A survey of my joint research with Gavin Brown[1]

Dedicated to Professor Gavin Brown on his 65th birthday

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Abstract I have been collaborating with Gavin Brown since 1990. Our research mainly concerns the positivity of trigonometric sums and Jacobi polynomial sums. It also concerns multiple trigonometric sums and the convergence of the linear means of multiple Fourier series and Fourier-Laplace series. The present report is a survey on our joint results.

§1 Positivity of trigonometric sums

In 1990, we considered the sine sums

$$T_n^\alpha(\theta) := \sum_{k=1}^{n} \frac{\sin k\theta}{k+\alpha}, \quad n \in \mathbb{N}_+.$$

We define three constants $\lambda_0, \mu_0, \alpha_0$ as follows. The constant λ_0 is the solution of the equation

$$(1+\lambda)\pi = \tan(\lambda\pi) \quad 0 < \lambda < \frac{1}{2}.$$

It is easy to see that $\lambda_0 = 0.4302967\cdots$ is the point at which the function $\frac{\sin\lambda\pi}{1+\lambda}$ attains its maximum on the interval $(0, \frac{1}{2})$. The constant $\mu_0 = 0.8128252\cdots$ is defined to be the solution of the equation

$$\frac{\sin\mu\pi}{\mu\pi} = \frac{\sin\lambda_0\pi}{(1+\lambda_0)\pi},$$

and $\alpha_0 = 2.1\cdots$ is the solution of the equation

$$\sum_{k=1}^{\infty} \frac{2k}{(2k-1+\alpha)(2k+\alpha)(2k+1+\alpha)} = \frac{\sin\lambda_0\pi}{2(1+\lambda_0)\pi}.$$

Our results are the following four theorems which were published in [1].

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Theorem 1 If $-1 < \alpha \leqslant \alpha_0$ then

$$T_{2n-1}^{\alpha}(\theta) > 0, \quad 0 < \theta < \pi, \ n \in \mathbb{N}_+.$$

Theorem 2 If $-1 < \alpha \leqslant \alpha_0$ then

$$T_{2n}^{\alpha}(\theta) > 0, \quad 0 < \theta < \pi - \frac{\mu_0 \pi}{2n + 0.5}, \ n \in \mathbb{N}_+.$$

Theorem 3 If $\alpha > \alpha_0$ then there exists an infinite subset $N \subset \mathbb{N}_+$ such that

$$T_{2n-1}^{\alpha}\Big(\pi - \frac{(1+\lambda_0)\pi}{2n-1}\Big) < 0, \quad n \in N.$$

Theorem 4 If $0 < \gamma < \mu_0$ then there exists an α near to but strictly smaller than α_0 such that

$$T_{2n}^{\alpha}\Big(\pi - \frac{\gamma \pi}{2n + 0.5}\Big) < 0, \quad \text{for an infinite number of } n.$$

Theorem 3 shows that α_0 is best possible in Theorem 1. Theorem 4 shows that μ_0 is best possible in Theorem 2.

The particular case $\alpha = 1$ has been considered by Brown and Wilson [2]. They obtained the following conclusion:

$$T_{2n-1}^{1}(\theta) > 0, \ 0 < \theta < \pi; \quad T_{2n}^{1}(\theta) > 0, \ 0 < \theta < \pi - \frac{\pi}{2n}.$$

These extended the Fejér-Jackson-Gronwall inequality [3],

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} > 0, \quad 0 < \theta < \pi,$$

Later, in 1991 we considered basic cosine sums with Dr. D.Wilson together. The sums we considered are

$$S_n^{\beta}(\theta) := 1 + \sum_{k=1}^{n} k^{-\beta} \cos k\theta, \ n \in \mathbb{N}_+, \ \beta > 0, \ \theta \in \mathbb{R}.$$

Define β_0 to be the unique root in $(0,1)$ of the equation

$$\int_0^{\frac{3}{2}\pi} \frac{\cos u}{u^{\beta}}\, du = 0.$$

It is easy to check that $0.308443 < \beta_0 < 0.308444$. Our main result is

Theorem 5[4] For all $\theta \in \mathbb{R}$ and $n > 1$ we have $S_n^{\beta_0}(\theta) > 0.0376908$.

