On Integrals of Third Degree in Momenta

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Abstract

Consider a Riemannian metric on a surface, and let the geodesic flow of the metric have a second integral that is a third degree polynomial in the momenta. Then we can naturally construct a vector field on the surface. We show that the vector field preserves the volume of the surface, and therefore is a Hamiltonian vector field. As examples we treat the Goryachev-Chaplygin top, the Toda lattice and the Calogero-Moser system, and construct their global Hamiltonians. We show that the simplest choice of Hamiltonian leads to the Toda lattice.

1 Introduction

Simple questions about the existence of integrable geodesic flows are still unresolved. In the review article by Bolsinov, Fomenko and Kozlov [1] a number of conjectures were stated about integrable geodesic flows on the torus and the sphere. Integrable systems on these surfaces with linear or quadratic integral (in the momenta) are well known. But there are only two systems known with integrals with higher degree: the Goryachev-Chaplygin case (degree 3) [4] and the Kovalevskaya case for zero angular momentum (degree 4), both inducing geodesic flows on the sphere. There are no examples of integrals of more than second degree on the torus.

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The existence of an integral, which is a polynomial in the momenta, naturally leads to a system of partial differential equations. The principal idea is due to Whittaker [5], although it has to be slightly modified [8]. Let us consider the integral F being a polynomial in the momenta with unknown coefficients that depend on the position. F is an integral of the geodesic flow iff the Poisson bracket of the Hamiltonian H of the geodesic flow and the integral F identically equals zero. The Hamiltonian of a geodesic flow is a quadratic polynomial in the momenta. Therefore the Poisson bracket of H and F is also polynomial in the momenta. Hence each coefficient of this polynomial must be equal to zero. Thus we have a system of partial differential equations which is equivalent to the geodesic flow being polynomialy integrable. If the degree of the integral is N, then the system has $N + 2$ equations. If we treat the coefficient of the metric as one more unknown function, then the system has $N + 2$ unknown functions.

Although it is not at all easy to solve this system, one can prove that locally the solution always exists. Because of this, it is possible to construct an integrable geodesic flow with arbitrarily high degree of the integral in a small disc [7]. But in general all direct attempts to solve the system of partial differential equations failed.

Birkhoff found that the system yields an additional structure on the surface. He showed that the highest coefficient A of F is a holomorphic function for the case of quadratic F [2]. This is also true for higher degree integrals. Under coordinate changes the coefficient A changes like the coefficient of a holomorphic form. Therefore, in the appropriate coordinates A identically equals 1. If we substitute 1 for A in the system, then in the case of quadratic integrals the system can be solved, and the solution gives the geodesic flows with separated variables.

Recent advances in this field were made by V.V. Kozlov, L.S. Hall, V.N. Kolokol'tzov, and N.V. Denisova. Kozlov [6] investigated the global behavior of geodesics. He proved that there are no (at least in the analytical case) integrable geodesic flows on surfaces of genus 2 and more. Hall [8] was extending the theory to more general Hamiltonian systems. For the case of geodesic flows in the coordinates in which $A = 1$, he introduced an auxiliary potential ψ. The coefficients of F are functions of ψ, and the condition on ψ for the existence of a third degree integral consists of only one partial differential equation. Kolokoltzov [9] investigated the structure of the Birkhoff form A for quadratic integrals of geodesic flows on the sphere. He proved that A is a polynomial of 3rd or 4th degree, and that the zeros of the polynomial lie symmetrically on a circle which goes through infinity if A is a polynomial of third degree. After covering the sphere by a torus via the appropriate Weierstrass $\wp$-function the coefficient A has no zeros. Then it is constant, and therefore the variables are separated. Thus Kolokoltzov completely describes quadratically integrable geodesic flows on the sphere. Denisova and Kozlov in [11] proved that if a metric on the torus reads $\lambda(x, y)(dx^2 + dy^2)$, where x, y are angular coordinates on the torus, and λ is a polynomial in $\cos(x), \sin(x), \cos(y), \sin(y)$, then any polynomial integral in the momenta is a function of integrals of first and second degree.

In this article we show that the existence of a third degree integral naturally allows the construction of a smooth vector field on the surface, related to the second highest coefficient B of F. This vector field preserves the Riemannian volume of the surface. Hence it is locally Hamiltonian with respect to Riemannian volume. If the 1-cohomology group $H^1(P^2, \mathbb{R})$ of the surface P^2 is trivial, then any locally Hamiltonian vector field is globally Hamiltonian. We construct the global Hamiltonians (denoted by K) for some known integrable geodesic flows with third degree integrals.

