Action Integrals and Energy Surfaces of the Kovalevskaya Top

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Abstract

The different types of energy surfaces are identified for the Kovalevskaya problem of rigid body dynamics, on the basis of a bifurcation analysis of Poincaré surfaces of section. The organization of their foliation by invariant tori is qualitatively described in terms of Poincaré-Fomenko stacks. The individual tori are then analysed for sets of independent closed paths, using a new algorithm based on Arnold's proof of the Liouville theorem. Once these paths are found, the action integrals can be calculated. Energy surfaces are constructed in the space of action variables, for six characteristic values of energy. The data are presented in terms of color graphs that give an intuitive access to this highly complex integrable system.

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1. Introduction

Among the integrable systems of classical mechanics, the Kovalevskaya case of rigid body dynamics has proved to be one of the most fascinating in its combination of mathematical beauty and physical complexity. Generations of mathematicians have investigated its numerous special cases, the emphasis gradually shifting from the integration of individual trajectories in terms of hyperelliptic functions to a more comprehensive analysis of its general structure. Starting with the celebrated demonstration of integrability by Kovalevskaya [1889] and the impressive achievements of Kötter [1893] with regard to explicit solutions, a steadily increasing body of knowledge about the system has accumulated, particularly in the Russian schools of mathematics and analytical mechanics. The achievements of the first fifty years after Kovalevskaya's original work were summarized in a memorial collection of papers in 1940 of which Appelrot's contribution [1940] was the central part. His classification of types of motion opened the way for the modern attempts to understand the principles of phase space foliation. Kharlamov's bifurcation analysis of the level sets of first integrals [1983] continued Appelrot's work to the point of virtual completion. Bobenko et al. surveyed the state of affairs 99 years after Kovalevskaya and identified the highlights of recent progress in the development of new methods of integration [Dubrovin *et al.*, 1988] and the construction of Lax pairs [Bobenko *et al.*, 1989] and [Haine & Horozov, 1987], [Horozov & van Moerbeke, 1989].

Among physicists, the recent revival of interest in classical mechanics relates to their desire to understand non-integrable systems, and the transition from classical to quantum mechanics. Perturbation theory and path integrals are the methodological tools used to leave the realm of integrable classical systems. They both require a thorough understanding of integrable limiting cases, and center around the concepts of action integrals and winding numbers as the most relevant characteristics of invariant tori, see Berry [1978] and Gutzwiller [1990] for reviews. They assume energy surfaces to be given in the canonical form $h = \mathcal{H}(\boldsymbol{I})$ and go on from there, $\mathcal{H}$ being the system's Hamiltonian and h the energy constant, $\boldsymbol{I} = (I_1, \ldots, I_n)$ a set of action variables. For many interesting systems, however, this knowledge is simply not available. During the active days of Bohr-Sommerfeld quantum theory, a number of systems were treated in this way, including the free symmetric and asymmetric rigid bodies (Schwarzschild [1916], Epstein [1919]). But for the Kovalevskaya top corresponding results have not been derived.

The present article fills this gap in terms of graphical representations of energy surfaces, based on extensive numerical calculations of action integrals. Two problems had to be solved in this context. The first was to develop a scheme that organizes the system's invariant tori for the allowed values (h, l, k) of the first integrals energy, angular momentum, and Kovalevskaya constant, respectively. We benefitted, of course, from the work of Appelrot and Kharlamov, but found it necessary to introduce modifications in cases where tori are not uniquely defined by the values (h, l, k). We base our bifurcation scheme on the analysis of critical points in a suitably chosen Poincaré surface of section. The second problem was to find a complete set of independent closed paths around each

torus. This could be solved by means of an algorithm that was recently devised by Dullin and Wittek [1993]. It is based on the constructive part of Arnold's proof for Liouville's theorem on the existence of action-angle variables [1978]. Integrating along the paths so determined, we generated data sets of actions for the various regions of continuity of the energy surfaces. These were then assembled in a kind of puzzle work to obtain the global pictures. Our energy surfaces, although of much higher complexity, resemble those calculated earlier [Richter, 1990] for the cases of Lagrange (at low and high energies) and Euler, using elliptic integrals for explicit expressions of the actions.

The article is organized as follows. In Sec. 2 we present two versions of the equations of motion of the Kovalevskaya system. One of these uses Euler angles and the canonical formalism of Hamiltonian mechanics; it provides the context in which physically meaningful actions are defined. The second version uses Euler variables and is not canonical (although a Lie-Poisson structure still holds); all calculations are done in these variables. Sec. 3 contains the bifurcation analysis of Poincaré surfaces of section. It leads to a qualitative ordering of the system's tori in terms of Fomenko graphs [Fomenko, 1991] and introduces the concept of Poincaré-Fomenko stacks as a semi-quantitative representation of energy surfaces. They are used as a guiding principle to take care of every torus in Sec. 4 where we find the independent paths, compute the action integrals, and present six pictures of qualitatively different energy surfaces.

2. Equations of Motion

The equations of motion of a rigid body with a fixed point have been formulated in numerous versions of which there are three major classes. The first and perhaps most natural version uses Euler's angles as coordinates of configuration space, and corresponding angular momenta; its advantage is the canonical structure, its disadvantage the cumbersome appearance of trigonometric functions. The second version starts from Euler's equations and uses non-canonical variables; this is ideal for analytical and numerical integration, but unsuitable for a discussion of actions and related properties. The third version employs Hamilton's quaternions, or Cayley-Klein parameters. This approach was promoted by Klein and Sommerfeld in their classical work on rigid body dynamics [1910]; it combines canonical structure with computational elegance and might indeed be the method of choice for the work presented here, were it not for the historical accident that Klein and Sommerfeld all but ignored the Kovalevskaya case, as a result of which almost all work on the Kovalevskaya top has ignored the Cayley-Klein variables. We shall conform with that tradition, and concentrate on Euler's variables for matters of visualization, and on Euler's equations for matters of computation. It will be important to have a clear view of the relationship between these two approaches.

2.1. Euler angles φ, ϑ, ψ and Euler variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$.

Euler's angles are a convenient choice of coordinates for the configuration space SO(3) of the rigid body with a fixed point. They describe the position of a body-fixed $(1,2,3)$-frame relative to an (x,y,z)-frame fixed in space, see Fig. 1 where we adopt the conventions of Landau, Lifshitz [1984], Goldstein [1950], and others. We assume the z-axis to point upward in the constant gravitational field, and the $(1,2,3)$-axes to coincide with the three major axes of inertia. A given vector $\vec{v}$ can then be expressed in the two frames; its components (v_1, v_2, v_3) and (v_x, v_y, v_z) are related by the familiar orthogonal matrix $\mathbf{D}$, the three-dimensional representation of SO(3):

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{D} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \tag{1}$$

where

$$\mathbf{D} = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\vartheta & \sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\vartheta & \sin\psi\sin\vartheta \\ -\cos\varphi\sin\psi - \sin\varphi\cos\psi\cos\vartheta & -\sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\vartheta & \cos\psi\sin\vartheta \\ \sin\varphi\sin\vartheta & -\cos\varphi\sin\vartheta & \cos\vartheta \end{pmatrix} \tag{2}$$

This transformation law characterizes physical vectors such as position $\vec{r}$, angular velocity $\vec{\Omega}$, or angular momentum $\vec{L}$. In contrast, the triples $\boldsymbol{\alpha}^{\boldsymbol{T}} := (\varphi, \vartheta, \psi)$ of Euler angles and

$\boldsymbol{L^T} := (L_\varphi, L_\vartheta, L_\psi)$ of canonical angular momenta do not represent physical vectors.

From the orthogonality relation $\mathbf{D}\mathbf{D}^T = \mathbf{1}$ we have $\mathbf{D}\dot{\mathbf{D}}^T + \dot{\mathbf{D}}\mathbf{D}^T = 0$ which defines the antisymmetric tensor $\boldsymbol{\Omega} := \mathbf{D}\dot{\mathbf{D}}^T$:

$$\boldsymbol{\Omega} \;=\; \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \;; \tag{3}$$

its components form an axial vector $\vec{\Omega}$ in moving frame representation:

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \;\equiv\; \begin{pmatrix} p \\ q \\ r \end{pmatrix} \;=\; \begin{pmatrix} \sin\psi\sin\vartheta & \cos\psi & 0 \\ \cos\psi\sin\vartheta & -\sin\psi & 0 \\ \cos\vartheta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\vartheta} \\ \dot{\psi} \end{pmatrix} \;=:\; \boldsymbol{\Phi}\,\dot{\boldsymbol{\alpha}} \;. \tag{4}$$

The notation (p, q, r) for the moving frame components of $\vec{\Omega}$ has been customary in most of the mathematical literature on rigid body dynamics; we shall adopt this convention.

Taking the time derivative of Eq. (1), we find for any vector $\vec{v}$ the kinematic relation

$$\mathbf{D} \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} \;=\; \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{pmatrix} \;+\; \boldsymbol{\Omega} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;. \tag{5}$$

This shows that velocities do not transform as ordinary vectors; there is an additional term due to the relative motion of the two frames. Its interpretation identifies $\vec{\Omega}$ as the (time dependent) vector of the body's angular velocity.

The components of $\vec{\Omega}$ are not the total derivatives of any angles; this makes them unsuitable as objects in a canonical formalism. Nevertheless they play a crucial role in the derivation of the Lagrangian and Hamiltonian functions, and turn out to be at the heart of the Euler equations.

The Euler equations are formulated in terms of the $(1, 2, 3)$-components of $\vec{\Omega}$ and $\vec{\gamma}$, the unit vector in z-direction of the fixed frame, $(\gamma_x, \gamma_y, \gamma_z) = (0, 0, 1)$. Its components in the moving frame, according to Eq. (1), are

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \;=\; \begin{pmatrix} \sin\psi\sin\vartheta \\ \cos\psi\sin\vartheta \\ \cos\vartheta \end{pmatrix} \;. \tag{6}$$

As this vector does not move in the fixed frame, the kinematic relation (5) becomes

$$\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \\ \dot\gamma_3 \end{pmatrix} \;=\; -\,\mathbf{\Omega} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \;=\; \begin{pmatrix} r\gamma_2 - q\gamma_3 \\ p\gamma_3 - r\gamma_1 \\ q\gamma_1 - p\gamma_2 \end{pmatrix} \;. \tag{7}$$

These equations are part of the set of Euler's equations for the variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$.

The variables $(\gamma_1, \gamma_2, \gamma_3)$ serve to avoid the computationally awkward trigonometric functions of Euler angles. The matrix $\mathbf{\Phi}$ and its inverse $\mathbf{\Phi}^{-1}$ can readily be expressed in these variables; introducing the notation

$$\gamma_\perp \;:=\; \sqrt{\gamma_1^2 + \gamma_2^2} \;=\; \sin\vartheta \;, \tag{8}$$

we have

$$\mathbf{\Phi} \;=\; \begin{pmatrix} \gamma_1 & \gamma_2/\gamma_\perp & 0 \\ \gamma_2 & -\gamma_1/\gamma_\perp & 0 \\ \gamma_3 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad {\gamma_\perp}^2 \mathbf{\Phi}^{-1} \;=\; \begin{pmatrix} \gamma_1 & \gamma_2 & 0 \\ \gamma_\perp\gamma_2 & -\gamma_\perp\gamma_1 & 0 \\ -\gamma_1\gamma_3 & -\gamma_2\gamma_3 & 1 \end{pmatrix} \;. \tag{9}$$

Another important vector is the body's center of mass $\vec c$, fixed in the moving frame. Its components will be denoted by (c_1, c_2, c_3).

