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ARTICLES

Symplectic Invariants Near Hyperbolic-Hyperbolic Points

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Abstract—We construct symplectic invariants for Hamiltonian integrable systems of 2 degrees of freedom possessing a fixed point of hyperbolic-hyperbolic type. These invariants consist in some signs which determine the topology of the critical Lagrangian fibre, together with several Taylor series which can be computed from the dynamics of the system. We show how these series are related to the singular asymptotics of the action integrals at the critical value of the energy-momentum map. This gives general conditions under which the non-degeneracy conditions arising in the KAM theorem (Kolmogorov condition, twist condition) are satisfied. Using this approach, we obtain new asymptotic formulae for the action integrals of the C. Neumann system. As a corollary, we show that the Arnold twist condition holds for generic frequencies of this system.

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1. INTRODUCTION

Normal forms are a powerful tool in the theory of Hamiltonian systems. The Birkhoff normal form of a Hamiltonian near an equilibrium point is a classical example of a local normal form. Here we are interested in semi-local normal forms for integrable Hamiltonian systems. The additional structure given by the integrability allows to work with semiglobal objects, namely the action variables of the integrable system, at the same time as with local objects obtained from the Birkhoff normal form. Combining the two approaches reveals the symplectic invariants of the Liouville foliation induced by the integrable system near the preimage of a particular critical value. For focus-focus points in two degrees of freedom this has been worked out in [1], and was used for the computation of non-degeneracy conditions in [2]. Here we present the semiglobal symplectic invariants attached to a hyperbolic-hyperbolic point, and use them to prove non-degeneracy. The theory is applied in the example of the C. Neumann system.

Hyperbolic-hyperbolic singularities appear in many other classical integrable systems, and most of them are described in the book [3]. The first systematic study of this kind of singularities started with the book [4], which explains the different possible topologies of the corresponding singular Lagrangian fibre. Simpler approaches appeared later; see [3, 5]. In this article we show that the symplectic geometry of such systems is governed by finer numerical invariants. We will first define these invariants in terms of the dynamics of the system, in all possible topological configurations; then we show how these invariants are related to the singular asymptotics of action integrals, and thus, how they can be used for KAM non-degeneracy. In the C. Neumann system, the leading terms if these invariants can be computed explicitly.

Let M be a smooth symplectic manifold of dimension 4 (smoothness is either C^∞ or analytic). Let H be a smooth function on M (the Hamiltonian). Assume that H is *Liouville integrable*: there exists a smooth function K on M such that $\{H, K\} = 0$ (for the symplectic Poisson bracket) and $dH \wedge dK \neq 0$

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almost everywhere on M. We shall always assume that the energy-momentum map $F = (H, K)$ is proper.

In this article we are interested in quantities defined by the dynamics of H and the symplectic geometry of the singular Lagrangian foliation $F = (H, K) : M \to \mathbb{R}^2$ *in a neighborhood of a singular leaf of hyperbolic-hyperbolic type, of complexity 1 (in the sense of Bolsinov–Fomenko)*, which means the following.

A point $m \in M$ is *critical* if $dH(m) \wedge dK(m) = 0$: the rank of $dF(m)$ is not maximal. If $dF(m) = 0$ one can consider the *hessians* $H''(m)$ and $K''(m)$. Such a point m is of *hyperbolic-hyperbolic* type whenever the linear space of quadratic forms spanned by $H''(m)$ and $K''(m)$ admits, in some canonical coordinates (x_1, x_2, ξ_1, ξ_2), the following basis (q_1, q_2):

$$q_1 = x_1\xi_1, \qquad q_2 = x_2\xi_2.$$

Let m_0 be such a hyperbolic-hyperbolic critical point, and let $c_0 = F(m_0)$ be the corresponding critical value. The associated *critical leaf* Λ_0 is the connected component of $F^{-1}(c_0)$ containing m_0. We shall assume that the complexity of the fibre is 1, which means that m_0 is the only critical point of rank 0 of Λ_0.

2. DYNAMICAL DEFINITION OF THE SYMPLECTIC INVARIANTS

In this section several semiglobal symplectic invariants for simple hyperbolic-hyperbolic foliations will be derived. Two such foliations are called semiglobally symplectically equivalent whenever there exists a symplectomorphism from one to the other, defined in a saturated neighborhood of the critical leaf Λ_0, and sending leaves to leaves. Then semiglobal symplectic invariants are by definition quantities that are defined on the set of equivalence classes.

In later sections, we will show that the invariants presented here are not all independent: several relations exist, depending on the topological type of the singular leaves.

We consider a saturated neighborhood Ω of the critical fibre Λ_0. Let Σ be the critical set of the map F in Ω:

$$\Sigma := \{m \in \Omega, \quad \text{rank}\, dF(m) < 2\},$$

and $\Sigma_0 = \Sigma \cap \Lambda_0$. The starting point for the definition of the symplectic invariants is Eliasson's normal form, which states that there exists a system of canonical coordinates near the critical point m_0 in which

$$\{H, q_i\} = \{K, q_i\} = 0 \text{ for } i = 1, 2. \tag{1}$$

In the analytic category, this is equivalent to saying that $F = g \circ (q_1, q_2)$ for some analytic local diffeomorphism g on $\mathbb{R}^2$.

It is well known that in the C^∞ setting things are a bit more involved due to the non-connectedness of the fibres of $\mathbf{q} = (q_1, q_2)$ (see [7] or [8]). Actually (1) is equivalent to the following statement: let H_i be the coordinate hyperplane $x_i = 0$. Then $\mathbb{R}^4 \setminus (H_1 \cup H_2)$ has four connected components; let $E_{\varepsilon_1,\varepsilon_2}$, $\varepsilon_i = \pm$ be their closures:

$$E_{\varepsilon_1,\varepsilon_2} := \{\varepsilon_1 x_1 \geqslant 0,\ \varepsilon_2 x_2 \geqslant 0\}. \tag{2}$$

Then Eq. (1) holds in a small ball B around the origin if and only if the following conditions are both fulfilled:

1. For each $\varepsilon_1, \varepsilon_2$ there is a local diffeomorphism $g_{\varepsilon_1,\varepsilon_2}$ of $(\mathbb{R}^2, 0)$ such that $F = g_{\varepsilon_1,\varepsilon_2} \circ (q_1, q_2)$ on $E_{\varepsilon_1,\varepsilon_2} \cap B$;
2. the differences $g_{\varepsilon_1,\varepsilon_2} - g_{\epsilon'_1,\epsilon'_2}$ are flat (all derivatives vanish) at each point of $\mathbf{q}(E_{\varepsilon_1,\varepsilon_2} \cap E_{\epsilon'_1,\epsilon'_2})$.

Of course in the analytic category we recover the simpler characterization cited above. In order to simplify the notation in the smooth case we define the space $\mathbb{X}$ whose smooth functions $C^\infty(\mathbb{X})$ are functions of two variables (x, ξ) commuting with $q = x\xi$. If f is such a function, instead of writing $f(x, \xi)$ we shall use the notation $f(\tilde{q})$. So (1) says that H and K are in $C^\infty(\mathbb{X} \times \mathbb{X})$: they are functions of $(\tilde{q_1}, \tilde{q_2})$. Notice that such functions have a well defined Taylor series at the origin.

The first consequence of this "normal form" is that the singular foliation defined by F near m_0 is symplectically equivalent to the foliation defined by $\mathbf{q} = (q_1, q_2)$ in $\mathbb{R}^4$. In particular it is immediate to check that $\Sigma \cap B$ in these coordinates is the union of the two planes $P_1 := \{x_2 = \xi_2 = 0\} \cap B$ and $P_2 := \{x_1 = \xi_1 = 0\} \cap B$. And $\Sigma_0 \cap B$ is the union of the two "crosses" given by the coordinate axis of each plane P_i.

Our starting point will be the structure of the "skeleton" Σ_0, which has been previously described in [9]:

Lemma 1. *Σ_0 is diffeomorphic a "four-leaf clover",* i.e. *the union of two "figures eight" intersecting transversally at their origin.*

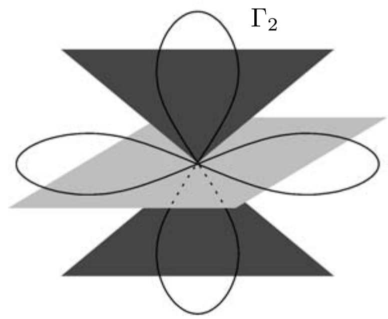


Fig. 1. The critical set $\Sigma_0 \subset \Lambda_0$.

Proof. The local structure near the singular point m_0 was discussed above. Now since m_0 is the only fixed point (for F) on Σ_0, each connected component of $\Sigma_0 \setminus \{m_0\}$ is an orbit of a locally free Hamiltonian $\mathbb{R}$ action. Hence these components are diffeomorphic to lines or circles. Assume for the moment that there are no circles. By the compactness of Λ_0 (and hence Σ_0), the lines must end at a fixed point; therefore they are homoclinic orbits for m_0. More precisely, the component starting on P_1 is an orbit of $\mathcal{X}_{q_1}$ and must return on a stable manifold of $\mathcal{X}_{q_1}$, and hence on P_1. This means that m_0 together with the union of all connected components of Σ_0 connecting P_1 is a figure eight. Of course the same holds for P_2.

It remains to prove that there cannot be any circle component. This could be obtained as a consequence of a theorem of Knörrer [10]; we give a simpler proof here. Let γ be such a circle. By the non-degeneracy condition, this circle must be isolated : there is a tubular neighborhood Ω of γ such that $(\Omega \setminus \gamma) \cap \Sigma_0 = \emptyset$. One can assume that there exists a path connecting γ to m_0 through $\Lambda_0 \setminus \Sigma_0$. Indeed, since Λ_0 is path-connected, there is a path on Λ_0 that starts on γ and ends at m_0. If the rest of the path does not cross Σ_0, we are done. If it crosses some circle component of Σ_0, then we keep replacing γ by this new circle, until the path does not cross any more circle. It it crosses a line component, then we modify the path by closely following the line, without touching it, until it reaches m_0.

Consider the connected component C of $\Lambda_0 \setminus \Sigma_0$ containing the interior points of this path. Since our integrable system defines a locally free $\mathbb{R}^2$ action on C, the latter is either a plane, a cylinder or a torus. It cannot be a torus since it is not compact. If it is a cylinder, then the isotropy of the action is isomorphic to $\mathbb{Z}$, which means that there is a constant linear combination of H and K whose hamiltonian flow on C is periodic (i.e. of constant period). This is impossible near m_0 due to the full hyperbolicity of the critical point. Hence C is a plane. Hence $\partial C \subset \Lambda_0$ is connected, which excludes the coexistence of m_0 and γ in ∂C. A contradiction.

The second consequence of the local normal form is that there exists two functions J_1 and J_2 defined in an invariant neighborhood of Λ_0, that coincide with q_1 and q_2, respectively, in suitable local canonical coordinates near m_0. In the analytic setting one just has to let $(J_1, J_2) = g^{-1} \circ F$, and in the C^∞ category one uses the corresponding formula in each orthant $E_{\varepsilon_1,\varepsilon_2}$.

Then one can check that $\mathcal{X}_{J_1}$ vanishes on one of the "eight figure" in Σ_0 — let's call it Γ_2, while $\mathcal{X}_{J_2}$ vanishes on the other eight figure, which we call Γ_1.

Consider for the moment the flow of J_1, denoted by $\phi_t^{J_1}$. Fix a small $\delta > 0$, such that $J_i = q_i$ in the ball of radius 2δ around m_0. Let U (for "unstable") be the point (on Γ_1) with coordinates $(x_1, \xi_1, x_2, \xi_2) = (\delta, 0, 0, 0)$ and $S = (0, \delta, 0, 0)$ on the local stable manifold. Let $\mathcal{S}_U \subset \mathbb{R}^4$ be the local hypersurface near U defined by $x_1 = \delta$, and $\mathcal{S}_S = \{\xi_1 = \delta\}$. Then $\mathcal{S}_U$ and $\mathcal{S}_S$ are smoothly foliated by level sets of J_1; we let $\mathcal{S}_{U,j_1} := \mathcal{S}_U \cap J_1^{-1}(j_1)$:

$$\mathcal{S}_{U,j_1} = \{(x_1 = \delta, \xi_1 = j_1/\delta, x_2, \xi_2)\}.$$

In the same way let

$$\mathcal{S}_{S,j_1} = \{(x_1 = j_1/\delta, \xi_1 = \delta, x_2, \xi_2)\}.$$

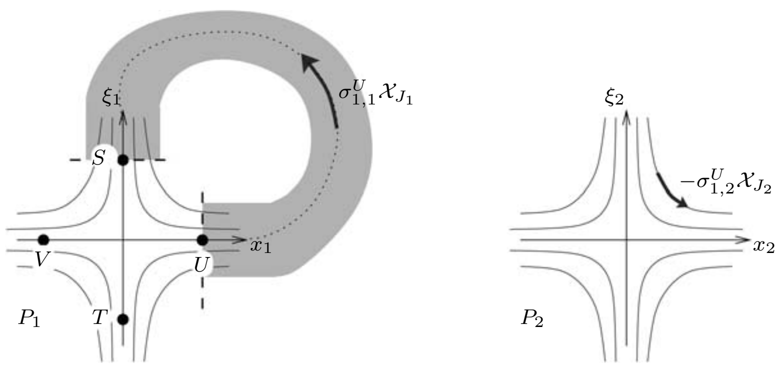


Fig. 2. Construction of the symplectic invariants.

Using if necessary the canonical transformation $(x_1, \xi_1) \to (\xi_1, -x_1)$ one can assume that the flow of J_1 takes U to S in positive time T. Hence there is a unique smooth function σ on $\mathcal{S}_{U,j_1}$ such that $\sigma(U) = T$ and $\phi_{\sigma(U')}^{J_1} \in \mathcal{S}_{S,j_1}$ for any $U' \in \mathcal{S}_{U,j_1}$. Of course, σ depends smoothly on j_1. Since ϕ^{J_1} and ϕ^{J_2} commute, and since $\mathcal{S}_{U,j_1}$ and $\mathcal{S}_{S,j_1}$ are both globally invariant by the flow of J_2, one can see that σ is actually a smooth function of $(J_1, \tilde{J}_2) \in \mathbb{R} \times \mathbb{X}$. Let's call it $\sigma_{1,1}$. Notice that $\mathcal{S}_{U,j_1}$ is naturally symplectomorphic to the local reduced manifold $J_1^{-1}(j_1)/\mathcal{X}_{J_1}$ at U; the same holds for $\mathcal{S}_{S,j_1}$ at the point S.

