35) is satisfied.

However, if we allow for negative h, apart from the shifting of x and y the type of the critical points changes from minimum to maximum, while a saddle stays a saddle. Thus, the (shifted) origin is now a maximum, therefore unstable. Most important, the primary points giving rise to the secondary points are now minima before they bifurcate, therefore potentially stable, similarly for the secondary points. In order to clarify the whole scenario, in Fig. 4 we draw the curves $(\tau(h),\Delta(h))$ for all the fixed points. For the primary fixed points we just find the straight lines

$$2\Delta = (s_1K_1 + s_2K_2)\tau - K_1^2 - K_2^2. \tag{39}$$

In Fig. 4 the position of zero coupling is indicated by a dot on the line. The secondary fixed points are described by

$$4\Delta = -\left(\frac{K_1^2 - K_2^2}{K_1^2 + K_2^2}\right)^2 \tau^2 + 2(K_1^2 + K_2^2) - (K_1^2 + K_2^2)^2/\tau^2, \tag{40}$$

and the relevant part of this curve connects the intersections of the lines (39) with the line $\Delta = 0$ at $\tau_{s_1s_2} = (K_1^2 + K_2^2)/(s_1K_1 + s_2K_2)$. For positive h these intersections are in the region with $\Delta < 0$, so that the secondary fixed points are of type $\mathcal{HH}$. However, for negative h the bifurcation scenario depends on the location of $\tau_{++} < \tau_{+-}$ relative to $\tau = 4$. If both are smaller than 4, the secondary fixed point is always stable. If $\tau_{++} < 4 < \tau_{+-}$, the secondary fixed point loses stability in a period-doubling bifurcation, as in Fig. 4. If both are larger than 4, the secondary fixed point is always unstable of type $\mathcal{EI}$. Note that in the latter case, for which $\tau_{++} > 4$ is sufficient, the map does not have stable fixed points for *any* value of h.

5 Discussion

We have presented a simplified method for obtaining linear stability criteria for natural symplectic maps of arbitrary dimension. We began by showing that linear stability is equivalent to the positivity of the Krein form, Ω. This led us to examine the spectrum of the Hessian of U and its complementary potential $V = \frac{1}{2}||\mathbf{q} - \mathbf{q_0}||^2 - \frac{1}{4}U$. Requiring that the eigenvalues of D^2U and D^2V be positive then yielded simple conditions linear in the invariants of D^2U, in contrast to the general case of a symplectic map with indefinite kinetic energy, where complicated nonlinear inequalities must be evaluated. In addition, a proof of the impossibility of Krein collisions in natural maps was obtained. Finally, stability boundaries were found for maps of dimension 2, 4, and 6 and applied to the 4D

Froeschlé map. With our new method we were able to calculate the complete bifurcation scenario, including a previously unrecognized pitchfork bifurcation.

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Figure Captions

Fig. 1. Stability diagram for 4D natural maps in the $\tau - \Delta$ plane. The stable region is the triangular area bounded by the τ axis, the line $\Delta = 4(\tau - 4)$, and the Krein boundary $\Delta = \frac{1}{4}\tau^2$, which cannot be crossed. The wedge regions to the right and left of the stable region are eliminated by the absolute bounds $0 \leq \tau \leq 8$.

Fig. 2. Stability diagram for 6D natural maps in the space of invariants τ, σ, and Δ. The stable region is bounded by the tangent bifurcation plane, Σ_{TB}, the period-doubling plane, Σ_{PD}, and the two-sheeted quartic discriminantal surface Σ_0, which is tangent to Σ_{PD} and transverse to Σ_{TB}. Only the portions of the surfaces actually bounding the stable region are shown.

Fig. 3. Level sets of Froeschlé potential (31) for $K_1 = 0.5$, $K_2 = 0.3$ and (a) $h = 0.2$, slightly above $h_+ = 0.1875$, so that the fixed point at (π, π) has undergone a pitchfork bifurcation, (b) $h = 0.3$, first reconnection, (c) $h = 0.5$, second reconnection, and (d) $h = 0.6$. The secondary fixed points are now approaching $(\pi, 0)$, where they will coalesce and disappear when $h = h_- = 0.75$.

Fig. 4. Stability of the fixed points of the Froeschlé map for $K_1 = 1.3$, $K_2 = 0.8$ in the plane $\tau - \Delta$. The dots indicate the position for $h = 0$. The straight lines correspond to the primary fixed point, the curves connecting their intersection with the τ-axis correspond to the secondary fixed points. The stability boundaries of Fig. 1 are also shown.