2.2. The canonical equations of motion

As we assumed the $(1, 2, 3)$-axes to be aligned along the major axes of inertia, the kinetic energy T of the rigid body is a diagonal quadratic form in the components of $\vec\Omega$. Let Θ_1, Θ_2, Θ_3 be the three elements of the diagonal tensor of inertia $\mathbf{\Theta}$.

$$T \;=\; \frac{1}{2}\,\vec\Omega^{\,T}\,\mathbf{\Theta}\,\vec\Omega \;\equiv\; \frac{1}{2}\,\Theta_1\,\Omega_1^2 \;+\; \frac{1}{2}\,\Theta_2\,\Omega_2^2 \;+\; \frac{1}{2}\,\Theta_3\,\Omega_3^2 \;. \tag{10}$$

The potential energy depends on the z-component of the center of mass, $c_z = \vec c \cdot \vec\gamma$:

$$V \;=\; Mg\,\vec c\cdot\vec\gamma + V_0 \;, \tag{11}$$

where M is the body's mass, g the gravitational field, and V_0 a constant such that the potential minimum is at $V = 0$. If we measure lengths in units of $c = |\vec c\,|$, energies in units of Mgc, time in units of $\sqrt{\Theta/Mgc}$ where Θ is a standard moment of inertia, the only parameters left are the relative moments of inertia Θ_i/Θ which will be denoted by Θ_i again, and the components of the unit vector $\vec c$ pointing to the center of mass. From T and V we obtain the system's Lagrangian $\mathcal{L}$ if we interpret $\vec\Omega$ and $\vec\gamma$ as functions of $(\varphi, \vartheta, \psi)$ and $(\dot\varphi, \dot\vartheta, \dot\psi)$, using Eqs. (4) and (6):

$$\mathcal{L} \;=\; \mathcal{L}(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}) \;=\; \frac{1}{2}\dot{\boldsymbol{\alpha}}^T \mathbf{T}(\boldsymbol{\alpha})\dot{\boldsymbol{\alpha}} \;-\; \left(1 + \vec c\cdot\vec\gamma\,(\boldsymbol{\alpha})\right) \;. \tag{12}$$

The matrix $\mathbf{T}(\boldsymbol{\alpha})$ is

$$\mathbf{T} \;=\; \boldsymbol{\Phi}^T\boldsymbol{\Theta}\boldsymbol{\Phi} \;=\; \begin{pmatrix} \Theta_{12} - B\cos^2\vartheta + A\cos 2\psi \sin^2\vartheta & -A\sin 2\psi\sin\vartheta & \Theta_3\cos\vartheta \\ -A\sin 2\psi\sin\vartheta & \Theta_{12} - A\cos 2\psi & 0 \\ \Theta_3\cos\vartheta & 0 & \Theta_3 \end{pmatrix} , \tag{13}$$

where the abbreviations

$$\Theta_{12} := \frac{\Theta_1 + \Theta_2}{2} , \qquad A := \frac{\Theta_2 - \Theta_1}{2} , \qquad B := \Theta_{12} - \Theta_3 \tag{14}$$

have been used.

The canonical angular momenta $\boldsymbol{L}$ are obtained as

$$\boldsymbol{L} \;\equiv\; \begin{pmatrix} L_\varphi \\ L_\vartheta \\ L_\psi \end{pmatrix} \;=\; \mathbf{T} \begin{pmatrix} \dot\varphi \\ \dot\vartheta \\ \dot\psi \end{pmatrix} , \tag{15}$$

and the Hamiltonian reads

$$\begin{aligned} \mathcal{H} \;&=\; \tfrac{1}{2}\boldsymbol{L}^{\boldsymbol{T}}\mathbf{T}^{-1}\boldsymbol{L} \,+\, V \\ &=\; \frac{((L_\varphi - L_\psi\cos\vartheta)\sin\psi + L_\vartheta\sin\vartheta\cos\psi)^2}{2\Theta_1\sin^2\vartheta} \\ &\quad + \frac{((L_\varphi - L_\psi\cos\vartheta)\cos\psi - L_\vartheta\sin\vartheta\sin\psi)^2}{2\Theta_2\sin^2\vartheta} + \frac{L_\psi^2}{2\Theta_3} + 1 + \vec{c}\cdot\vec{\gamma}\,(\boldsymbol{\alpha}) \; . \end{aligned} \tag{16}$$

The canonical equations of motion are then given as

$$\begin{pmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{L}} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \partial\mathcal{H}/\partial\boldsymbol{\alpha} \\ \partial\mathcal{H}/\partial\boldsymbol{L} \end{pmatrix} \;=:\; \mathbf{M}\nabla\mathcal{H} \; . \tag{17}$$

It is evident that the explicit form of these equations will be quite cumbersome and inconvenient for numerical or analytical integration.

2.3. The Euler equations

The simple expression of energies T and V in terms of the Euler variables suggests to use the components γ_i of $\vec{\gamma}$ and l_i of the vector $\vec{l} = \boldsymbol{\Theta}\vec{\Omega}$ as independent variables. The transformation $(\boldsymbol{\alpha}, \boldsymbol{L}) \mapsto (\vec{\gamma}, \vec{l})$ can be carried out by means of Eqs. (4) and (6). It is non-canonical, but preserves the Lie-Poisson structure of the equations, cf. [Holmes & Marsden, 1982]. Defining the Jacobian $\mathbf{J}$ as

$$\mathbf{J} \;\equiv\; \begin{pmatrix} \partial\vec{\gamma}\,/\partial\boldsymbol{\alpha} & \partial\vec{\gamma}\,/\partial\boldsymbol{L} \\ \partial\vec{l}\,/\partial\boldsymbol{\alpha} & \partial\vec{l}\,/\partial\boldsymbol{L} \end{pmatrix} \;\equiv\; \begin{pmatrix} \mathbf{J}_{\boldsymbol{\gamma\alpha}} & \mathbf{0} \\ \mathbf{J}_{\boldsymbol{l\alpha}} & \mathbf{J}_{\boldsymbol{lL}} \end{pmatrix} \;, \tag{18}$$

the canonical equations (17) transform into

$$\begin{pmatrix} \dot{\vec{\gamma}} \\ \dot{\vec{l}} \end{pmatrix} \;=\; \mathbf{J}\mathbf{M}\mathbf{J}^T \begin{pmatrix} \partial\mathcal{H}/\partial\vec{\gamma} \\ \partial\mathcal{H}/\partial\vec{l} \end{pmatrix} \;=\; \begin{pmatrix} \mathbf{0} & \mathbf{\Gamma} \\ -\mathbf{\Gamma}^T & \mathbf{\Lambda} \end{pmatrix} \begin{pmatrix} \partial\mathcal{H}/\partial\vec{\gamma} \\ \partial\mathcal{H}/\partial\vec{l} \end{pmatrix} \;=:\; \mathbf{M}_E \nabla_E \mathcal{H} \;. \tag{19}$$

The partial Jacobians $\mathbf{J}_{\boldsymbol{\gamma\alpha}}$, $\mathbf{J}_{\boldsymbol{l\alpha}}$, $\mathbf{J}_{\boldsymbol{lL}}$ are

$$\gamma_\perp \mathbf{J}_{\boldsymbol{\gamma\alpha}} \;=\; \begin{pmatrix} 0 & \gamma_1\gamma_3 & \gamma_2\gamma_\perp \\ 0 & \gamma_2\gamma_3 & -\gamma_1\gamma_\perp \\ 0 & -{\gamma_\perp}^2 & 0 \end{pmatrix} \;, \qquad {\gamma_\perp}^2 \mathbf{J}_{\boldsymbol{lL}} \;=\; \begin{pmatrix} \gamma_1 & \gamma_2\gamma_\perp & -\gamma_1\gamma_3 \\ \gamma_2 & -\gamma_1\gamma_\perp & -\gamma_2\gamma_3 \\ 0 & 0 & {\gamma_\perp}^2 \end{pmatrix} \;, \tag{20}$$

$${\gamma_\perp}^2 \mathbf{J}_{\boldsymbol{l\alpha}} \;=\; \begin{pmatrix} 0 & (L_\psi - L_\varphi \cos\vartheta)\,\dfrac{\gamma_1}{\gamma_\perp} & l_2{\gamma_\perp}^2 \\ 0 & (L_\psi - L_\varphi \cos\vartheta)\,\dfrac{\gamma_2}{\gamma_\perp} & -l_1{\gamma_\perp}^2 \\ 0 & 0 & 0 \end{pmatrix} \;. \tag{21}$$

It is now straightforward to compute the antisymmetric matrices $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$:

$$\mathbf{\Gamma} \;=\; -\mathbf{\Gamma}^T \;=\; \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix} \;, \qquad \mathbf{\Lambda} \;=\; -\mathbf{\Lambda}^T \;=\; \begin{pmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{pmatrix} \;. \tag{22}$$

The equations of motion (19) are the well known Euler equations

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} \;=\; \begin{pmatrix} r\gamma_2 - q\gamma_3 \\ p\gamma_3 - r\gamma_1 \\ q\gamma_1 - p\gamma_2 \end{pmatrix} \tag{23}$$

and

$$\begin{pmatrix} \dot{l}_1 \\ \dot{l}_2 \\ \dot{l}_3 \end{pmatrix} \;=\; \begin{pmatrix} c_3\gamma_2 - c_2\gamma_3 \\ c_1\gamma_3 - c_3\gamma_1 \\ c_2\gamma_1 - c_1\gamma_2 \end{pmatrix} \;+\; \begin{pmatrix} rl_2 - ql_3 \\ pl_3 - rl_1 \\ ql_1 - pl_2 \end{pmatrix} \;. \tag{24}$$

With $(l_1, l_2, l_3) = (\Theta_1 p, \Theta_2 q, \Theta_3 r)$, the last set can be written as

$$\begin{pmatrix} \Theta_1\dot{p} \\ \Theta_2\dot{q} \\ \Theta_3\dot{r} \end{pmatrix} = \begin{pmatrix} c_3\gamma_2 - c_2\gamma_3 \\ c_1\gamma_3 - c_3\gamma_1 \\ c_2\gamma_1 - c_1\gamma_2 \end{pmatrix} + \begin{pmatrix} (\Theta_2 - \Theta_3)qr \\ (\Theta_3 - \Theta_1)rp \\ (\Theta_1 - \Theta_2)pq \end{pmatrix} . \tag{25}$$

The Euler equations (23) and (25) will be used for purposes of integration, whereas the canonical equations (17) form the basis of our description of phase space. The relation between the two formulations of the rigid body dynamics is expressed in terms of the Eqs. (17) - (19).

The transformation $(\boldsymbol{\alpha}, \boldsymbol{L}) \mapsto (\vec{\gamma}, \vec{l})$ is not invertible, as is obvious from Eqs. (20) and (21). The cyclic variable φ is ignored in the transformation; thus it cannot be recovered by inverse transformation. Of course, we can go back to ϑ and ψ by inverting Eq. (6), and to the canonical momenta $\boldsymbol{L} = \boldsymbol{\Phi}^T\vec{l}$ with $\boldsymbol{\Phi}$ as given by Eq. (9):

$$L_\varphi = \vec{\gamma} \cdot \vec{l} , \tag{26}$$

$$L_\vartheta = (l_1\gamma_2 - l_2\gamma_1)/\gamma_\perp , \tag{27}$$

$$L_\psi = l_3 . \tag{28}$$

In order to find the angle φ at a given time, we have to integrate the φ-part of the canonical equations (17)

$$\dot{\varphi} = \frac{1}{\Theta_1}\frac{\gamma_1 l_1}{{\gamma_\perp}^2} + \frac{1}{\Theta_2}\frac{\gamma_2 l_2}{{\gamma_\perp}^2} = \frac{\gamma_1 p + \gamma_2 q}{{\gamma_\perp}^2} . \tag{29}$$

For completeness, we give the other two components of $\dot{\boldsymbol{\alpha}} = \boldsymbol{\Phi}^{-1}\boldsymbol{\Theta}^{-1}\vec{l}$ as well:

$$\dot{\vartheta} = \frac{1}{\Theta_1}\frac{\gamma_2 l_1}{\gamma_\perp} - \frac{1}{\Theta_2}\frac{\gamma_1 l_2}{\gamma_\perp} = \frac{\gamma_2 p - \gamma_1 q}{\gamma_\perp} , \tag{30}$$

$$\dot{\psi} = r - \gamma_3\dot{\varphi} . \tag{31}$$

The matrix $\mathbf{M}_E$ defines the Lie-Poisson structure of the Euler equations. Just as the Poisson bracket of the canonical formalism is defined by

$$\{F, G\} = \nabla F^T \mathbf{M} \nabla G \tag{32}$$

with $\nabla \equiv (\partial/\partial\boldsymbol{\alpha}, \partial/\partial\boldsymbol{L})$, the matrix $\mathbf{M}_E$ defines a Poisson bracket

$$\{F, G\}_E = \nabla_E F^T \mathbf{M}_E \nabla_E G \tag{33}$$

with $\nabla_E \equiv (\partial/\partial\vec{\gamma}, \partial/\partial\vec{l})$, in terms of which the time evolution of any function $F = F(\vec{\gamma}, \vec{l})$ is given as

$$\frac{dF}{dt} = \{F, \mathcal{H}\}_E . \tag{34}$$

It is easy to check from Eqs. (23) and (24) that the values of energy, $h = \mathcal{H}$, angular momentum $l = L_\varphi = \vec{\gamma} \cdot \vec{l}$, and of $\vec{\gamma} \cdot \vec{\gamma}$ are general constants of the motion. One more constant is needed to ensure integrability.

