Harish-Chandra images of quantum Gelfand invariants

Alexander Molev

University of Sydney

Harish-Chandra isomorphism

Harish-Chandra isomorphism

We will regard the basis elements E_{ij} of $\mathfrak{gl}_n$ as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$.

Harish-Chandra isomorphism

We will regard the basis elements E_{ij} of $\mathfrak{gl}_n$ as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$.

We will think of $\mathrm{U}(\mathfrak{gl}_n)$ as the associative algebra with these generators subject to the defining relations

$$E_{ij}\,E_{kl} - E_{kl}\,E_{ij} = \delta_{kj}E_{il} - \delta_{il}E_{kj}, \qquad i,j,k,l \in \{1,\dots,n\}.$$

Harish-Chandra isomorphism

We will regard the basis elements E_{ij} of $\mathfrak{gl}_n$ as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$.

We will think of $\mathrm{U}(\mathfrak{gl}_n)$ as the associative algebra with these generators subject to the defining relations

$$E_{ij}\,E_{kl} - E_{kl}\,E_{ij} = \delta_{kj}E_{il} - \delta_{il}E_{kj}, \qquad i,j,k,l \in \{1,\dots,n\}.$$

The center $\mathrm{Z}(\mathfrak{gl}_n)$ of $\mathrm{U}(\mathfrak{gl}_n)$ is defined by

$$\mathrm{Z}(\mathfrak{gl}_n) = \{z \in \mathrm{U}(\mathfrak{gl}_n) \mid zu = uz \quad \text{for all} \;\; u \in \mathrm{U}(\mathfrak{gl}_n)\}.$$

Any element of the center is called a Casimir element.

Given an n-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_n$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Given an n-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_n$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Any element $z \in \mathrm{Z}(\mathfrak{gl}_n)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$.

Given an n-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_n$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Any element $z \in \mathrm{Z}(\mathfrak{gl}_n)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. When regarded as a function of the highest weight, $\chi(z)$ is a symmetric polynomial in the variables $\ell_1, \ldots, \ell_n$, where $\ell_i = \lambda_i + n - i$.

The Harish-Chandra isomorphism is the map

$$\chi : \mathrm{Z}(\mathfrak{gl}_n) \to \mathbb{C}[\ell_1, \ldots, \ell_n]^{\mathfrak{S}_n},$$

where $\mathbb{C}[\ell_1, \ldots, \ell_n]^{\mathfrak{S}_n}$ denotes the algebra of symmetric polynomials in $\ell_1, \ldots, \ell_n$.

The Harish-Chandra isomorphism is the map

$$\chi : \mathrm{Z}(\mathfrak{gl}_n) \to \mathbb{C}[\ell_1, \dots, \ell_n]^{\mathfrak{S}_n},$$

where $\mathbb{C}[\ell_1, \dots, \ell_n]^{\mathfrak{S}_n}$ denotes the algebra of symmetric polynomials in $\ell_1, \dots, \ell_n$.

[A. Okounkov and G. Olshanski, 1996]:

The quantum immanants $\mathbb{S}_\mu$ form a basis of $\mathrm{Z}(\mathfrak{gl}_n)$ as μ runs over Young diagrams with at most n rows.

The Harish-Chandra isomorphism is the map

$$\chi : \mathrm{Z}(\mathfrak{gl}_n) \to \mathbb{C}[\ell_1, \dots, \ell_n]^{\mathfrak{S}_n},$$

where $\mathbb{C}[\ell_1, \dots, \ell_n]^{\mathfrak{S}_n}$ denotes the algebra of symmetric polynomials in $\ell_1, \dots, \ell_n$.

[A. Okounkov and G. Olshanski, 1996]:

The quantum immanants $\mathbb{S}_\mu$ form a basis of $\mathrm{Z}(\mathfrak{gl}_n)$ as μ runs over Young diagrams with at most n rows. Moreover,

$$\chi : \mathbb{S}_\mu \mapsto s^*_\mu,$$

the s^*_μ are the shifted Schur polynomials.

The Capelli determinant [1890] is defined by

$$C(u) = \operatorname{cdet} \begin{bmatrix} u+n-1+E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u+n-2+E_{22} & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u+E_{nn} \end{bmatrix}.$$

The Capelli determinant [1890] is defined by

$$C(u) = \mathrm{cdet} \begin{bmatrix} u+n-1+E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u+n-2+E_{22} & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u+E_{nn} \end{bmatrix}.$$

This is a polynomial in u,

$$C(u) = u^n + C_1\, u^{n-1} + \dots + C_n$$

The Capelli determinant [1890] is defined by

$$C(u) = \operatorname{cdet} \begin{bmatrix} u+n-1+E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u+n-2+E_{22} & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u+E_{nn} \end{bmatrix}.$$

This is a polynomial in u,

$$C(u) = u^n + C_1\, u^{n-1} + \dots + C_n$$

with

$$\chi : C(u) \mapsto (u+\ell_1)\dots(u+\ell_n).$$

The Capelli determinant [1890] is defined by

$$C(u) = \operatorname{cdet} \begin{bmatrix} u+n-1+E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u+n-2+E_{22} & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u+E_{nn} \end{bmatrix}.$$

This is a polynomial in u,

$$C(u) = u^n + C_1\, u^{n-1} + \dots + C_n$$

with

$$\chi : C(u) \mapsto (u+\ell_1)\dots(u+\ell_n).$$

The coefficients $C_1, \dots, C_n$ are free generators of $\mathrm{Z}(\mathfrak{gl}_n)$.