The following result is standard (see for instance [11, §22]) :

Lemma 2. *The map $U' \to \phi_{\sigma_{1,1}(U')}^{J_1}$ is symplectic for the natural symplectic form $d\xi_2 \wedge dx_2$ on $\mathcal{S}_{U,j_1}$ and $\mathcal{S}_{S,j_1}$.*

Let us identify $\mathcal{S}_{U,j_1}$ and $\mathcal{S}_{S,j_1}$ using the coordinates x_2, ξ_2. Then $\phi_{\sigma_{1,1}(U')}^{J_1}$ is a symplectic map from $\mathbb{R}^2$ to itself which preserves the quadratic foliation defined by $q_2 = x_2\xi_2$. Hence it is equal near the origin to its linear part at the origin composed (in any order) by the time-1 Hamiltonian flow of a smooth function commuting with q_2 (this is a theorem of Miranda–Zung [12]). But symplectic matrices preserving q_2 are just the diagonal matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $\lambda \in \mathbb{R}^*$. This group has two connected components, corresponding to the sign of λ. The positive component consists of matrices that are

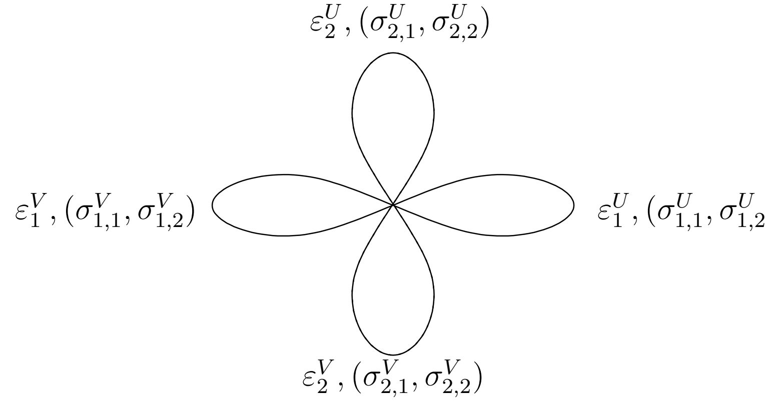


Fig. 3. The four signs with the eight σ-invariants.

exponentials of diagonal Hamiltonian matrices. Therefore there is a sign $\epsilon = \pm 1$ such that $\epsilon\phi^{J_1}_{\sigma_{1,1}}$ is the time-1 hamiltonian flow of a map σ' on $\mathbb{R}^2$ which commutes with q_2. Since everything here depends smoothly on the values of J_1 we have proven the following lemma:

Lemma 3. *There is a sign $\varepsilon_1 = \pm 1$ and a smooth function $\sigma_{1,2}$ of $(J_1, \tilde{J}_2)$ such that near the origin $\phi^{J_1}_{\sigma_{1,1}}$ (as acting on the coordinates (x_2, ξ_2)) is equal to the flow of $\mathcal{X}_{q_2}$ at the time $-\sigma_{1,2}$, composed with ε_1 times the identity.*

Notice that our construction uses only one half of the dynamics of J_1, namely the neighborhood of the "lobe" of the figure eight connecting U to S. One can also start again with the other lobe, connecting $V = (-\delta, 0, 0, 0)$ to $T = (0, -\delta, 0, 0)$ (see figure 2). To distinguish between the invariants obtained, we shall use the upperscripts U and V to refer to the starting points. More precisely, in order to have quantities that do not depend on δ, we shall make now the following normalization:

Definition 1. *If $\sigma_{1,i}$ are the functions defined in the construction above, we let*

$$\sigma_{1,1}^{U/V} := \sigma_{1,1} + \ln(\delta^2) \tag{3}$$

$$\sigma_{1,2}^{U/V} := \sigma_{1,2}, \tag{4}$$

where the exponent U refers to the quantities defined using the joint flow of (J_1, J_2) from U to S and the exponent V refers to the analogous construction from V to T.

Furthermore we can now perform the same construction using the flow of J_2 instead of the flow of J_1. Of course, this just amounts to swapping the indices $1, 2$. Hence we are left with eight functions $\sigma_{i,j}^{U/V}$ of J_i and $\tilde{J}_j$ and four signs $\varepsilon_i^{U/V}$ $(i, j = 1, 2)$, associated by their construction to the four parts of the skeleton (see figure 3).

Proposition 1. *The Taylor series $[\![\sigma_{i,j}^{U/V}]\!]$ of the eight functions $\sigma_{i,j}^{U/V}$, $i, j = 1, 2$, together with the four signs $\varepsilon_i^{U/V}$, $i = 1, 2$, are symplectic invariants of the foliation near the critical leaf Λ_0.*

In order to prove this proposition, we first notice that the only arbitrariness used in the construction of the invariants is the freedom offered by Eliasson's normal form. This freedom is characterized in the lemma below.

Lemma 4. *Suppose there is a local diffeomorphism g of $(\mathbb{R}^2, 0)$, and a local symplectomorphism ψ of $(\mathbb{R}^4, 0)$, such that*

$$q = g \circ q \circ \psi. \tag{5}$$

Then there is a transformation $M \in GL(2,\mathbb{Z})$ of the form

$$M = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \quad or \quad M = \begin{pmatrix} 0 & \epsilon_1 \\ \epsilon_2 & 0, \end{pmatrix} \tag{6}$$

with $\epsilon_j \in \{-1,1\}$, such that the map $Mg - \mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2$ is flat (all derivatives vanish at the origin).

Proof. Using the linearized version of (5), it is easy prove that there exist a matrix M such that Mg is tangent to the identity. This is a small exercise in symplectic linear algebra which we leave to the reader. So we can assume that g is tangent to the identity.

Consider the Liouville 1-form $\alpha = \xi dx = \xi_1 dx_1 + \xi_2 dx_2$. Since ψ is symplectic, $\alpha - \psi^*\alpha$ is closed. Hence there exists a smooth function $h \in C^\infty(\mathbb{R}^4, 0)$ such that

$$\psi^*\alpha = \alpha + dh.$$

For $c \in \mathbb{R}^2$, let Λ_c be a simply connected open subset of $q^{-1}(c)$. Since the restriction $\alpha_{\upharpoonright\Lambda_c}$ is closed (Λ_c is a Lagrangian submanifold of $\mathbb{R}^4$), its integral along a path drawn on Λ_c only depends on the extremities A and B of the path. We denote it by

$$\int_A^B \alpha.$$

The image of Λ_c under ψ^{-1} is again a smooth Lagrangian submanifold of $\mathbb{R}^4$, and $q(\psi^{-1}(\Lambda_c)) = g(c)$. We wish to examine the consequence of the formula

$$\int_A^B \alpha = \int_{\psi^{-1}(A)}^{\psi^{-1}(B)} \psi^*\alpha, \tag{7}$$

for some properly chosen A and B. Notice already that

$$\int_A^B \alpha = \int_{\psi^{-1}(A)}^{\psi^{-1}(B)} \alpha + dh = \int_{\psi^{-1}(A)}^{\psi^{-1}(B)} \alpha + h(\psi^{-1}(B)) - h(\psi^{-1}(A)).$$

It remains to calculate $\int_A^B \alpha$ for arbitrary points A, B in Λ_c. This is in fact explicit : let $(x_1^A, \xi_1^A, x_2^A, \xi_2^A)$ be the coordinates of A, and a similar notation for B. For simplicity we shall assume that Λ_c is convex with respect to the $\mathbb{R}^2$ hamiltonian action generated by q (this is enough for our purposes). Then we can choose the path

$$x_j(t) = x_j^A e^{\alpha_j t}, \quad \xi_j(t) = \xi_j^A e^{-\alpha_j t},$$

where $t \in [0,1]$, and α_j is chosen accordingly :

$$\alpha_j = \ln(x_j^B/x_j^A) = \ln(c_j/x_j^A\xi_j^B) = \ln(x_j^B\xi_j^A/c_j) = \ln(\xi_j^A/\xi_j^B).$$

(These equalities hold as soon as they are well-defined. One of them will be preferred depending on our choice of A and B.) Then

$$\begin{aligned} \int_A^B \xi dx &= \int_0^1 \xi_1(t)x_1'(t) + \xi_2(t)x_2'(t)dt \\ &= \alpha_1 c_1 + \alpha_2 c_2. \end{aligned} \tag{8}$$

As we did in the construction of the invariants, let us now fix some small $\delta > 0$ and choose A in a (even smaller) neighborhood of the point $S = (\delta, 0, 0, 0)$, and B in a neighborhood of the point $U = (0, \delta, 0, 0)$. Namely let

$$A = A(c) = (\delta, c_1/\delta, 0, 0), \qquad B = B(c) = (c_1/\delta, \delta, 0, 0).$$

Then for c in a neighborhood of the origin, one has

$$\int_A^B \alpha = c_1(\ln|c_1| - \ln(|x_1^A \xi_1^B|)) = c_1 \ln|c_1| - c_1 \ln \delta^2.$$

On the other hand, one computes also

$$\int_{\psi^{-1}(A)}^{\psi^{-1}(B)} \alpha = \hat{c}_1 \left(\ln|\hat{c}_1| - \ln\left|x_1^{\psi^{-1}(A)} \xi_1^{\psi^{-1}(B)}\right| \right),$$

with the notation $(\hat{c}_1, \hat{c}_2) = g(c_1, c_2)$. For δ small enough there is a neighborhood of U whose image by ψ^{-1} does not contain the origin; the same holds for a neighborhood of V. Hence the function

$$c \mapsto x_1^{\psi^{-1}(A)} \xi_1^{\psi^{-}(B)}$$

is smooth and never vanishing, for small c.

Hence formula (7) says that the function

$$c \mapsto c_1 \ln c_1 - \hat{c}_1 \ln \hat{c}_1 \tag{9}$$

is C^∞ for c in a small neighborhood of the origin. Since g is tangent to the identity, one has

$$\hat{c}_1 = c_1 + \mathcal{O}(|c|^2).$$

Moreover, it follows from (5) that the subset

$$\{z = (z_1, z_2) \in \mathbb{R}^4; \quad \operatorname{rank}(dq(z)) \leqslant 1\}$$

is invariant by ψ. This critical set is the union of the two 2-planes $\{z_1 = 0\}$ and $\{z_2 = 0\}$. Since g is tangent to the identity, each of these planes is invariant under ψ. Therefore $\psi^{-1}(0, z_2) = (0, \hat{z}_2(z_2))$, which implies, using (5),

$$g \circ q(0, z_2) = q(0, \hat{z}_2) = (0, q_2(0, \hat{z}_2)).$$

This says that for any c_2 in a neighborhood of 0, $\hat{c}_1(0, c_2) = 0$. Hence one can write

$$\hat{c}_1 = c_1(1 + \mathcal{O}(|c|)).$$

Finally, if follows from this and (9) that the function

$$c \mapsto (c_1 - \hat{c}_1) \ln|c_1|$$

is smooth in a neighborhood of the origin. This is possible only when the function $c \mapsto c_1 - \hat{c}_1$ is flat along the axis $c_1 = 0$. Repeating the argument for the second component of c, we see that g must be flat along both coordinate axis, and in particular it must be flat at the origin.

Proof of Proposition 1. The symplectic invariants are defined by the dynamics of the functions J_1 and J_2, which, in some symplectic coordinates near m_0, coincide with q_1 and q_2, respectively. Suppose one chooses now another set of canonical coordinates near m_0 where the singular foliation is again reduced to the standard model (q_1, q_2), yielding new Hamiltonians $\hat{J}_1$ and $\hat{J}_2$. It means that there is a local symplectomorphism ψ of $\mathbb{R}^4$, such that $\{q_i \circ \psi, q_j\} = 0$, for all i, j in $\{1, 2\}$. In the analytic setting, we are then immediately in the situation of lemma 4. In the C^∞ setting, we are also in entitled to apply the lemma, provided we restrict to an orthant $E_{\varepsilon_1, \varepsilon_2}$ (see formula (2)). In the new canonical coordinates, the ordering of indices 1, 2 is arbitrary : therefore one can always assume that the matrix M of the lemma has the form

$$M = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2. \end{pmatrix} \tag{10}$$

Now, remember that in the definition of the invariants, it was assumed that the flow of J_1 takes U to S in positive time, and similarly for J_2. This fixes the signs $\epsilon_1 = \epsilon_2 = 1$.

The result of the lemma is that $(\hat{J}_1, \hat{J}_2)$ differ from (J_1, J_2) by a function which is flat along Λ_0 (and still invariant by the flow of the system), entailing that all symplectic invariants are indeed the same.

Remark 1. As will be shown below, the four signs $\varepsilon_i^{U/V}$, $i = 1, 2$ completely determine the *topology* of the critical fibre Λ_0 (and of the whole semiglobal foliation as well). For each value of these signs, the σ-invariants $[\![\sigma_{i,j}^{U/V}]\!]$, which were defined using dynamical considerations, have to do with the finer symplectic structure of the foliation.

Remark 2. We have proved that the eight $[\![\sigma_{i,j}^{U/V}]\!]$'s are symplectic invariants. This does not mean that they are independent. Actually we shall see in the next section that in general they are partial derivatives of only four Taylor series, and depending on the topological type of the critical fibre, further relations may exist. We conjecture that these four Taylor series, together with the signs $\varepsilon_i^{U/V}$ and the aforementioned relations are a complete and minimal set of invariants for the foliation near Λ_0, in the analytic category. It is probably false in the C^∞ category. In the last section we will see that in examples with a non-generic large discrete symmetry group like the C. Neumann system additional relations between the invariants may exist.

3. RELATIONS BETWEEN THE σ-INVARIANTS AND ACTION INTEGRALS

The topological type of the foliation near Λ_0 is not unique. Actually, according to Lerman and Umanskii, there are 4 possible types. The goal of this section is to show that, for each topological type, the invariants that we have defined above can be properly combined in order to be expressed in terms of action integrals along cycles on the Liouville tori (theorem 1 below). As a corollary, we obtain general formulas for the singular asymptotics of the action integral close the the singular leaf.