2.4. The Kovalevskaya case

Up to this point the description applies to any rigid body. We are now going to specify parameters to the Kovalevskaya case [Kowalewski, 1889], [Golubev, 1953]. Its defining feature is that two of the moments of inertia be equal and twice as large as the third; the center of gravity must lie in the plane of the two equal moments of inertia. This leaves a number of formal choices which, of course, are physically equivalent. Considering the matrix $\mathbf{T}$ in Eq. (13), the simplest formulae – without any loss of generality – are obviously obtained with the traditional choice

$$\Theta_1 = \Theta_2 = 2 \ , \qquad \Theta_3 = 1 \ , \qquad (c_1, c_2, c_3) = (-1, 0, 0) \ . \tag{35}$$

This means that Θ_3 is chosen as the standard moment of inertia, and the potential assumes its minimum for $(\vartheta, \psi) = (\pi/2, \pi/2)$. The parameters in $\mathbf{T}$ take on the values

$$\Theta_{12} = 2 \ , \qquad A = 0 \ , \qquad B = 1 \ , \tag{36}$$

and the Hamiltonian is

$$\mathcal{H} \ = \ \frac{1}{4} l_1^2 + \frac{1}{4} l_2^2 + \frac{1}{2} l_3^2 + 1 - \gamma_1 \ = \ \frac{1}{2} \boldsymbol{L}^{\boldsymbol{T}} \mathbf{T}^{-1} \boldsymbol{L} \ + \ 1 - \sin\psi \sin\vartheta \tag{37}$$

with

$$\mathbf{T}^{-1} \ = \ \frac{1}{2\sin^2\vartheta} \begin{pmatrix} 1 & 0 & -\cos\vartheta \\ 0 & \sin^2\vartheta & 0 \\ -\cos\vartheta & 0 & 1+\sin^2\vartheta \end{pmatrix} \ . \tag{38}$$

In explicit terms we have

$$\mathcal{H} \ = \ \frac{1}{4\sin^2\vartheta} (L_\varphi - L_\psi \cos\vartheta)^2 \ + \ \frac{1}{4} L_\vartheta^2 \ + \ \frac{1}{2} L_\psi^2 \ + \ 1 - \ \sin\psi \sin\vartheta \ . \tag{39}$$

The Euler equations of motion are (23) and

$$\begin{pmatrix} 2\dot{p} \\ 2\dot{q} \\ \dot{r} \end{pmatrix} \ = \ \begin{pmatrix} 0 \\ -\gamma_3 \\ \gamma_2 \end{pmatrix} \ + \ \begin{pmatrix} qr \\ -rp \\ 0 \end{pmatrix} \ . \tag{40}$$

The integrability of this system was demonstrated by S. Kovalevskaya in 1889 [1889]. It derives from the existence of four independent integrals (plus the fact that time t does not enter the equations explicitly):

energy: $$h \ = \ p^2 + q^2 + \tfrac{1}{2} r^2 + 1 - \gamma_1 \tag{41}$$

angular momentum: $$l = 2p\gamma_1 + 2q\gamma_2 + r\gamma_3 \tag{42}$$

unity: $$1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \tag{43}$$

Kovalevskaya constant: $$k^2 = (p^2 - q^2 + \gamma_1)^2 + (2pq + \gamma_2)^2 \tag{44}$$

To check the constancy of these four functions F of $\vec{\gamma}$ and $\vec{l}$, one uses Eq. (34) and shows that the gradients $\nabla_E F$ are orthogonal to

$$\mathbf{M}_E \nabla_E \mathcal{H} = \begin{pmatrix} r\gamma_2 - q\gamma_3 \\ p\gamma_3 - r\gamma_1 \\ q\gamma_1 - p\gamma_2 \\ qr \\ -\gamma_3 - rp \\ \gamma_2 \end{pmatrix} . \tag{45}$$

We note that the system is not symmetric under time reversal unless $l = 0$. By time reversal we mean the transformation

$$\begin{aligned} T : (\varphi, \vartheta, \psi, L_\varphi, L_\vartheta, L_\psi, t) &\mapsto (\varphi, \vartheta, \psi, -L_\varphi, -L_\vartheta, -L_\psi, -t) \\ (\gamma_1, \gamma_2, \gamma_3, p, q, r, t) &\mapsto (\gamma_1, \gamma_2, \gamma_3, -p, -q, -r, -t) \ ; \end{aligned} \tag{46}$$

it leaves h and k^2 constant but changes the sign of l. Of course, this behavior allows us to derive the results for $l < 0$ from those for $l > 0$. Therefore we shall assume $l \geq 0$ throughout. There are, however, two symmetries S_2 and S_3 closely connected to time reversal. The first reverses the signs of φ and ψ together with time:

$$\begin{aligned} S_2 : (\varphi, \vartheta, \psi, L_\varphi, L_\vartheta, L_\psi, t) &\mapsto (\pi - \varphi, \vartheta, \pi - \psi, L_\varphi, -L_\vartheta, L_\psi, -t) \\ (\gamma_1, \gamma_2, \gamma_3, p, q, r, t) &\mapsto (\gamma_1, -\gamma_2, \gamma_3, p, -q, r, -t) \ . \end{aligned} \tag{47}$$

It is easy to see that this transformation leaves all constants of motion invariant; much less obvious is the observation that this symmetry is also respected by each individual orbit. The symmetry S_3, on the other hand, reverses the signs of φ and ϑ together with time:

$$\begin{aligned} S_3 : (\varphi, \vartheta, \psi, L_\varphi, L_\vartheta, L_\psi, t) &\mapsto (-\varphi, \pi - \vartheta, \psi, L_\varphi, L_\vartheta, -L_\psi, -t) \\ (\gamma_1, \gamma_2, \gamma_3, p, q, r, t) &\mapsto (\gamma_1, \gamma_2, -\gamma_3, p, q, -r, -t) \ . \end{aligned} \tag{48}$$

This again leaves the constants of motion invariant, but we shall find individual orbits to break this symmetry. In such cases the system's symmetry is restored by the occurrance

of these orbits in pairs related through S_3.

As a consequence of the existence of the three constants of motion h, l, and k, the $(\boldsymbol{\alpha}, \boldsymbol{L})$–phase space is foliated by 3-tori which can be labelled – although not necessarily uniquely! – by the values of energy, angular momentum, and Kovalevskaya constant. The aim of the present work is to find, for each torus, the transformation $(h, l, k) \mapsto (I_1, I_2, I_3)$ to action variables I_i,

$$I_i := \frac{1}{2\pi} \oint_{C_i} \boldsymbol{L} \cdot d\boldsymbol{\alpha} \ , \tag{49}$$

where C_i $(i = 1, 2, 3)$ are three topologically different closed paths around the given torus. The main problem turns out to be the identification of these paths. As the (φ, L_φ)-motion separates from the rest of the dynamics, the problem reduces to analyzing the 2-tori in the $(\vartheta, \psi, L_\vartheta, L_\psi)$-part of phase space, with $l = L_\varphi$ taken as a parameter. This task is complicated by the fact that there are eight different types of energy surfaces, depending on the value of h. Before we can turn to calculating the actions, we need to identify this structure of parameter space.

3. The Phase Diagram of Energy Surfaces of the Kovalevskaya Top

M. P. Kharlamov [1983] computed the (h, l, k)-phase diagram of the Kovalevskaya top by analyzing the critical points of the map from $(\vec{\gamma}, \vec{l})$-space into the four integrals (41) – (44). We propose here to achieve the same goal by considering the dominant periodic orbits of appropriate Poincaré sections. This method not only reproduces the bifurcation results but gives direct insight into the types of orbits that exist in each domain of the phase diagram. It suggests a slight change in the classification of types in cases where different tori have the same values of (h, l, k). We use it to locate paths through phase space that are guaranteed to cross every torus just once. As this is one of the most difficult pieces in the puzzle of generating a complete set of actions, we recommend the method as generally useful in the analysis of integrable systems.

3.1. A comprehensive Poincaré section

In order to define and analyze a suitable Poincaré section, we introduce the Kovalevskaya variables x_i, y_i, and ξ_i $(i = 1, 2)$:

$$\begin{array}{rclcrcl} x_1 &=& p &+& iq & x_2 &=& p &-& iq \\ y_1 &=& \gamma_1 &+& i\gamma_2 & y_2 &=& \gamma_1 &-& i\gamma_2 \\ \xi_1 &=& x_1^2 &+& y_1 & \xi_2 &=& x_2^2 &+& y_2 \end{array} \tag{50}$$

It is convenient to express the four integrals in terms of these variables, and to eliminate r and γ_3 with the help of Eqs.(42) and (43). This transforms the energy equation (41) into

$$\mathcal{AC} - \mathcal{B}^2 + R(x_2)\xi_1 + R(x_1)\xi_2 + k^2(x_1 - x_2)^2 \;=\; 0 \;, \tag{51}$$

and the Kovalevskaya constant (44) into

$$k^2 \;=\; \xi_1\xi_2 \;, \tag{52}$$

where we follow the traditional notation:

$$\begin{array}{rcl} \mathcal{A} &=& 2(h-1) - (x_1 + x_2)^2 \\ \mathcal{B} &=& l + x_1 x_2 (x_1 + x_2) \\ \mathcal{C} &=& 1 - k^2 - x_1^2 x_2^2 \\ R(x_i) &=& \mathcal{A}x_i^2 + 2\mathcal{B}x_i + \mathcal{C} = 1 - k^2 + 2lx_i + 2(h-1)x_i^2 - x_i^4 \;. \end{array} \tag{53}$$

Eq. (51) defines the 3-D energy surface in the 4-D space of variables (x_1, x_2, y_1, y_2). The additional Eq. (52) characterizes the invariant 2-D tori that foliate the energy surface.

We now add the condition that defines our Poincaré surface of section:

$$\gamma_2 \;=\; 0 \qquad \text{and} \qquad \dot{\gamma}_2 \geq 0 \;. \tag{54}$$

In terms of Euler angles, this has the two components (cf. Eq. (6))

$$\begin{cases} \psi \;=\; \dfrac{\pi}{2} & \text{and} \quad \dot{\psi} \leq 0 \\ \psi \;=\; \dfrac{3\pi}{2} & \text{and} \quad \dot{\psi} \geq 0 \;. \end{cases} \tag{55}$$

This condition is suggested by the observation that it can be met for all values of the energy h, since the potential minimum occurs at $\psi = \pi/2$. It is not easy to prove that every orbit (up to symmetry transformation S_3) has an intersection with this Poincaré surface; we take it as a consequence of the agreement between our bifurcation diagram and Kharlamov's. Note that the condition (54) or (55) is invariant under the symmetry transformation S_2 (47) whereas S_3 (48) transforms the condition $\dot{\gamma}_2 \geq 0$ into $\dot{\gamma}_2 \leq 0$, and vice versa. This can be used to obtain points in our Poincaré section from *every* intersection $\gamma_2 = 0$: if $\dot{\gamma}_2 < 0$, apply S_3 to the phase space data.

Condition (54) defines a 2-D surface in the 3-D energy surface. As usual, we consider its projection onto a plane of conjugate variables, in order to have an area preserving Poincaré map. A convenient choice is the (ϑ, L_ϑ)-plane, as $\sin\vartheta = \gamma_1$, and $L_\vartheta = -l_2 = -2q$ in the Poincaré surface of section. Because of the two components (55) this projection is 1:2 and must be unfolded. We achieve this by letting ϑ vary, formally, from $-\pi/2$ to $3\pi/2$, i. e. we take $\vartheta = \arcsin\gamma_1$ if $\gamma_3 \geq 0$, and $\vartheta = \pi - \arcsin\gamma_1$ if $\gamma_3 \leq 0$. The resulting 1:1 projection is continuous at $\vartheta = 0$ and $\vartheta = \pi$ because of the symmetry S_2, which for the Poincaré surface of section reduces to its invariance under $(\vartheta, L_\vartheta) \mapsto (\vartheta, -L_\vartheta)$.