The paper is organized as follows. In Sec. 2 we introduce the vector field $\lambda\vec{B}$ and prove that the vector field preserves Riemannian volume. In Sec. 3 we illustrate the theory by way of some examples. Besides the linearly integrable geodesic flow we treat the case of Goryachev-Chaplygin [4] on S^2, the Toda lattice [17] and the Calogero-Moser system [15]. The latter two systems are initially defined on $\mathbb{R}^3$, reduction with respect to the total momentum and restriction to momentum 0 gives a natural system on $\mathbb{R}^2$. In Sec. 4 we prove that if the metric reads $\lambda(x, y)(dx^2 + dy^2)$, and if the Hamiltonian K of the vector field $\lambda\vec{B}$ equals $\lambda^2(x, y)$, then either the geodesic flow is linearly integrable or the metric coincides with the reduced Toda lattice.

2 Theory

Let P^2 be a smooth surface with a Riemannian metric g. Since the metric g allows one to identify the tangent and cotangent bundle of P^2, we have a scalar product and a norm on every cotangent plane. The *geodesic flow* of the metric g is the Hamiltonian system with the Hamiltonian $H = \frac{1}{2}|\vec{p}|^2$, where $\vec{p}$ is the vector of momenta and $|.|$ is the norm.

A geodesic flow is called *integrable* if it is integrable as a Hamiltonian system. That is, there exists a function on the cotangent bundle $F : T^*P^2 \to$

$\mathbb{R}$ such that it is constant on the trajectories of the Hamiltonian system, and H and F are functionally independent. Recall that two functions are called *functionally independent* if their differentials are linearly independent almost everywhere. The function F is called an *integral*.

Now suppose our integral is *an integral of third degree*. That is, in local coordinates x, y, p_x, p_y, where x, y are local coordinates on the surface, p_x, p_y are the corresponding momenta, the integral is a homogeneous polynomial of third degree in the momenta given by the formula

$$F(x, y, p_x, p_y) = a(x, y)p_x^3 + b(x, y)p_x^2 p_y + c(x, y)p_x p_y^2 + d(x, y)p_y^3. \quad (1)$$

Evidently, the degree does not depend on the choice of the coordinates x, y.

Recall that if an integral is a sum of homogeneous polynomials of different degrees, then each polynomial is also an integral. Since that we consider only homogeneous polynomials.

There always exists a system of local coordinates x, y on the surface P^2 such that the metric g is given by $\lambda(x, y)(dx^2 + dy^2)$, the so called isothermal coordinates, see e.g. [16]. Consider the complex variable $z \overset{\text{def}}{=} x + iy$. In this complex variable the metric g has the form $\lambda(z, \bar{z})dzd\bar{z}$. The Hamiltonian of the metric is given by $2p\bar{p}/\lambda(z\bar{z})$, where $p \overset{\text{def}}{=} (p_x - ip_y)/2$ defines the corresponding momentum.

Let the integral F in the coordinates $z, \bar{z}, p, \bar{p}$ be given by

$$F(z, \bar{z}, p, \bar{p}) = A(z, \bar{z})p^3 + B(z, \bar{z})p^2\bar{p} + \bar{B}(z, \bar{z})p\bar{p}^2 + \bar{A}(z, \bar{z})\bar{p}^3. \quad (2)$$

Since F is a real-valued function, the coefficient of $p^i\bar{p}^j$ is conjugate to the coefficient of $p^j\bar{p}^i$.

Remark 1. If in coordinates x, y the integral F is given by (1), then $B(z, \bar{z}) = 3a(x, y) + c(x, y) + i(b(x, y) + 3d(x, y))$.

Proposition. *λB is a vector field.*

In other words, if we consider a new coordinate system $z_1, \bar{z}_1$, rewrite the metric and the integral in the new coordinates, then the coefficient B multiplied by λ transforms like a vector field.

Proof. For simplicity consider the linear transformation $z = Cz_1$, where C is a complex constant. Then the relation between the momenta is given by $p = p_1/C$. The metric g in the new coordinates $z_1, \bar{z}_1$ reads $C\bar{C}\lambda(z, \bar{z})dz_1 d\bar{z}_1$. Hence, $\lambda_{new} = C\bar{C}\lambda$.