3.1. The Four Topological Types

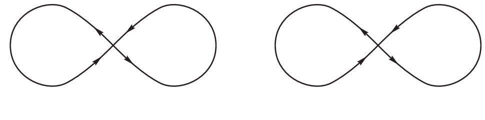
In order to recover the different topological types, the σ invariants can be neglected. Actually one can think of them as constants: $\sigma_{i,1}^{U/V} = 1$ and $\sigma_{i,2}^{U/V} = 0$ ($i = 1, 2$). So one just has to deal with the signs $\varepsilon_i^{U/V}$. In the discussion below we shall group these four signs as follows: $(\varepsilon_1^U \varepsilon_1^V)(\varepsilon_2^U \varepsilon_2^V)$. The determination of what the exponents ^{U} and ^{V} refer to was arbitrary, so the order in each parenthesis is unimportant; for instance $(+-)(++)$ and $(-+)(++)$ define the same system. Of course the indices $i = 1$ or 2 play the same role as well; so the order of the two parenthesis is also irrelevant; for instance $(+-)(++)$ is the same as $(++)(+-)$. According to these rules one is left with six possibilities:

$$(++)(++),\ (++)(+-),\ (++)(--),\ (--)(--),\ (+-)(+-),\ (+-)(--).$$

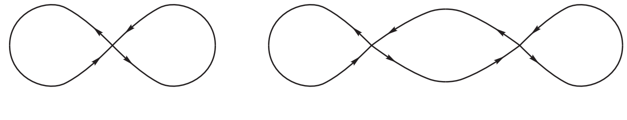
It turns out that the last two cases are impossible; we will explain this below (lemma 5). So we are left with four cases which are all realizable. In fact, there are explicit models that we describe now.

These models are also described in the book by Lerman and Umanskii [6], and in the book by Bolsinov and Fomenko [3, p. 345]. We recall here the latter description, both for the sake of completeness and because it helps understanding the construction of action integrals, but nothing will rely on it. This description is a particular case of a general theorem by Zung [5] concerning non-degenerate singularities.

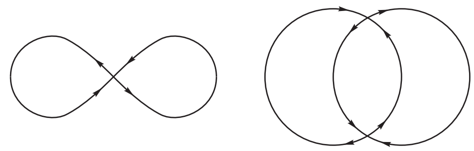
$(++)(++)$ Λ_0 is a direct product of two figures eight (or B atoms in the terminology of Fomenko).



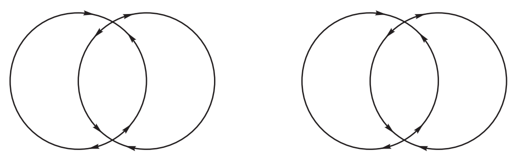
$(++)(+-)$ Λ_0 is the product of a figure eight and a "figure eight with two nodes" (a D_1 atom), quotiented by $\mathbb{Z}/2\mathbb{Z}$ acting on each factor as the central symmetry.



$(++)(--)$ Λ_0 is the product of a figure eight and the union of two circles intersecting at two points (a C_2 atom), quotiented by $\mathbb{Z}/2\mathbb{Z}$ acting on each factor as the central symmetry.



$(--)(--)$ Λ_0 is the product of two C_2 atoms quotiented by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where the first generator acts on the first factor as the central symmetry and on the second factor as the symmetry with respect to the vertical axis (Oz) when the C_2 atom is drawn on a sphere as in Fig. 4. Conversely, the second generator acts on the first factor as the (Oz) symmetry and on the second factor as the central symmetry.



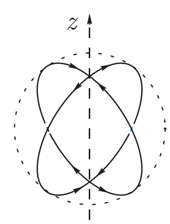


Fig. 4. Representation of the C_2 atom as orthogonal vertical circles on the sphere.

3.2. Cycles

We shall systematically define basis of cycles on the Liouville tori close to the critical fibre Λ_0 using the following idea.

We use the notation of Section 2 (see Fig. 2).

We will start at a regular point near U and as in the construction of the symplectic invariants, we will let it evolve under the flow of J_1 until it comes back close to S. If it is possible then to close up the path using a local flow of J_1 and J_2 near the critical point m_0, we are done. If not, we will continue with the flow of J_1 until we come back a second time near m_0. This means that we can arrive close to S again, or close to T. After a finite number (bounded by 4) of such iterations, we shall see that it is always possible to close the path using the local flow of J_1 and J_2.

Then we shall repeat the procedure starting from V instead of U.

Finally we will start again by switching the indices $i = 1, 2$ (that is, we use the flow of J_2). Notice that in this way we don't need to know *a priori* the topological type of the singular fibre to perform this construction (it will be determined by the signs $\varepsilon_i^{U/V}$).

Recall that, for our construction, we fix a small $\delta > 0$, such that the points S,T, U and V are in a small ball around the origin where Eliasson's normal form applies.

a. Case $(\varepsilon_1^U \varepsilon_1^V) = (++)$. Let $U' = (\delta, \xi_1, x_2, \xi_2)$ with small ξ_1, x_2, ξ_2 and $\xi_1 > 0$. The value of J_1 on U' is $j_1 = \delta\xi_1$. We let U' evolve under the flow of J_1 at time $\sigma_{1,1}^U(U') - \ln(\delta^2)$, so that we arrive at $S' = (x_1, \delta, x_2', \xi_2')$ with

$$x_1 = j_1/\delta, \qquad (x_2', \xi_2') = \phi^{q_2}_{-\sigma_{1,2}^U}(x_2, \xi_2)\,.$$

In this case we can close up using local flows. Namely acting with the flow of $q_1 = J_1$ at the time $\ln(\delta^2/j_1)$ and with the flow of $q_2 = J_2$ at the time $\sigma_{1,2}^U(U')$ will take S' back to U'.

In other words, we have proven that the vector field

$$Y_{1,+}^U := \left(\sigma_{1,1}^U + \ln(1/j_1)\right)\mathcal{X}_{J_1} + (\sigma_{1,2}^U)\mathcal{X}_{J_2}$$

is periodic of primitive period 1 on the Liouville torus containing U'. (Notice how the $\ln(\delta^2)$ have cancelled out.) Here and in what follows the subscripts $1, +$ indicate that we are in a region where $J_1 > 0$.

Now suppose we start at U' with $\xi_1 < 0$ (so $j_1 < 0$). Then when arriving near S, we are not on the initial local connected component of the hyperbolic foliation of q_1. Hence we have to start again with the flow of J_1 at the time $\ln(\delta^2/|j_1|)$ (to be close to V) plus $\sigma_{1,1}^V$ (to come back close to T). Now we can close up with the flow of J_2, and obtain the second periodic vector field:

$$Y_{1,-}^U := \left(\sigma_{1,1}^U + \sigma_{1,1}^V + 2\ln(1/|j_1|)\right)\mathcal{X}_{J_1} + (\sigma_{1,2}^U + \sigma_{1,2}^V)\mathcal{X}_{J_2}\,.$$

The corresponding smooth family of cycles will be denoted by $\gamma_{1,-}^U$.

Similarly if we start near V with $x_1 = -\delta$ and $\xi_1 < 0$ we obtain

$$Y_{1,+}^V := \left(\sigma_{1,1}^V + \ln(1/j_1)\right)\mathcal{X}_{J_1} + (\sigma_{1,2}^V)\mathcal{X}_{J_2}$$

and finally if we start near V with $x_1 = -\delta$ and $\xi_1 > 0$ we obtain

$$Y_{1,-}^V = Y_{1,-}^U\,.$$

Here again, $\gamma_{1,\pm}^{U/V}$ denote the corresponding families of cycles. See Fig. 5.

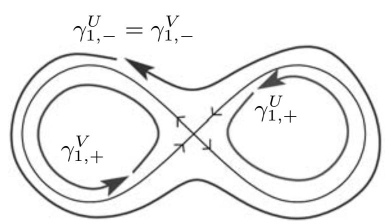


Fig. 5. Schematic representation of the cycles obtained using the flow of J_1 in the $(++)$ case.

Notice that in defining these cycles we have made no assumption on (x_2, ξ_2), except that they must be small. In particular they are defined even for $J_2 = 0$ (but not $J_1 = 0$). The total number of different cycles obtained for fixed J_1 and J_2 depends on the signs $\varepsilon_2^{U/V}$, as we shall see in the next subsection.

b. Case $(\varepsilon_1^U \varepsilon_1^V) = (+-)$. Since $\varepsilon_1^U > 0$, if we start near U with $\xi_1 > 0$ then we can close up after the first iteration, as before; so the corresponding periodic vector field is

$$Y_{1,+}^U := \left(\sigma_{1,1}^U + \ln(1/j_1)\right)\mathcal{X}_{J_1} + (\sigma_{1,2}^U)\mathcal{X}_{J_2}\,.$$

If we start near U with $\xi_1 < 0$ then as before we have to make at least two iterations. But for the second one, since $\varepsilon_1^V < 0$, the coordinates (x_2, ξ_2) have jumped to the antipodal quadrant. So we have to go on. The next iteration involves ε_1^U, so we need a fourth iteration to apply ε_1^V again and finally close up in the initial quadrant for both (x_1, ξ_1) and (x_2, ξ_2). We have then

$$Y_{1,-}^U := \left(2\sigma_{1,1}^U + 2\sigma_{1,1}^V + 4\ln(1/|j_1|)\right)\mathcal{X}_{J_1} + (2\sigma_{1,2}^U + 2\sigma_{1,2}^V)\mathcal{X}_{J_2}\,.$$

Of course we still have

$$Y^V_{1,-} = Y^U_{1,-} .$$

Finally if we start near V with $\xi_1 < 0$, then one iteration (i.e. the flow of J_1 at time $\sigma^V_{1,1}$) flips the coordinates (x_2, ξ_2), because $\varepsilon^V_1 < 0$. So we have to iterate again, and

$$Y^V_{1,+} := \left(2\sigma^V_{1,1} + 2\ln(1/j_1)\right) \mathcal{X}_{J_1} + (2\sigma^V_{1,2})\mathcal{X}_{J_2} .$$

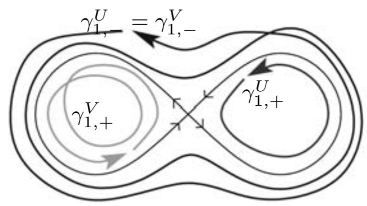


Fig. 6. Schematic representation of the cycles obtained using the flow of J_1 in the $(+-)$ case.

Lemma 5. *If* $(\varepsilon^U_1 \varepsilon^V_1) = (+-)$ *then one must have* $(\varepsilon^U_2 \varepsilon^V_2) = (++)$.

Proof. This is due to the fact that, roughly speaking, the period of $Y^V_{1,+}$ is approximately *twice* the period of $Y^U_{1,+}$.

If ε^U_2 or ε^V_2 is negative then there is a path on a Liouville torus which connects $\gamma^U_{1,+}$ to $\gamma^V_{1,+}$ (for instance, use the same construction as the one we just used for defining $Y^U_{1,+}$, but now with the flow of J_2). Such a path can always be realized as a joint flow of J_1 and J_2 at appropriate times. Using such a flow, $\gamma^U_{1,+}$ is transformed into a closed path which shares the same properties as $\gamma^V_{1,-}$ but which approaches only once the critical point. This is a contradiction.

c. Case $(\varepsilon^U_1 \varepsilon^V_1) = (--)$. The periodic vector fields in this case are the following:

$$Y^U_{1,+} := \left(2\sigma^U_{1,1} + 2\ln(1/j_1)\right) \mathcal{X}_{J_1} + (2\sigma^U_{1,2})\mathcal{X}_{J_2} ;$$

$$Y^U_{1,-} := \left(\sigma^U_{1,1} + \sigma^V_{1,1} + 2\ln(1/|j_1|)\right) \mathcal{X}_{J_1} + (\sigma^U_{1,2} + \sigma^V_{1,2})\mathcal{X}_{J_2} ;$$

$$Y^V_{1,-} = Y^U_{1,-} ;$$

$$Y^V_{1,+} := \left(2\sigma^V_{1,1} + 2\ln(1/j_1)\right) \mathcal{X}_{J_1} + (2\sigma^V_{1,2})\mathcal{X}_{J_2} .$$

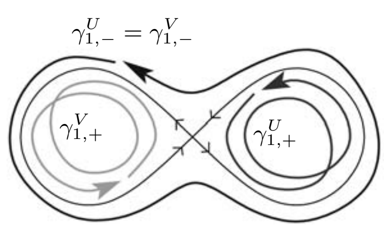


Fig. 7. Schematic representation of the cycles obtained using the flow of J_1 in the $(--)$ case.

Proposition 2. *Let* $U' \in M$ *be a point near* m_0 *such that the leaf* Λ *of the foliation* F *containing* U' *is a regular Liouville torus. For* $i = 1, 2$ *let* γ_i *be the unique cycle amongst the* $\gamma^{U/V}_{i,\pm}$*'s containing* U'*. Then provided* U' *is close enough to* m_0, (γ_1, γ_2) *is a basis of the homology* $H_1(\Lambda, \mathbb{Z}) \simeq \mathbb{Z}^2$.

Proof. We use the fact that $H_1(\Lambda,\mathbb{Z})$ is isomorphic to the space of linear combinations of $\mathcal{X}_{J_1}$ and $\mathcal{X}_{J_2}$ forming 1-periodic vector fields on Λ (see eg.[13]).

γ_1 is the trajectory of a vector field Y_1 of the form

$$Y_1 = (f_1 + \alpha_1 \ln|j_1|)\mathcal{X}_{J_1} + g_1\mathcal{X}_{J_2},$$

where f_1 and g_1 depend smoothly on J_1 and J_2 and $\alpha_1 \neq 0$ is constant. Similarly γ_2 is the trajectory of a vector field Y_2 of the form

$$Y_2 = (f_2 + \alpha_2 \ln|j_2|)\mathcal{X}_{J_2} + g_2\mathcal{X}_{J_1},$$

where f_2 and g_2 depend smoothly on J_1 and J_2 and $\alpha_2 \neq 0$ is constant. But for (j_1, j_2) small enough, the matrix

$$\begin{pmatrix} (f_1 + \alpha_1 \ln|j_1|) & g_1 \\ g_2 & (f_2 + \alpha_2 \ln|j_2|) \end{pmatrix} \tag{11}$$

is invertible. Let Z_1, Z_2 be two 1-periodic vector fields whose orbits form a basis of $H_1(\Lambda,\mathbb{Z})$ (they exists in virtue of the action-angle theorem). Since $\mathcal{X}_{J_1}$ and $\mathcal{X}_{J_2}$ are independent on Λ, there is an invertible matrix N such that $(\mathcal{X}_{J_1}, \mathcal{X}_{J_2}) = N \cdot (Z_1, Z_2)$. Hence there is an invertible matrix $\tilde{N}$ (necessarily of integer coefficients) such that $(Y_1, Y_2) = \tilde{N} \cdot (Z_1, Z_2)$, which proves our claim.