This projection of the Poincaré surface of section onto the (ϑ, L_ϑ)-plane (or equivalently, the (γ_1, q)-plane) will be called $\mathcal{P}$ in the following. Remember that $\mathcal{P}$ depends on the values of h and l.

With $\gamma_2 = 0$ we have $y_1 = y_2 =: y = \gamma_1$, so that Eq. (51) reduces to

$$\mathcal{F}(p,q,\gamma_1) := \mathcal{AC} - \mathcal{B}^2 + x_1^2 R(x_2) + x_2^2 R(x_1) + y(R(x_1)+R(x_2)) + k^2(x_1-x_2)^2 \;=\; 0 \,, \tag{56}$$

and and (52) to

$$k^2 \;=\; x_1^2 x_2^2 + y(x_1^2 + x_2^2) + y^2 \;. \tag{57}$$

The lines that Eq. (57) defines in $\mathcal{P}$ are the projections of the intersections of invariant tori with the Poincaré surface of section.

The boundary $\partial\mathcal{P}$ of $\mathcal{P}$ is defined by $\dot{\gamma}_2 = 0$ which implies $p\gamma_3 = r\gamma_1$ by Eq. (23). Using Eqs. (41)-(43) it is straightforward to express this as

$$\partial\mathcal{P} \ : \qquad 2(h-1+\gamma_1-q^2)(1+\gamma_1^2)-l^2 \ = \ 0 \ . \tag{58}$$

The symmetry with respect to $q \to -q$, or $L_\vartheta \to -L_\vartheta$ is obvious, and the symmetry of $\gamma_1 = \pm\sin\vartheta$ (at $\gamma_2 = 0$) with respect to $\vartheta = \pi/2$ implies that $\partial\mathcal{P}$ has a vertical symmetry axis as well. This is not true for the interior of $\mathcal{P}$ except at $l = 0$ where the system is symmetric under time reversal (cf. Fig. 3K and L).

The topology of $\mathcal{P}$ depends on the values of h and l. It can be an empty set, a disk, an annulus, or a disconnected set of two or three disks. The transitions are governed by three conditions on the $q = 0$ axis. The first requires that the point $\gamma_1 = 1$ be on $\partial\mathcal{P}$. This implies

$$h \ = \ l^2/4 \ . \tag{59}$$

If this condition is met, $\mathcal{P}$ reduces to a point; the rigid body performs a pure rotation about the vertical axis, its center of mass resting in the stable equilibrium position. At given l, no motion is possible with lower energy. The second transition occurs when $\partial\mathcal{P}$ reaches the point $(\gamma_1, q) = (-1, 0)$. This happens when

$$h \ = \ 2 + l^2/4 \ . \tag{60}$$

Again, the rigid body rotates about its vertical axis, but with the center of mass in the unstable equilibrium position. At given l and energy lower than (60), the potential maximum cannot be reached. As long as $\mathcal{P}$ is connected, the condition discriminates between disk and annulus topology.

The third condition governs the connectedness of $\mathcal{P}$ which depends on the number of zeroes of the polynomial (58) on the line $q = 0$, in the physical range $\gamma_1^2 \le 1$. This number changes when the discriminant Δ of the polynomial is zero,

$$\Delta \ \equiv \ \frac{27}{16}l^4 - \frac{1}{2}(h-1)(9+(h-1)^2)l^2 + (1+(h-1)^2)^2 \ = \ 0 \ ; \tag{61}$$

the location of the corresponding double zeroes is

$$\gamma_1^* \ = \ -\frac{1}{3}(h-1) \pm \frac{1}{3}\sqrt{h^2-2h-2} \ . \tag{62}$$

The results of an elementary analysis are summarized in Fig. 4T. The zeroes of Δ define the line with a cusp, the other two lines corresponds to Eqs. (59) and (60). The bifurcation scheme of the topology of $\mathcal{P}$ is part of Appelrot's more detailed classification as given in Fig. 5. With reference to the notation of Fig. 5 for identification of regions in the (h, l^2)-plane, the following holds: $\mathcal{P}$ is a disk in region **1d** which comprises A, B, C, and J. It is an annulus in region **a** of which D, E, and H are parts. The region **2d** where the annulus is disrupted into two disks consists of parts F and G, and in the tiny region **3d** where we have three disks coincides with the Appelrot region I. Parameters on the line $\Delta = 0$ allow for a particularly simple type of motion, if initial conditions are chosen at the point $(\gamma_1, q) = (\gamma_1^*, 0)$: all Euler variables are constant in time! The only motion left is in φ and

ψ which according to Eqs. (29) and (31) change as

$$\dot{\varphi} \;=\; \frac{l}{1+\gamma_1^{*\,2}}\;, \qquad\qquad \dot{\psi} \;=\; r - \gamma_3\dot{\varphi}\;. \tag{63}$$

This motion is reminiscent of a merry-go-round. It is stable along the line separating regions G and H, or I and J; $(\gamma_1^*, 0)$ is then a point where two disks are about to break apart. The motion is unstable along the line separating regions B and I; $(\gamma_1^*, 0)$ is then a point where a new disk appears out of nothing, together with a tangent bifurcation of critical orbits. The unstable partner of these orbits is the merry-go-round motion.

An example of a Poincaré section with $h = 2.2$ and $l = 1.259$ is given in Fig. 2. It shows a collection of orbits with k-values in the range $0 \le k \le 2.339$. The line $\vartheta = \pi/2$ is the vertical symmetry axis of the boundary, but clearly not of the interior of $\mathcal{P}$. The angle ϑ extends approximately from $-\pi/4$ to $5\pi/4$; the potential maximum at $\vartheta = 3\pi/2$ is not accessible, and the topology of $\mathcal{P}$ is that of a disk. The lines $\vartheta = 0$ and $\vartheta = \pi$ are always lines of constant k, see Sec. 3.2.2; with $k = h - 1 - l^2/2$ from Eq. (80) we have $k = 0.408$.

There are five elliptic and three hyperbolic critical points, plus four boundary points where the separatrix is tangent to the surface of section (see Sec. 3.2.2). This arrangement of critical points varies with parameters h and l; we shall identify 10 different types. A thorough analysis of critical orbits was first given by Appelrot [1940] who divided them into four classes. Class I orbits have $k = 0$ and are marked here by full circles; they are always elliptic. The two points on the horizontal axis belong to the same orbit of period 2; the two off-axis centers belong to another such orbit. The empty circle marks an elliptic orbit of class IV; it has the maximum value of the Kovalevskaya constant, $k = 2.339$. The squares mark points of class III, three hyperbolic orbits (full squares) and four boundary points (empty squares). The three saddles and the boundary points are connected by a separatrix, of which the lines $\vartheta = 0$ or π are a part. There are no critical orbits of class II at the given parameter choice $(h, l) = (2.2, 1.259)$. The foliation of $\mathcal{P}$ by intersections of invariant tori is indicated by a few examples. It is organized by the system of critical orbits and the associated separatrices. Fig. 2 contains two segments of horizontal lines that connect the critical points of minimum and maximum k so as to intersect every torus once (except for the tori around the off-axis centers). These segments can be viewed as the backbone of the Poincaré surface of section; they will be important in the discussion of Fomenko graphs (Sec. 3.3) and for the computation of action integrals.

The series of Figs. 3 gives a survey on the ten different types of Poincaré surfaces of section. A color code is used to identify invariant tori (equal color means equal k) and regions between separatrices. Within a region of given color, the gradation from dark to light follows increasing values of k. The five main colors red, green, yellow, blue, and purple characterize connected components of topologically equivalent tori in (h, l, k)-space. The orange region in Fig. 3C is connected to the red region in full (h, k, l)-space, but not at given energy h. Similarly, the turquois regions in Figs. 3I and J have a connection to the green region, but at fixed h there is a separatrix between them.

Our next goal is to present the scheme of bifurcations that organizes these pictures.

3.2. The different types of Poincaré sections

Fig. (2) can be viewed as a contour diagram with lines of constant values of k^2. Its structure is determined by the arrangement of critical points in $\mathcal{P}$: minima and maxima representing elliptic periodic orbits, and saddle points as hyperbolic centers connected to separatrices. Starting from Eq. (56), we can express p as a function of q and γ_1, and insert this into Eq. (57) to obtain k^2 as a function of (q, γ_1) or (ϑ, L_ϑ). The critical points are then defined by

$$\frac{\partial k^2}{\partial q} \;=\; 0 \qquad \text{and} \qquad \frac{\partial k^2}{\partial \gamma_1} \;=\; 0\;. \tag{64}$$

It is straightforward though lengthy to compute and distinguish all possible cases. We sketch the main steps and support the results by a set of graphs. Our results reproduce the classification of Appelrot [1940] and Kharlamov [1983], and in addition provide formulae for the location of critical tori in $\mathcal{P}$.

We start with the first requirement in (64). With

$$dk^2 \;=\; (\xi_1+\xi_2)d\gamma_1 + 2(x_1\xi_2+x_2\xi_1)dp + 2i(x_1\xi_2-x_2\xi_1)dq \tag{65}$$

from Eq. (57), and

$$\left.\frac{\partial p}{\partial q}\right|_{\gamma_1} \;=\; -\frac{\partial \mathcal{F}}{\partial q}\Big/\frac{\partial \mathcal{F}}{\partial p}\;, \tag{66}$$

we find from Eq. (56) that $\partial k^2/\partial q$ contains a factor q. This suggests to look for critical points by solving the second Eq. (64) on the symmetry line $q=0$. It turns out that most critical points are indeed obtained in this way. Some others are related to them through the symmetry operation S_3; the rest will be calculated in Sec. 3.2.2.

With $q=0$ we have $x_1 = x_2 =: x = p$ and $\xi_1 = \xi_2 =: \xi$. This implies

$$\left.\frac{\partial k^2}{\partial \gamma_1}\right|_{q=0} \;=\; 2\xi\left(1+2x\left.\frac{\partial p}{\partial \gamma_1}\right|_{q=0}\right) \;\stackrel{!}{=}\; 0\;. \tag{67}$$

Depending on which factor is taken to be zero, we get different classes of critical orbits. In Appelrot's notation, condition $\xi = 0$ defines class I whereas $1+2x\partial p/\partial\gamma_1|_{q=0} = 0$ leads to three separate classes II-IV.

Appelrot's classification scheme may appear somewhat artificial from a modern point of view; it emphasizes the algebra from which the location of bifurcation points is derived rather than the aspect of their unfolding. Nevertheless it serves its purpose well to identify the order in the bewildering complexity of this integrable system.

3.2.1. Appelrot class I

The first class of critical orbits obeys $\xi = 0$ and therefore

$$k^2 \;=\; 0\;. \tag{68}$$

This has been termed Appelrot class I, or Delauney class [Kharlamov, 1983]. It characterizes tori with the smallest possible value of the Kovalevskaya constant. In order to locate the corresponding critical points in $\mathcal{P}$ we insert $\xi = 0$ and $k = 0$ in Eq. (56) to find $\mathcal{AC} - \mathcal{B}^2 = 0$ or

$$2(h-1)(1-p^4) - 4p^2 - l^2 - 4lp^3 \;=\; 0\;. \tag{69}$$

This fourth order polynomial for p can be turned into a polynomial for γ_1 with $\gamma_1 = -p^2$:

$$(2(h-1)(1-\gamma_1^2) + 4\gamma_1 - l^2)^2 + 16l^2\gamma_1^3 \;=\; 0\;. \tag{70}$$

We thus have an explicit formula for the critical points with $k^2 = 0$.

The polynomial (70) may have 0, 2, or 4 roots depending on the values of h and l. These cases are separated by the zeroes of the discriminant of which there are two types:

$$h - 1 - \frac{1}{2}l^2 \;=\; 0 \qquad \text{and} \qquad \Delta \;=\; 0 \tag{71}$$

where Δ is the discriminant (61) that appeared already in the topological bifurcations. The corresponding lines in the (h, l^2)-plane are shown in Fig. 4I. Outside the cone $h \geq 1 + l^2/2$ there are no roots of (70), while there are 2 or 4 roots inside. The line $\Delta = 0$ circumscribes a small region with four roots of Appelrot class I. It has a cusp at $(h, l^2) = (1 + \sqrt{3}, 16/3\sqrt{3})$ and joins the cone at $(h, l^2) = (3, 4)$; for larger l^2 the corresponding roots of (70) are in the unphysical range $\gamma_1^2 > 1$.

