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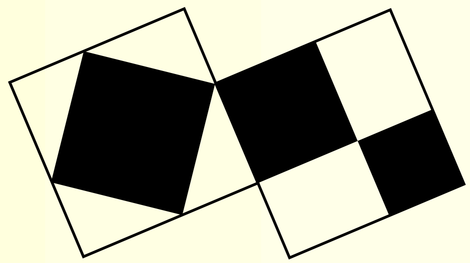
An Awful Problem about Integers in Base Four

(d'après J H Loxton and A J vdP, *Acta Arith.* 49 (1987), 192–203)



Alf van der Poorten

ceNTRe for Number Theory Research, Sydney

Gavin Brown 65 **Sydney University, March 5, 2007**

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The matter is troublesome. For instance, given an odd integer k it is not at all obvious how to find a nonzero multiplier m in $\mathcal{L}$ so that also km is in $\mathcal{L}$. Indeed, the only method we found is not an algorithm at all: it happens always to work, but there's no good a priori reason why it must work.

Roughly, the strategy at each step in the computations below is to multiply by 4 and to add or subtract k or to do nothing, all the while ensuring that no digit 2 remains trapped on the left.

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$$\begin{array}{rcl} 2\ \bar{1}\ 2\ \bar{1}\ 1\ \ & & + \\ 2\ \bar{1}\ 2\ \bar{1}\ 1 & & - \\ \hline 1\ 1\ 2\ 1\ 2\ \bar{1} & & \end{array}$$

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$$
\begin{array}{ccccccccc}
2 & \overline{1} & 2 & \overline{1} & 1 & & & & + \\
 & 2 & \overline{1} & 2 & \overline{1} & 1 & & & - \\
\hline
1 & 1 & 2 & 1 & 2 & \overline{1} & & & \\
 & & 2 & \overline{1} & 2 & \overline{1} & 1 & & - \\
\hline
1 & 1 & 0 & 2 & 0 & 0 & \overline{1} & & -
\end{array}
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$$
\begin{array}{cccccccccc}
2 & \overline{1} & 2 & \overline{1} & 1 & & & & \qquad & + \\
 & 2 & \overline{1} & 2 & \overline{1} & 1 & & & & \\
\hline
1 & 1 & 2 & 1 & 2 & \overline{1} & & & & - \\
 & & 2 & \overline{1} & 2 & \overline{1} & 1 & & & \\
\hline
1 & 1 & 0 & 2 & 0 & 0 & \overline{1} & & & - \\
 & & & 2 & \overline{1} & 2 & \overline{1} & 1 & & \\
\hline
1 & 1 & 1 & 0 & \overline{1} & 1 & 2 & 1 & & +
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$$
\begin{array}{ccccccccccc}
2 & \overline{1} & 2 & \overline{1} & 1 & & & & & & + \\
 & 2 & \overline{1} & 2 & \overline{1} & 1 & & & & & \\
\cline{1-6}
1 & 1 & 2 & 1 & 2 & \overline{1} & & & & & - \\
 & & 2 & \overline{1} & 2 & \overline{1} & 1 & & & & \\
\cline{1-7}
1 & 1 & 0 & 2 & 0 & 0 & \overline{1} & & & & - \\
 & & & 2 & \overline{1} & 2 & \overline{1} & 1 & & & \\
\cline{1-8}
1 & 1 & 1 & 0 & \overline{1} & 1 & 2 & 1 & & & + \\
 & & & & 2 & \overline{1} & 2 & \overline{1} & 1 & & \\
\cline{1-9}
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & & +
\end{array}
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$$
\begin{array}{ll}
2\ \bar{1}\ 2\ \bar{1}\ 1 & + \\
\quad 2\ \bar{1}\ 2\ \bar{1}\ 1 & \\
\hline
1\ 1\ 2\ 1\ 2\ \bar{1} & - \\
\qquad 2\ \bar{1}\ 2\ \bar{1}\ 1 & \\
\hline
1\ 1\ 0\ 2\ 0\ 0\ \bar{1} & - \\
\qquad\quad 2\ \bar{1}\ 2\ \bar{1}\ 1 & \\
\hline
1\ 1\ 1\ 0\ \bar{1}\ 1\ 2\ 1 & + \\
\qquad\qquad 2\ \bar{1}\ 2\ \bar{1}\ 1 & \\
\hline
1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1 & +
\end{array}
\qquad\qquad
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\end{array}
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$$
\begin{array}{rl}
2\,\overline{1}\,2\,\overline{1}\,1 & + \\
2\,\overline{1}\,2\,\overline{1}\,1 & \\
\hline
1\,1\,2\,1\,2\,\overline{1} & - \\
2\,\overline{1}\,2\,\overline{1}\,1 & \\
\hline
1\,1\,0\,2\,0\,0\,\overline{1} & - \\
2\,\overline{1}\,2\,\overline{1}\,1 & \\
\hline
1\,1\,1\,0\,\overline{1}\,1\,2\,1 & + \\
2\,\overline{1}\,2\,\overline{1}\,1 & \\
\hline
1\,1\,1\,0\,1\,1\,0\,0\,1 & +
\end{array}
\qquad\qquad
\begin{array}{rl}
2\,\overline{1}\,1\,1 & + \\
2\,\overline{1}\,1\,1 & \\
\hline
1\,1\,2\,0\,\overline{1} & -
\end{array}
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\begin{array}{lllllllllc}
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 & 2 & \overline{1} & 2 & \overline{1} & 1 & & & & \\
\hline
1 & 1 & 2 & 1 & 2 & \overline{1} & & & & - \\
 & & 2 & \overline{1} & 2 & \overline{1} & 1 & & & \\
\hline
1 & 1 & 0 & 2 & 0 & 0 & \overline{1} & & & - \\
 & & & 2 & \overline{1} & 2 & \overline{1} & 1 & & \\
\hline
1 & 1 & 1 & 0 & \overline{1} & 1 & 2 & 1 & & + \\
 & & & & 2 & \overline{1} & 2 & \overline{1} & 1 & \\
\hline
1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & +
\end{array}
\qquad\qquad
\begin{array}{llllllc}
2 & \overline{1} & 1 & 1 & & & + \\
 & 2 & \overline{1} & 1 & 1 & & \\
\hline
1 & 1 & 2 & 0 & \overline{1} & & - \\
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 & & & 2 & \bar{1} & 2 & \bar{1} & 1 & & + \\
 & & & & 2 & \bar{1} & 2 & \bar{1} & 1 & \\
\hline
 & & & 1 & 1 & 2 & 1 & 2 & \bar{1} & - \\
 & & & & & 2 & \bar{1} & 2 & \bar{1} \; 1 & \\
\hline
 & & & 1 & 1 & 0 & 2 & 0 & 0 \; \bar{1} & - \\
\end{array}
$$