We shall use this basis as a 'canonical' basis for Liouville tori close the the critical fibre. It is characterized by the following property, in terms of the behavior of the period lattice.

Proposition 3. *If c is close enough to the critical value c_0, and $\Lambda \subset F^{-1}(c)$ is a regular Liouville torus, then the basis (γ_1, γ_2) given by proposition 2 is the unique basis of $H_1(\Lambda,\mathbb{Z})$ such that the matrix N which gives the corresponding 1-periodic vector fields Y_1, Y_2 on Λ in terms if $\mathcal{X}_{J_1}$ and $\mathcal{X}_{J_2}$: $(Y_1, Y_2) = N \cdot (\mathcal{X}_{J_1}, \mathcal{X}_{J_2})$ has the following asymptotic behavior:*

$$\lim_{c\to 0} \begin{pmatrix} 1/\ln|J_1| & 0 \\ 0 & 1/\ln|J_2| \end{pmatrix} \cdot N(c) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

where c varies in a quadrant of regular values and $N(c)$ is the unique smooth extension of N in this region.

Moreover $\alpha_i \in \{1, 2, 4\}$.

Proof. The fact that N admits a unique smooth extension is due to the discreteness of the fibre of the fibre bundle $H_1(F^{-1}(c),\mathbb{Z}) \to c$, which implies that (γ_1, γ_2) extends uniquely to a flat section of this bundle. The rest of the proof is a corollary of the previous proposition 2, and especially (11).

Of course using all the formulae we have obtained for the periodic vector fields $Y^{U/V}_{i,\pm}$, it is easy to find the values of α_i, which are the prefactors of the log terms. In fact, we may denote by $\alpha^{U/V}_{i,\pm}$ the integer obtained when γ_i is the orbit of $Y^{U/V}_{i,\pm}$. The results are the following:

- If $(\varepsilon^U_i \varepsilon^V_i) = (++)$ then $\alpha^{U/V}_{i,+} = 1$, $\alpha^{U/V}_{i,-} = 2$;
- if $(\varepsilon^U_i \varepsilon^V_i) = (--)$ then $\alpha^{U/V}_{i,\pm} = 2$;
- if $(\varepsilon^U_i \varepsilon^V_i) = (+-)$ then $\alpha^{U}_{i,+} = 1$, $\alpha^{U}_{i,-} = 4$, and $\alpha^{U/V}_{i,-} = 2$.

3.3. Tori

Proposition 2 allows us to count the effective number of independent cycles we have found by associating them to the tori they belong to. And indeed, the number of different Liouville tori having a common value of $F = (H, K)$ depends on the topological type of Λ_0 (that is, on the signs $\varepsilon_i^{U/V}$). Here again we shall work with the new momentum map $\mathbf{J} = (J_1, J_2)$ and examine each quadrant delimited by the axis $J_1 = 0$ or $J_2 = 0$. For each quadrant there are four starting positions for our cycles: close to U or close to V in the plane P_1, combined with the analogous possibilities in the plane P_2.

d. Tori in the $(++)(++)$ *case.* Let us first examine the tori for which we fix $J_1 = j_1 > 0$ and $J_2 = j_2 > 0$. It is clear that the cycles $\gamma_{1,+}^U$, $\gamma_{1,+}^V$, $\gamma_{2,+}^U$ and $\gamma_{2,+}^U$ are the only ones that satisfy these inequalities. The four starting positions are $U^+ = (\delta, j_1/\delta, \delta, j_2/\delta)$, $U^- = (\delta, j_1/\delta, -\delta, -j_2/\delta)$, $V^+ = (-\delta, -j_1/\delta, \delta, j_2/\delta)$ and $V^- = (-\delta, -j_1/\delta, -\delta, -j_2/\delta)$ (see Fig. 8). Because all the signs $\varepsilon_i^{U/V}$ are

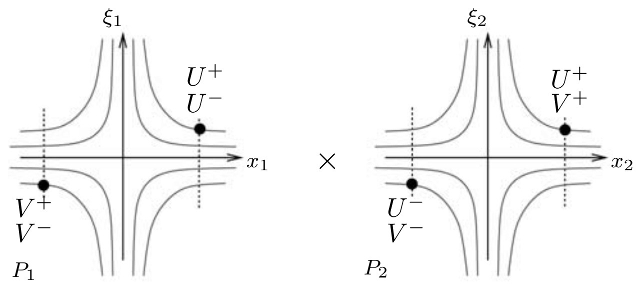


Fig. 8. Starting positions for the cycles in the case $J_1 > 0$, $J_2 > 0$.

positive, one can see that the joint flow of (J_1, J_2) starting at one of these positions can never intersect the other positions; so all pairings $(\gamma_{1,+}^{U/V}, \gamma_{2,+}^{U/V})$ define different connected components, and we have four different Liouville tori.

We turn now to the case $J_1 < 0$ and $J_2 > 0$. The starting positions are again U^+, U^-, V^+, V^- (but of course with $j_1 < 0$ in the definition !). However it is clear that U^+ can be connected to V^+ using the joint flow of $\mathbf{J}$ and similarly U^- can be connected to V^-. For instance to connect U^+ to V^+ one can use the flow at time 1 of the vector field:

$$\left(\sigma_{1,1}^U + \ln(1/|j_1|)\right)\mathcal{X}_{J_1} + \sigma_{1,2}^U \mathcal{X}_{J_2}\ . \tag{12}$$

So we have only two tori which correspond to the pairings of $\gamma_{1,-}^U$ with either $\gamma_{2,+}^V$ or $\gamma_{2,+}^U$.

Similarly we have two Liouville tori if $J_1 > 0$ and $J_2 < 0$. And if $J_1 < 0$ and $J_2 < 0$ we can only pair $\gamma_{1,-}^U$ with $\gamma_{2,-}^U$ and all starting positions are connected to each other so we have only one torus.

To summarize we can represent the number of tori in this case by the obvious diagram: $\begin{array}{c|c} 2 & 4 \\ \hline 1 & 2 \end{array}$.

e. Tori in the $(++)(+-)$ *case.* Assume $J_1 > 0$ and $J_2 > 0$ and use the notation of figure 8. Using that $\varepsilon_2^V < 0$ one can see that U^- is connected to V^- using the flow of J_2 and a local adjustment by the flow of J_1 (the precise formula is similar to (12), with J_1 exchanged with J_2; in the remaining of the discussion, such a local adjustment will be implicit). Hence amongst the four pairings mentioned above, this means that $(\gamma_{1,+}^U, \gamma_{2,+}^V)$ and $(\gamma_{1,+}^V, \gamma_{2,+}^V)$ define the same torus. There is no other self-intersection between the remaining cycles, so we have three tori.

Assume now $J_1 < 0$ and $J_2 > 0$. As before U^- is connected to V^- and U^+ to V^+; we can pair $\gamma_{1,-}^U$ with either $\gamma_{2,+}^V$ or $\gamma_{2,+}^U$. Here the fact that $\gamma_{2,+}^V$ has two iterations does not matter, since the flip of the (x_1, ξ_1) coordinates after the first iteration preserves the cycle $\gamma_{1,-}^U$. Hence we have two tori.

If $J_1 > 0$ and $J_2 < 0$ things are now different. Similarly to the previous case one can see using the flow of J_2 starting at U^+ or V^+ that U^+ is connected to U^-, and V^+ to V^-. But if we start at U^- or V^-

one has to take into account that $\varepsilon_2^V < 0$ and we find that U^- is connected to V^-, and U^+ to V^+. Hence we have only one torus.

The case $J_1 < 0$ and $J_2 < 0$ has nothing more particular and is left to the reader: there is only one torus. So the diagram in this case is $\begin{array}{c|c} 2 & 3 \\ \hline 1 & 1 \end{array}$.

f. Tori in the $(++)(--)$ *case.* Assume $J_1 > 0$ and $J_2 > 0$. Using the flow of J_2 starting at U^+ or U^- one checks that U^+ is connected to V^+, and U^- to V^-. We have only two tori, corresponding for instance to the pairings of $\gamma_{1,+}^U$ with either $\gamma_{2,+}^U$ or $\gamma_{2,+}^V$.

If $J_1 < 0$ and $J_2 > 0$ then again U^+ is connected to V^+, and U^- to V^-, and we have only two tori. This time $\gamma_{1,-}^U$ is paired with either $\gamma_{2,+}^U$ or $\gamma_{2,+}^V$.

If $J_1 > 0$ and $J_2 < 0$ then U^+ is connected to V^- and U^- to V^+, so we have two tori.

In the case $J_1 < 0$ and $J_2 < 0$ one can check that all starting positions are connected, so there is only one torus.

Finally the diagram is $\begin{array}{c|c} 2 & 2 \\ \hline 1 & 2 \end{array}$.

g. Tori in the $(--)(--)$ *case.* We leave the details to the reader except for the more interesting case $J_1 < 0$ and $J_2 < 0$. Here the flow of J_1 starting at U^+ or U^- shows that U^+ is connected to V^-, and U^- to V^+. But the flow of J_2 starting for instance at V^+ or U^+ yield the *same* connections. Hence there are two tori in this case. In all other cases all starting points are connected, which leads to the diagram $\begin{array}{c|c} 1 & 1 \\ \hline 2 & 1 \end{array}$.

Remark 3. The four multiplicity diagrams introduced in this section encode a piece of information (the number of connected components of the momentum map) which is obviously a topological invariant. This implies that the unordered grouping of unordered signs $(\varepsilon_1^U \varepsilon_1^V)(\varepsilon_2^U \varepsilon_2^V)$ is indeed a topological invariant. If we use the topological classification of Lerman and Umanskii (recalled in Section 3.1) together with the fact that our multiplicity diagrams are all different, we see that this topological invariant actually completely determines the topological type of the foliation.

3.4. Topological Relations Between the σ-Invariants

The symplectic invariants defined in Section 2 are Taylor series in the variables J_1, J_2. So, to define them, one can always restrict to a region where $J_1 > 0$, $J_2 > 0$ which is, in the terminology of Section 3.2, the region where we defined the four cycles $\gamma_{i,+}^{U/V}$, $i = 1, 2$.

Each such cycle comes along with a sign $\varepsilon_i^{U/V}$ and a pair of invariants $([\![\sigma_{i,1}^{U/V}]\!], [\![\sigma_{i,2}^{U/V}]\!])$. But in the previous Section 3.3 we just saw that some of these cycles are not independent on some Liouville tori, because there exists a trajectory of the system connecting two starting points $U^\pm, V^\pm$. This will affect the corresponding invariants, in the following way.

Lemma 6. *For A and A' in $\{U, V\}$ and $i \in \{1, 2\}$ suppose $\gamma_{i,+}^A$ is on the same Liouville torus as $\gamma_{i,+}^{A'}$. Then for $j = 1, 2$, $\sigma_{i,j}^A = \sigma_{i,j}^{A'}$ on $J_1 > 0$, $J_2 > 0$ and hence $[\![\sigma_{i,j}^A]\!] = [\![\sigma_{i,j}^{A'}]\!]$.*

Proof. To simplify the notation assume $i = 1$. Then by proposition 2 there is a symbol $B \in \{U, V\}$ such that $(\gamma_{1,+}^A, \gamma_{2,+}^B)$ is a basis of $H_1(\Lambda, \mathbb{Z})$, where Λ is the corresponding Liouville; the same holds for $(\gamma_{1,+}^{A'}, \gamma_{2,+}^{B'})$ with some symbol B'. Because these basis have the same asymptotic behavior in J_1, J_2, proposition 3 asserts that they are actually the same: in particular, as elements of $H_1(\Lambda, \mathbb{Z})$, $\gamma_{1,+}^A = \gamma_{1,+}^{A'}$. This implies that the corresponding periodic vector fields $Y_{1,+}^A$ and $Y_{1,+}^{A'}$ are equal. Since the σ-invariants are uniquely defined in terms of the coefficients of these vector fields in the basis $(\mathcal{X}_{J_1}, \mathcal{X}_{J_2})$, we must have $\sigma_{1,j}^A = \sigma_{1,j}^{A'}$.

Thus the σ-invariants may not be independent. We conjecture that in general this lemma is the only way to produce relations between the four pairs $([\![\sigma_{i,1}^{U/V}]\!], [\![\sigma_{i,2}^{U/V}]\!])$. Of course, it remains now to check the possibilities of applying the lemma. But this is exactly what we have done in section 3.3 where we counted the number of tori. As an example, let us recall the $(++)(--)$ case. Assume again that $J_1 > 0$, $J_2 > 0$; we saw that U^+ was connected to V^+. Hence $\gamma_{1,+}^U$ and $\gamma_{1,+}^V$ are on the same Liouville torus (and hence, by the proof above, are equal as homology cycles). Therefore the lemma says that $\sigma_{1,j}^U = \sigma_{1,j}^V$, $j = 1, 2$. The fact that U^- is connected to V^- produces the same relation. Inspecting in the same way the other cases, one obtains:

Proposition 4. *The relations between the σ-invariants are listed as follows.*

- $(++)(++)$ ***case.*** *The lemma 6 produces no relation;*
- $(++)(+-)$ ***case.*** *The lemma 6 produces one relation:*

$$(\sigma_{1,1}^U, \sigma_{1,2}^U) = (\sigma_{1,1}^V, \sigma_{1,2}^V);$$

- $(++)(--)$ ***case.*** *The lemma 6 produces one relation:*

$$(\sigma_{1,1}^U, \sigma_{1,2}^U) = (\sigma_{1,1}^V, \sigma_{1,2}^V);$$

- $(--)(--)$ ***case.*** *The lemma 6 produces two relations:*

$$(\sigma_{i,1}^U, \sigma_{i,2}^U) = (\sigma_{i,1}^V, \sigma_{i,2}^V) \qquad \textit{for } i = 1, 2.$$

3.5. Asymptotics of Action Integrals

The σ-invariants we have derived are closely related to action integrals. Recall that in a tubular neighborhood of Λ_0 the cohomology class of the symplectic form ω must vanish (because Λ_0 is Lagrangian). Let α be a primitive for ω, i.e. $d\alpha = \omega$ near Λ_0. Given a cycle γ on a Lagrangian fibre Λ close to Λ_0 the *action* of γ is by definition the integral $A_\gamma = \int_\gamma \alpha$. When c varies in a subset of regular values of F and Λ_c is a smooth family in $F^{-1}(c)$, then $A_\gamma = A_\gamma(c)$ is a smooth function of c, which is defined up to an additive constant (relative to the choice of α). The whole point here is to investigate the singularity of $A_\gamma(c)$ as c approaches a critical value of F. In Eliasson coordinates $(j_1, j_2) \in \mathbb{R}^2$, the critical values of F are the axis $j_1 = 0$ and $j_2 = 0$.