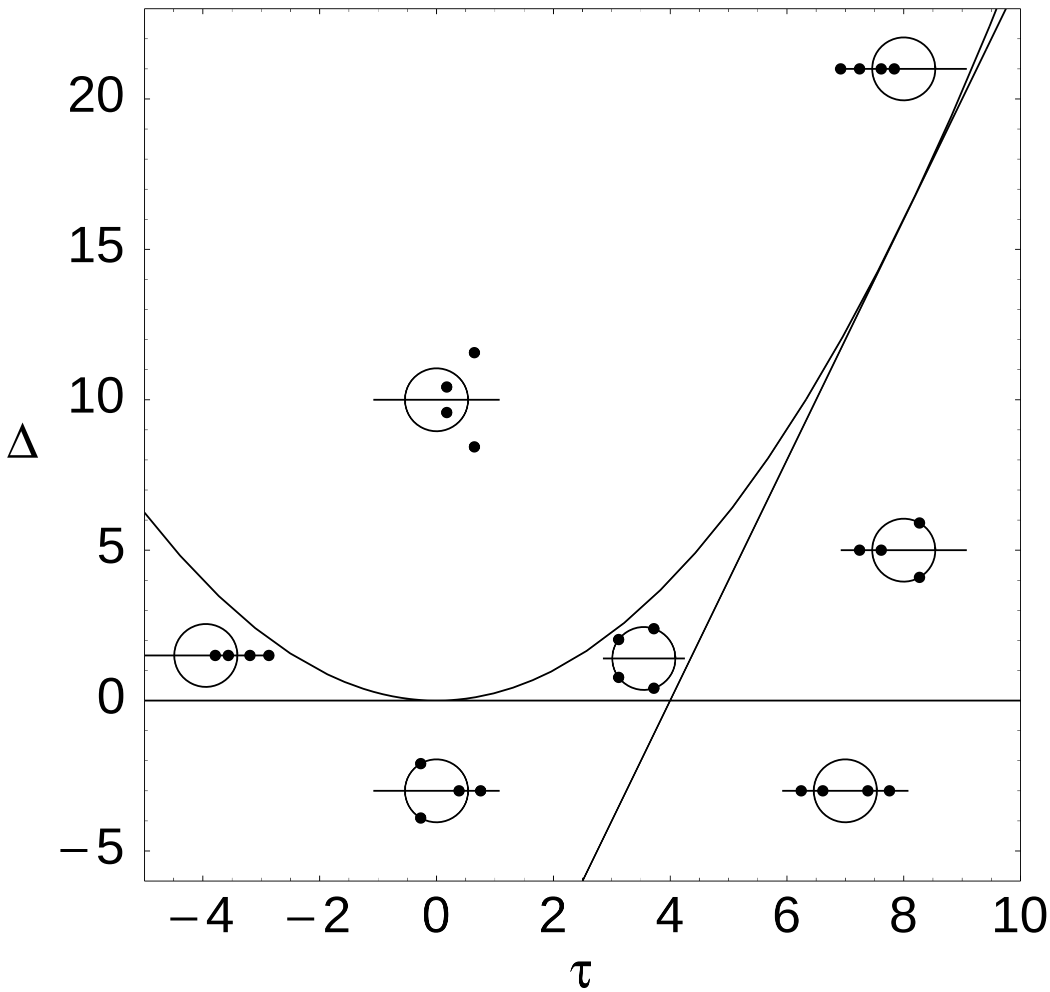

Figure 1: Howard/Dullin

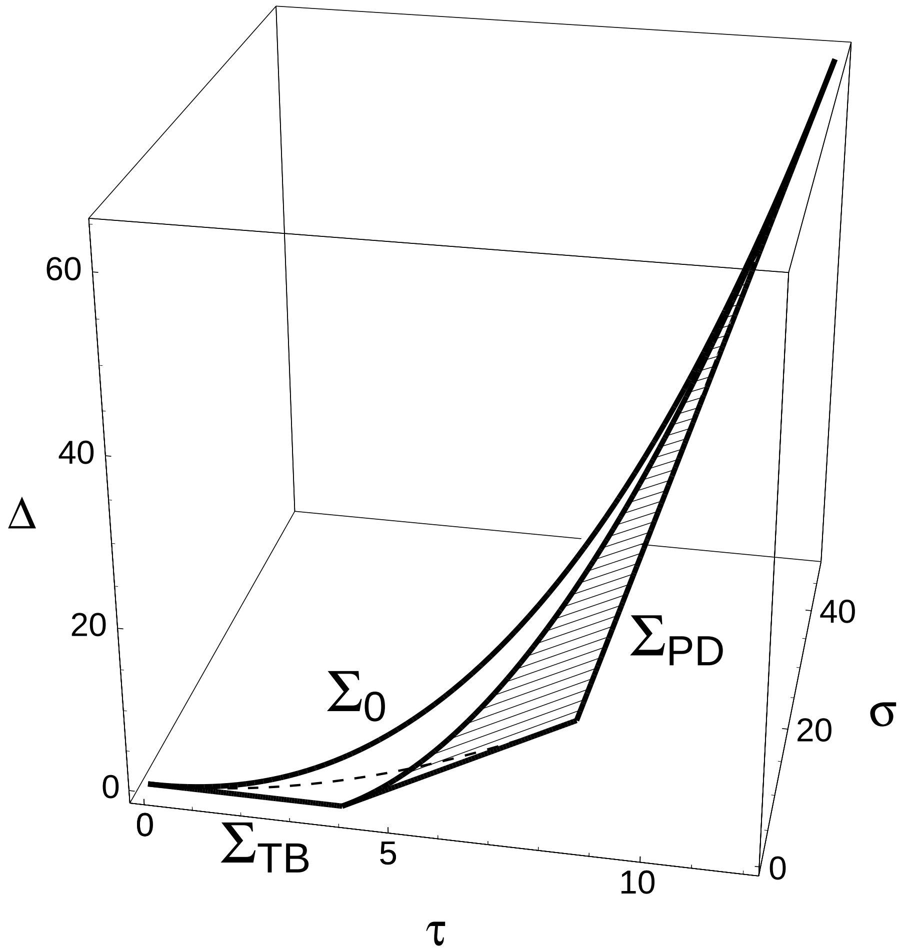