$$
\begin{array}{r}
2\,\bar{1}\,2\,\bar{1}\,1 \qquad + \\
2\,\bar{1}\,2\,\bar{1}\,1 \\
\hline
1\,1\,2\,1\,2\,\bar{1} \qquad - \\
2\,\bar{1}\,2\,\bar{1}\,1 \\
\hline
1\,1\,0\,2\,0\,0\,\bar{1} \qquad - \\
2\,\bar{1}\,2\,\bar{1}\,1 \\
\hline
1\,1\,1\,0\,\bar{1}\,1\,2\,1 \qquad + \\
2\,\bar{1}\,2\,\bar{1}\,1 \\
\hline
1\,1\,1\,0\,1\,1\,0\,0\,1 \qquad +
\end{array}
\qquad\qquad
\begin{array}{r}
2\,\bar{1}\,1\,1 \qquad + \\
2\,\bar{1}\,1\,1 \\
\hline
1\,1\,2\,0\,\bar{1} \qquad - \\
2\,\bar{1}\,1\,1 \\
\hline
1\,1\,0\,0\,2\,\bar{1} \qquad - \\
\qquad 0 \\
2\,\bar{1}\,1\,1 \\
\hline
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\end{array}
$$

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Now denote by $\mathcal{S}$ the set of integers which can be written in base four using just the digits 0 and 1, and for $n = 0,\ 1,\ 2,\ \ldots$, denote by $\mathcal{S}_n$ the subset of words in $\mathcal{S}$ of at most n letters.

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$$\mathcal{S}_n + k\mathcal{S}_n = \{s + ks' \mid s,\ s' \text{ in } \mathcal{S}_n\}$$

has fewer than 4^n elements.

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The so what of this result is of course that, necessarily, if some element of $\mathcal{S}_n + k\mathcal{S}_n$ has two representatives, say $s_1 + ks'_1 = s_2 + ks'_2$, then

$$k(s'_1 - s'_2) = s_2 - s_1$$

displays a multiplier $s'_1 - s'_2$ in $\mathcal{L}$ yielding $s_2 - s_1$ in $\mathcal{L}$.

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Congruence Classes and Types

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Given k, say $k \equiv 1 \pmod 4$, the set $\mathcal{S}_1 + k\mathcal{S}_1$ yields three groups $\{0\}$, $\{1, k\}$, $\{k+1\}$ consisting of its four elements grouped in congruence classes mod 4.