We see that orbits of type I occur at energies h no lower than 1. They are necessarily stable because k^2 is at its minimum value 0. As to their symmetry, we observe graphically that they are not invariant under S_3. The two critical points in the major cone in Fig. 4I belong in fact to the same orbit of period 2; the image of this orbit under S_3 is a period 2 orbit with intersections outside the ϑ-axis. (The notation 2+2 in Fig. 4I indicates 2 on-axis and 2 off-axis points.) These two orbits and their neighborhoods are shown in green color in the series of Figs. 3. The two additional critical points in the small upper right part of Fig. 4I are two distinct orbits of period 1; they are mirror images of each other under S_3 and have no off-axis counterparts. (Therefore the total number of critical points of this class is 4+2.) In Figs. 3G-J they are the centers of the purple regions.

3.2.2. Appelrot classes II and III

Eq. (67) can also be fulfilled by $1 + 2x\partial p/\partial\gamma_1|_{q=0} = 0$. Taking the differential of $\mathcal{F}$, see Eq. (56), this is equivalent to

$$\left.\frac{\partial \mathcal{F}}{\partial p}\right|_{q=0} \;=\; 2x \left.\frac{\partial \mathcal{F}}{\partial \gamma_1}\right|_{q=0} \;. \tag{72}$$

The condition is most conveniently evaluated by reintroducing the variables r and γ_3 which leads to the factorization

$$(r\gamma_1 - 2p\gamma_3)(pr + \gamma_3) \;=\; 0 \;. \tag{73}$$

The two factors define classes II/III and IV respectively. We begin with

$$r\gamma_1 - 2p\gamma_3 \;=\; 0 \;. \tag{74}$$

which leads to Appelrot's classes II and III. Eliminating γ_3, r, and p by means of Eqs. (41) - (43) we get the following polynomial in γ_1:

$$(1-\gamma_1^2)(h - 1 - \frac{1}{2}l^2 + \gamma_1 + l^2\gamma_1^2) \;=\; 0 \;. \tag{75}$$

The possibility $\gamma_1^2 = 1$ implies $\gamma_3 = 0$ and by Eq. (74) $r = 0$. This holds only in the special cases where the 1-axis is aligned along the z-axis, and the top performs a pure rotation – the cases (59) and (60) discussed in connection with the topological bifurcations of $\mathcal{P}$. The corresponding values of h and k are

$$\begin{aligned}
&\gamma_1 = 1 \quad : && h = \frac{1}{4}l^2 && k = 1 + h \;, \\
&\gamma_1 = -1 : && h = 2 + \frac{1}{4}l^2 && k = \begin{cases} h - 3 & (h \le 1 + l^2/2) \\ 3 - h & (h \ge 1 + l^2/2) \end{cases} \;.
\end{aligned} \tag{76}$$

The second factor in Eq. (75) describes the critical points if $r \neq 0$:

$$h - 1 - \frac{1}{2}l^2 + \gamma_1 + l^2\gamma_1^2 \;=\; 0 \;. \tag{77}$$

The corresponding values of k are

$$k = \begin{cases} 1 - h + \frac{1}{2}l^2 & (h \le 1 + l^2/2) \qquad \text{class II} \\ h - 1 - \frac{1}{2}l^2 & (h \ge 1 + l^2/2) \qquad \text{class III} \end{cases} \tag{78}$$

The distinction made in Eqs. (76) and (78) defines the Appelrot classes II and III.

Eq. (77) defines 0, 1, or 2 critical points. Their bifurcation scheme in the (h, l^2)-diagram is governed by the lines defined in (76) – where points may be entering or leaving the physical range $\gamma_1^2 \le 1$ – and by the discriminant of the polynomial in Eq. (77):

$$2 - 2(h-1)l^2 + l^4 \;=\; 0\;. \tag{79}$$

This gives the hyperbola shown in Fig. 4II/III; it touches the line (60) in the point $(h, l^2) = (2.5, 2)$.

The distinction of Appelrot classes II and III refers to their stability. Orbits of class II are elliptic, those of class III are hyperbolic. The change of stability occurs along the line $h = 1 + l^2/2$, i. e. in connection with the appearance of orbits of class I. Fig. 4II/III contains the relevant information; see also Fig. 5 for a combined picture.

There is just one orbit of class II/III in the energy range $l^2/4 \leq h \leq 2 + l^2/4$. It has period 1 and possesses both symmetries S_2 and S_3. In region A of Fig. 5, it has the lowest possible value of k. A typical Poincaré section is shown in Fig. 3A. The critical point of class II marks the dark center near the right boundary. When the parameters (h, l^2) cross over into region B, the lowest k belong to class I ($k = 0$), and the class II orbit moves to the neighborhood of the left boundary where it becomes class III. In Fig. 3B it is the left saddle point. The scenario is more involved if $h > 3$. Before class II turns into class III there is a bifurcation from 1 to 2 orbits of class II. The corresponding region is termed F in Fig. 5, and a Poincaré section is shown in Fig. 3F. The new class II orbits, of lowest k, are the dark centers of the yellow region. The individual orbits violate S_3-symmetry but are images of each other under S_3. The saddle point near $\gamma_1 = -1$ appears together with these orbits but is class IV, see below. Both orbits lose their stability simultaneously upon transition into parameter region G, see Fig. 3G. This happens in connection with the occurance of period 1 and period 2 orbits of class I. Each critical point of class II gives way to two points of class III. Two of these lie on the axis $q = 0$ (in the left part of Fig. 3G), their S_3-images are off-axis. The orbits persist in region H (cf. Fig. 3H) but disappear at the bifurcation line (79) which marks the transition to region E.

At this stage we digress from our computation of critical points with $q = 0$, and add information about critical points with $\gamma_1 = 0$. Our Poincaré sections suggest that such orbits should exist. They are most easily discussed directly from Eqs. (41)-(44). With $\gamma_1 = \gamma_2 = 0$, we have immediately

$$h - 1 - \frac{1}{2}\,l^2 \;=\; p^2 + q^2 \;=\; k \tag{80}$$

which is the definition of Appelrot's class III, see Eq. (78). Considering, as before, k^2 as a function of q, and γ_1, we find that

$$\left.\frac{\partial k^2}{\partial q}\right|_{\gamma_1 = 0} \;=\; 0 \tag{81}$$

identically. This peculiar feature is clearly borne out in the series of Fig. 3. The other condition for critical points is

$$\left.\frac{\partial k^2}{\partial \gamma_1}\right|_q (\gamma_1 = 0) \;=\; 2p(p + l(p^2+q^2)) \;\stackrel{!}{=}\; 0\;. \tag{82}$$

This can be fulfilled in two ways. One is to take $p = 0$. Inserting this into Eqs. (41)-(44) we get

$$h - 1 - \frac{1}{2}\,l^2 \;=\; q^2\;. \tag{83}$$

By comparison with Eq. (58), we see that this defines four points on the boundary of the Poincaré section. The adequate interpretation is here not that of a critical torus; rather we have a tangency of the separatrix with the Poincaré surface of section. The other possibility $p + l(p^2+q^2) = 0$ leads to $p < 0$ and

$$q^2 \;=\; (h - 1 - \frac{1}{2}\,l^2)(1 - (h-1)l^2 + \frac{1}{2}\,l^4) \;=\; k(1 - kl^2) \tag{84}$$

which by comparison with Eq. (79) gives real values for q in the whole parameter range of Appelrot class III, except in region E. With $\gamma_1 = 0$ the Euler Eqs. (23) give $\dot\gamma_2 = p\gamma_3$, and because of $p < 0$ the section condition $\dot\gamma_2 \geq 0$ is only fulfilled at $\gamma_3 < 0$, implying $\vartheta = \pi$; it does not hold at $\vartheta = 0$.

To sum up, there are 4 or 6 off-axis points of Appelrot class III. The 4 boundary points are a special feature of the particular section condition (54) and not to be viewed as critical tori. This is especially obvious in region E where the line $\gamma_1 = 0$ is not part of a separatrix. The 2 off-axis critical points of class III exist in regions B, C, D, G, and H. They are included in the scheme of Fig. 4II/III.

3.2.3. Appelrot class IV

The last Appelrot class derives from the factor

$$pr + \gamma_3 \;=\; 0\;. \tag{85}$$

in Eq. (73). Proceeding as before, we eliminate the variables r and γ_3 with the help of Eqs. (56) and (57). The result is more involved than in the other cases:

$$\begin{aligned} 1 - \gamma_1^2 + p(l - 2\gamma_1 p) &= 0\;, \\ 2(h - 1 + \gamma_1 - p^2)(1 - \gamma_1^2) - (l - 2\gamma_1 p)^2 &= 0\;. \end{aligned} \tag{86}$$

We do not care to write down the sixth order polynomial for γ_1 that is obtained by eliminating p; it is easier to first eliminate γ_1 and compute p from

$$l + 2p(h - 1 - p^2) \;=\; 0\;. \tag{87}$$

γ_1 can then easily be obtained from the first of Eqs. (86). The corresponding k-values are

$$k^2 \;=\; 1 + lp + p^4 \;. \tag{88}$$

Depending on the values of (h, l^2), we find up to 5 physically possible solutions of Eqs. (86) and (87). Their bifurcation scheme is organized by the discriminant of the sixth order γ_1-polynomial, and by the two lines defined in (76) which again mark the situation $\gamma_1^2 = 1$ for some solutions. The discriminant has two relevant parts. One is identical with the topological discriminant Δ, see (61) and (71); the other is

$$(h-1)^3 - \frac{27}{16}\, l^2 \;=\; 0 \;. \tag{89}$$

The complete scheme for class IV is presented in Fig. 4IV. For identification of certain parts we refer to the notation in Fig. 5.

In regions A and B there is just 1 critical orbit of class IV. It has period 1 and is stable because it assumes the maximum possible value of k. It has both S_2 and S_3 symmetry. In the series of Poincaré sections (Figs. 3) this orbit is characterized by the lightest color in the red region. It is the only orbit that exists for all parameter values (h, l^2).

Crossing from region B into region C, two new orbits of class IV appear by tangent bifurcation, one elliptic, the other hyperbolic. Both orbits possess complete S_2, S_3 symmetry. The elliptic orbit corresponds to a local maximum of k; its neighborhood is shown in blue. The separatrix associated with the hyperbolic point divides the red region into two parts, with orbits of the same symmetry. The orange color was chosen as a reminder of this similarity in character. An alternative choice would have been to continue the red color across the separatrix; we prefer to enhance the separatrix.

A major change occurs at the transition from C to D, as the topology of $\mathcal{P}$ changes from disk to annulus through a connection at $\gamma_1 = -1$ (the potential maximum). A new orbit of class IV (yellow) enters from the boundary at $\gamma_1 = -1$; it can be considered as the continuation of the vanishing orange orbit of class III, and has the same symmetry (only S_3). At the same time there is a rearrangement of the separatrices around the green and blue regions. This scenario can be observed by comparison of Figs. 3C and D, or for the special case $l = 0$ in Figs. 3K and L.

The transition from parameter region D to E does not change the number of critical orbits of class IV, but the blue elliptic orbit becomes hyperbolic via collision with two off-axis saddle points of class III; the blue region vanishes in that collision. A similar pitchfork bifurcation, without change in number of class IV orbits, occurs at the transition from region E to H. In Fig. 3H the elliptic centers of the purple regions are class IV, the saddles between green and purple are class III. These changes of stability occur along the bifurcation line (79) of class III; in Fig. 4IV this is marked as a dotted line.

Going from region H to G in parameter space, $\mathcal{P}$ separates into two pieces along $\gamma_1 = \gamma_1^*$, see (62). In this connection, the two elliptic orbits in the purple regions change from class IV to class I. As a result, there are only two class IV orbits in regions G and F: the stable center of the red region, and the saddle between the two yellow lobes. The tiny regions I and J have 3 and 5 orbits of class IV, respectively. Going from H to J, the

annulus structure of $\mathcal{P}$ is broken at $\gamma_1 = -1$, and the class III orbit at the left boundary of Fig. 3H turns into class IV. This makes for 5 class IV orbits altogether, 3 elliptic and 2 hyperbolic. Going from J to I, $\mathcal{P}$ is again disrupted at $\gamma_1 = \gamma_1^*$, and the purple centers become class I.