Gelfand invariants

Gelfand invariants

Combine the generators E_{ij} into the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \dots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}.$$

Gelfand invariants

Combine the generators E_{ij} into the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \dots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}.$$

The traces $\operatorname{tr} E^m$ are Casimir elements known as the Gelfand invariants [1950].

Gelfand invariants

Combine the generators E_{ij} into the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \dots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}.$$

The traces $\operatorname{tr} E^m$ are Casimir elements known as

the Gelfand invariants [1950].

The Harish-Chandra images $\chi(\operatorname{tr} E^m)$ were first calculated by

[A. Perelomov and V. Popov, 1966]:

Gelfand invariants

Combine the generators E_{ij} into the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \dots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}.$$

The traces $\operatorname{tr} E^m$ are Casimir elements known as

the Gelfand invariants [1950].

The Harish-Chandra images $\chi(\operatorname{tr} E^m)$ were first calculated by

[A. Perelomov and V. Popov, 1966]:

$$\chi(\operatorname{tr} E^m) = \sum_{k=1}^{n} \ell_k^m \frac{(\ell_1 - \ell_k + 1) \dots (\ell_n - \ell_k + 1)}{(\ell_1 - \ell_k) \dots \wedge \dots (\ell_n - \ell_k)}.$$

A short proof is based on the formula

$$1 + \sum_{m=0}^{\infty} \frac{(-1)^m \operatorname{tr} E^m}{u^{m+1}} = \frac{C(u+1)}{C(u)},$$

A short proof is based on the formula

$$1+\sum_{m=0}^{\infty}\frac{(-1)^m\,\mathrm{tr}\,E^m}{u^{m+1}}=\frac{C(u+1)}{C(u)},$$

generalizing both the Newton formula and Liouville formula.

A short proof is based on the formula

$$1+\sum_{m=0}^{\infty}\frac{(-1)^m \operatorname{tr} E^m}{u^{m+1}}=\frac{C(u+1)}{C(u)},$$

generalizing both the Newton formula and Liouville formula.

Under the Harish-Chandra isomorphism,

$$\chi:\frac{C(u+1)}{C(u)}\mapsto\frac{(u+\ell_1+1)\dots(u+\ell_n+1)}{(u+\ell_1)\dots(u+\ell_n)}.$$

Drinfeld–Jimbo quantum group $\mathrm{U}_q(\mathfrak{gl}_n)$

Drinfeld–Jimbo quantum group $\mathrm{U}_q(\mathfrak{gl}_n)$

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by $e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}$

and invertible elements $t_1, \dots, t_n$, subject to relations

Drinfeld–Jimbo quantum group $\mathrm{U}_q(\mathfrak{gl}_n)$

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by $e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}$ and invertible elements $t_1, \dots, t_n$, subject to relations

$$t_i\, t_j = t_j\, t_i,$$

$$t_i e_j t_i^{-1} = e_j\, q^{\delta_{ij}-\delta_{i,j+1}}, \qquad t_i f_j t_i^{-1} = f_j\, q^{-\delta_{ij}+\delta_{i,j+1}},$$

$$[e_i, f_j] = \delta_{ij}\, \frac{k_i - k_i^{-1}}{q - q^{-1}} \qquad \text{with } k_i = t_i t_{i+1}^{-1},$$

$$[e_i, e_j] = [f_i, f_j] = 0 \qquad \text{if } |i-j| > 1,$$

$$e_i^2 e_j - (q+q^{-1})\, e_i e_j e_i + e_j e_i^2 = 0,$$

$$f_i^2 f_j - (q+q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \qquad \text{if } |i-j| = 1.$$

Reshetikhin–Takhtajan–Faddeev presentation

Reshetikhin–Takhtajan–Faddeev presentation

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by entries of the matrices

$$L^+ = \begin{bmatrix} l_{11}^+ & l_{12}^+ & \dots & l_{1n}^+ \\ 0 & l_{22}^+ & \dots & l_{2n}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & l_{nn}^+ \end{bmatrix}$$

Reshetikhin–Takhtajan–Faddeev presentation

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by entries of the matrices

$$L^+ = \begin{bmatrix} l_{11}^+ & l_{12}^+ & \dots & l_{1n}^+ \\ 0 & l_{22}^+ & \dots & l_{2n}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & l_{nn}^+ \end{bmatrix}$$

and

$$L^- = \begin{bmatrix} l_{11}^- & 0 & \dots & 0 \\ l_{21}^- & l_{22}^- & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}^- & \dots & \dots & l_{nn}^- \end{bmatrix}.$$

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by elements l_{ij}^{+} and l_{ij}^{-} with $1 \leqslant i, j \leqslant n$ subject to the relations