Now substitute p_1/C for p in the formula for the integral. We have

$$F(z,\bar z,p_1,\bar p_1)=\frac{A(z,\bar z)}{C^3}p_1^3+\frac{B(z,\bar z)}{C^2\bar C}p_1^2\bar p_1+\frac{\bar B(z,\bar z)}{C\bar C^2}p_1\bar p_1^2+\frac{\bar A(z,\bar z)}{\bar C^3}\bar p_1^3.$$

Then $B_{new}=B/(C^2\bar C)$. Combining the formula for B_{new} with the formula for λ_{new}, we have, $(\lambda B)_{new}=\lambda B/C$. The proof for a general transformation is essentially the same.
Q.E.D.

Theorem 1. *The vector field $\lambda\vec{B}$ preserves the volume $\lambda dx\wedge dy$.*

Proof. Since the function F is an integral, the Poisson bracket $\{H,F\}$ is equal to 0.

$$\begin{aligned}
\{H,F\} &= \frac{\partial H}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial H}{\partial p}\frac{\partial F}{\partial z}+\frac{\partial H}{\partial \bar z}\frac{\partial F}{\partial \bar p}-\frac{\partial H}{\partial \bar p}\frac{\partial F}{\partial \bar z}\\
&= \frac{1}{\lambda^2}\left\{\lambda\frac{\partial A}{\partial \bar z}p^4+\left(\lambda\frac{\partial A}{\partial z}+\lambda\frac{\partial B}{\partial \bar z}+3\frac{\partial \lambda}{\partial z}A+\frac{\partial \lambda}{\partial \bar z}B\right)p^3\bar p+\right.\\
&\qquad +\lambda\frac{\partial \bar A}{\partial z}\bar p^4+\left(\lambda\frac{\partial \bar A}{\partial \bar z}+\lambda\frac{\partial \bar B}{\partial z}+3\frac{\partial \lambda}{\partial \bar z}\bar A+\frac{\partial \lambda}{\partial z}\bar B\right)\bar p^3 p+\\
&\qquad\left.+\left(\lambda\frac{\partial B}{\partial z}+\lambda\frac{\partial \bar B}{\partial \bar z}+2\frac{\partial \lambda}{\partial z}B+2\frac{\partial \lambda}{\partial \bar z}\bar B\right)p^2\bar p^2\right\}. \qquad (3)
\end{aligned}$$

In order for the homogeneous polynomial $\{H,F\}$ to be zero, every coefficient has to be zero. For later use we remark that the coefficients of p^4, $\bar p^4$ and $p^3\bar p$, $p\bar p^3$ are conjugate to each other, and we obtain

$$0 = \frac{\partial A}{\partial \bar z} \qquad (4)$$

$$0 = \lambda\frac{\partial A}{\partial z}+\lambda\frac{\partial B}{\partial \bar z}+3\frac{\partial \lambda}{\partial z}A+\frac{\partial \lambda}{\partial \bar z}B, \qquad (5)$$

such that A has to be a holomorphic function. The coefficient of $p^2\bar p^2$ is self conjugate, in particular

$$0=\lambda\frac{\partial B}{\partial z}+\lambda\frac{\partial \bar B}{\partial \bar z}+2\frac{\partial \lambda}{\partial z}B+2\frac{\partial \lambda}{\partial \bar z}\bar B,$$

or equivalently

$$0=\frac{\partial(\lambda^2 B)}{\partial z}+\frac{\partial(\lambda^2\bar B)}{\partial \bar z}. \qquad (6)$$

This is the condition for the vector field $\lambda\vec{B}$ to preserve the volume $\lambda dz \wedge d\bar{z}$. Returning to real coordinates gives the desired result $dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = dx \wedge dx + idy \wedge dy - 2idx \wedge dy = -2i(dx \wedge dy)$.
Q.E.D.

Remark 2. Since $\lambda\vec{B}$ is area preserving, there exists a real valued Hamiltonian $K(x, y)$ such that with respect to the symplectic structure $\lambda dx \wedge dy$ we have

$$\lambda\vec{B} = -2i\frac{\partial K}{\partial \bar{z}}\frac{1}{\lambda}. \tag{7}$$

In real variables x, y instead of (7) we have

$$\lambda\vec{B} = \frac{1}{\lambda}\left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x}\right). \tag{8}$$

The expression for $\lambda\vec{B}$ in terms of the coefficients of the integral (1) with respect to the coordinates x, y is $\lambda\vec{B} = (3a + c, b + 3d)\lambda$.