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The point to notice is that numbers belonging to different classes of course cannot give rise to numbers in the same class at higher level, so it suffices to follow the career of a typical class.

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The point to notice is that numbers belonging to different classes of course cannot give rise to numbers in the same class at higher level, so it suffices to follow the career of a typical class. More, take a group $\{t_1, t_2, \ldots, t_h\}$, with $t_1 < t_2 < \ldots < t_h$, of elements congruent mod 4^n and set $t_i - t_1 = 4^n r_i$.

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Obviously the r_i are bounded in terms of k; in fact by $(k+1)/3$. Since the r_i must be distinct it follows that for each k only finitely many different types can occur in the construction.

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$$\sum_{t \text{ in } \mathcal{S}_n + k\mathcal{S}_n} N_{i-t}^{(n)} = M \quad (0 < i \leq 4^n), \text{ and the given } \sum_{i \bmod 4^n} N_i^{(n)} = M\,.$$

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So our attention should turn to the the $4^n \times 4^n$ matrix $C = (c_{j-i}^{(n)} - 1)$. It is a circulant and those who know such things well well know that it is diagonalisable and that its eigenvalues are given by the 4^n resolvent sums

$$\varphi^{(n)}(\theta) = \sum_{i \bmod 4^n} c_i^{(n)} \theta^i \quad \text{for } \theta^{4^n} = 1 \text{ and } \theta \neq 1; \text{ but } \varphi^{(n)}(1) = 0.$$

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The general solution for $N_i^{(n)}$ is given by $4^{-n}M$ from $\theta = 1$ plus some linear combination of solutions coming from the other θ for which $\varphi^{(n)}(\theta)$ vanishes.

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Hence, $\varphi^{(n)}(\theta) \neq 0$ if θ has exact order 4^h; and $\varphi^{(n)}(\theta) = 0$ if θ has exact order $2 \cdot 4^{h-1}$ $(0 < h \leq n)$; here we really do explicitly use the condition k odd. The almost magical result $N_i^{(n)} = N_{i+2\cdot 4^{n-1}}^{(n)}$ $(0 \leq i < 2 \cdot 4^{n-1})$ follows, and that is enough to give us our contradiction.

Indeed, it is easy to see by induction that at least 2^n of the $N_i^{(n)}$ are non-zero: for each i mod 4^n for which $N_i^{(n)}$ is non-zero, at least two of

$$N_i^{(n+1)}, \quad N_{i+4^n}^{(n+1)}, \quad N_{i+2\cdot 4^n}^{(n+1)}, \quad N_{i+3\cdot 4^n}^{(n+1)}$$

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Painting the Lilly

'To gild refined gold, to paint the lilly, ... is', as Salisbury warns King John, 'wasteful and ridiculous excess'. Nonetheless, we add some remarks on the number of congruence classes of $\mathcal{S} + k\mathcal{S}$ mod 4^n, and therefore an alternate proof, primarily, I guess, because that was our original line of argument.

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Thus if we follow a type of say h elements to level n we either find fewer than $4^n h$ elements or we find a congruence class mod 4^n with more than h elements. In either case at least one of the 4^n congruence classes must contain fewer than h elements. So each type leads eventually to the singleton type.

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Incidentally, the argument fails if the last nonzero digit of k is a 2, because T then has an irreducible component in which all row sums are 4.

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It follows that if almost all the $r_n(k)$ are zero then some $r_n(k)$ must exceed 1, again solving our problem. Our arguments in fact show, if k is odd, that there are $r_n(k)$ that are arbitrarily large.

Notes and References

Gavin Brown, William Moran, and Robert Tijdeman, 'Riesz products are basic measures', *J. London Math. Soc.* **30** (1984), 105–109.

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D. H. Lehmer, K. Mahler and A. J. vdP, 'Integers with digits 0 or 1', *Math. Comp.* **46** (1986), 683–689.

We knew that $\mathcal{S} - \mathcal{S} = \mathcal{L}$ because of this work.

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This talk, though without my spoken commentary, can be found at `http://www.maths.mq.edu.au/~alf/AwfulTalk.pdf`.

Gavin

$$\mathrm{V} \times \mathrm{XIII} = \mathrm{XIII} \times \mathrm{V}$$

Gavin

Happy

$$\mathrm{V} \times \mathrm{XIII} = \mathrm{XIII} \times \mathrm{V}$$

Gavin

Many Happy

$$\mathrm{V} \times \mathrm{XIII} = \mathrm{XIII} \times \mathrm{V}$$

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Many Happy Returns

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