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by elements l_{ij}^{+} and l_{ij}^{-} with $1 \leqslant i, j \leqslant n$ subject to the relations

$$l_{ij}^{-} = l_{ji}^{+} = 0, \qquad 1 \leqslant i < j \leqslant n,$$

$$l_{ii}^{-}\, l_{ii}^{+} = l_{ii}^{+}\, l_{ii}^{-} = 1, \qquad 1 \leqslant i \leqslant n,$$

$$R\, L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R, \qquad R\, L_1^{+} L_2^{-} = L_2^{-} L_1^{+} R,$$

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by elements l_{ij}^{+} and l_{ij}^{-} with $1 \leqslant i,j \leqslant n$ subject to the relations

$$l_{ij}^{-} = l_{ji}^{+} = 0, \qquad 1 \leqslant i < j \leqslant n,$$

$$l_{ii}^{-}\, l_{ii}^{+} = l_{ii}^{+}\, l_{ii}^{-} = 1, \qquad 1 \leqslant i \leqslant n,$$

$$R\, L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R, \qquad R\, L_1^{+} L_2^{-} = L_2^{-} L_1^{+} R,$$

where

$$R = q \sum_{i} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}$$

The algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by elements l_{ij}^{+} and l_{ij}^{-} with $1 \leqslant i,j \leqslant n$ subject to the relations

$$l_{ij}^{-} = l_{ji}^{+} = 0, \qquad 1 \leqslant i < j \leqslant n,$$

$$l_{ii}^{-}\, l_{ii}^{+} = l_{ii}^{+}\, l_{ii}^{-} = 1, \qquad 1 \leqslant i \leqslant n,$$

$$R\, L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R, \qquad R\, L_1^{+} L_2^{-} = L_2^{-} L_1^{+} R,$$

where

$$R = q \sum_{i} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}$$

with subscripts of $L^{\pm}$ indicating the copies of $\mathrm{End}\,\mathbb{C}^n$ as in

$$L_1^{\pm} = \sum_{i,j} e_{ij} \otimes 1 \otimes l_{ij}^{\pm} \in \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}_q(\mathfrak{gl}_n).$$

Explicitly,

$$q^{\delta_{ij}}\, l^{\pm}_{ia}\, l^{\pm}_{jb} - q^{\delta_{ab}}\, l^{\pm}_{jb}\, l^{\pm}_{ia} = (q - q^{-1})\,(\delta_{b<a} - \delta_{i<j})\, l^{\pm}_{ja}\, l^{\pm}_{ib}$$

Explicitly,

$$q^{\delta_{ij}}\, l^{\pm}_{ia}\, l^{\pm}_{jb} - q^{\delta_{ab}}\, l^{\pm}_{jb}\, l^{\pm}_{ia} = (q - q^{-1})\,(\delta_{b<a} - \delta_{i<j})\, l^{\pm}_{ja}\, l^{\pm}_{ib}$$

and

$$q^{\delta_{ij}}\, l^{+}_{ia}\, l^{-}_{jb} - q^{\delta_{ab}}\, l^{-}_{jb}\, l^{+}_{ia} = (q - q^{-1})\,(\delta_{b<a}\, l^{-}_{ja}\, l^{+}_{ib} - \delta_{i<j}\, l^{+}_{ja}\, l^{-}_{ib}).$$

Explicitly,

$$q^{\delta_{ij}}\, l^{\pm}_{ia}\, l^{\pm}_{jb} - q^{\delta_{ab}}\, l^{\pm}_{jb}\, l^{\pm}_{ia} = (q - q^{-1})\,(\delta_{b<a} - \delta_{i<j})\, l^{\pm}_{ja}\, l^{\pm}_{ib}$$

and

$$q^{\delta_{ij}}\, l^{+}_{ia}\, l^{-}_{jb} - q^{\delta_{ab}}\, l^{-}_{jb}\, l^{+}_{ia} = (q - q^{-1})\,(\delta_{b<a}\, l^{-}_{ja}\, l^{+}_{ib} - \delta_{i<j}\, l^{+}_{ja}\, l^{-}_{ib}).$$

Isomorphism between presentations:

$$l^{-}_{ii} \mapsto t_i, \qquad l^{+}_{ii} \mapsto t_i^{-1},$$

$$l^{+}_{i,i+1} \mapsto -(q - q^{-1})\, e_i\, t_i^{-1}, \qquad l^{-}_{i+1,i} \mapsto (q - q^{-1})\, t_i f_i.$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is recovered from $\mathrm{U}_q(\mathfrak{gl}_n)$ in the limit $q \to 1$ by the formulas

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is recovered from $\mathrm{U}_q(\mathfrak{gl}_n)$ in the limit $q \to 1$ by the formulas

$$\frac{l_{ij}^-}{q-q^{-1}} \to E_{ij}, \qquad \frac{l_{ji}^+}{q-q^{-1}} \to -E_{ji} \qquad \text{for} \quad i > j,$$

$$\frac{l_{ii}^- - 1}{q-1} \to E_{ii}, \qquad \frac{l_{ii}^+ - 1}{q-1} \to -E_{ii} \qquad \text{for} \quad i = 1, \dots, n.$$