Remark 3. If we find a solution $(\lambda(z, \bar{z}), A(z, \bar{z}), B(z, \bar{z}))$ of the equations (4-6), then the function (2) is an integral for the geodesic flow of the metric $\lambda(z, \bar{z})dzd\bar{z}$. But we should not forget that if $\lambda \le 0$, then the metric is not positive definite, and that possibly there exists an integral of degree less than 3.

3 Examples

3.1 Linearly integrable geodesic flows

Let the geodesic flow of a metric g admit an integral linear in the momenta. Then from Birkhoff's results [2], see also [9], it easily follows that in appropriate coordinates x, y the metric has the form $f(x)(dx^2 + dy^2)$, and a linear integral is given by $F(x, y, p_x, p_y) = C_0 p_y$, where $C_0 \neq 0$ is a constant. The Hamiltonian of the geodesic flow is $H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2)/f(x)$. Using the constancy of p_y we can construct an integral of the third degree $F_3(x, y, p_x, p_y) = C_1 p_y^3 + C_2 p_y H(x, y, p_x, p_y)$. If the surface P^2 is closed, and if the metric is not the metric of constant curvature, then any third degree integral is a linear combination of p_y^3 and $p_y H(x, y, p_x, p_y)$, see [9, 10, 14].

Let $F_3 = C_1 p_y^3 + C_2 p_y H(x, y, p_x, p_y)$ be an integral of the geodesic flow of the metric $f(x)(dx^2 + dy^2)$. Then,

$$\lambda\vec{B} = (0, 3C_1 f(x) + 2C_2),$$

and K is a linear combination of functions

$$K_1 = \int_{x_0}^{x} f^2(\xi)d\xi \quad , \quad K_2 = \int_{x_0}^{x} f(\xi)d\xi$$

The proof is by direct calculation. Thus for linear integrable geodesic flows the vector field $\lambda\vec{B}$ is a sum of two vector fields, one of which preserves not only the volume, but even the metric. Recall that if there exists a vector field preserving the metric, then it follows from the Noether theorem that the geodesic flow of the metric is linearly integrable.

3.2 Toda lattice

Consider the Toda lattice with three particles. The three degree of freedom Hamiltonian and the integrals are given by

$$\begin{aligned} H_2 &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + d_3 e^{x_2 - x_1} + d_1 e^{x_3 - x_2} + d_2 e^{x_1 - x_3} \\ F_1 &= p_1 + p_2 + p_3 \\ F_3 &= p_1 p_2 p_3 - p_1 d_1 e^{x_3 - x_2} - p_3 d_3 e^{x_2 - x_1} - p_2 d_2 e^{x_1 - x_3}. \end{aligned}$$

By reduction with respect to the linear momentum F_1 we construct a family of natural Hamiltonian systems which is integrable by a third degree integral. In order for the reduced system to give a geodesic flow we must set F_1 equal to zero. By a linear change of coordinates the kinetic energy of the reduced system can be diagonalized, and after an additional scaling of variables we find

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V \tag{9}$$

$$V = d_2 e^{-\sqrt{3}y_1 + y_2} + d_3 e^{\sqrt{3}y_1 + y_2} + d_1 e^{-2y_2} \tag{10}$$

$$\begin{aligned} F &= -\frac{p_1^3}{3} + p_1 p_2^2 + p_2\sqrt{3}\left(d_3 e^{\sqrt{3}y_1 + y_2} - d_2 e^{\sqrt{3}y_1 - y_2}\right) \\ &\quad - p_1\left(d_2 e^{-\sqrt{3}y_1 + y_2} - 2d_1 e^{-2y_2} + d_3 e^{\sqrt{3}y_1 + y_2}\right) \end{aligned}$$

This Hamiltonian system is also known under the name 'Hamiltonian system with exponential interaction', [12].