Figure 2: Howard/Dullin

Figure 3: Howard/Dullin

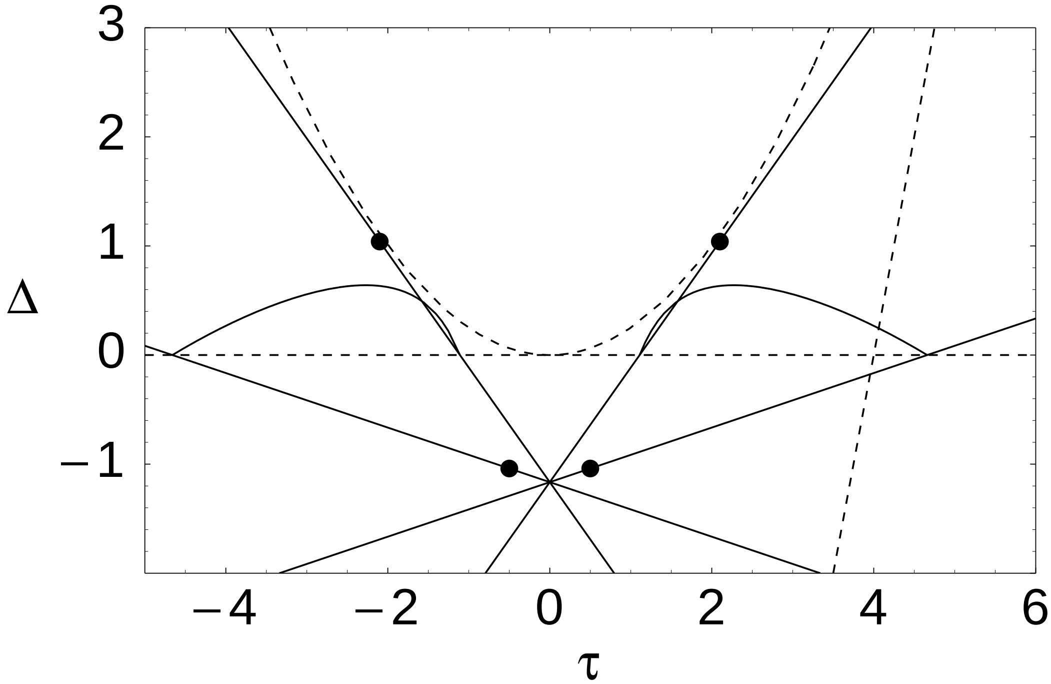

Figure 4: Howard/Dullin