Quantum Gelfand invariants

Quantum Gelfand invariants

Set $M = L^-(L^+)^{-1}$ with

Quantum Gelfand invariants

Set $M = L^-(L^+)^{-1}$ with

$$L^+ = \begin{bmatrix} l_{11}^+ & l_{12}^+ & \dots & l_{1n}^+ \\ 0 & l_{22}^+ & \dots & l_{2n}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & l_{nn}^+ \end{bmatrix} \quad \text{and} \quad L^- = \begin{bmatrix} l_{11}^- & 0 & \dots & 0 \\ l_{21}^- & l_{22}^- & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}^- & \dots & \dots & l_{nn}^- \end{bmatrix}.$$

Quantum Gelfand invariants

Set $M = L^-(L^+)^{-1}$ with

$$L^+ = \begin{bmatrix} l_{11}^+ & l_{12}^+ & \dots & l_{1n}^+ \\ 0 & l_{22}^+ & \dots & l_{2n}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & l_{nn}^+ \end{bmatrix} \quad \text{and} \quad L^- = \begin{bmatrix} l_{11}^- & 0 & \dots & 0 \\ l_{21}^- & l_{22}^- & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}^- & \dots & \dots & l_{nn}^- \end{bmatrix}.$$

The quantum traces are defined by

$$\mathrm{tr}_q M^m = \mathrm{tr}\, DM^m,$$

with

$$D = \mathrm{diag}\big[q^{n-1}, q^{n-3}, \dots, q^{-n+1}\big].$$

Theorem [RTF 1989].

Theorem [RTF 1989].

- All elements $\mathrm{tr}_q M^m$ belong to the center $\mathrm{Z}_q(\mathfrak{gl}_n)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$.

Theorem [RTF 1989].

- All elements $\mathrm{tr}_q M^m$ belong to the center $\mathrm{Z}_q(\mathfrak{gl}_n)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$.

- The elements $\mathrm{tr}_q M^m$ with $m = 1, \dots, n$ together with $l_{11}^{\pm} \dots l_{nn}^{\pm}$ generate the center.

Theorem [RTF 1989].

- All elements $\mathrm{tr}_q M^m$ belong to the center $\mathrm{Z}_q(\mathfrak{gl}_n)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$.

- The elements $\mathrm{tr}_q M^m$ with $m = 1, \dots, n$ together with $l_{11}^{\pm} \dots l_{nn}^{\pm}$ generate the center.

- The center is also generated by the coefficients $d_0, \dots, d_n$ of the quantum determinant

$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)} \big(l^+_{\sigma(1)1} - l^-_{\sigma(1)1} u q^{2n-2}\big) \cdots \big(l^+_{\sigma(n)n} - l^-_{\sigma(n)n} u\big)$$

$$= d_0 + d_1 u + \dots + d_n u^n.$$

Harish-Chandra isomorphism

Harish-Chandra isomorphism

The representation $L_q(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ is generated by a nonzero vector ξ such that

$$l_{ij}^{+}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n,$$

$$l_{ii}^{-}\,\xi = q^{\lambda_i}\,\xi \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Harish-Chandra isomorphism

The representation $L_q(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ is generated by a nonzero vector ξ such that

$$l_{ij}^{+}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n,$$

$$l_{ii}^{-}\,\xi = q^{\lambda_i}\,\xi \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Any element $z \in \mathrm{Z}_q(\mathfrak{gl}_n)$ acts in $L_q(\lambda)$ by multiplying each vector by a scalar $\chi(z)$.

Harish-Chandra isomorphism

The representation $L_q(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ is generated by a nonzero vector ξ such that

$$l_{ij}^{+}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n,$$
$$l_{ii}^{-}\,\xi = q^{\lambda_i}\,\xi \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Any element $z \in \mathrm{Z}_q(\mathfrak{gl}_n)$ acts in $L_q(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. Recall that $\ell_i = \lambda_i + n - i$.

Harish-Chandra isomorphism

The representation $L_q(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ is generated by a nonzero vector ξ such that

$$l_{ij}^{+}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n,$$
$$l_{ii}^{-}\,\xi = q^{\lambda_i}\,\xi \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

Any element $z \in \mathrm{Z}_q(\mathfrak{gl}_n)$ acts in $L_q(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. Recall that $\ell_i = \lambda_i + n - i$.

We have the Harish-Chandra isomorphism

$$\chi : \mathrm{Z}_q(\mathfrak{gl}_n) \to \Big\langle \mathbb{C}\,[q^{2\ell_1}, \dots, q^{2\ell_n}]^{\mathfrak{S}_n},\ q^{\pm(\ell_1+\dots+\ell_n)} \Big\rangle.$$

Theorem [N. Jing, M. Liu, A. M. 2023].

Theorem [N. Jing, M. Liu, A. M. 2023].

We have

$$\chi : \mathrm{tr}_q M^m \mapsto \sum_{k=1}^{n} q^{2\ell_k m} \frac{[\ell_1 - \ell_k + 1]_q \dots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \dots \wedge \dots [\ell_n - \ell_k]_q},$$

Theorem [N. Jing, M. Liu, A. M. 2023].