Using Maupertuis' principle the Hamiltonian system with the Hamiltonian (9) is orbitally equivalent to the geodesic flow of the metric

$$\lambda(y_1, y_2)(dy_1^2 + dy_2^2), \qquad \text{where} \quad \lambda(y_1, y_2) = h - V(y_1, y_2), \tag{11}$$

which is positive definite wherever $h > V(y_1, y_2)$. Therefore, on the energy surface $\{H = h\}$ the function F is an integral of the geodesic flow of the metric (11).

Let us multiply the terms in F which are linear in momenta, by $\frac{H-V}{h-V}$. We obtain a homogeneous polynomial in momenta of third degree denoted by $\tilde{F}$. Since the factor $\frac{H-V}{h-V}$ identically equals 1 on the energy surface, at least on the energy surface the function $\tilde{F}$ is an integral of the geodesic flow. Since the Poisson bracket of two homogeneous functions in the momenta is again a homogeneous function in the momenta, $\tilde{F}$ is an integral of the geodesic flow not only on the energy surface, but everywhere [1].

Introducing complex variables as in the general theorem we find the vector field $\lambda\vec{B}$

$$\left(2d_2 e^{-\sqrt{3}y_1+y_2} - 4d_1 e^{-2y_2} + 2d_3 e^{\sqrt{3}y_1+y_2}, 2\sqrt{3}(d_2 e^{-\sqrt{3}y_1+y_2} - d_3 e^{\sqrt{3}y_1+y_2})\right)$$

which is given by (8) with the Hamiltonian K

$$K(y_1, y_2) = -\sqrt{2}\lambda^2(y_1, y_2).$$

In the last section we will prove that conversely if the Hamiltonian K equals λ^2, then the corresponding integrable metric is given by (11) and (10).

3.3 Calogero-Moser system

The Calogero-Moser system of three particles [15] is given by the Hamiltonian H_2 and integrals F_1, F_3 as

$$\begin{aligned}
H_2 &= \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{\gamma^2}{2}\left(\frac{1}{(x_2-x_1)^2} + \frac{1}{(x_3-x_2)^2} + \frac{1}{(x_1-x_3)^2}\right) \\
F_1 &= p_1 + p_2 + p_3 \\
F_3 &= p_1^3 + p_2^3 + p_3^3 - \frac{3}{2}\gamma^2\left(\left(\frac{1}{(x_1-x_2)^2} + \frac{1}{(x_1-x_3)^2}\right)p_1 + \left(\frac{1}{(x_2-x_1)^2}\right.\right. \\
&\quad \left.\left. + \frac{1}{(x_2-x_3)^2}\right)p_2 + \left(\frac{1}{(x_3-x_1)^2} + \frac{1}{(x_3-x_2)^2}\right)p_3\right).
\end{aligned}$$

Arguing as above, by setting $F_1 = 0$ we construct a family of geodesic flows with a third degree integral. The Hamiltonian and the integral of the corresponding natural system are

$$\begin{aligned}
H &= \frac{1}{2}(p_1^2 + p_2^2) + V \\
V &= -\left(\frac{3\gamma(y_1^2 + y_2^2)}{2y_2(y_2^2 - 3y_1^2)}\right)^2 \\
F &= p_1^3 - 3p_1p_2^2 + \frac{9\gamma^2(3y_1^4 - 6y_1^2y_2^2 - y_2^4)}{2y_2^2(y_2^2 - 3y_1^2)^2}p_1 - 36\frac{\gamma^2 y_1 y_2}{(y_2^2 - 3y_1^2)^2}p_2.
\end{aligned}$$

As before, $\lambda = h - V$, and the Hamiltonian K of the vector field $\lambda\vec{B}$ given by (8) reads

$$K(y_1, y_2) = 2\gamma\Big(\lambda(y_1, y_2) + 2h\Big)\sqrt{\lambda(y_1, y_2) - h}.$$

3.4 Goryachev Chaplygin top

As an example with nontrivial configuration space we treat the Goryachev Chaplygin top, which is a heavy top with ratio of moments of inertia 1 : 1 : 1/4, which is integrable for zero angular momentum. We use Euler angles (ϕ, θ, ψ) and the corresponding canonical momenta $(p_\phi, p_\theta, p_\psi)$ as a symplectic coordinate system, see e.g. [3, 13]. The angle ϕ of rotation around the axis of gravity is cyclic and the angular momentum p_ϕ is conserved. Reduction leads to a flow on $S^2 = SO(3)/S^1$, and restricting to $p_\phi = 0$ gives a geodesic flow on S^2 by Maupertuis' principle. The reduced Hamiltonian with $p_\phi = 0$ for ratios of moments of inertia being $\Theta : \Theta : \Theta/r$ reads

$$\begin{aligned}
H &= \frac{p_\theta^2}{2\Theta} + \frac{p_\psi^2}{2\Theta}G^{-1} + V \\
G^{-1} &= \frac{\cos^2\theta}{\sin^2\theta} + r \\
V &= s_1 \sin\theta \sin\psi + s_2 \sin\theta \cos\psi + s_3 \cos\theta.
\end{aligned}$$