It is know (see [5], V.2, 29) that for $\beta < \beta_0$ the sums $S_n^{\beta}(\theta)$ are not uniformly bounded below.

An immediate corollary to Theorem 5 is

Theorem 6[4] Let $\beta \geqslant \beta_0$ and suppose $\{a_k\}_{k=0}^{\infty}$ is a non-increasing sequence of non-negative numbers satisfying $k^{\beta} a_k \geqslant (k+1)^{\beta} a_{k+1}$ for all $k \geqslant 1$. Then

$$\sum_{k=0}^{n} a_k \cos k\theta \geqslant 0$$

for all $\theta \in \mathbb{R}$ and $n \geqslant 0$.

These results extend the Young's inequality [6] which is a particular case of Theorem 6 when $\beta = 1$ and $a_0 = 1, a_k = k^{-1}$ for all $k \geqslant 1$.

Remark There is no analogue of Theorem 5 for sine sums. Indeed, for any $\beta < 1$,

$$\limsup_{n\to\infty} \min_{\theta\in\mathbb{R}} n^{\beta} \sum_{k=1}^{n} k^{-\beta} \sin k\theta \leqslant -\frac{1}{2}.$$

L. Vietoris [7] proved in 1958 that if $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n > 0$ and $a_{2k} \leqslant (1 - \frac{1}{2k})a_{2k-1}, \quad k \geqslant 1$, then for $n \geqslant 1$ and $\theta \in (0, \pi)$,

$$\sum_{k=1}^{n} a_k \sin k\theta > 0 \quad \text{and} \quad \sum_{k=0}^{n} a_k \cos k\theta > 0.$$

These inequalities of Vietoris extend both the above mentioned Fejér-Jackson-Gronwall inequality and W. H. Young's inequality. Moreover it was shown by Askey and Steinig [8] in 1974 that these inequalities can be applied to yield various new results, including improved estimates for the localization of zeros of a class of trigonometric polynomials and new positive sums of ultraspherical polynomials. The significance of Vietoris's inequalities was illustrated once again by Askey in his report [9], where he discussed some problems suggested by these inequalities and showed how one of them leaded to the derivation of the hypergeometric summation formula and to other summation formulas.

As shown by Vietoris himself, his result is actually equivalent to the following

Theorem A If $a_0 = a_1 = 1,\quad a_{2k} = a_{2k+1} = \left(1 - \frac{1}{2k}\right)a_{2k-1} = \frac{\Gamma(k+0.5)}{\Gamma(0.5)\Gamma(k+1)},\quad k \in \mathbb{N}$, then for any $n \in \mathbb{N}_+$ and $\theta \in (0, \pi)$,

$$\sum_{k=1}^{n} a_k \sin k\theta > 0, \quad \sum_{k=0}^{n} a_k \cos k\theta > 0.$$

One extension of Theorem A was proved by Brown and Hewitt [10]:

Theorem B If $a_0 = a_1 = 1, a_{2k} = a_{2k+1} = \left(1 - \frac{1}{2k+1}\right)a_{2k-1},\quad k \in \mathbb{N}_+$, then the Vietoris' cosine inequality $\sum_{k=0}^{n} a_k \cos k\theta > 0$ remains true.

Additional interest of such an extension was noted by Askey [11].

In 1998 we investigated some extensions of Vietoris's cosine inequalities. For $\beta \in (0, 1)$ we define $c_j(\beta),\ j \geqslant 0$, inductively, by

$$c_0(\beta) = c_1(\beta) = 1, \quad c_{2k}(\beta) = c_{2k+1}(\beta) = \left(1 - \frac{\beta}{k}\right)c_{2k-1}(\beta) = \frac{\Gamma(k+1-\beta)}{\Gamma(1-\beta)\Gamma(k+1)}, \quad k \geqslant 1$$

and for $n \geqslant 1$, define

$$V_n^{\beta}(\theta) := \sum_{k=0}^{n} c_k(\beta) \cos k\theta.$$

Our main results are following two theorems which have been announced in [12].