We have

$$\chi : \mathrm{tr}_q M^m \mapsto \sum_{k=1}^{n} q^{2\ell_k m} \frac{[\ell_1 - \ell_k + 1]_q \ldots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \ldots \wedge \ldots [\ell_n - \ell_k]_q},$$

where

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

Theorem [N. Jing, M. Liu, A. M. 2023].

We have

$$\chi : \mathrm{tr}_q M^m \mapsto \sum_{k=1}^{n} q^{2\ell_k m} \frac{[\ell_1 - \ell_k + 1]_q \dots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \dots \wedge \dots [\ell_n - \ell_k]_q},$$

where

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

Remark. A different family of quantum Gelfand invariants together with their Harish-Chandra images was given by [M. Gould, R. Zhang and A. Bracken 1991].

In the limit $q \to 1$ we have

$$\frac{M-1}{q-q^{-1}} \to E.$$

In the limit $q \to 1$ we have

$$\frac{M-1}{q-q^{-1}} \to E.$$

The element $\operatorname{tr} E^m \in \mathrm{U}(\mathfrak{gl}_n)$ is obtained as the limit value as $q \to 1$ of the expression

$$\frac{1}{(q-q^{-1})^m} \operatorname{tr}_q (M-1)^m = \frac{1}{(q-q^{-1})^m} \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} \operatorname{tr}_q M^r.$$

In the limit $q \to 1$ we have

$$\frac{M-1}{q-q^{-1}} \to E.$$

The element $\operatorname{tr} E^m \in \mathrm{U}(\mathfrak{gl}_n)$ is obtained as the limit value as $q \to 1$ of the expression

$$\frac{1}{(q-q^{-1})^m} \operatorname{tr}_q (M-1)^m = \frac{1}{(q-q^{-1})^m} \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} \operatorname{tr}_q M^r.$$

Hence, the Perelomov–Popov formulas follow from the theorem.

Quantum loop algebra

Quantum loop algebra

Consider the R-matrix

$$R(x) = R - x\widetilde{R} \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n,$$

Quantum loop algebra

Consider the R-matrix

$$R(x) = R - x\widetilde{R} \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n,$$

where

$$R = q\sum_{i} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}$$

Quantum loop algebra

Consider the R-matrix

$$R(x) = R - x\widetilde{R} \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n,$$

where

$$R = q\sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}$$

and

$$\widetilde{R} = q^{-1}\sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} - (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}.$$

The quantum loop algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad l_{ij}^{-}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant n, \qquad r = 0, 1, \dots,$$

The quantum loop algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad l_{ij}^{-}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant n, \qquad r = 0, 1, \dots,$$

subject to the defining relations

$$l_{ji}^{+}[0] = l_{ij}^{-}[0] = 0 \qquad \text{for} \qquad 1 \leqslant i < j \leqslant n,$$

$$l_{ii}^{+}[0]\, l_{ii}^{-}[0] = l_{ii}^{-}[0]\, l_{ii}^{+}[0] = 1 \qquad \text{for} \qquad i = 1, \dots, n,$$

The quantum loop algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ is generated by elements

$$l_{ij}^{+}[-r], \qquad l_{ij}^{-}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant n, \qquad r = 0, 1, \dots,$$

subject to the defining relations

$$l_{ji}^{+}[0] = l_{ij}^{-}[0] = 0 \qquad \text{for} \qquad 1 \leqslant i < j \leqslant n,$$

$$l_{ii}^{+}[0]\, l_{ii}^{-}[0] = l_{ii}^{-}[0]\, l_{ii}^{+}[0] = 1 \qquad \text{for} \qquad i = 1, \dots, n,$$

and

$$R(u/v)L_1^{\pm}(u)L_2^{\pm}(v) = L_2^{\pm}(v)L_1^{\pm}(u)R(u/v),$$

$$R(u/v)L_1^{+}(u)L_2^{-}(v) = L_2^{-}(v)L_1^{+}(u)R(u/v).$$

Here we consider the matrices $L^{\pm}(u) = \big[\, l_{ij}^{\pm}(u) \big]$, whose entries are formal power series in u and u^{-1},

$$l_{ij}^{+}(u) = \sum_{r=0}^{\infty} l_{ij}^{+}[-r]\, u^{r}, \qquad l_{ij}^{-}(u) = \sum_{r=0}^{\infty} l_{ij}^{-}[r]\, u^{-r}.$$

Here we consider the matrices $L^{\pm}(u) = \big[\, l_{ij}^{\pm}(u) \big]$, whose entries are formal power series in u and u^{-1},

$$l_{ij}^{+}(u) = \sum_{r=0}^{\infty} l_{ij}^{+}[-r]\, u^{r}, \qquad l_{ij}^{-}(u) = \sum_{r=0}^{\infty} l_{ij}^{-}[r]\, u^{-r}.$$

We regard the matrices as elements

$$L^{\pm}(u) = \sum_{i,j=1}^{n} e_{ij} \otimes l_{ij}^{\pm}(u) \in \operatorname{End} \mathbb{C}^{n} \otimes \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)[[u^{\pm 1}]]$$

and use subscripts to indicate copies of the matrix in the multiple tensor product algebra.