Theorem 1. *Let $\Lambda_c \subset F^{-1}(c)$ be a regular Liouville torus, $(\gamma_1(c), \gamma_2(c))$ the corresponding 'canonical basis' of proposition 3, and (α_1, α_2) the corresponding prefactors. Then, as c varies in a connected component of regular values of F, the action integrals $A_i(c) := \int_{\gamma_i(c)} \alpha$ $(i = 1, 2)$ admit the following decomposition:*

$$A_i(c) = \alpha_i(J_i \ln |J_i| - J_i) + g_i(c), \qquad (13)$$

where g_i extends to a smooth function on the set of Lagrangian leaves (i.e. $g_i = g_i(\tilde{J}_1, \tilde{J}_2)$ in the terminology of section 2).

Moreover the partial derivatives of g_i are linear combinations of σ-invariants (and α_i is the number of σ-invariants involved), as follows. We use the notation $g_{i,\pm}^{U/V}$ to indicate that the cycle γ_i was obtained as the flow of the corresponding vector field $Y_{i,\pm}^{U/V}$.

- *If $(\varepsilon_i^U \varepsilon_i^V) = (++)$ then for $k = 1, 2$ one has*

$$\frac{\partial g_{i,+}^{U/V}}{\partial J_k} = \sigma_{i,k}^{U/V}, \qquad \frac{\partial g_{i,-}^{U/V}}{\partial J_k} = \sigma_{i,k}^U + \sigma_{i,k}^V.$$

• *If* $(\varepsilon_i^U \varepsilon_i^V) = (--)$ *then*

$$\frac{\partial g_{i,+}^{U/V}}{\partial J_k} = 2\sigma_{i,k}^{U/V}, \qquad \frac{\partial g_{i,-}^{U/V}}{\partial J_k} = \sigma_{i,k}^U + \sigma_{i,k}^V.$$

• *If* $(\varepsilon_i^U \varepsilon_i^V) = (+-)$ *then*

$$\frac{\partial g_{i,+}^{U}}{\partial J_k} = \sigma_{i,k}^U, \qquad \frac{\partial g_{i,+}^{V}}{\partial J_k} = 2\sigma_{i,k}^V, \qquad \frac{\partial g_{i,-}^{U/V}}{\partial J_k} = 2\sigma_{i,k}^U + 2\sigma_{i,k}^V.$$

Proof. If a Hamiltonian L commutes with J_1 and J_2 and has a 1-periodic hamiltonian vector field $\mathcal{X}_L$, then the action integral along the corresponding cycle γ_L gives back L, plus a constant:

$$\int_{\gamma_L} \alpha = L + \text{const}. \tag{14}$$

This is readily verified in action-angle coordinates.

Because of proposition 2, a cycle γ_i in a canonical basis corresponds to a 1-periodic vector field of the form

$$Y_i = (f_i + \alpha_i \ln |J_i|)\mathcal{X}_{J_i} + h_i \mathcal{X}_{J_{(i \bmod 2)+1}}, \tag{15}$$

where f_i and h_j are smooth functions of $(\tilde{J}_1, \tilde{J}_2)$, and Y_i in turn corresponds to the Hamiltonian $L_i = \int_{\gamma_i} \alpha = A_i$. Near a regular torus one can write $A_i = A_i(J_1, J_2)$, which defines A_i modulo a constant by the equations

$$\frac{\partial A_i}{\partial J_i} = f_i + \alpha_i \ln |J_i| \qquad \frac{\partial A_i}{\partial J_k} = h_k,$$

where we have denoted $k = (i \bmod 2) + 1$. The solution of this system is indeed of the form (13), with $f_i = \frac{\partial g_i}{\partial J_i}$ and $h_k = \frac{\partial g_i}{\partial J_k}$.

The precise formulae for $\frac{\partial g_i}{\partial J_k}$ $(i, k = 1, 2)$ are easily obtained by replacing f_i and h_i in (15) by their correct values as determined for the various cases in section 3.2.

As a corollary, the situation for a general cycle γ can be described as follows:

Corollary 1. *1. If $\gamma(c) \subset F^{-1}(c)$ is a smooth family of loops on regular Liouville tori around Λ_0: c varies in a set of regular values of F near the origin, then there exists unique integers n_1 and n_2 in $\mathbb{Z}$ such that the action*

$$A(c) := \int_{\gamma(c)} \alpha$$

is of the form

$$A(c) = n_1 J_1 \ln(1/|J_1|) + n_2 J_2 \ln(1/|J_2|) + g(\tilde{J}_1, \tilde{J}_2),$$

where g is smooth.

2. *The integers n_i can be computed using the homology class of γ, the topological type of the hyperbolic-hyperbolic singularity, and the signs of J_1 and J_2 on these tori.*

3. *The partial derivatives $\frac{\partial g}{\partial c_1}$ and $\frac{\partial g}{\partial c_2}$ are linear combinations of the $\sigma_{i,j}^{U/V}$ invariants (restricted to the relevant tori).*

3.6. Symplectic Invariants from Action Integrals

As a side-effect of the theorem, we remark that the eight σ-invariants may actually be grouped in pairs and appear as partial derivatives of only four functions $\Sigma_i^{U/V}$ $(i = 1, 2)$: $\sigma_{i,k}^{U/V} = \frac{\partial \Sigma_i^{U/V}}{\partial J_k}$. Up to an additive constant, one has $\Sigma_i^{U/V} = g_{i,+}^{U/V}$ in the $(++)$ case, $\Sigma_i^{U/V} = \frac{1}{2} g_{i,+}^{U/V}$ in the $(--)$ case, whereas in the $(+-)$ case $\Sigma_i^U = g_{i,+}^U$ and $\Sigma_i^V = \frac{1}{2} g_{i,+}^V$. In all cases one can see that $\Sigma_i^{U/V}$ can be regarded as a kind of regularized action integral. These functions are smooth on the set of leaves of the foliation; in other words they are functions of $(\tilde{J}_1, \tilde{J}_2)$. Now, since functions of $(\tilde{J}_1, \tilde{J}_2)$ have a well-defined Taylor series at the origin, we conclude that the eight symplectic invariants of proposition 1 can be replaced by the four Taylor series of the functions $\Sigma_i^{U/V}$ without their constant terms, which shall be denoted by $[\![\Sigma_i^{U/V}]\!]_0$. Taking into account the relations given by proposition 4, we obtain the following result, which is probably the most compact way to present the symplectic invariants of general hyperbolic-hyperbolic singularities.

Theorem 2. *Without loss of information, the eight σ-invariant $[\![\sigma_{i,j}^{U/V}]\!]$ can be reduced to*

- *four Taylor series in the $(++)(++)$ case: $[\![\Sigma_1^U]\!]_0$, $[\![\Sigma_1^V]\!]_0$, $[\![\Sigma_2^U]\!]_0$ and $[\![\Sigma_2^V]\!]_0$;*
- *three Taylor series in the $(++)(+-)$ and $(++)(--)$ cases: $[\![\Sigma_1^U]\!]_0 = [\![\Sigma_1^V]\!]_0$, $[\![\Sigma_2^U]\!]_0$ and $[\![\Sigma_2^V]\!]_0$;*
- *two Taylor series in the $(--)(--)$ case: $[\![\Sigma_1^U]\!]_0 = [\![\Sigma_1^V]\!]_0$ and $[\![\Sigma_2^U]\!]_0 = [\![\Sigma_2^V]\!]_0$.*

We conjecture that no further relation exist in general between these new invariants. However we shall see that in the case of the C. Neumann system they do exist (theorem 6), due to a hidden symmetry of the commuting integrals.

4. FREQUENCIES

From the form of the actions A in terms of the local normal form momenta J the frequencies and their ratio can be computed. Recall that we use the symbol c to denote the value of the normalized momentum map (J_1, J_2), and the notation $\tilde{c}$ to denote a point in the set of Lagrangian leaves with momentum value c; the latter set being endowed with the smooth structure defined in section 2. Recall also that there is no point in making such a distinction in the analytic category.

Lemma 7. *The rotation number $W(\tilde{c})$ near the origin $c = 0$ of the momentum map J is given by*

$$W = \frac{-\Phi_{11} n_2 \ln |c_2| + \rho_2}{-\Phi_{12} n_1 \ln |c_1| + \rho_1},$$

where Φ_{1j} and ρ_i are smooth functions on the set of Lagrangian leaves and $n_i \in \mathbb{N}$. They are determined from the normal form $(H_1, H_2) = H = \Phi \circ \tilde{J}$ by $\Phi_{ij} = \partial_i \partial_j \Phi$, where $\Phi_{1j}(0) = \lambda_j$ are the eigenvalues of the Hamiltonian H_1 at the hyperbolic-hyperbolic equilibrium point.

Proof. According to Corollary 1 the actions near a hyperbolic-hyperbolic point can be be written as

$$A_1 = n_1(c_1 \ln 1/|c_1| + c_1) + g_1(\tilde{c}), \quad A_2 = n_2(c_2 \ln 1/|c_2| + c_2) + g_2(\tilde{c}),$$

where g_i are smooth functions of $\tilde{c}$ and n_i are integers. Therefore the period lattice is given by

$$\tau_{ij} = \frac{\partial(A_1, A_2)}{\partial(c_1, c_2)} = \begin{pmatrix} n_1 \ln 1/|c_1| + g_{11} & g_{12} \\ g_{21} & n_2 \ln 1/|c_2| + g_{22} \end{pmatrix}$$

where the second index of g denotes the derivative with respect to c_j. The normal form is given by $(H_1, H_2) = (\Phi_1(\tilde{c}), \Phi_2(\tilde{c}))$. We write $\partial \Phi_1 / \partial c_i = \Phi_{1i}$ where $H = H_1$ is the Hamiltonian such that

$\Phi_{1i}(0) = \lambda_i$ are the eigenvalues of the equilibrium. The following identities hold where τ, ω, and $D\Phi$ are 2×2 matrices,

$$X_A = \tau X_J, \quad X_H = \omega X_A, \quad X_H = D\Phi\, X_J\,,$$

and $X_A = (\mathcal{X}_{A_1}, \mathcal{X}_{A_2})^t$, etc. Together this gives $\omega = D\Phi\,\tau^{-1}$. The first row of the matrix ω gives the frequencies of the flow of $H = H_1$. All the four frequencies are now

$$\omega(c) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} -n_2 \ln|c_2| + g_{22} & -g_{12} \\ -g_{21} & -n_1 \ln|c_1| + g_{11} \end{pmatrix} \frac{1}{\det \tau}.$$

The ratio of the frequencies ω_{11} and ω_{12} gives the rotation number W. The smooth functions $\rho_i(\tilde{c})$ in the statement of the theorem are therefore given by $\rho_2 = \Phi_{11} g_{22} - \Phi_{12} g_{21}$ and $\rho_1 = \Phi_{12} g_{11} - \Phi_{11} g_{12}$.

Now we can show that the twist is non-vanishing for every regular value near a hyperbolic-hyperbolic point provided a transversality condition holds. The main task is to understand the level lines of constant rotation number W near the critical value. Then one merely needs to check whether the line of constant energy $H_1(\Phi_1(c)) = h_1$ is tangent to a level line of $W(c)$, which would imply vanishing twist. Changing if necessary H_j into $-H_j$, we may assume from now on that the eigenvalues λ_j are positive.

Theorem 3. *With a single exception every level line of $W(c)$ approaches the origin $c = 0$ either with horizontal or vertical tangent. The exception occurs for $W(c) = n_2\lambda_1/(n_1\lambda_2)$ which has slope $\exp(\rho_2(0)/(n_2\lambda_1) - \rho_1(0)/(n_1\lambda_2))$.*

Proof. Using the Lemma in the limit $c \to 0$ the leading order term in W is

$$W(c) = \frac{-\lambda_1 n_2 \ln|c_2| + \rho_2(0)}{-\lambda_2 n_1 \ln|c_1| + \rho_1(0)} + O(c \ln|c|)\,.$$

Consider the positive quadrant in the c-plane. The curve $W(c) = w$ is tangent at the origin to the curve given by

$$c_2 = \exp((\rho_2(0) - w\rho_1(0))/(n_2\lambda_1)) c_1^{w n_1 \lambda_2/(n_2\lambda_1)},$$

whose tangent is the c_1-axis when $w > n_2\lambda_1/(n_1\lambda_2)$ or the c_2-axis when $w < n_2\lambda_1/(n_1\lambda_2)$ or the line with slope $\exp(\rho_2(0)/(n_2\lambda_1) - \rho_1(0)/(n_1\lambda_2))$ when $w = n_2\lambda_1/(n_1\lambda_2)$. In the other quadrants the situation is similar, except that in the $+-$ and $-+$ quadrants the slope of the critical line is negative.

Corollary 2. *Near a simple hyperbolic-hyperbolic critical value in the image of the energy-momentum map the twist is non-vanishing for every regular value near the origin provided that $\lambda_2/\lambda_1 \neq \exp(\rho_2(0)/(n_2\lambda_1) - \rho_1(0)/(n_1\lambda_2))$.*

Proof. Recall that vanishing twist means that locally the rotation number $W(c)$ is not changing in the energy surface $H_1(\Phi_1(c)) = h_1$, hence the twist vanishes when the level lines of the rotation number are tangent to the energy surface. Since the level lines of the rotation number only have 3 possible slopes (0, ∞, and the slope of the exceptional line, see above) when approaching the origin, we only need to exclude that the energy surface at the origin has these slopes. The Hamiltonian cannot be parallel to either axis, because then it's quadratic part would be degenerate. Since $H_1 = \lambda_1 J_1 + \lambda_2 J_2 + \ldots$ the slope of the energy surface $H_1(\Phi_1(c)) = 0$ at the origin is $-\lambda_2/\lambda_1$. Hence in the $+-$ and $-+$ quadrant a tangency with the exceptional line $W(c) = n_2\lambda_1/(n_1\lambda_2)$ may only occur when the slope of this line at the origin (see the previous Theorem) coincides with that of the energy surface.