Theorem 7[12] For $0 < \theta < \pi$ and $n \geqslant 1$, the inequality,

$$V_n^{\beta}(\theta) = \sum_{k=0}^{n} c_k(\beta) \cos k\theta > 0$$

is true whenever $\beta_0 \leqslant \beta < 1$, where $\beta_0 = 0.30844\cdots$ is the same number as in Theorem 5. Furthermore, the number β_0 is best possible in the sense that for any $\beta \in (0, \beta_0)$ we have

$$\lim_{n\to\infty} \min_{\theta\in(0,\pi)} V_n^{\beta}(\theta) = -\infty.$$

As a direct corollary we get

Theorem 8[12] Let $\{a_k\}_{k=0}^{n}$ be a sequence of real numbers satisfying $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n > 0$ and $a_{2k} \leqslant \left(1 - \frac{\beta_0}{k}\right)a_{2k-1},\quad k \geqslant 1$. Then for $\theta \in (0, \pi)$ we have

$$\sum_{k=0}^{n} a_k \cos k\theta > 0.$$

In 2003, we extended Vietoris's inequality in another way. For $\alpha > -1$ and $k \geqslant 0$, we define

$$d_{2k}(\alpha) = d_{2k+1}(\alpha) = \frac{\Gamma(\frac{2+\alpha}{2})\Gamma(k+\frac{1+\alpha}{2})}{\Gamma(\frac{1+\alpha}{2})\Gamma(k+\frac{2+\alpha}{2})}$$

and define for $n \geqslant 1$,

$$U_n^\alpha(x) = \sum_{k=0}^{n} d_k(\alpha)\cos(kx).$$

It can be shown that the equation

$$\psi(\alpha) := \min_{x\in(0,\pi)} U_6^\alpha(x) = 0$$

has a unique solution $\tilde{\alpha}_0$ in $(-1,\infty)$. And mechanical computation shows $\tilde{\alpha}_0 \in (2.3308, 2.3309)$. Our main results are the following two theorems which were announced in [13]

Theorem 9[13] If $-1 < \alpha \leqslant \tilde{\alpha}_0$, then

$$U_n^\alpha(x) \geqslant 0$$

for all $x \in (0,\pi)$ and $n \in \mathbb{N}_+$, where the equality holds for some $x \in (0,\pi)$ if and only if $n = 6$ and $\alpha = \tilde{\alpha}_0$. Moreover, for $\alpha > \tilde{\alpha}_0$, $\min\limits_{x\in(0,\pi)} U_6^\alpha(x) < 0$.

It is clear that Theorem 9 for $\alpha = 0$ corresponds to Vietoris's cosine inequality. We remark that Theorem 9 for $\alpha = 1$ is due to Brown and Hewitt [10] while for $\alpha = 2$ is due to a recent paper [14] by Brown and Yin.

As an immediate consequence of Theorem 9, we obtain, by Abel's transform,

Theorem 10[13] Let $\{a_k\}_{k=0}^\infty$ be a sequence of real numbers satisfying $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n > 0$ and $(2k+\tilde{\alpha}_0)a_{2k} \leqslant (2k-1+\tilde{\alpha}_0)a_{2k-1}$, $k \geqslant 1$. Then we have

$$\sum_{k=0}^{n} a_k \cos kx > 0, \quad x \in (0,\pi), \quad n \in \mathbb{N}_+,$$

where the equality holds for some $x \in (0,\pi)$ if and only if $n = 6$ and

$$a_0 = a_1 = 1, \quad a_2 = a_3 = \frac{1+\tilde{\alpha}_0}{2+\tilde{\alpha}_0},$$

$$a_4 = a_5 = \frac{(1+\tilde{\alpha}_0)(3+\tilde{\alpha}_0)}{(2+\tilde{\alpha}_0)(4+\tilde{\alpha}_0)},$$

$$a_6 = \frac{(1+\tilde{\alpha}_0)(3+\tilde{\alpha}_0)(5+\tilde{\alpha}_0)}{(2+\tilde{\alpha}_0)(4+\tilde{\alpha}_0)(6+\tilde{\alpha}_0)}.$$

Also, we have proved the following theorem as well.