Quantum determinants

Quantum determinants

The quantum determinants $\operatorname{qdet} L^{+}(u)$ and $\operatorname{qdet} L^{-}(u)$ are series in u and u^{-1}, respectively, whose coefficients belong to the center of the quantum loop algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$:

Quantum determinants

The quantum determinants $\operatorname{qdet} L^{+}(u)$ and $\operatorname{qdet} L^{-}(u)$ are series in u and u^{-1}, respectively, whose coefficients belong to the center of the quantum loop algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$:

$$\operatorname{qdet} L^{\pm}(u) = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)}\, l^{\pm}_{\sigma(1)1}(uq^{2n-2}) \cdots l^{\pm}_{\sigma(n)n}(u).$$

Liouville formula

Liouville formula

Theorem. All coefficient of the series $z^{\pm}(u)$ defined by

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr}_q L^{\pm}(uq^{2n}) L^{\pm}(u)^{-1}$$

belong to the center of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

Liouville formula

Theorem. All coefficient of the series $z^{\pm}(u)$ defined by

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr}_q L^{\pm}(uq^{2n}) L^{\pm}(u)^{-1}$$

belong to the center of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

Moreover, we have the relations

$$z^{\pm}(u) = \frac{\operatorname{qdet} L^{\pm}(uq^2)}{\operatorname{qdet} L^{\pm}(u)}.$$

Liouville formula

Theorem. All coefficient of the series $z^{\pm}(u)$ defined by

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr}_q L^{\pm}(uq^{2n}) L^{\pm}(u)^{-1}$$

belong to the center of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

Moreover, we have the relations

$$z^{\pm}(u) = \frac{\operatorname{qdet} L^{\pm}(uq^2)}{\operatorname{qdet} L^{\pm}(u)}.$$

Remark. The Yangian version is due to M. Nazarov, 1991,

q-version – S. Belliard and E. Ragoucy, 2009 (without proof).

Classical limit

Classical limit

Rewrite the formula in the form

$$\frac{1}{[n]_q}\,\mathrm{tr}_q\,\big(L(uq^{2n})-L(u)\big)L(u)^{-1}=\frac{\mathrm{qdet}\,L(uq^2)-\mathrm{qdet}\,L(u)}{\mathrm{qdet}\,L(u)}.$$

Classical limit

Rewrite the formula in the form

$$\frac{1}{[n]_q}\,\mathrm{tr}_q\,\big(L(uq^{2n}) - L(u)\big)L(u)^{-1} = \frac{\mathrm{qdet}\,L(uq^2) - \mathrm{qdet}\,L(u)}{\mathrm{qdet}\,L(u)}.$$

Divide both sides by $uq^2 - u$ and let $q \to 1$ to get

Classical limit

Rewrite the formula in the form

$$\frac{1}{[n]_q}\,\mathrm{tr}_q\,\big(L(uq^{2n})-L(u)\big)L(u)^{-1}=\frac{\mathrm{qdet}\,L(uq^2)-\mathrm{qdet}\,L(u)}{\mathrm{qdet}\,L(u)}.$$

Divide both sides by uq^2-u and let $q\to 1$ to get

$$\mathrm{tr}\Big(L^{-1}(u)\frac{d}{du}L(u)\Big)=\frac{1}{\det L(u)}\cdot\frac{d}{du}\det L(u).$$

Classical limit

Rewrite the formula in the form

$$\frac{1}{[n]_q}\,\mathrm{tr}_q\,\big(L(uq^{2n})-L(u)\big)L(u)^{-1}=\frac{\mathrm{qdet}\,L(uq^2)-\mathrm{qdet}\,L(u)}{\mathrm{qdet}\,L(u)}.$$

Divide both sides by uq^2-u and let $q\to 1$ to get

$$\mathrm{tr}\Big(L^{-1}(u)\frac{d}{du}L(u)\Big)=\frac{1}{\det L(u)}\cdot\frac{d}{du}\det L(u).$$

This is the Liouville formula for matrix-valued functions:

$$L'(u)=A(u)L(u)\implies\big(\det L(u)\big)'=\mathrm{tr}\,A(u)\,\det L(u).$$

Proof of the quantum Liouville formula

Proof of the quantum Liouville formula

Introduce the quantum comatrix $\widehat{L}(u)$ by the relation

$$\widehat{L}(uq^2)\,L(u) = \operatorname{qdet} L(u)\,1.$$

Proof of the quantum Liouville formula

Introduce the quantum comatrix $\widehat{L}(u)$ by the relation

$$\widehat{L}(uq^2)\, L(u) = \text{qdet}\, L(u)\, 1.$$

Then we also have

$$D\widehat{L}(u)^t D^{-1} L(uq^{2n-2})^t = \text{qdet}\, L(u)\, 1.$$

Proof of the quantum Liouville formula

Introduce the quantum comatrix $\widehat{L}(u)$ by the relation

$$\widehat{L}(uq^2)\,L(u) = \operatorname{qdet} L(u)\,1.$$

Then we also have

$$D\widehat{L}(u)^t D^{-1} L(uq^{2n-2})^t = \operatorname{qdet} L(u)\,1.$$

Hence

$$L(uq^{2n})^t D(L(u)^{-1})^t = \frac{\operatorname{qdet} L(uq^2)}{\operatorname{qdet} L(u)}\,D.$$