For the Goryachev Chaplygin top we have $r = 4$ and $s_3 = 0$. For $h > \sqrt{s_1^2 + s_2^2}$ the whole S^2 is accessible the motion, such that on the energy surface $H = h$ we find the nonsingular Hamiltonian $\tilde{H}$ of the corresponding geodesic flow

$$\tilde{H} = \frac{1}{2\Theta(h - V)G}\left(p_\psi^2 + p_\theta^2 G\right) = \frac{1}{2\lambda}\left(p_\psi^2 + p_\theta^2 G\right)$$

with h as a parameter. The spherical coordinates θ and ψ are not isothermal coordinates for this problem. Since G only depends on θ they could, however, be introduced by a change of coordinates with

$$\frac{d\theta}{\sqrt{G}} = d\tilde{\theta}. \tag{12}$$

The second constant of motion is

$$\begin{aligned} F &= p_\psi \left(L_1^2 + L_2^2\right) - \Theta \cos\theta (s_1 L_1 + s_2 L_2) \\ L_1 &= p_\theta \cos\psi - p_\psi \sin\psi \frac{\cos\theta}{\sin\theta} \\ L_2 &= -p_\theta \sin\psi - p_\psi \cos\psi \frac{\cos\theta}{\sin\theta}. \end{aligned}$$

As in the previous case we multiply the linear terms of F by $\tilde{H}$ in order to make it a homogeneous polynomial. Since we set $\tilde{H} = 1$ we can ignore this factor in the following.

Instead of introducing isothermal coordinates and complex variables in this example it is better to perform the equivalent real calculation, which means the Fourier series of the value of the integral F on the momentum circle. Let the integral be given by the formula (2), and consider the energy surface $\{\tilde{H} = 1\}$. In a cotangent plane the momenta are located on an ellipse, or on a circle in isothermal coordinates. Parameterizing this momentum circle by an angle α and substituting $\sqrt{\lambda/2}e^{-i\alpha}$ for p in (2), we have

$$\begin{aligned} F(\alpha) &= \frac{\lambda^{3/2}}{\sqrt{2}} \left(Ae^{-3i\alpha} + Be^{-i\alpha} + \bar{B}e^{i\alpha} + \bar{A}e^{3i\alpha}\right) \\ &= \frac{\lambda^{3/2}}{\sqrt{2}} \left(\Re(A)\cos(3\alpha) + \Re(B)\cos(\alpha) + \Im(B)\sin(\alpha) + \Im(A)\sin(3\alpha)\right) \end{aligned}$$

The restriction of the integral to the ellipse is a finite Fourier series, and the coefficients of $\cos(\alpha)$ and $\sin(\alpha)$ give $\lambda\vec{B}\sqrt{\lambda/2}$.

In the present case we have to take into account the change of coordinates (12) and therefore we introduce

$$p_\psi(\alpha) = \sqrt{2\lambda}\cos\alpha, \quad p_\theta(\alpha) = \sqrt{2\lambda/G}\sin\alpha. \tag{13}$$

Note that the coordinates $\theta, \psi \in S^2$ and the angle α parameterize $\mathbb{R}P^3$, the unit tangent bundle of S^2, which is the energy surface $\{\tilde{H} = 1\}$ of the geodesic flow.