Notice that violating the above condition is necessary in order to have vanishing twist near the origin, but it is not sufficient.

The Kolmogorov non-degeneracy condition on the frequency map can be checked in a similar way.

Theorem 4. *The Jacobian determinant of the frequency map near a simple hyperbolic-hyperbolic critical value is non-zero.*

Proof. The frequency map from actions to frequencies (of H_1) is invertible when $\det \partial(\omega_{11}, \omega_{12})/\partial(A_1, A_2) \neq 0$. Hence we need to compute

$$\det \frac{\partial(\omega_{11}, \omega_{12})}{\partial(c_1, c_2)} \frac{\partial(J_1, J_2)}{\partial(A_1, A_2)} .$$

Following the form of ω in Lemma 7 we write $\omega_{1i} = W_i/d$ where $d = \det \tau$ so that $W = W_1/W_2$. The second index on W_{ij} and the first on d_i denotes a partial derivative. The entries of the first jacobian matrix are

$$\frac{\partial(\omega_{11}, \omega_{12})}{\partial(c_1, c_2)} = \frac{1}{d^2} \begin{pmatrix} W_{11}d - W_1 d_1 & W_{12}d - W_1 d_2 \\ W_{21}d - W_2 d_1 & W_{22}d - W_2 d_2 \end{pmatrix} .$$

The terms W_{ii}/d are of $O(1/\ln|c|)$ while all other terms are $O(1/|c|/\ln^2|c|)$. The leading order of the determinant (provided it is non-zero) therefore is

$$\frac{W_1 d_2 W_{21} + W_2 d_1 W_{12} - d W_{12} W_{21}}{d^3} = \frac{\lambda_1 \lambda_2 \ln c_1 \ln c_2}{c_1 c_2\, d^3} + O(\frac{1}{|c| \ln^4 |c|}) .$$

The leading order term is non-vanishing near the origin. The second matrix from the change of variables from J to A is the inverse of τ, and so the final result is that the Kolmogorov non-degeneracy condition near the origin behaves like

$$\det \frac{\partial(\omega_{11}, \omega_{12})}{\partial(A_1, A_2)} = \frac{\lambda_1 \lambda_2}{c_1 c_2 \, \ln^3 c_1 \ln^3 c_2} + O(\frac{1}{|c| \ln^6 |c|}) .$$

5. THE C. NEUMANN SYSTEM

The C. Neumann System [14] is an analytically integrable system of a particle at $x \in \mathbb{R}^3$ constrained to move on the two-sphere $||x|| = 1$ in the presence of a harmonic potential. A good introduction is given in [15], and the action variables of the system are discussed in [16]. A possible Hamiltonian to be restricted to the tangent bundle of the sphere $||x|| = 1$, $x \cdot y = 0$ is

$$H = T + V = \frac{1}{2}(||x||^2||y||^2 - (x.y)^2) + \frac{1}{2}(a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2) ,$$

with $y \in \mathbb{R}^3$, symplectic structure $dx \wedge dy$ and parameters $a_0 < a_1 < a_2$. The Hamiltonian is invariant under the three rotations $R_i : (x_i, y_i) \to (-x_i, -y_i)$. There are two hyperbolic-hyperbolic equilibrium points at the maxima of the potential $x = (0, 0, \pm 1)$, $y = (0, 0, 0)$. The eigenvalues of this equilibrium are

$$\lambda_1^2 = a_2 - a_0 \text{ and } \lambda_2^2 = a_2 - a_1, \quad \lambda_1 > \lambda_2.$$

In general the equilibria of the Neumann system are non-degenerate when the spring constants a_i are distinct, see [8, Prop. 2.4]. The two equilibria are mapped into each other by R_2 and are connected by heteroclinic orbits that are contained in the fixed sets of the discrete symmetries R_0 and R_1, respectively. As noted by Devaney [17] these orbits are transverse intersections of stable and unstable manifolds, which nevertheless does *not* imply that the system is non-integrable.

The second integral is any one of

$$F_\nu = x_\nu^2 + \sum_{\mu \neq \nu, \mu = 0}^{2} \frac{(x_\nu y_\mu - x_\mu y_\nu)^2}{a_\nu - a_\mu}, \quad \nu = 0, 1, 2 .$$

These integrals are not independent, but satisfy the relations

$$F_0 + F_1 + F_2 = ||x||^2, \quad H = \frac{1}{2}(a_0 F_0 + a_1 F_1 + a_2 F_2) .$$

In this example the approach is very different from the general theory of the previous sections. A global expression for the actions is obtained in terms of complete hyperelliptic integrals, and the problem is to extract the symplectic invariants from these integrals in a singular limit.

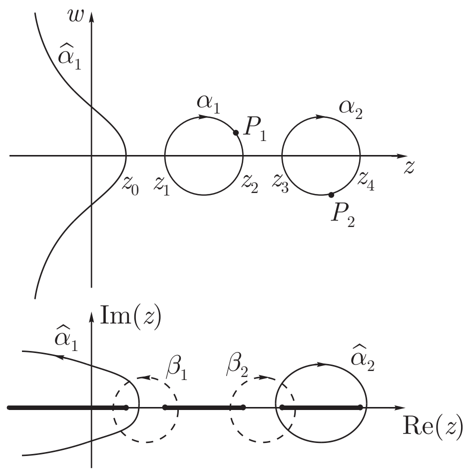


Fig. 9. Top: The real part of the hyperelliptic curve $w^2 = -Q(z)A(z)$ with the real cycles α_i. bottom: Complex z-plane with branch cuts and choice of canonical basis $(\hat{\alpha}_i, \beta_i)$. Half of the β_i cycles must be imagined to run on another copy of $\mathbb{C}[z]$ corresponding to the negative square-root. At the hyperbolic-hyperbolic point both β-cycles vanish.

5.1. Separation of Variables

The separation of variables was found by Neumann [14] as a student of Jacobi, see, e.g., [16] for the details. At the hyperbolic-hyperbolic (HH) equilibrium point the values of the constants of motion are $(h, f_0, f_1, f_2) = (a_2/2, 0, 0, 1)$. Separation of variables leads to an explicit expression for the actions in terms of hyperelliptic integrals on the curve

$$\mathcal{C} = \{(z, w) \in \mathbb{C}^2 : w^2 = -Q(z)A(z)\},$$

where

$$Q(z) = (z - r_1)(z - r_2), \quad A(z) = (z - a_0)(z - a_1)(z - a_2).$$

Globally the constants of motion r_i are not smooth, that is why in general the coefficients of $Q(z) = z^2 + 2\eta_1 z + 2\eta_2$ are chosen as separation constants. However, near the HH point with $(r_1, r_2) = (a_0, a_1)$ the r_i are smooth constants of motion as well. A convenient choice of constants of motion therefore is $\varepsilon = (\varepsilon_1, \varepsilon_2)$ where $r_1 = a_0 + \varepsilon_1$, $r_2 = a_1 + \varepsilon_2$. The relation to the original constants of motion is

$$H = (a_2 - \varepsilon_1 - \varepsilon_2)/2, \quad F_2 = (\varepsilon_1 - \lambda_2^2)(\varepsilon_2 - \lambda_1^2)/(\lambda_1^2\lambda_2^2). \tag{16}$$

For regular values of the energy-momentum map the curve is non-singular, and the actions are given by ([16], Theorem 1)

$$I_i = \frac{1}{2\pi}\oint_{\hat{\gamma}_i} \frac{Q(z)}{w} \mathrm{d}z. \tag{17}$$

Here $\hat{\gamma}_i = m\alpha_i$, where the integer $m = 1$ or 2 depending on the values of the integrals. These actions are natural from the point of view of separation of variables, but they are not the actions A that were constructed in the previous sections. Similarly the cycles $\hat{\gamma}_i$ described here are *not* the cycles γ_i of the previous sections. In Proposition 8 the correct cycles and corresponding actions A_i will be given. Here we continue describing the cycles used in [16]. The cycle α_1 surrounds the 2nd and 3rd root of w^2, while α_2 surrounds the 4th and 5th root of w^2 (the roots are labelled in increasing order). We denote the cycles by $[z_0, z_1]$ where z_i are the corresponding branch points on the real axis. It follows that, in the four regions adjacent to the HH critical value arranged in the coordinate system given by $(\varepsilon_1, \varepsilon_2)$, the cycles α_1, α_2 are given by

$[a_0, a_1], [r_2, a_2]$	$[r_1, a_1], [r_2, a_2]$
$[a_0, r_2], [a_1, a_2]$	$[r_1, r_2], [a_1, a_2]$

When $\alpha_i = [r_1, r_2]$, then $m = 1$, otherwise $m = 2$. The notation is taken from [16]. The reduction of the system by the group generated by R_0, R_1, and R_2 gives a system with globally continuous actions $\tilde{I}_i$ corresponding to the cycles α_i and only one torus in the preimage of the energy-momentum map [16]. Here we extend the results of [16] by studying the flow of the reduced actions $\tilde{I}_i$ in the full system.

Lemma 8. *The time 2π map of the flow generated by the reduced action*

$$\tilde{I}_i = \frac{1}{2\pi} \oint_{\alpha_i} \frac{Q(z)}{w} \, \mathrm{d}z$$

is an element R of the group $G \simeq \mathbb{Z}_2^3$ generated by R_0, R_1, and R_2. The element R is the product of those R_j, for which a_j is contained as a branch point of the projection of α_i to the real axis.

Proof. The group G acts freely on the set of regular points of the energy-momentum map. For the reduced system the flow $\tilde{\phi}_i^t$ generated by $\tilde{I}_i$ is 2π periodic [16]. The reduced system is obtained by symmetry reduction with respect to the discrete group G. Thus for the full system the time 2π map of the flow generated by $\tilde{I}_i$ will be an element R of the symmetry group G. Which element can be read off from the boundary of the projection of the cycles α_i onto the real axis. The map $\tilde{\phi}_i^{2\pi}$ of the corresponding action $\tilde{I}_i$ has the following property: Whenever a boundary point a_i is reached along the projection of α_i to the real axis the corresponding variables x_i, y_i in the full system change sign (which is the action of the symmetry R_i). The reason is [16] that the elliptical spherical coordinates in which the system separates only determine the squares x_i^2, but not the signs of x_i. Now, $x_\nu = 0$ when one of these coordinates equals a_ν. Thus, when one of the endpoints of the segment $[z_0, z_1]$ equals a_ν, then the coordinate x_ν changes sign along the cycle α_i. Thus the corresponding maps $(\tilde{\phi}_1^{2\pi}, \tilde{\phi}_2^{2\pi})$ are simply the discrete symmetries

$(R_0 R_1, R_2)$	(R_1, R_2)
$(R_0, R_1 R_2)$	$(id, R_1 R_2)$

,

where the four cases near the HH point are arranged according to $(\varepsilon_1, \varepsilon_2)$ as before.

Let (g_1, g_2) be the two group elements corresponding to $(\tilde{I}_1, \tilde{I}_2)$. The number of tori in the corresponding region of the energy-momentum map can be computed as follows. Divide the total order 8 of the symmetry group G by the order of the subgroup generated by $\{g_1, g_2\}$. The order of the subgroup generated by g_1 and g_2 gives us the number of points on the torus in different quadrants that can be reached by using the flows generated by $\tilde{I}_1$ and $\tilde{I}_2$. The number of cosets therefore is the number of disjoint tori in the preimage of the energy momentum map. Elements that are not in the subgroup generated by g_1 and g_2 will map an invariant torus into a disjoint invariant torus. E.g. in the upper left we have $(g_1, g_2) = (R_0 R_1, R_2)$ and the subgroup generated by g_1 and g_2 is of order 4, and hence there are 2 tori. Either R_0 or R_1 map the disjoint tori into each other. Similarly, in the lower right we have $(g_1, g_2) = (id, R_1 R_2)$ and the subgroup generated by g_1 and g_2 is of order 2, and hence there are 4 disjoint tori. The disjoint tori are be mapped into each other by R_0 and R_1 or R_2. For all 4 regions near HH we arrange the answer again in the form of a little diagram and find

2	2
2	4

. This diagram is not among the four cases found in the general theory because there are two equilibrium points in the singular fibre. In [16] it was also shown that when reducing by the group $G = \mathbb{Z}_2^3$ generated by R_i the reduced actions are given by the integrals over α_i instead of γ_i. The reduced actions are globally continuous, because the number of tori in the preimage of a regular value of the energy-momentum map is always 1 for the reduced system. The action of G is free on the regular points. Here we are interested in the critical point of the HH equilibrium, and therefore cannot use this reduction, because the action of G it is not free at the HH point. However, we need to reduce some discrete symmetry in order to have only a single equilibrium point in the singular leave, such that the general theory of the previous sections applies. The HH equilibrium is contained in the fixed set of R_0 and R_1, while R_2 maps the two equilibria at $x_2 = \pm 1$ into each other. Thus there are four different reductions of the full system that are free near the HH point and its separatrix. They correspond to the 4 subgroups $\mathbb{Z}_2$ of G generated by R_2, R_0R_2, R_1R_2 and $R_0R_1R_2$.

Proposition 5. *The reduction of a neighborhood of the singular HH level of the Neumann system with respect to $\mathbb{Z}_2$ generated by R_2, R_0R_2, R_1R_2, or by $R_0R_1R_2$ is of type $(++)(--)$, $(--)(--)$, $(++)(++)$, or $(++)(--)$, respectively.*

Proof. The rotation $R_2 : (x_2, y_2) \to (-x_2, -y_2)$ is free outside the set $x_0^2 + x_1^2 = 1$, $x_0y_0 + x_1y_1 = 0$. This set is an invariant subsystem given by the Neumann system on the equator of the sphere. This set is not contained in a sufficiently small neighborhood of the singular level of the HH point. Similar arguments show that the action of the other subgroups is also free near the critical HH level.