Proof of the quantum Liouville formula

Introduce the quantum comatrix $\widehat{L}(u)$ by the relation

$$\widehat{L}(uq^2)\,L(u) = \operatorname{qdet} L(u)\,1.$$

Then we also have

$$D\widehat{L}(u)^t D^{-1} L(uq^{2n-2})^t = \operatorname{qdet} L(u)\,1.$$

Hence

$$L(uq^{2n})^t D(L(u)^{-1})^t = \frac{\operatorname{qdet} L(uq^2)}{\operatorname{qdet} L(u)}\,D.$$

The formula follows by taking trace.

Evaluation homomorphism

Evaluation homomorphism

The mapping

$$\mathrm{ev} : L^{+}(u) \mapsto L^{+} - L^{-}u, \qquad L^{-}(u) \mapsto L^{-} - L^{+}u^{-1},$$

Evaluation homomorphism

The mapping

$$\mathrm{ev}: L^{+}(u) \mapsto L^{+} - L^{-}u, \qquad L^{-}(u) \mapsto L^{-} - L^{+}u^{-1},$$

defines a homomorphism $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \to \mathrm{U}_q(\mathfrak{gl}_n)$.

Evaluation homomorphism

The mapping

$$\mathrm{ev}: L^{+}(u) \mapsto L^{+} - L^{-}u, \qquad L^{-}(u) \mapsto L^{-} - L^{+}u^{-1},$$

defines a homomorphism $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \to \mathrm{U}_q(\mathfrak{gl}_n)$.

Apply it to both sides of the Liouville formula

$$z^{+}(u) = \frac{\operatorname{qdet} L^{+}(uq^2)}{\operatorname{qdet} L^{+}(u)}.$$

Evaluation homomorphism

The mapping

$$\mathrm{ev}: L^{+}(u) \mapsto L^{+} - L^{-}u, \qquad L^{-}(u) \mapsto L^{-} - L^{+}u^{-1},$$

defines a homomorphism $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \to \mathrm{U}_q(\mathfrak{gl}_n)$.

Apply it to both sides of the Liouville formula

$$z^{+}(u) = \frac{\mathrm{qdet}\, L^{+}(uq^2)}{\mathrm{qdet}\, L^{+}(u)}.$$

The image of the quantum determinant $\mathrm{qdet}\, L^{+}(u)$ is found by

$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)} \big(l^{+}_{\sigma(1)1} - l^{-}_{\sigma(1)1} uq^{2n-2}\big) \cdots \big(l^{+}_{\sigma(n)n} - l^{-}_{\sigma(n)n} u\big).$$

The eigenvalue on the highest vector ξ of $L_q(\lambda)$ is

$$\big(q^{-\lambda_1} - q^{\lambda_1+2n-2}u\big)\dots\big(q^{-\lambda_n} - q^{\lambda_n}u\big)$$
$$= q^{n(n-1)/2}(q^{-\ell_1} - q^{\ell_1}u)\dots(q^{-\ell_n} - q^{\ell_n}u).$$

The eigenvalue on the highest vector ξ of $L_q(\lambda)$ is

$$\big(q^{-\lambda_1} - q^{\lambda_1+2n-2}u\big)\dots\big(q^{-\lambda_n} - q^{\lambda_n}u\big)$$
$$= q^{n(n-1)/2}(q^{-\ell_1} - q^{\ell_1}u)\dots(q^{-\ell_n} - q^{\ell_n}u).$$

Hence

$$\frac{\operatorname{qdet} L^+(uq^2)}{\operatorname{qdet} L^+(u)} \to C + \frac{a_1}{1-q^{2\ell_1}u} + \dots + \frac{a_n}{1-q^{2\ell_n}u}$$

The eigenvalue on the highest vector ξ of $L_q(\lambda)$ is

$$\big(q^{-\lambda_1} - q^{\lambda_1+2n-2}u\big)\dots\big(q^{-\lambda_n} - q^{\lambda_n}u\big)$$
$$= q^{n(n-1)/2}(q^{-\ell_1} - q^{\ell_1}u)\dots(q^{-\ell_n} - q^{\ell_n}u).$$

Hence

$$\frac{\operatorname{qdet} L^+(uq^2)}{\operatorname{qdet} L^+(u)} \to C + \frac{a_1}{1-q^{2\ell_1}u} + \dots + \frac{a_n}{1-q^{2\ell_n}u}$$

to find that the constants a_k are given by

$$a_k = (q^{n-1} - q^{n+1})\,\frac{[\ell_1-\ell_k+1]_q\dots[\ell_n-\ell_k+1]_q}{[\ell_1-\ell_k]_q\dots\wedge\dots[\ell_n-\ell_k]_q}.$$