Theorem 11[13] For any $\alpha > -1$ there's an $M = M(\alpha)$ depending only on α such that for $n \geqslant M$ and all $x \in (0, \pi)$, $U_n^\alpha(x) > 0$. Furthermore, given an integer $n \geqslant 2$ there is a number $\gamma = \gamma(n)$ depending only on n such that for $\alpha > \gamma(n)$,

$$\min_{x \in (0,\pi)} U_n^\alpha(x) < 0.$$

For $\alpha > -1$, we denote by $N(\alpha)$ the smallest positive integer for which $U_n^\alpha(x) > 0$ holds for all $x \in (0, \pi)$ whenever $n \geqslant N(\alpha)$. Theorem 11 means that for each $\alpha > -1$, $N(\alpha)$ is finite and that

$$\lim_{\alpha \to \infty} N(\alpha) = \infty.$$

It would be interesting if one could find a better upper estimate of $N(\alpha)$ for each specific α. However, this seems to be quite difficult.

In 2005, we considered cosine sums of another type with Dr. Dai together. This investigation relates also to the extension of Young's inequality.

Given $0 < \beta < 1$, we set

$$a_k(\beta) = \frac{\Gamma(k+1-\beta)}{\Gamma(2-\beta)\Gamma(k+1)}, \quad k = 1, 2, \cdots$$

and define

$$S_n(x, \beta) := 1 + \sum_{k=1}^{n} a_k(\beta) \cos kx, \quad x \in [0, \pi], \; n = 1, 2, \cdots.$$

We remember that $\beta_0 = 0.308443\cdots$ is the number defined in Theorem 5. Define β^* to be the unique solution $\beta \in (0, 1)$ of the equation

$$\min_{x \in [0,\pi]} S_7(x, \beta) = 0.$$

Numerical evaluation shows that $\beta^* = 0.33542\cdots$. For convenience, we set, for $\beta \in (0, 1)$,

$$M(\beta) := \inf \Big\{ N \geqslant 0 : \min_{x \in [0,\pi]} S_n(x, \beta) > 0 \;\text{ whenever } n \geqslant N \Big\},$$

where it is agreed that $\inf \emptyset = \infty$.

Our main result is

Theorem 12[15] (i) If $\beta^* \leqslant \beta < 1$ then

$$S_n(x,\beta) \geqslant 0, \quad x \in [0,\pi], \quad n = 1,2,\cdots,$$

with equality being true for some $x \in [0,\pi]$ if and only if $\beta = \beta^*$ and $n = 7$.

(ii) If $\beta_0 < \beta < \beta^*$ then

$$8 \leqslant M(\beta) \leqslant M_0 \cdot (\beta - \beta_0)^{-\frac{1}{\beta_0(1-\beta_0)}} = N_0 \cdot (\beta - \beta_0)^{-4.6881\cdots},$$

where M_0 is a positive absolute constant.

(iii) If $\beta = \beta_0$ then

$$\limsup_{n\to\infty} \Big(\min\{S_n(x,\beta_0) : x \in [0,\pi]\} \Big) = -\frac{\beta_0}{1-\beta_0} = -0.446014\cdots.$$

(iv) If $0 < \beta < \beta_0$ then

$$\liminf_{n\to\infty} \Big(\min\{S_n(x,\beta) : x \in [0,\pi]\} \Big) = -\infty.$$

§2 Positivity of Jacobi sums

In 1993 we investigates some Jacobi sums with Koumandos together.

Let $P_k^{(\alpha,\beta)}, \alpha > -1, \beta > -1, k = 0, 1, 2, \cdots$ denote Jacobi polynomials (see [16]) which are orthogonal on $[-1, 1]$ with the weight function $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ and are normalized by

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)\Gamma(n+1)}.$$

In the special case $\alpha = \beta = 0$, $P_k^{(0,0)}$ are written as P_k and called Legendre polynomials. Define

$$S_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} P_{2k}(\cos\theta),\ T_n(\theta) = \frac{\sqrt{15}}{45} + \sum_{k=1}^{n} P_{2k-1}(\cos\theta),\ U_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} P_{4k}(\cos\theta)$$

for $\theta \in [0, \pi]$ and $n = 1, 2, \cdots$. We have proved

Theorem 13[17]

$$S_n(\theta) \geqslant 0 \ \ \forall\, \theta \in [0, \pi],\ n = 1, 2, \cdots ; \quad T_n(\theta) \geqslant 0 \ \ \forall\, \theta \in [0, \frac{\pi}{2}],\ n = 1, 2, \cdots .$$