3.3. Phase diagram and Poincaré-Fomenko stacks

Fig. 5 summarizes the information about the bifurcation scheme of critical orbits. We call it a phase diagram of Poincaré surfaces of section. Each point of the (h, l^2)-plane (with $h \geq l^2/4$) is understood to carry one of the ten different types of $\mathcal{P}$ shown in Fig. 3. The projections of the various bifurcation lines onto this plane are the loci of transitions between the various types, or "phases", of dynamical behavior.

A more refined phase diagram might be obtained by taking the full three dimensional (h, l^2, k)-parameter space, i. e. to attach to each point of the (h, l^2)-plane the allowed range of k-values. Connecting the critical points in this 3-D representation, one obtains surfaces which divide the parameter space into five connected regions. These regions were introduced and numbered i through v by Kharlamov [1983] who provided 2-D sections of the 3-D parameter space at constant l. With (h, l^2)-values from region A in Fig. 5, all allowed values of k give the same type of orbit (colored red in the series of Fig. 3), and the corresponding Kharlamov type is i. There are two types over region B, i at high values of k (red orbits), and ii at low k (green orbits). The situation over region C shows, however, that the scheme of Kharlamov regions is not yet sufficient to distinguish all types of tori: there is a range of k-values with orbits of two different types (red and blue). In order to keep track of this kind of distinction, we propose a more detailed color scheme that characterizes tori rather than (h, l^2, k)-values. Moreover, we take the rather physical point of view that it is most interesting to look at the system at fixed values of the energy h. It is then possible that orbits of the same Kharlamov type (red and orange) are separated by a separatrix (in analogy to the separation of fluids into liquid and gaseous phases at constant temperature). The relationship of the two classification schemes can best be discussed in terms of Fomenko graphs [Fomenko, 1991].

Consider Fig. 6 which assigns each type of $\mathcal{P}$ a Fomenko graph. The graphs carry vertical order: the value of k increases from bottom to top. (This feature is absent in Fomenko's version of his graphs.) The construction principle is to give every torus a unique representation, to identify its type by color, and to preserve connections. Color changes occur at separatrices. End points of an edge correspond to local or global extrema of k; they represent elliptic critical orbits. At the left side of each graph we indicate the Appelrot classes of bifurcations, at the right we identify the Kharlamov regions.

The overall correspondence between Kharlamov's regions and our colors is the following: red for region i, green for region ii, yellow for region iii, blue for region iv, and purple for region v. The difference comes in where there are two types of tori for the same set of (h, l^2, k): Kharlamov's region iv contains blue *and* red orbits; his region v contains purple *and* green orbits; on the other hand he identifies our orange region as i (red), and

turquois as ii (green). We need these additional colors because the Poincaré surfaces of section contain a separatrix between red and orange tori, as well as between green and turquois.

Let us discuss the individual graphs. Graph A has only one edge: for each k-value between minimum (class II) and maximum (class IV) there exists exactly one torus of full symmetry and Kharlamov region i. All other graphs carry a red edge at their top. Graphs B and F are the next simplest, and of identical structure. Two S_3-symmetry related edges have developped at the lower end, of Kharlamov regions ii and iii respectively. Graph E can be viewed as a natural extension of B or F, with two segments to its symmetry breaking parts. But note that there is no direct transition from E to B or F, except through the special point $(h, l^2) = (2.5, 2)$. Graph C has one arm attached to the i-region of graph B. Being without partner, the new arm must represent tori with full symmetry. The point of attachment corresponds to a separatrix of class IV. The appearance of the arm does not change the character of the red orbits; thus we retain their color in contrast to Kharlamov who counts them as part of region iv. The change of color from red to orange (which is not reflected in Kharlamov's typology) is required to take care of the new separatrix. Graph D has an interesting separatrix of class III at which four different tori meet. Graph J has a similar relationship to B as C has, except that the tangent bifurcation has occured at the two lower edges. Going from J to H there is an exchange of classes III and IV of critical tori; this implies a transition of Kharlamov type from region ii to region iii. The graphs G and I are very similar to graphs H and J respectively, the only difference being the length of the purple edges.

The Fomenko graphs are valuable because they condense the information contained in the Poincaré surfaces of section to their very essence; they are a qualitatively correct representation of the organization of invariant tori for given parameters (h, l^2). To obtain a representation of a complete energy surface with given h, we have to superimpose all Fomenko graphs for that particular h. At $h = 1.9$, e. g., we would have to stick together graphs of type C, B, and A, following increasing values of l. This gives a first idea of the structure of the energy surfaces.

To obtain a quantitative version of the Fomenko representation, we observe that the ϑ-axes of the $\mathcal{P}$'s contain already the relevant information. Fig. 2 which is typical of phase B, contains two segments of the ϑ-axis which lead from minimum to maximum k: the left piece from $k = 0$ to the separatrix value, the right piece from the separatrix to the maximum. If we keep in mind that the two green edges of Fomenko graph 6B are related by S_3-symmetry (and thus have the same action integrals), we may ignore the second edge, and take the combination of the two horizontal segments in Fig. 2 as a quantitative realization of the Fomenko graph for $(h, l) = (2.2, 1.259)$. A superposition of corresponding segments for all possible values of l, $0 \leq l^2 \leq 4h$, could be taken as a vizualization of the energy surface in which each (l, k)-torus is represented with just one point (up to symmetry S_3).

Fig. 7 is essentially that representation for the different types of energy surfaces, corresponding to the qualitatively different ways to cut through the phase diagram (Fig. 5)

at constant h. There is redundancy in the pictures of Fig. 7 in that the tori are represented with two or even four points (in case of the green and yellow orbits). This allows us to select from each torus the most convenient representative point for matters of integration. The pictures of Fig. 2 help us keep track of what belongs together.

Consider now the "Poincaré-Fomenko stack" of Fig. 7a. Its energy is $h = 0.5$, and the picture is typical for energies $h \leq 1$. All tori belong to the same Kharlamov region i, and are represented by two points. Both critical orbits are elliptic; their location is indicated by two black lines. Class IV runs near the vertical axis and belongs to the highest k, class II stays close to the right boundary and represents the tori of lowest k. The region between these two lines contains one and only one point of each torus; thus it is a complete representation of the energy surface and may be called a "minimal Poincaré-Fomenko stack", or simply a Fomenko stack. Along the boundary of this Fomenko stack (including the bottom line $l = 0$), one of the three action integrals is zero as the corresponding torus has zero width. At the three intersections of these lines, the motion is particularly simple in that two action integrals vanish: The top of the figure represents pure rotation, with the center of mass in the stable equilibrium position; the center of the bottom line represents a pendulum oscillation in γ_2 (i. e. in angle ψ, with $\vartheta = \pi/2$), while the right corner point represents a pendulum oscillation in γ_3 (i. e. in angle ϑ, with $\psi = \pi/2$).

The Poincaré-Fomenko stack of Fig. 7b ($h = 1.9$) is typical for energies in the range $1 < h < 2$. Its upper part (large values of l^2) resembles the previous stack, but at intermediate l^2, Fomenko graphs of type B are added, and near the bottom, graphs of type C. Three Kharlamov regions contribute: i, ii, and iv. The black lines in the red, dark green, and blue regions give the location of elliptic critical tori: classes IV and II in the red, class I in the dark green, and class IV in the blue region. Separatrices are identified as a change in color; if that is not possible as in the light part of the green region near the left boundary, saddle points are also marked black. Again the stack contains two points for every torus (but note that only one of the two symmetry related green tori is shown). A minimal Poincaré-Fomenko stack would contain the following parts: the red region between the line of class IV orbits at left and the line of class II orbits and separatrices at right; the adjoining part of the green region up to the orbits of class I; the blue region between the separatrix at its left and the central orbit of class IV; the orange region between green and blue. This leaves us with a hole in the bottom part of the stack. We remark that at $l = 0$, the red and blue orbits are images of each other under S_3; this symmetry suggests to select the left part of the blue region rather than the right, though this is of course an arbitrary choice. With regard to the eventual goal of our analysis, which is the determination of action integrals for each torus, we cannot in general expect continuity across separatrices. We shall attempt to preserve as much continuity as possible, and use the Poincaré-Fomenko stacks to guide us. At this stage it appears impossible to connect the blue part of a Fomenko stack to both red and orange.

As to the physical interpretation of special tori, such as the beginning and end of the lines of critical tori in the green and blue regions, a large body of knowledge has accumulated ever since the original work of Kovalevskaya, notably in the Russian math-

ematical literature. Most of this has remained unread by Western physicists to this very day. Nowadays, even though that literature has become much more accessible, the fastest way to develop familiarity with the Kovalevskaya system is probably by working with an interactive graphics oriented computer program which allows the user to select any conceivable torus, and display it in whatever 3-D projection wanted. We have written such a program, and suggest that the interested reader obtain a copy for his or her own studies. A comprehensive description of all these details in pictures or words is definitely outside the scope of this paper.

The yellow region (Kharlamov region iii) adds further complexity at energies above 2, see Fig. 7c ($h = 2.2$) as typical for the energy range $2 < h < 1 + \sqrt{2}$. Both symmetry related yellow tori appear with two points each. If we include them both in the minimal Poincaré-Fomenko stack, a convenient choice would be to add to the stack of Fig. 7b the rightmost yellow strip, and the right part of the central yellow lobe.

Increasing the energy further, the yellow region becomes more and more prominent; eventually, in the Euler–Lagrange limit $h \to \infty$, yellow is the only color to survive. Fig. 7d ($h = 2.45$) is typical for the small energy range $1 + \sqrt{2} < h < 2.5$. Its new feature is that the stack contains Fomenko graphs of type E, at intermediate levels of l. Because of the lack of blue in that range, the yellow part of the energy surface connects to both green and red (compare the Fomenko graph 6E). In the energy range $2.5 \leq h \leq 3$ this feature develops into a major characteristics of the stacks. The fine details of the phase diagram in the parameter range $2.5 \leq h \leq 3$, $2 \leq l^2 \leq 4$, give rise to a complicated scenario which is not well resolved on the scale of Figs. 7. Thus we refrain from presenting Poincaré-Fomenko stacks for these energies, and leave the discussion of the details related to regions I and J to a forthcoming publication where we plan to attain higher resolution. Figs. 7e and f are the same type of Poincaré-Fomenko stacks, at intermediate and high energies $h = 3.3$ and $h = 10$. The blue and purple parts are still present, but with vanishing relative weight. As before, every torus is represented by two points, and all tori are visible, except for the green of which only one S_3-partner is shown, but with four representative points. There are several obvious possibilities to select a minimal Poincaré-Fomenko stack.

This completes the ordering of all invariant tori of the Kovalevskaya system. Note that for every torus with parameters (h, l, k) there is a partner under time reversal $l \to -l$ if $l \neq 0$. This requires to take every stack twice, but we shall not consider this extension any further. We now proceed to computing the relevant physical properties of the Kovalevskaya tori, viz., their action integrals and energy surfaces.

4. Calculation of Action Integrals

The Arnold–Liouville theorem [Arnold, 1978] guarantees the existence of action-angle variables in the Kovalevskaya system. There are three independent closed paths C_i around each invariant 3-torus, and the integrals

$$I_i \;:=\; \frac{1}{2\pi}\oint_{C_i} L_\varphi d\varphi \;+\; \frac{1}{2\pi}\oint_{C_i} L_\vartheta d\vartheta \;+\; \frac{1}{2\pi}\oint_{C_i} L_\psi d\psi \tag{90}$$

define actions $\boldsymbol{I} := (I_1, I_2, I_3)$ which are invariant under deformations of the paths, and under canonical transformations of the variables $(\boldsymbol{\alpha}, \boldsymbol{L})$. The particular canonical transformation that chooses the (I_1, I_2, I_3) to be the new momenta, introduces cyclic angles $(\theta_1, \theta_2, \theta_3)$ whose time dependence is trivial:

$$\dot\theta_i \;=\; \frac{\partial \mathcal{H}}{\partial I_i} \;=:\; \omega_i \;. \tag{91}$$

The $\boldsymbol{\omega} := (\omega_1, \omega_2, \omega_3)$ are the set of characteristic frequencies of each torus, and represent its physically most relevant properties. As Eq. (91) shows, this information is contained in the function $\mathcal{H}(\boldsymbol{I})$. Unfortunately, this function has never been determined. An explicit analytical derivation seems out of reach. The following is a numerical calculation of energy surfaces $h = \mathcal{H}(\boldsymbol{I})$ for six selected values of the energy constant h. This work was made possible by the recent development of an algorithm for the efficient calculation of actions, based on Arnold's constructive proof of the Liouville theorem (Dullin and Wittek [1993]).