On the other hand, recalling that $M = L^{-}(L^{+})^{-1}$,

for the image of $z^{+}(u)$ we get

$$\frac{1}{[n]_q} \operatorname{tr} D(L^{+} - L^{-}uq^{2n})(L^{+} - L^{-}u)^{-1}$$

$$= \frac{1}{[n]_q} \operatorname{tr} D(1 - Muq^{2n})(1 - Mu)^{-1}$$

On the other hand, recalling that $M = L^-(L^+)^{-1}$,

for the image of $z^+(u)$ we get

$$\frac{1}{[n]_q} \operatorname{tr} D(L^+ - L^- u q^{2n})(L^+ - L^- u)^{-1}$$
$$= \frac{1}{[n]_q} \operatorname{tr} D(1 - Muq^{2n})(1 - Mu)^{-1}$$

which equals

On the other hand, recalling that $M = L^{-}(L^{+})^{-1}$,

for the image of $z^{+}(u)$ we get

$$\frac{1}{[n]_q}\,\mathrm{tr}\,D(L^{+} - L^{-}uq^{2n})(L^{+} - L^{-}u)^{-1}$$

$$= \frac{1}{[n]_q}\,\mathrm{tr}\,D(1 - Muq^{2n})(1 - Mu)^{-1}$$

which equals

$$1 + (q^{n-1} - q^{n+1}) \sum_{m=1}^{\infty} \mathrm{tr}_q M^m u^m.$$

On the other hand, recalling that $M = L^-(L^+)^{-1}$,

for the image of $z^+(u)$ we get

$$\frac{1}{[n]_q}\operatorname{tr} D(L^+ - L^- u q^{2n})(L^+ - L^- u)^{-1}$$
$$= \frac{1}{[n]_q}\operatorname{tr} D(1 - M u q^{2n})(1 - M u)^{-1}$$

which equals

$$1 + (q^{n-1} - q^{n+1}) \sum_{m=1}^{\infty} \operatorname{tr}_q M^m u^m.$$

Thus,

$$\chi : \operatorname{tr}_q M^m \mapsto \sum_{k=1}^{n} q^{2\ell_k m} \frac{[\ell_1 - \ell_k + 1]_q \dots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \dots \wedge \dots [\ell_n - \ell_k]_q}.$$

More central elements

More central elements

We also have

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr}_{q^{-1}} L^{\pm}(u)^{-1} L^{\pm}(uq^{2n}).$$

More central elements

We also have

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr}_{q^{-1}} L^{\pm}(u)^{-1} L^{\pm}(uq^{2n}).$$

By applying the evaluation homomorphism, we get four families of central elements in $\mathrm{U}_q(\mathfrak{gl}_n)$.

More central elements

We also have

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr}_{q^{-1}} L^{\pm}(u)^{-1} L^{\pm}(uq^{2n}).$$

By applying the evaluation homomorphism, we get four families of central elements in $\mathrm{U}_q(\mathfrak{gl}_n)$.

They are related by

$$\operatorname{tr}_{q^{-1}} \left((L^{+})^{-1} L^{-}\right)^m = \operatorname{tr}_q M^m$$

More central elements

We also have

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr}_{q^{-1}} L^{\pm}(u)^{-1} L^{\pm}(uq^{2n}).$$

By applying the evaluation homomorphism, we get four families of central elements in $\mathrm{U}_q(\mathfrak{gl}_n)$.

They are related by

$$\operatorname{tr}_{q^{-1}} ((L^{+})^{-1} L^{-})^m = \operatorname{tr}_q M^m$$

and

$$\operatorname{tr}_{q^{-1}} ((L^{-})^{-1} L^{+})^m = \operatorname{tr}_q (L^{+} (L^{-})^{-1})^m.$$

The symmetries come from the isomorphism

$$\mathrm{U}_q(\mathfrak{gl}_n) \to \mathrm{U}_{q^{-1}}(\mathfrak{gl}_n), \qquad L^{\pm} \mapsto (L^{\pm})^{-1}.$$

The symmetries come from the isomorphism

$$\mathrm{U}_q(\mathfrak{gl}_n) \to \mathrm{U}_{q^{-1}}(\mathfrak{gl}_n), \qquad L^{\pm} \mapsto (L^{\pm})^{-1}.$$

This implies the formulas for the eigenvalues of the other central elements of $\mathrm{U}_q(\mathfrak{gl}_n)$ in $L_q(\lambda)$:

The symmetries come from the isomorphism

$$\mathrm{U}_q(\mathfrak{gl}_n) \to \mathrm{U}_{q^{-1}}(\mathfrak{gl}_n), \qquad L^{\pm} \mapsto (L^{\pm})^{-1}.$$

This implies the formulas for the eigenvalues of the other central elements of $\mathrm{U}_q(\mathfrak{gl}_n)$ in $L_q(\lambda)$:

$$\chi : \mathrm{tr}_q \, (L^{+}(L^{-})^{-1})^{m} \mapsto \sum_{k=1}^{n} q^{-2\ell_k m} \, \frac{[\ell_1 - \ell_k + 1]_q \dots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \dots \wedge \dots [\ell_n - \ell_k]_q}.$$