Each non-trivial element g of the $\mathbb{Z}_2$ subgroups contains R_2 as a factor. Thus we can choose as a fundamental region the part of phase space with $x_2 \geqslant 0$. An element g_k in (g_1, g_2) will be changed by the reduction if it contains R_2 as a factor. Thus the map g_k in the reduced system will be given by gg_k if g_k contains R_2. The reduction acts by either reducing the number of tori by mapping them into each other, or by shortening a full cycle $\hat{\gamma}_i = m\alpha_i$ of the torus by half, $m \to m/2$. E.g. for the upper left (R_0R_1, R_2) reduction by the group generated by R_1R_2 gives (R_0R_1, R_1) and there is only one torus remaining. In a similar way we obtain

$$R_2 \text{ reduction: } \begin{array}{c|c} (R_0R_1, id) & (R_1, id) \\ \hline (R_0, R_1) & (id, R_1) \end{array}, \qquad \text{multiplicity: } \begin{array}{c|c} 2 & 2 \\ \hline 1 & 2 \end{array}$$

$$R_0R_2 \text{ reduction: } \begin{array}{c|c} (R_0R_1, R_0) & (R_1, R_0) \\ \hline (R_0, R_0R_1) & (id, R_0R_1) \end{array}, \qquad \text{multiplicity: } \begin{array}{c|c} 1 & 1 \\ \hline 1 & 2 \end{array}$$

$$R_1R_2 \text{ reduction: } \begin{array}{c|c} (R_0R_1, R_1) & (R_1, R_1) \\ \hline (R_0, id) & (id, id) \end{array}, \qquad \text{multiplicity: } \begin{array}{c|c} 1 & 2 \\ \hline 2 & 4 \end{array}$$

$$R_0R_1R_2 \text{ reduction: } \begin{array}{c|c} (R_0R_1, R_0R_1) & (R_1, R_0R_1) \\ \hline (R_0, R_0) & (id, R_0) \end{array}, \qquad \text{multiplicity: } \begin{array}{c|c} 2 & 1 \\ \hline 2 & 2 \end{array}.$$

Matching the multiplicity diagrams to the ε_i signs using the results of Section 3.3 completes the proof.

A special basis that will be used later on is given by $\hat{\alpha}_1 = \alpha_1 + \alpha_2$ and $\hat{\alpha}_2 = \alpha_2$. Addition of cycles corresponds to the composition of flows, and hence symmetries. The diagram corresponding to the 2π maps of the flows generated by the integrals over these cycles is

$$\text{actions } \hat{I}\text{: } \begin{array}{c|c} (R_0R_1R_2, R_2) & (R_1R_2, R_2) \\ \hline (R_0R_1R_2, R_1R_2) & (R_1R_2, R_1R_2) \end{array}, \qquad \text{multiplicity: } \begin{array}{c|c} 2 & 2 \\ \hline 2 & 4 \end{array}.$$

This diagram has the property that g_1 does not change when moving up/down, while g_2 does not change when moving left/right. The diagrams of the 4 reductions corresponding to the choice of cycles $(\hat{\alpha}_1, \hat{\alpha}_2)$ are obtained by simply replacing any entry (g_1, g_2) by the entry (g_1g_2, g_2). All the resulting diagrams have the structure

$$\begin{array}{c|c} (g_1, g_3) & (g_2, g_3) \\ \hline (g_1, g_4) & (g_2, g_4) \end{array} = (g_1, g_2) \otimes (g_3, g_4)^t .$$

When one g_i of (g_1, g_2) or of $(g_3, g_4)^t$ equals id this corresponds to a $(++)$ in the notation of Section 3.1, while if both are different from id this corresponds to a $(--)$.

5.2. Local Normal Form I

The normal form of Eliasson can be computed order by order from a Birkhoff normal form expansion. This gives the following result:

Proposition 6. *There are local symplectic coordinates* (q_i, p_i) *near the HH equilibrium point such that*

$$\begin{aligned} H =& \frac{1}{2}a_2 + \lambda_1 J_1 + \lambda_2 J_2 + \frac{1}{4}(J_1^2 + J_2^2) + O(J^3) \\ F_2 =& 1 + 2\frac{J_1}{\lambda_1} + 2\frac{J_2}{\lambda_2} + \frac{J_1^2}{2\lambda_1^2} + \frac{2J_1J_2}{\lambda_1\lambda_2} + \frac{J_2^2}{2\lambda_2^2} + O(J^3)\,, \end{aligned} \tag{18}$$

where the local normal form momenta are $J_1 = q_1p_1$ *and* $J_2 = q_2p_2$.

Proof. To obtain the Birkhoff normal form near the equilibrium point with $x = (0, 0, 1)$ local standard symplectic coordinates (q_1, q_2, p_1, p_2) are introduced. They are related to the global symplectic coordinates by

$$\begin{aligned} x =& (q_1, q_2, \sqrt{1 - q_1^2 - q_2^2}), \\ y =& ((1 - q_1^2)p_1 - q_1q_2p_2, (1 - q_2^2)p_2 - q_1q_2p_1, -(q_1p_1 + q_2p_2)x_2)\,. \end{aligned}$$

At the linear level simply $x_i = q_i$, $y_i = p_i$, $i = 1, 2$. In these variables we find

$$\begin{aligned} H =& \frac{1}{2}(a_2 + p_1^2 + p_2^2 - q_1^2\lambda_1^2 - q_2^2\lambda_2^2) - \frac{1}{2}(q_1p_1 + q_2p_2)^2 \\ F_2 =& 1 + \left(\frac{p_1^2}{\lambda_1^2} + \frac{p_2^2}{\lambda_2^2}\right)(1 - q_1^2 - q_2^2) - q_1^2 - q_2^2\,. \end{aligned}$$

A linear transformation that normalizes the quadratic terms of H and F_2 is given by

$$(q_i, p_i) = (\tilde{q}_i - \frac{\tilde{p}_i}{2\lambda_i}, \lambda_i\tilde{q}_i + \frac{\tilde{p}_i}{2})\,.$$

We drop the tildes. The higher order terms are removed by the Lie transform method, in which a flow is generated by $W_d(q, p)$ of degree d which is chosen in order to remove terms of degree d in H and F_2. There are no cubic terms to be removed. For the quartic terms we find

$$\begin{aligned} W_4(p, q) = & \frac{p_1^2p_2^2}{32\lambda_1\lambda_2(\lambda_1 + \lambda_2)} - \frac{\lambda_1\lambda_2q_1^2q_2^2}{2(\lambda_1 + \lambda_2)} - \frac{\lambda_2p_1^2q_2^2}{8\lambda_1(\lambda_1 - \lambda_2)} + \frac{\lambda_1p_2^2q_1^2}{8\lambda_2(\lambda_1 - \lambda_2)} + \\ & + \frac{p_1^4}{128\lambda_2^3} + \frac{p_2^4}{128\lambda_1^3} - \frac{1}{8}\lambda_1q_1^4 - \frac{1}{8}\lambda_2q_2^4\,. \end{aligned}$$

Transforming H and F_2 with the transformation generated by W_4 gives the stated result (after dropping the tildes once more). In fact the transformation W_4 generates already the correct coefficients in front of the unremovable terms of order 6 in H and F_2. They can simply be read of and are given by

$$\begin{aligned} H^{(6)} =& -\frac{J_1^3}{16\lambda_1} + \frac{J_1J_2}{4(\lambda_1^2 - \lambda_2^2)}(\lambda_2J_1 - \lambda_1J_2) - \frac{J_2^3}{16\lambda_2} \\ F_2^{(6)} =& -\frac{J_1^3}{8\lambda_1^3} + \frac{J_1J_2}{2(\lambda_1^2 - \lambda_2^2)}\left(-\frac{J_1}{\lambda_2} + \frac{J_2}{\lambda_1}\right) - \frac{J_2^3}{8\lambda_2^3}\,. \end{aligned}$$

5.3. Local Normal Form II

Alternatively the normal form near the hyperbolic-hyperbolic point can be computed from the action integral. The main observation is that the residues associated to the vanishing cycles of the hyperelliptic curve can be easily computed and they give (J_1, J_2) as functions of $(\varepsilon_1, \varepsilon_2)$. This is more familiar near elliptic-elliptic equilibrium points, where the local normal form actions can be obtained by evaluating the closed loop integrals over the vanishing cycles. The main difference is that in the elliptic case this computes the Liouville–Arnold actions I_i, while in the hyperbolic case it does not compute these actions, but instead the momenta of the Eliasson normal form J_i. The main advantage of J_i over the true action in the hyperbolic case is that the integral over a vanishing cycle can be computed by Taylor expansion of the integrand because it is a local quantity.

Proposition 7. *The inverse of the Eliasson normal form is given by*

$$J_i(\varepsilon) = \frac{1}{2\pi \mathrm{i}} \oint_{\beta_i} \frac{Q(z)}{w} \mathrm{d}z\,, \tag{19}$$

where β_i are the cycles of the hyperelliptic curve that vanish for the critical value. The expansion near $\varepsilon = 0$ is given by

$$-2\lambda_1 J_1(\varepsilon) = \varepsilon_1 \left(1 + \frac{\varepsilon_1}{8\lambda_1^2} + \frac{\varepsilon_2}{2\lambda_2^2} + O(\varepsilon^2)\right)\,. \tag{20}$$

The inverse near $\varepsilon = 0$ is given by

$$\varepsilon_1(J) = -2\lambda_1 J_1 \left(1 + \frac{J_1}{4\lambda_1} + \frac{\lambda_2 J_2}{\lambda_1^2 - \lambda_2^2} + O(J^2)\right) \tag{21}$$

and a similar expression with all indices 1 and 2 exchanged.

Proof. Let β_1 be the cycle around the double root of w^2 forming at a_0 and β_2 the cycle around the double root forming at a_1. A series of the integrand at the critical value $\varepsilon = 0$ is

$$\frac{Q(z)}{w} = \sqrt{\frac{1 + 2\xi(z)}{a_2 - z}} = \frac{1}{\sqrt{a_2 - z}} \left(1 + \xi - \frac{1}{2}\xi^2 + \frac{1}{2}\xi^3 + \dots\right) \tag{22}$$

where

$$\xi(z) = -\frac{\varepsilon_1(z - a_1) + \varepsilon_2(z - a_2) - \varepsilon_1\varepsilon_2}{2(z - a_0)(z - a_1)}.$$

Now the integral over β_i in the limit of vanishing cycle can be computed by the residue theorem. In particular J_i is obtained term by term from the series computing the residue at $z = a_{i+1}$. The result for the first terms gives (20). The series for J_2 is obtained by exchanging all the indices $1 \leftrightarrow 2$. The series was expanded in powers of ξ instead of in powers of ε because powers of ξ give the series by order of poles. After computing the residues the series is ordered by ε. The result is a mapping from $(\varepsilon_1, \varepsilon_2)$ to (J_1, J_2) that fixes the origin. The coordinate axes are mapped into each other.

The inverse of this mapping can be obtained by formal inversion order by order, the result of which gives (21). When these ε are substituted into (16) the result is (18), which shows that near the HH equilibrium the inverse of the Birkhoff normal form is given by the hyperelliptic integral (19).

For many integrable systems the actions are given by complete (hyper)elliptic integrals, and for these systems the local Eliasson normal form at an equilibrium is given by the inverse of a mapping whose components are complete (hyper)elliptic integrals evaluated near vanishing cycles. Unlike the normal form itself, the inverse thus has a simple representation.

5.4. Semi-Global Invariants

The computation of the non-local terms is much harder. From the explicit equation for the action $I = I(\varepsilon_1, \varepsilon_2)$ given in (17) we can find the period lattice with respect to the flows of the global smooth integrals $H_1 = H$ and $H_2 = F_2$ as

$$\mathcal{X}_{I_i} = \frac{\partial I_i}{\partial h_1} \mathcal{X}_{H_1} + \frac{\partial I_i}{\partial h_2} \mathcal{X}_{H_2}\,,$$

which is valid for both actions, the only difference is the integration path. The period lattice is therefore obtained by the following hyperelliptic integrals of the first kind

$$\begin{aligned} \frac{\partial I_i}{\partial h_1} &= \frac{1}{2\pi} \oint_{\hat{\gamma}_i} \frac{2z - 2a_2}{w}\, \mathrm{d}z, \\ \frac{\partial I_i}{\partial h_2} &= \frac{1}{2\pi} \oint_{\hat{\gamma}_i} \frac{\lambda_1^2 \lambda_2^2}{w}\, \mathrm{d}z\,. \end{aligned} \tag{23}$$

Note that these are integrals of first kind, since the numerators are of degree one. From the local normal form $(H_1, H_2) = \Phi(J_1, J_2)$ we find

$$\mathcal{X}_{H_1} = \frac{\partial H_1}{\partial J_1} \mathcal{X}_{J_1} + \frac{\partial H_1}{\partial J_2} \mathcal{X}_{J_2}$$

and similarly for $\mathcal{X}_{H_2}$, which altogether is written as $X_H = D\Phi\, \mathcal{X}_J$. Inserting the flows X_H into the expression for the flows X_I gives the period lattice in terms of the vector fields of the local normal form momenta as

$$\mathcal{X}_{I_i} = \tau_{i1} \mathcal{X}_{J_1} + \tau_{i2} \mathcal{X}_{J_2}$$

where

$$\tau_{ij} = \frac{\partial I_i}{\partial h_1} \frac{\partial H_1}{\partial J_j} + \frac{\partial I_i}{\partial h_2} \frac{\partial H_2}{\partial J_j} = \frac{\lambda_j}{2\pi} \oint_{\hat{\gamma}_i} \frac{z - a_{2-j}}{w}\, \mathrm{d}z + O(J_1, J_2)\,. \tag{24}$$

From Proposition 2 we know that there is a choice of cycles for which τ_{ij} with $i \neq j$ is finite in the limit $J \to 0$. The cycles used to define I_i do not have this property. We are now going to find such cycles, and denote the corresponding actions by A_i, as in the previous sections.

The cycle $\hat{\gamma}_2$ is fine, since the pole that develops at a_1 in w when $J \to 0$ is cancelled by the numerator in (24). But for the cycle $\hat{\gamma}_1$ there are two poles nearby at a_0 and at a_1, and only the one at a_0 is cancelled, so that τ_{12} would diverge in the limit $J \to 0$. This leads to the following choice of cycles:

Proposition 8. *The unique basis of cycles near the HH equilibrium of the Neumann system is given by* $(\hat{\alpha}_1, \hat{\alpha}_2) = (\alpha_1 + \alpha_2, \alpha_2)$. *Together with the complex cycles* (β_1, β_2) *this is a canonical basis of cycles for the curve* $\mathcal{C}$.