Theorem 14[17]

$$U_n(\theta) > 0 \ \ \forall\, \theta \in [0, \pi],\ n = 1, 2, \cdots .$$

Our another work is on the Cotes numbers at Jacobi abscissas. Expand

$$(1-x)^{-\gamma}(1+x)^{-\delta} \sim \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x) \quad -1 < x < 1.$$

The values of the partial sum of the above expansion, i.e.

$$\sum_{k=0}^{n} a_k P_k^{(\alpha,\beta)}(x) \tag{$*$}$$

at zeros of $P_n^{(\alpha,\beta)}(x)$ are (positive multiples) of the Cotes numbers for integration with respect to $(1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}\,dx$. Our main results are

Theorem 15[18] Let $\delta = 0, \beta = -\frac{1}{2}$. If $\alpha \geqslant 0$ and $0 < \gamma \leqslant \frac{1}{2}$ then all the partial sums in $(*)$ are strictly positive for $x \in [-1, 1]$.

Theorem 16[18] If $\alpha = 0, \beta = -\frac{1}{2}, \gamma = \frac{1}{2}, \delta = \frac{1}{4}$ then all the partial sums in $(*)$ are strictly positive for $x \in [-1, 1]$.

Theorem 17[18] Suppose $\alpha+\beta+1>0, -1<\gamma<0, \delta=0$. All the partial sums in $(*)$ are strictly positive for $x\in[-1,1]$ when $\alpha\geqslant\beta$. Moreover, they are decreasing when (i)$\alpha\geqslant\beta$ or (ii)$|\alpha|<\beta+1, -1<\gamma\leqslant\frac{1}{2}(\alpha-\beta-1)$.

Theorem 18[18] When $-\beta-1<\alpha<\beta, -1<\gamma\leqslant\frac{1}{2}(\alpha-\beta-1)$ and $\delta=0$ all the partial sums in $(*)$ are strictly positive for $x\in[-1,1]$.

If $\alpha=\beta$, Jacobi polynomial $P_n^{(\alpha,\beta)}$ is called an ultraspherical polynomial or Gegenbauer polynomial. The customary notation by [16] is

$$P_n^{\lambda}(x)=\frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(2\lambda)}\frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+\frac{1}{2})}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x),\quad \lambda>-\frac{1}{2}.$$

Define

$$S_n^{\lambda}(x)=\sum_{k=0}^{n}\frac{P_n^{\lambda}(x)}{P_n^{\lambda}(1)},\quad -1<x<1,\ n\in\mathbb{N}_+.$$

It has long been known that for $\lambda\geqslant\frac{1}{2}$ all the sums are positive (see [19], for example) but determination of the best lower for λ has been a recurring source of speculation. Szegö commented in [20] that "there exists a critical value λ', $0<\lambda'<\frac{1}{2}$, such that $S_n^{\lambda}>0$ for $\lambda>\lambda'$ but $S_n^{\lambda}(x)$ takes negative values for appropriate x and n when $\lambda<\lambda'$." He noted that " evaluation of this number (viz. λ') seems to be difficult." We solved this problem posed by Szegö forty six years ago.

Define α' to be the solution of the equation

$$\int_0^{j_{\alpha,2}}t^{-\alpha}J_\alpha(t)\,dt=0$$

where $j_{\alpha,2}$ is the second positive root of the Bessel function J_α of the first kind of order α. Our result is

Theorem 19[21] Let $\lambda'=\alpha'+\frac{1}{2}$. Then

$$\begin{aligned}&\inf\{S_n^{\lambda}(x): -1\leqslant x\leqslant 1,\ n\in\mathbb{N}_+\}=-\infty &&\text{when}\quad \lambda<\lambda';\\ &S_n^{\lambda}(x)>0,\ \forall\ x\in(-1,1) &&\text{when}\quad \lambda\geqslant\lambda'.\end{aligned}$$

Numerical calculation shows that $\lambda'=0.23061297\cdots$.