After some manipulations we find

$$\begin{aligned}
\lambda\vec{B} &= \frac{1}{\lambda}\left(\sqrt{G}\frac{\partial K}{\partial\theta}, -\frac{\partial K}{\partial\psi}\right) \\
K &= -\frac{1}{\sqrt{2}}\frac{\lambda^2}{G^{3/2}}\frac{\cos\theta}{\sin\theta} \quad .
\end{aligned}$$

Using (12), we see that K is the Hamiltonian of the vector field $\lambda\vec{B}$ on S^2. The function K has exactly two critical points, one minimum and one maximum, as shown in Fig. 1. They are located at $\psi = \pi + \arctan(s_1/s_2)$, $\theta = \arcsin x$, and x determined by the zeroes of the polynomial

$$3x^4 + 2cx - 1 = 0, \quad c = h(s_1^2 + s_2^2)^{-1/2}.$$

Since $\theta \in [0,\pi]$ we require $x \in [0,1]$, and for $c > -1$ there is a unique root in this interval, as can be shown using the Sturm sequence. Both solutions for $\theta = \arcsin x$ in the range $[0,\pi]$ are valid, the larger θ gives the maximum of K. For large c, i.e. large h, we have $2x \approx 1/c$, such that the zeroes move towards the poles $\theta = 0,\pi$ of S^2.

4 Third degree integrable geodesic flows with $K = \lambda^2$

Let the first cohomology group $H^1(P^2,\mathbb{R})$ of the surface P^2 be trivial. Then any closed 1-form is the differential of a function. Therefore any locally Hamiltonian vector field is globally Hamiltonian. Denote by K the Hamiltonian of the vector field $\lambda\vec{B}$ (with respect to the symplectic form $\lambda dx \wedge dy$). By theorem 1 the function F given by (2) is an integral if the coefficients of (3) vanish. Equation (4) holds iff the function A is holomorphic. Since by assumption $H^1(P^2,\mathbb{R})$ is trivial (6) is equivalent to the vector field $\lambda\vec{B}$ being Hamiltonian. Let us assume that the function A is bounded. Then $A \equiv \text{const} \in \mathbb{C}$. The aim of this section is to prove the following theorem.

Theorem 2. *Suppose the geodesic flow of a metric $\lambda(x,y)(dx^2+dy^2)$ on $\mathbb{R}^2$ is integrable by a third degree integral (2) with the coefficient A being constant and the function K equal to λ^2. Then either the geodesic flow of the metric is linearly integrable or in appropriate coordinates y_1, y_2 the metric is given by formulas (10) and (11).*

Proof. Let A equal $A_1 + iA_2$. There exists a coordinate system $\hat{x}, \hat{y}$, in which the metric reads $\hat{\lambda}(\hat{x}, \hat{y})(d\hat{x}^2 + d\hat{y}^2)$, the coefficient $\hat{A}$ equals $A_0 \in \mathbb{R}$, and $K(\hat{x}, \hat{y}) = \hat{\lambda}^2(\hat{x}, \hat{y})$. Indeed, consider the linear transformation $\hat{z} = Cz$, where $z = x + iy$, $\hat{z} = \hat{x} + i\hat{y}$, and C is a complex constant, $|C| = 1$. Then the coefficient A transforms as $\hat{A} = C^3 A$. Obviously, for the appropriate constant C we have $\hat{A} = A_0 \in R$. Since $|C| = 1$, in the corresponding coordinate system $\hat{x}, \hat{y}$ the new factor $\hat{\lambda}$ equals the old factor λ. Hence the function K still equals $\hat{\lambda}^2$. Without loss of generality it can therefore be assumed that in coordinates x, y the coefficient A is real.

Now substitute A_0 for $A(z, \bar{z})$ into the equations (4-6). The first equation is now automatically fulfilled. Substitute (8) for B in the remaining equations. By the Hamiltonian property of the the vector filed λB equation (6) is identically fulfilled. The remaining equation (5) in real variables x, y reads

$$0 = 2\frac{\partial^2 \lambda(x,y)}{\partial y^2} - 2\frac{\partial^2 \lambda(x,y)}{\partial x^2} - 3A_0 \frac{\partial \lambda(x,y)}{\partial y} \tag{14}$$

$$0 = \frac{\partial}{\partial x}\left(4\frac{\partial \lambda(x,y)}{\partial y} + 3A_0 \lambda(x,y)\right) \tag{15}$$