Proof. We need to show that for $\hat{\alpha}_1$ the integral τ_{12} is finite. Notice that on the Riemann surface $\mathcal{C}$ the cycle $\hat{\alpha}_1$ can be deformed to a cycle that projects onto $[-\infty, a_0]$ on the real z-axis, because the integral is of 2nd kind, i.e. there are no residues when we are outside critical values. We take this as the standard representation of the cycle $\hat{\alpha}_1$. Now the integral τ_{12} remains finite because the pole that develops in w at $z = a_0$ is cancelled by the numerator in (24), while the other pole at $z = a_1$ is outside the (real) integration range, and hence the integral is finite. The basis of cycles is canonical because their intersection numbers give the standard symplectic matrix. When we require that the period lattice has two non-diverging off-diagonal entries in the matrix τ_{ij} then the choice of cycles is uniquely determined, because any transformation from $SL(2, \mathbb{Z})$ would destroy this property. At least there is unique choice in each quadrant, but in the analytic category it is unique for all of them.

The actions corresponding to the new cycles are given by a unimodular transformation of the previous ones, namely $(A_1, A_2) = (I_1 + I_2, I_2)$. From now on we always work with the canonical basis, and denote the new action integrals by

$$\hat{A}_i = \frac{1}{2\pi} \oint_{\hat{\alpha}_i} \frac{Q(z)}{w}\, \mathrm{d}z\,.$$

Notice that we also removed the numerical factor m_i which has to do with the covering number of the separating coordinate system. For the C. Neumann system this makes sense because of the additional symmetry. The actions A_i have the property that they are 'as smooth as possible'. Not only is τ_{ij}, $i \neq j$, finite, but it is smooth when crossing the line $J_j = 0$ (not for crossing $J_i = 0$, though) outside the origin, and hence also the action A_i itself is smooth in this case.

Using (22) the series of the action integrand in ε is

$$\frac{Q}{w} = \frac{1}{\sqrt{a_2 - z}}\left(1 + \frac{\varepsilon_1/2}{a_0 - z} + \frac{\varepsilon_2/2}{a_1 - z} + O(\varepsilon^2)\right) \tag{25}$$

Now using (21), ε can be expressed in terms of J. Thus the derivative of the action with respect to J can be computed directly. The finite parts of the action $\hat{A}_i$ and of its J_j derivatives τ_{ij} at the critical value are therefore given by

$$2\pi\hat{A}_1(0,0) = \oint_{\hat{\alpha}_1} \frac{1}{\sqrt{a_2 - z}}\,\mathrm{d}z = 2\lambda_1 \tag{26}$$

$$2\pi\hat{A}_2(0,0) = \oint_{\hat{\alpha}_2} \frac{1}{\sqrt{a_2 - z}}\,\mathrm{d}z = 2\lambda_2 \tag{27}$$

$$2\pi\tau_{12}(0,0) = \lambda_2 \oint_{\hat{\alpha}_1} \frac{1}{(z - a_1)\sqrt{a_2 - z}}\,\mathrm{d}z = 2\tanh^{-1}\frac{\lambda_2}{\lambda_1} \tag{28}$$

$$2\pi\tau_{21}(0,0) = \lambda_1 \oint_{\hat{\alpha}_2} \frac{1}{(z - a_0)\sqrt{a_2 - z}}\,\mathrm{d}z = 2\tanh^{-1}\frac{\lambda_2}{\lambda_1} \tag{29}$$

The complete expansion of the actions needs more sophisticated methods, which lead to our main result about the HH equilibrium of the Neumann system:

Theorem 5. *Near the HH equilibrium point of the Neumann system the actions A as a function of the normal form momenta J up to $O(J^3)$ are given by*

$$\pi\hat{A}_2 = 2\lambda_2 + J_2 \ln\frac{8\lambda_2}{|J_2|} + J_2 + 2J_1\tanh^{-1}\frac{\lambda_2}{\lambda_1} + \frac{3J_2^2}{8\lambda_2} + \frac{\lambda_1 J_1 J_2}{\lambda_1^2 - \lambda_2^2} - \frac{\lambda_2 J_1^2}{2(\lambda_1^2 - \lambda_2^2)}$$

$$\pi\hat{A}_1 = 2\lambda_1 + J_1 \ln\frac{8\lambda_1}{|J_1|} + J_1 + 2J_2\tanh^{-1}\frac{\lambda_2}{\lambda_1} + \frac{3J_1^2}{8\lambda_1} + \frac{\lambda_1 J_2 J_1}{\lambda_2^2 - \lambda_1^2} - \frac{\lambda_2 J_2^2}{2(\lambda_2^2 - \lambda_1^2)}$$

Proof. Consider the action A_2. It is to be integrated over the cycle $\hat{\alpha}_2$, which projects onto $[\max(a_1 + \varepsilon_2, a_1), a_2]$. We separate the divergent contributions to the integral in the following way:

$$\frac{Q}{w} = \mathcal{F}_2\mathcal{E}_2, \qquad \mathcal{F}_2 = \sqrt{1 - \frac{\varepsilon_1}{z - a_0}}, \quad \mathcal{E}_2 = \sqrt{\frac{z - a_1 - \varepsilon_2}{(z - a_1)(a_2 - z)}}$$

Since $\mathcal{F}_2$ is uniformly smooth on the integration interval, it can be expanded in ε_1. The result is a series of complete elliptic integrals with integrands $\mathcal{E}_2/(z - a_0)^l$. For $l = 0$ the integral is of 2nd kind, otherwise of 3rd kind. The first few terms are

$$2\pi\hat{A}_2 = \oint_{\hat{\alpha}_2}\left(\mathcal{E}_2 - \frac{\varepsilon_1}{2(z - a_0)}\mathcal{E}_2 - \frac{\varepsilon_1^2}{8(z - a_0)^2}\mathcal{E}_2 + O(\varepsilon_1^3)\right)\mathrm{d}z\,.$$

The series can be integrated term by term. For each term the known series expansions for the elliptic integrals in the complementary modulus $1 - k^2 \to 0$ are used. For $\varepsilon_2 > 0$ we find $1 - k^2 = \varepsilon_2/\lambda_2^2$ and parameter $n = (\lambda_2^2 - \varepsilon_2)/\lambda_1^2$ and the Legendre normal forms for $\varepsilon_2 > 0$ are

$$\oint_{\hat{\alpha}_2} \mathcal{E}_2\,\mathrm{d}z = 4\lambda_2 E(k) - 4\varepsilon_2 K(k)/\lambda_2 \tag{30}$$

$$\oint_{\hat{\alpha}_2} \frac{1}{z - a_0}\mathcal{E}_2\,\mathrm{d}z = \frac{4}{\lambda_2}\left(K(k) - (1 - n)\Pi(n,k)\right) \tag{31}$$

The expansion in $1 - k^2$ gives terms of the form $\ln(16\lambda_2^2/\varepsilon_2)$ and $\sqrt{n}\tanh^{-1}\sqrt{n}$ (note that $n < k^2$) in addition to polynomials in $1 - k^2$ and n. Now the momenta J are introduced using (21), and expanding again in J gives the above series. When $\varepsilon_2 < 0$ the computation is similar but different, however, the results are exactly the same. We find $1 - k^2 = -\varepsilon_2/(\lambda_2^2 - \varepsilon_2)$ and the parameter n satisfies the equation $1 - (1 - k^2)/(1 - n) = \lambda_2^2/\lambda_1^2$. The Legendre form of the basic integrals for the case $\varepsilon_2 < 0$ is given by

$$\oint_{\hat{\alpha}_2} \mathcal{E}_2\,\mathrm{d}z = 4\sqrt{\lambda_2^2 - \varepsilon_2}\,E(k) \tag{32}$$

$$\oint_{\hat\alpha_2} \frac{1}{z-a_0}\mathcal{E}_2\,\mathrm{d}z = \frac{4\varepsilon_2}{\sqrt{\lambda_2^2-\varepsilon_2}(\lambda_2^2-\lambda_1^2}\Pi(n,k) \tag{33}$$

with similar expansions as before.

For the action A_1 a similar calculation can be done, where

$$\mathcal{F}_1 = \sqrt{1-\frac{\varepsilon_2}{z-a_1}}, \quad \mathcal{E}_1 = \sqrt{\frac{z-a_0-\varepsilon_1}{(z-a_a)(a_2-z)}}$$

and the integration path is $\hat\alpha_1$ which projects onto $(-\infty, \min(a_0, a_0+\varepsilon_1)]$. In this case $1-k^2 = \varepsilon_1/\lambda_1^2$ and $n = \lambda_2^2/\lambda_1^2$ assuming $\varepsilon_1 > 0$. The integral of $\mathcal{E}_1$ is more complicated because it diverges when treated as a real integral. For the integral $\mathcal{E}_1$ the integration path can be changed to $[\max(a_1+\varepsilon_2, a_1), a_2]$ (i.e. back to $\hat\alpha_2$!) which is simply changing the sign of the result because the integral is elliptic and of 2nd kind. This gives (30) with each index 1 replaced by 2. For the 3rd kind integral no such regularization is needed, and the Legendre normal form for $\varepsilon_1 > 0$ is

$$\oint_{\hat\alpha_1} \frac{1}{z-a_1}\mathcal{E}_1\,\mathrm{d}z = \frac{4}{\lambda_1}\left(\frac{k^2}{n}K(k^2) + \left(1-\frac{k^2}{n}\right)\Pi(n,k^2)\right).$$

The results for $\varepsilon_1 < 0$ are similar and not shown here. We note that in the four cases of different signs of ε_i the intermediate formulas, moduli, and parameter are all different, while the end result (up to exchange of indices) is the same. The elliptic integrals with poles of order $l > 1$ can all be reduced to linear combinations of K, E, and Π by well known recursion formulas, see e.g. [18]. The intermediate results of the tedious but straightforward computations are not shown here.

Combining this result with corollary 2 we can show that the twist condition is satisfied everywhere near the HH equilibrium of the C. Neumann system for almost all parameters:

Corollary 3. *If $\lambda_1/\lambda_2 \neq 1+\sqrt{2}$ then the twist is non-vanishing near the hyperbolic-hyperbolic critical value in the Neumann system.*

Proof. Instead of the full Neumann system we consider the global reduction by the antipodal map $R_0R_1R_2$. We have $g_{ii}(0) = 2\ln 8\lambda_i$ and $g_{ij}(0) = 4\tanh^{-1}\lambda_2/\lambda_1$, $i \neq j$ and $n_1 = n_2 = 2$ for the reduced system in the $-+$ and $-+$ quadrant of J. For the other quadrants the slopes of $W = const$ and $H = 0$ have different signs. Now the slope of the exceptional line $W(c) = \lambda_1/\lambda_2$ is

$$-\exp\left(\ln 8\lambda_2 - \ln 8\lambda_1 + 2\frac{\lambda_1^2-\lambda_2^2}{\lambda_1\lambda_2}\tanh^{-1}\frac{\lambda_2}{\lambda_1}\right) = -\frac{1}{r}\left(\frac{r+1}{r-1}\right)^{r-1/r}$$

where $r = \lambda_1/\lambda_2 > 1$. The slope of the energy surface at the origin is $-r$, and this equation has unique solution $r = 1+\sqrt{2}$.

A trivial solution $r = 1$ occurs for the degenerate case $\lambda_1 = \lambda_2$.

As a corollary of theorem 5 above we see that the period lattice τ_{ij} takes the form

$$\pi\tau_{ij} = \begin{pmatrix} \ln\frac{8\lambda_1}{|J_1|} + \frac{3J_1}{4\lambda_1} - \frac{\lambda_2 J_2}{\lambda_1^2-\lambda_2^2} + O(J^2) & 2\tanh^{-1}\frac{\lambda_2}{\lambda_1} + \frac{\lambda_1 J_2 - \lambda_2 J_1}{\lambda_1^2-\lambda_2^2} + O(J^2) \\ 2\tanh^{-1}\frac{\lambda_2}{\lambda_1} + \frac{\lambda_1 J_2 - \lambda_2 J_1}{\lambda_1^2-\lambda_2^2} + O(J^2) & \ln\frac{8\lambda_2}{|J_2|} + \frac{3J_2}{4\lambda_2} + \frac{\lambda_1 J_1}{\lambda_1^2-\lambda_2^2} + O(J^2) \end{pmatrix}$$

The off-diagonal terms τ_{12} and τ_{21} appear to be equal. They are given by the unique complete integral of the first kind that stays finite and smooth when all other integrals of first kind blow up in the limit $J \to 0$. Consider a basis for the space of differentials of first kind. The curve $\mathcal{C}$ has genus 2, so this space has dimension two, and a basis is $b_0 = \mathrm{d}z/w$ and $b_1 = z\mathrm{d}z/w$. The limit $J \to 0$ implies that the β cycles vanish. The integrals over the $\hat\alpha$ cycles become singular, because each one develops a pole at its boundary. Because the cycles form a canonical basis the other pole is far away from the cycle. If there is a pole at $z = a_i$, then the differential $b_1 - a_i b_0$ will be finite. Thus there would be two finite integrals of first kind in the limit $J \to 0$, which is not possible because the basis only has two elements and one of them must diverge since the curve becomes singular, so τ_{12} and τ_{21} must be proportional. Together with the above expansion this proves that $\tau_{12} = \tau_{21}$ to all orders. It means that the cross-derivative of the actions are equal, and hence there exists a 'pre-potential' G whose gradient with respect to J gives the actions, and the period lattice is the Hessian matrix of this function. The expansion of G is obtained by integrating the actions A_i. Thus we have proved

Theorem 6. *Near the HH point of the Neumann system there exists a function $G(J)$ such that $\partial G/\partial J_i = A_i$. The series of G at $J = 0$ is given by*

$$\pi G = \sum_{i=1}^{2}\left(2\lambda_i J_i + \frac{3}{4}J_i^2 + \frac{1}{2}J_i^2 \ln\frac{8\lambda_i}{|J_i|} + \frac{J_i^3}{8\lambda_i} + \dots\right) + 2J_1J_2\tanh^{-1}\frac{\lambda_2}{\lambda_1} + J_1J_2\frac{\lambda_1 J_2 - \lambda_2 J_1}{2(\lambda_1^2 - \lambda_2^2)} + O(J^4)$$

It would be interesting to understand why the cubic terms in πG are negative twice the cubic terms in the Hamiltonian H.

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