Our another result is

Theorem 20[22] Let $\beta_0 = 0.308443\cdots$ be the number defined in Theorem 5. If $\lambda \geqslant \beta_0$ then

$$\sum_{k=0}^{n}(-1)^k \frac{P_{2k}^{\lambda}(\cos\theta)}{P_{2k}^{\lambda}(1)} \geqslant 0,\ n \in \mathbb{N}_+,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

The only cases of equality are when $\theta = 0$ and n is odd.

It is easy to check that the inequality in Theorem 20 is equivalent to

$$\sum_{k=0}^{n} \frac{P_k^{(-\frac{1}{2},\lambda-\frac{1}{2})}(x)}{P_k^{(\lambda-\frac{1}{2},-\frac{1}{2})}(1)} \geqslant 0,\ -1 \leqslant x \leqslant 1.$$

§3 Multiple trigonometric sums and trigonometric series

Let $a_{i,j}, i, j \in \mathbb{N}_+$, be real numbers satisfying the condition

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\{\sum_{k=i}^{\infty}\sum_{\ell=j}^{\infty}a_{k,\ell}^2\right\}^{\frac{1}{2}} < \infty \tag{1}$$

Suppose $q > 1$ and m_i, n_j are positive numbers satisfying the condition

$$\frac{m_{i+1}}{m_i} \geqslant q, \quad \frac{n_{j+1}}{n_j} \geqslant q, \quad m_1 = n_1 = 1. \tag{2}$$

Define (in L^2 sense)

$$f(x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} \sin m_i x \sin n_j y,$$

$$g_j(x) = \sum_{i=1}^{\infty} a_{i,j} \sin m_i x, \quad h_i(y) = \sum_{j=1}^{\infty} a_{i,j} \sin n_j y.$$

F. Móricz [22] proved in the case $m_i = n_i = 2^{i-1}, i \in \mathbb{N}_+$ that the condition (1) is equivalent to

$$\frac{f(x,y)}{xy} \in L(0,1)^2, \qquad \frac{g_i(x)}{x} \in L(0,1), \ \frac{h_i(y)}{y} \in L(0,1) \quad i \in \mathbb{N}_+. \tag{3}$$

He proposed that in the general case when m_i, n_j are positive integers satisfying condition (2) then (3) is satisfied if and only if

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \log\frac{m_{i+1}}{m_i}\log\frac{n_{j+1}}{n_j}\left\{\sum_{k=i}^{\infty}\sum_{\ell=j}^{\infty}a_{k,\ell}^2\right\}^{\frac{1}{2}} < \infty. \tag{4}$$

Our result is

Theorem 21[23] Let $a_{i,j}, m_i, n_j$ satisfy (1) and (2). Let f, g_i, h_j be as above. Define

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{i,j}|,$$

$$T = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \log\frac{m_{i+1}}{m_i}\left\{\sum_{k=i+1}^{\infty}a_{k,j}^2\right\}^{\frac{1}{2}}, \quad U = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \log\frac{n_{j+1}}{n_j}\left\{\sum_{\ell=j+1}^{\infty}a_{i,\ell}^2\right\}^{\frac{1}{2}},$$

$$V = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \log\frac{m_{i+1}}{m_i}\log\frac{n_{j+1}}{n_j}\left\{\sum_{k=i+1}^{\infty}\sum_{\ell=j+1}^{\infty}a_{k,\ell}^2\right\}^{\frac{1}{2}}.$$

The condition (3) is equivalent to the condition

$$S + T + U + V < \infty. \tag{5}$$

We point out that in our theorem, m_i, n_j need not be integers. When $m_i = n_i = 2^{i-1}, i \in \mathbb{N}_+$, (4) is equivalent to (5). But in general, (4) is stronger than $V < \infty$, and (4) is not equivalent to (5).