The first action integral presents no problem, as φ is a cyclic variable and $L_\varphi = l$ a constant. Defining path C_1 by ϑ =const and ψ =const, we have

$$I_1 \;\equiv\; I_\varphi \;=\; \frac{1}{2\pi}\int_0^{2\pi} L_\varphi d\varphi \;=\; l \;. \tag{92}$$

The non-trivial problem is to find independent paths $C_{2,3}$ around the 2-tori in $(\vartheta, \psi, L_\vartheta, L_\psi)$-space, with l considered a parameter and $\varphi =$ constant. The action integrals are then

$$I_{2,3} \;:=\; \frac{1}{2\pi}\oint_{C_{2,3}} L_\vartheta d\vartheta \;+\; \frac{1}{2\pi}\oint_{C_{2,3}} L_\psi d\psi \;. \tag{93}$$

As the 2-tori remain 2-tori after the transformation $(\boldsymbol{\alpha}, \boldsymbol{L}) \mapsto (\vec\gamma, \vec l)$ to Euler variables, we attack the problem in these more convenient though non-canonical variables. To give the reader an impression of how hopeless it would be to find suitable paths by means of intuitive guessing, Fig. 8 shows the torus $(h, l, k) = (2.2, 1.259, 0.492)$ in three different 3-D projections of $(\vec\gamma, \vec l)$-space. A systematic procedure is clearly indispensable.

4.1. Independent paths around the 2-tori

The key to the algorithm of [Dullin & Wittek, 1993] is the observation that Hamiltonian and Kovalevskaya constant generate two independent flows on the invariant tori. Just as the Hamiltonian flow of a quantity F is given by Eqs. (34) and (45), the Kovalevskaya flow is defined as

$$\frac{dF}{dt_K} \;=\; \{F, k^2\}_E \;=\; \nabla_E F^T \mathbf{M}_E \nabla_E k^2 \tag{94}$$

with

$$\mathbf{M}_E \nabla_E k^2 \;=\; \begin{pmatrix} m_2\gamma_3 & & \\ m_1\gamma_3 & & \\ -m_2\gamma_1 - m_1\gamma_2 & & \\ m_2 r & - & 2k_2\gamma_3 \\ m_1 r & + & 2k_1\gamma_3 \\ -2m_2 p - 2m_1 q & + & 2k_2\gamma_1 - 2k_1\gamma_2 \end{pmatrix} , \tag{95}$$

where the abbreviations

$$\begin{aligned} k_1 &= p^2 - q^2 + \gamma_1 \,, \\ k_2 &= 2pq + \gamma_2 \end{aligned} \tag{96}$$

and

$$\begin{aligned} m_1 &= k_1 l_1 + k_2 l_2 \;=\; 2p(p^2 + q^2 + \gamma_1) + 2q\gamma_2 \,, \\ m_2 &= k_1 l_2 - k_2 l_1 \;=\; -2q(p^2 + q^2 - \gamma_1) - 2p\gamma_2 \end{aligned} \tag{97}$$

have been used.

A given point P on the torus, if subject to the Hamiltonian flow g_H, is carried in time t_H to another point $\mathrm{P}(t_H) = g_H^{t_H}\mathrm{P}$ on the same torus. In the case of irrational winding number, it never returns to the original point, as t_H increases. So g_H in itself does not produce a closed path. Likewise, if P is taken up by the Kovalevskaya flow g_K, its trajectory $\mathrm{P}(t_K) = g_K^{t_K}\mathrm{P}$ in general does not close upon itself. But as the two flows are linearly independent, it is possible to construct closed paths by combining them. Fig. 9 shows how this is done in principle. Following the Hamiltonian dynamics from P_0 to A, and then the Kovalevskaya dynamics from A to P_1, one arrives at $\mathrm{P}_1 \equiv \mathrm{P}_0$ along a nontrivial closed path. Similarly, going from P_0 to B with g_H, and then backward in time t_K with g_K, one arrives at $\mathrm{P}_2 \equiv \mathrm{P}_0$ along an independent closed path. Furthermore, as Hamiltonian and Kovalevskaya constant are in involution, their flows commute, and the closed paths from P_0 to P_1 or P_2 may be generated by the flow $g_H^{t_H} g_K^{t_K} = g_K^{t_K} g_H^{t_H}$ which is generated by a linear combination of the Hamiltonian and Kovalevskaya vector fields,

with suitably chosen times t_H and t_K:

$$\frac{d}{dt}\begin{pmatrix}\vec{\gamma}\\ \vec{l}\end{pmatrix} \;=\; t_H\,\mathbf{M}_E\nabla_E\mathcal{H} \;+\; t_K\,\mathbf{M}_E\nabla_E k^2 \;. \tag{98}$$

So the problem is reduced to finding the times t_H and t_K. The algorithm achieves this in the following way. For all values of t_H and t_K in a sufficiently large and dense lattice, the distance between P and $g_H^{t_H} g_K^{t_K}$P in phase space is measured, and recorded as a contour plot in the (t_H, t_K)-plane. The result is a doubly periodic lattice, each cell of which covers the torus just once. If the corners of a cell are chosen to be the points equivalent to P, then their distance values to point P$= g_H^0 g_K^0$P are zero; at all other points, the distance is positive. In a real computation, the distance zero is of course never found exactly, and a search for minima sufficiently close to zero must be performed. Once this has been achieved for a torus with parameters (h, l, k), one proceeds in small increments to neighboring tori, using Newton's method to find their minima from the old.

The procedure has proven to be efficient and accurate in [Dullin & Wittek, 1993]. In the present analysis of the Kovalevskaya system, we show 6 energy surfaces with some 200 times 200 tori each. After identification of the two independent paths, the integration of Eqs. (98) was done with standard integration routines of variable step size, the minimum number of steps being 4000. The action integrals (90) were computed along with that integration, using the transformation Eqs. (6) and (26) - (28) to recover the canonical variables from Euler's. The calculations were done on an R4400 64 bit processor of Silicon Graphics (100 MHz); they required about 3 hours CPU time for energy $h = 0.5$, and 16 hours for $h = 3.3$.

Let us comment on a number of difficulties, one systematic, the others numerical. The systematic problem pertains to the φ-motion which cannot quite be ignored inspite of the fact that it has been separated from the rest. The action integrals (93) assume φ to be constant, but in fact it changes according to Eq. (29) which we integrate along with the determination of the γ_i and l_i, in order to keep track of its behavior. Let $\Delta\varphi$ be the total change of φ along a given closed path on the 2-torus. Now vary the parameters; it sometimes happens that $\Delta\varphi$ jumps by $\pm 2\pi$ because the trajectory encircles the points $\vartheta = 0$ or π once more or less than before. This jump is then accompanied by a jump of the first part of the action integral (90), and since the total integral (90) is continuous, the integral of interest (93) jumps by $\mp l$. Such jumps are compensated by addition of $\pm l$ to have the $I_{2,3}$ continuous within a region of continuity of the energy surface.

The first numerical problem arises in the vicinity of elliptic critical orbits where one of the two diameters of the torus tends to zero. As a consequence, the distances along one of the two fundamental paths never get very big, so the minima are ill defined. The second problem appears in connection with separatrices where one of the fundamental frequencies tends to zero. This implies long integration times with accumulating errors; moreover, the φ-variable tends to make many turns in these cases, and it becomes difficult to keep track of the number of jumps it makes between neighboring parameter values.

Finally, it is not trivial to secure a smooth connection of pieces that belong together, but are computed with different Fomenko graphs. Consider, e. g., the Poincaré-Fomenko stack of Fig. 7d. Starting at the top, its succession of graph types is A, B, E, D. At the transition from type B to E, our data did not quite allow to present a smooth red surface, see Fig. (10d). This may be related to the very thin strips of type C and D graphs coming in, see Fig. (5).

4.2. Action integrals and energy surfaces

The computation described in the last section provides a set of raw data that have to be turned into the desired energy surfaces $\mathcal{H}(\boldsymbol{I}) = h$ in action space. For the simpler integrable cases of rigid body dynamics, i. e. Euler and Lagrange tops (including the modifications introduced by a cardanic suspension) this goal was first achieved in 1990 [Richter, 1990], using explicit expressions for the I_i in terms of elliptic integrals. There it was possible to associate the actions I_2 and I_3 with the motion of Euler angles ψ and ϑ respectively. As the character of φ- and ψ-motion was rotational (except for the low energy regime of the Lagrange top), and that of the ϑ-motion oscillatory, it was natural to have I_φ and I_ψ of both signs, corresponding to the two senses of the rotation, and only positive I_ϑ. The shape of the energy surfaces in $(I_\varphi, I_\psi, I_\vartheta)$-space turned out to be that of pyramids with bottom in the (I_φ, I_ψ)-plane and top on the I_ϑ-axis.

In the very low energy range $h \ll 1$, the actions can easily be computed analytically. Expanding the Hamiltonian (39) to second order in the small dynamical variables $\delta\vartheta = \vartheta - \pi/2$, $\delta\psi = \psi - \pi/2$, and L_φ, L_ϑ, L_ψ, we get

$$\mathcal{H} \;=\; \frac{1}{4}L_\varphi^2 \;+\; \frac{1}{4}L_\vartheta^2 \;+\; \frac{1}{2}(\delta\vartheta)^2 \;+\; \frac{1}{2}L_\psi^2 \;+\; \frac{1}{2}(\delta\psi)^2 \;, \tag{99}$$

which can be separated into a φ-rotation, a ϑ-oscillation with frequency $\omega_\vartheta = 1/\sqrt{2}$, and a ψ-oscillation with frequency $\omega_\psi = 1$. The action representation is therefore

$$h \;=\; \frac{1}{4}I_\varphi^2 \;+\; \frac{1}{\sqrt{2}}I_\vartheta \;+\; I_\psi \;. \tag{100}$$

The energy surface in $(I_\varphi, I_\psi, I_\vartheta)$-space intersects the plane $I_\varphi = 0$ in a straight line $I_\vartheta = \sqrt{2}(h - I_\psi)$, representing the possible superpositions of the two oscillations. The other two planes are intersected in parabolas, corresponding to superpositions of φ-rotation and one of the two oscillations. The Kovalevskaya constant reduces to

$$k^2 \;=\; 1 + \frac{1}{2}L_\varphi^2 \;-\; \frac{1}{2}L_\vartheta^2 \;-\; (\delta\vartheta)^2 \;=\; 1 - 2h + I_\varphi^2 + I_\psi \;, \tag{101}$$

and ranges from $1 - 2h$ at $I_\varphi = I_\psi = 0$ to $1 + 2h$ at $I_\vartheta = I_\psi = 0$.

Fig. 10a shows a very similar energy surface for $h = 0.5$, calculated with data from the above procedure. The I_φ-axis points to the right, the I_ψ-axis to the left, and the I_ϑ-axis upwards. As I_ψ and I_ϑ correspond to oscillatory motion, only positive values

are meaningful whereas $I_\varphi = l$ may have both signs. Because of the symmetry of the Hamiltonian under time reversal, we ignore the $I_\varphi < 0$ part of the energy surface in all the following. The correspondence of the Poincaré-Fomenko stack of Fig. 7a and the surface of Fig. 10a is obvious.