From (15) we see that the solution has the form $X(x)Y(y) + \beta(y)$. Substituting $X(x)Y(y) + \beta(y)$ for λ in the second equation, we have

$$4Y'(y)X'(x) + 3A_0 Y(y)X'(x) = 0.$$

In the trivial case $X \equiv \text{const}$ the metric depends only on y, and the geodesic flow is even linearly integrable. Assume that $X \neq \text{const}$. Then

$$Y(y) = C_0 e^{-3A_0 y/4}.$$

Substitute $X(x)e^{-3A_0 y/4} + \beta(y)$ for λ in (14). We obtain

$$2\beta''(y)e^{3/4A_0 y} + 3A_0\beta'(y)e^{3/4A_0 y} - \frac{27}{8}X(x) + 2A_0^2 X''(x),$$

and the variables in the last equations are separated. Therefore

$$\begin{cases} 2\beta''(y)e^{-3/4y} + 3A_0\beta'(y)e^{-3/4y} = C \\ 2X''(x) - 27/8X(x) = -C, \end{cases}$$

where C is a constant. Thus

$$X(x) = -\frac{8C}{27A_0^2} + C_1 e^{-3/4\sqrt{3}A_0 x} + C_2 e^{3/4\sqrt{3}A_0 x},$$

$$\begin{aligned}\beta(y) &= \frac{8C}{27A_0^2}e^{-3/4A_0y} + E + C_3e^{3/2A_0y}, \quad \text{and finally} \\ \lambda(x,y) &= E + C_1e^{-3/4A_0(y+\sqrt{3}x)} + C_2e^{-3/4A_0(y-\sqrt{3}x)} + C_3e^{3/2y}.\end{aligned}$$

Q.E.D.

5 Summary

We have shown that the existence of a third degree polynomial integral for a geodesic flow on a surface naturally allows to construct a vector field on the surface. This vector field preserves the Riemannian volume $\sqrt{\det(g)}dx \wedge dy$ of the surface, and therefore for surfaces with trivial 1-cohomology group is Hamiltonian. We constructed the Hamiltonians for a number of known examples of third degree polynomial integrable geodesic flows. The Hamiltonians for these examples have a simple relation with the conformal coefficient λ of the metric. The Hamiltonian for the reduced Toda system reads λ^2, for the reduced Calogero-Moser system it is $K = 2\gamma(\lambda + 2h)\sqrt{\lambda - h}$, and for the Goryachev-Chaplygin system we found $K = -\frac{1}{\sqrt{2}}\lambda^2\frac{\cos\theta}{\sin\theta}G(\theta)^{-3/2}$. Thus, the Hamiltonians for Toda and Calogero-Moser cases are functions of λ, the Hamiltonian for Goryachev-Chaplygin case is a function of λ, multiplied by a function that depends on one variable only. For the first case the converse statement is also true. We have shown that if the $K = \lambda^2$, then the corresponding integrable system coincides with the reduced Toda lattice.

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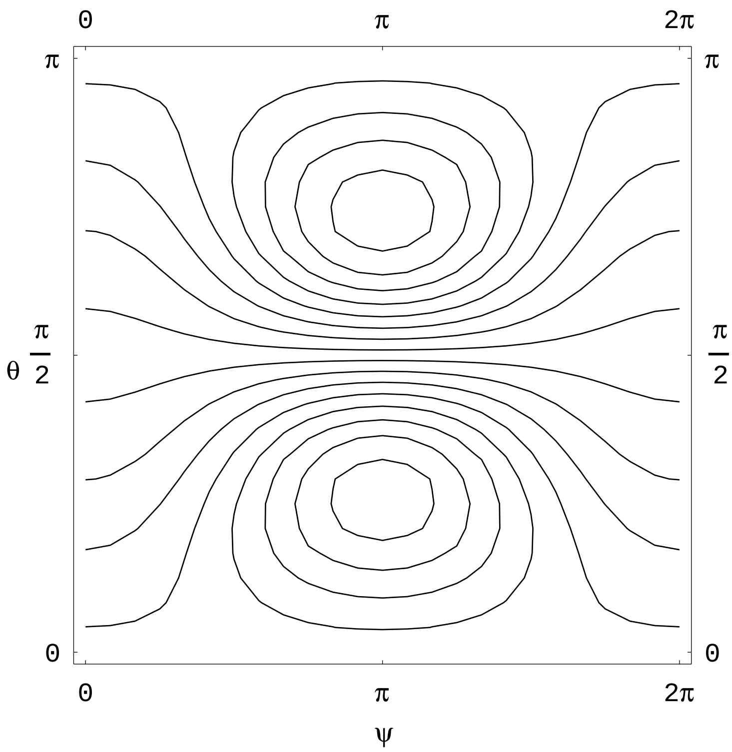

Figure 1: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case