We also considered the problem of strong uniform approximation of multivariate periodic continuous functions by their Bochner-Riesz means of critical order. The Bochner-Riesz means of order α of a function $f \in L(\mathbb{T}^n)(n \geqslant 2)$ $(\mathbb{T} = [-\pi, \pi]$) are defined as

$$S_R^\alpha(f)(x) = \sum_{|m|<R} c_m(f) e^{imx}(1 - |m|^2 r^{-2})^\alpha, x \in \mathbb{R}^n,\ R > 0,$$

where $c_m(f)$ denotes the $m-$th Fourier coefficient, $m \in \mathbb{Z}^n$, mx denotes the usual inner product of m and x. The order α has a critical value $\frac{n-1}{2}$. The means $S_R^{\frac{n-1}{2}}$ at critical order are regarded as analogues of partial sums of single Fourier series in some sense. Denote by $\omega_2(f, t), (t > 0)$, the second modulus of continuity of $f \in C(\mathbb{T}^n)$. Our result is

Theorem 22[24] Let $q > 0$. Then for all $f \in C(\mathbb{T}^n)$

$$\left\| \frac{1}{R} \int_0^R |f - S_r^{\frac{n-1}{2}}(f)|^q \, dr \right\| \leqslant \frac{C}{R} \int_0^R |\omega_2(f, \frac{1}{r})|^q \, dr, \quad R > 0$$

where the norm is in the sense of $C(\mathbb{T}^n)$ and C denotes constant independent of f and R.

§4 Convergence of the linear means of Fourier-Laplace series

For investigating the almost everywhere convergence of the linear means of Fourier-Laplace series, we considered Jacobi polynomials with complex indices. By the formula ([16], p.62, (4.21.2))

$$P_k^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+\beta+k+1)} \sum_{j=0}^{k} \frac{\Gamma(\alpha+\beta+k+j+1)}{\Gamma(j+1)\Gamma(k-j+1)\Gamma(\alpha+j+1)} \Big(\frac{x-1}{2}\Big)^j$$

we naturally extend the definition of the Jacobi polynomials to the case of complex indices (α, β) with $\Re\alpha > -1$, $\Re\beta > -1$.

We established the following estimates for the Jacobi polynomials with complex indices.

Theorem 23[25] Let $\alpha \in [0, 2n],\ \beta \in [0, n],\ n \in \mathbb{N}_+$ and let $\mu = \frac{1}{2} + i\tau,\ \tau \in \mathbb{R}$. The for $k \in \mathbb{N}_+$

$$|P_k^{(\alpha+\mu,\beta)}(\cos\theta)| \leqslant \begin{cases} B_n e^{3|\tau|} k^{\alpha+\frac{1}{2}}, & \text{when } 0 < \theta < 2k^{-1}; \\ B_n e^{3|\tau|} k^{-\frac{1}{2}} \theta^{-\alpha-1}(\pi-\theta)^{-\beta-1} & \text{when } 2k^{-1} < \theta < \pi - k^{-1}; \\ B_n e^{3|\tau|} k^{\beta+\frac{1}{2}}, & \text{when } \pi - k^{-1} < \theta < \pi. \end{cases}$$

Let Σ_{n-1} denote the unit sphere of $\mathbb{R}^n$, $n \geqslant 3$. Any function $f \in L(\Sigma_{n-1})$ corresponds a Fourier-Laplace expansion

$$f \sim \sum_{k=0}^{\infty} Y_k(f),$$

where $Y_k(f)$ denotes the projection of f into the function space of spherical harmonics of degree k. The Cesàro means of order $\delta, (\Re\delta > -1)$, of f are the sums

$$\sigma_N^\delta(f)(x) := \frac{1}{A_N^\delta} \sum_{k=0}^{N} A_{N-k}^\delta Y_k(f)(x), \quad x \in \Sigma_{n-1}$$

where $A_k^\delta := \frac{\Gamma(\delta+k+1)}{\Gamma(\delta+1)\Gamma(k+1)}$ are Cesàro numbers. The critical value of order δ is $\frac{n-2}{2}$. And $\sigma_N^{\frac{n-2}{2}}$ is regarded as an analogue of partial sum of single Fourier series in certain sense. Theorem 23 provides necessary estimates for investigating maximal operator $S_*(f)(x) = \sup\{|\sigma_N^{\frac{n-2}{2}}(f)(x)| : N \in \mathbb{N}\}$ by applying Stein's interpolation method for the analytic class of operators (see [26]). Then we get the following convergence theorem.

Theorem 24[25] If $f \in L\log^2 L(\Sigma_{n-1})(n \geqslant 3)$ then

$$\lim_{N\to\infty} \sigma_N^{\frac{n-2}{2}}(f)(x) = f(x) \quad \text{a.e } x \in \Sigma_{n-1}.$$

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