The relation of Figs. 7b and 10b is more complicated and needs a number of comments. One may take the point of view that actions should be defined separately for each part of the energy surface which is confined by a separatrix, the reason being logarithmic singularities at separatrices that prevent analytic continuation. This would suggest to consider five independent sheets in the energy range $1 < h < 2$: one each for the red, orange, and blue regions of the Fomenko stack, and two for the two symmetry related green. On the other hand, one may try to preserve as much continuity as possible, using the sum rules obeyed across separatrices, and the freedom to transform actions by matrices from $\mathrm{SL}_2(\mathbf{Z})$. This was a guiding principle in our construction of energy surfaces, together with the rules formulated in [Dullin & Wittek, 1993] that elliptic periodic orbits should have one action zero, and that two different elliptic periodic orbits, if connected by a smooth path of non-critical tori, should have different zero actions. Comparison of Figs. 10a and 10b presents no problem as far as the red and orange parts are concerned. The rupture between the two is caused by the blue region creeping in at low l. But where to put the blue sheet? We mentioned that at $l = 0$, the blue and red orbits are related by S_3 symmetry. Their actions are therefore the same, so the red and blue sheets coincide along a line in the $I_\varphi = 0$-plane. Furthermore, the elliptic tori at the center of the blue region in Fomenko stack 7b should have another action zero, according to the above rules; so they must be mapped to the plane $I_2 = 0$ where they lie slightly above the red sheet. The action integrals of the green region connect smoothly to the red or orange, if we take the sum for the two symmetry related green orbits. This was done in Fig. 10b to obtain maximum continuity. One may argue whether that is convenient. An alternative would be to give separate representations to the two green orbits, and to distinguish them by the sign of I_2. With that choice, the blue sheet should also lie on the side of negative I_2, because of its relationship to the red. Fig. 11 illustrates this alternative for the case of $h = 10$.

Fig. 10c gives the energy surface for $h = 2.2$. The orange part has almost disappeared, and yellow shows up for the first time. Its symmetry properties are the same as those of green, so in the alternative representation, the two would undergo identical changes. At energy $h = 2.45$, see Fig. 10d, the orange piece is absent, and yellow connects to red along a separatrix. A new feature is observed at energy $h = 3.3$, see Fig. 10e: an edge has formed within the yellow sheet, and a pyramid structure reminiscent of those in [Richter, 1990] has formed. The constructional principle that leads to the edge is a change in the linear combination of raw data, necessitated by the requirement of continuity with both green and red. The locus of the edge on the Poincaré-Fomenko stack 7h is the intersection of graphs of type E with the line $\gamma_1 = 0$, where the condition for Appelrot's class III holds. A physical interpretation is that corresponding orbits have the rigid body's 3-axis go through the z-axis of the fixed system. This implies a jump in the rotation of the

φ-variable with respect to the fixed axis. Purple and turquois pieces should be added where the green sheet meets the yellow edge, but at the resolution of the picture they are too small.

Our last picture, Fig. 10f, illustrates the high energy end, $h = 10$. The yellow part assumes most of the energy surface and resembles the high energy surfaces of the Lagrange case in ref. [Richter, 1990]. The green seam marks the low k end of the energy surface (Appelrot class I) while the red tori survive at the high k end. In the limit $h \gg 1$ the Kovalevskaya constant can be expressed in terms of the total angular momentum $l_{tot}^2 = l_1^2 + l_2^2 + l_3^2$ as

$$k \;=\; \frac{1}{2} l_{tot}^2 - h \;. \tag{102}$$

The lines of constant k can then be interpreted as lines of constant total angular momentum. For comparison, we show in Fig. 11 the alternative version where the S_3-symmetry related green and yellow orbits are distinguished by the sign of I_2. The red part of the energy surface represents tori which possess this symmetry individually, and has only positive values of I_2; in this rendering, it is no longer continuously connected to the yellow part.

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Figure captions

Fig. 1: Definition of Euler's angles as used in the text. The (x, y, z)-system is fixed in space, gravity pointing downward. The $(1, 2, 3)$-system is the rigid body's frame of reference. Its position relative to the fixed system is given uniquely by angles $0 \leq \varphi < 2\pi$, $0 \leq \vartheta \leq \pi$, $0 \leq \psi < 2\pi$. The Kovalevskaya top has its center of mass on the negative 1-axis, so its stable equilibrium position has the 1-axis aligned with the z-axis. This means $\vartheta = \psi = \pi/2$; the angle φ is undetermined.

Fig. 2: Projection $\mathcal{P}$ of a Poincaré surface of section onto the (ϑ, L_ϑ)-plane, for $h = 2.2$, $l = 1.259$. The angle ϑ ranges from approximately $-\pi/4$ to $5\pi/4$, the momentum L_ϑ between $\pm\sqrt{4h - l^2}$. The lines are intersections of invariant tori (except for the two horizontal pieces). The open circle marks an elliptic fixed point of period 1, representing a critical torus of class IV, of maximum k. The four full circles mark two stable orbits of period 2, the pair on the ϑ-axis being the S_3-symmetry partner of the off-axis pair. They are class I ($k = 0$). The full squares are hyperbolic critical orbits of class III; they are the centers of a system of separatrices which has a tangency with the boundary at the four empty squares. The horizontal segments intersect every torus of the system in exactly one point (except for the tori in the off-axis lobes); they form a possible Fomenko graph for this particular Poincaré section.

Fig. 3: Series of typical Poincaré surfaces of section from the 10 phases defined in Fig. 5. Regions with topologically equivalent tori are coded by colors; they are closely related to Kharlamov's regions, although there are discrepancies as explained in the text: i - red/orange, ii - green/turquois, iii - yellow, iv - blue, v - purple; within each region, the change from dark to light codes for increasing values of k. The invariant tori are given as level lines of equal color. The ϑ-scale extends from $-\pi/2$ to $3\pi/2$ in each case, the maximum of $|L_\vartheta|$ is $\sqrt{4h - l^2}$ and changes from picture to picture. A: Low energy $h = 0.5$, $l^2 = 1$. The darkest point near the right boundary is a stable periodic orbit of class II, the lightest point near the center a stable periodic orbit of class IV. B: $h = 2.2$, $l^2 = 1.58$. The class II point of Fig. A has developped into 3 unstable points of class III and two stable orbits of period 2, the centers of the green lobes. C: $h = 1.9$, $l^2 = 0.34$. Two new orbits of class IV have appeared through a tangent bifurcation; the unstable partner introduces a separatrix in Kharlamov's region i, dividing it into a red and an orange part. There is a local maximum of k in the blue region. D: $h = 2.2$, $l^2 = 0.40$. The system of separatrices around the green and blue lobes is rearranged, and the orange region has bifurcated into two S_3-related yellow regions. E: $h = 2.85$, $l^2 = 1.43$. The blue orbits have disappeared, their stable center has collided with the two unstable off-axis point to give way to just one unstable orbit. F: $h = 3.5$, $l^2 = 5.30$. Comparing with Fig. A, a new disk has formed around the unstable equilibrium point at $\vartheta = 3\pi/2$. This disk contains a pair of orbits, a stable one of class II and an unstable of class IV. The stable class

IV orbit near the right boundary of Fig. A has turned into a stable class II orbit, the S_3-related partner of the other class II orbit. G: $h = 3.5,\ l^2 = 4.82$. The two yellow centers have each bifurcated into two stable points of period 2 and class I (green), two unstable points of class III, and one stable point of period 1 and class I (purple). H: $h = 2.85,\ l^2 = 3.29$. The two disks of Fig. G have merged into an annulus. The two purple centers change their Appelrot class from I to IV, otherwise G and H are not very different and might be considered physically equivalent. I: $h = 2.8,\ l^2 = 3.26$. This Poincaré section should be compared to B. Two new disks have appeared, each containing a pair of orbits, one stable, the other unstable. The corresponding separatrix introduces a new division in the previously green region, part of which is therefore colored in turquois now. J: $h = 2.8,\ l^2 = 3.23$. The relation of J to I is the same as that of H to G. The two small disks have joined the big one, and the two purple centers have changed their class from I to IV, without change in stability. There is no physical distinction between region I and J. K: $h = 1.9,\ l = 0$. The topology is the same as in C, but there is now an additional symmetry with respect to the axis $\vartheta = \pi/2$; this reflects the time reversal symmetry. L: $h = 2.2,\ l = 0$. Topology as in D, but with time reversal symmetry as in K.

Fig. 4: Bifurcation lines in the (h, l^2) parameter plane, $0 \le h \le 4$, $0 \le l^2 \le 8$. Part T of the figure shows the lines defined by changes in the topology of the Poincaré surfaces $\mathcal{P}$. They consist of 1, 2, and 3 disks in the regions marked **1d**, **2d**, **3d** respectively; in region **a**, $\mathcal{P}$ is an annulus. Part I gives the numbers of critical orbits of class I in the various regions, 4+2 meaning 4 on-axis and 2 off-axis orbits; they are all elliptic. Part II/III is the bifurcation scheme for orbits of classes II (above the dotted line) and III (below). Orbits of class II are elliptic, class III orbits are hyperbolic. Part IV gives the numbers of class IV orbits; they all lie on the ϑ-axis. The dotted line marks a change of stability of one class IV orbit.

Fig. 5: Phase diagram of Poincaré surfaces of section, combining the information of the four parts of Fig. 4. The bifurcation scheme defines 10 different types of surfaces $\mathcal{P}$, denoted by letters A through J. Some "phases" are very tiny and require a blowup (see the inset). The open circles mark the parameter values chosen for the series in Fig. 3.

Fig. 6: Fomenko graphs for the 10 different phases of Fig. 5. There is a 1:1 correspondence of points in the graph and tori in the Poincaré surface of section. The vertical ordering follows the Kovalevskaya constant: low k at the bottom, high k at the top. The colors of the edges code for the type of tori. The ends of an edge represent critical tori; free ends are elliptic periodic orbits, branch points represent separatrices. The Appelrot classes of critical tori are indicated at the left, the Kharlamov regions at the right of each graph.

Fig. 7: Poincaré-Fomenko stacks, generated by superimposing the ϑ-axes of all Poincaré surfaces of given energy h. The horizontal axis of the pictures is ϑ, the vertical axis

is l^2, with maximum value $4h$. Stable critical points are indicated in black. Separatrices appear as changes in color; if that is not the case (because the enclosed region is entirely off-axis), they are also marked in black. These stacks are a qualitative representation of energy surfaces, with some redundancy because tori have two or four intersections with the ϑ-axis. The redundancy is used to select minimal stacks that are most convenient as a set of initial points in the numerical path finding algorithm. The energy values of the eight qualitatively different stacks are 0.5 (a), 1.9 (b), 2.2 (c), 2.45 (d), 3.3 (e). The last stack (f) with $h = 10$ has the same structure as stack (e), but illustrates how the yellow orbits dominate the Lagrange-Euler limit of high energies.

Fig. 8: Three 3-D projections of the same torus $(h, l, k) = (2.2, 1.259, 0.492)$. a: (γ_1, p, q)-space, b: $(\gamma_1, \gamma_2, \gamma_3)$-space, c:$(p, q, r)$-space. The pictures were generated by a single trajectory. In case b the projection is really 2-D rather than 3-D because of the restriction $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ to a sphere; therefore we followed the trajectory only for a short time. The figure is meant to illustrate the formidable difficulty to identify independent paths around the Kovalevskaya tori.

Fig. 9: Schematic description of the determination of fundamental paths P_0P_1 and P_0P_2 around a given torus. The line from P_0 to A, B, and parallel lines, represent the Hamiltonian flow; the line from P_2 to B, and parallel lines, indicate the flow generated by the Kovalevskaya constant. The thick lines are combinations of the two flows leading back to the original point in two inequivalent ways.

Fig. 10: Six energy surfaces in action representation $h = h(I_1, I_2, I_3)$. The axis pointing the right represents the first action $I_1 \equiv I_\varphi = l$, I_2 points to the left, and I_3 upwards. The axes are scaled according to the maximum value of l which is $2\sqrt{h}$. The energy values are a: $h = 0.5$, b: $h = 1.9$, c: $h = 2.2$, d: $h = 2.45$, e: $h = 3.3$, f: $h = 10$. Surfaces of this kind contain the relevant physics of an integrable mechanical system; besides the ordering of tori into types of different behavior, their slopes give frequencies and winding ratios. They are the basis for perturbation theory and semiclassical quantization.

Fig. 11: Another rendering of Fig. 10f. The I_2-axis is extended to negative values; positive values pointing towards the back. Two S_3-symmetry related yellow or green tori are distinguished by the sign of I_2.

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Figure captions

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Fig. 2 Example of a Poincaré section

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