Casimir elements for classical Lie algebras and affine Kac–Moody algebras

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Plan of lectures

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- Casimir elements for the classical Lie algebras from the Schur–Weyl duality.

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- Affine Kac–Moody algebras: center at the critical level.

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- Affine Kac–Moody algebras: center at the critical level.
- Affine Harish-Chandra isomorphism and classical $\mathcal{W}$-algebras.

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where both products are taken in the lexicographical order on the set of pairs (a, b).

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where $P_{a\,b}$ is the permutation operator

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and $e_{ij} \in \operatorname{End}\mathbb{C}^N$ are the matrix units.

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which we regard as elements of the algebra

$$\mathrm{End}\,(\mathbb{C}^N)^{\otimes m} \cong \underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^N}_{m}.$$

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We will combine the generators into the matrix $E = [E_{ij}]$ which will also be regarded as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{gl}_N).$$

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Note the property $P_{ab} E_a = E_b P_{ab}$.

Key Lemma. The defining relations of $\mathrm{U}(\mathfrak{gl}_N)$ are equivalent to the single relation

$$E_1 E_2 - E_2 E_1 = (E_1 - E_2) P_{12}.$$

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Theorem. For any $s \in \mathbb{C}[\mathfrak{S}_m]$ and $u_1, \ldots, u_m \in \mathbb{C}$ the element

$$\mathrm{tr}_{1,\ldots,m}\, S\,(u_1 + E_1) \ldots (u_m + E_m)$$

belongs to the center $\mathrm{Z}(\mathfrak{gl}_N)$ of $\mathrm{U}(\mathfrak{gl}_N)$.

Proof. Consider the tensor product

$$\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,(\mathbb{C}^N)^{\otimes m} \otimes \mathrm{U}(\mathfrak{gl}_N)$$

with the copies of the algebra $\mathrm{End}\,\mathbb{C}^N$ labelled by $0, 1, \dots, m$.

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We will show that

$$\big[E_0, \operatorname{tr}_{1,\ldots,m} S\,(u_1 + E_1)\ldots(u_m + E_m)\big] = 0.$$

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By the Key Lemma,

$$[E_0, u_a + E_a] = P_{0\,a}(u_a + E_a) - (u_a + E_a)P_{0\,a},$$

where we used the relations $P_{a\,b}\,E_b = E_a\,P_{a\,b}$.

Hence

$$\left[E_0, S\,(u_1+E_1)\ldots(u_m+E_m)\right]$$
$$= S\sum_{a=1}^{m} P_{0\,a}\,(u_1+E_1)\ldots(u_m+E_m)$$
$$- S\,(u_1+E_1)\ldots(u_m+E_m)\sum_{a=1}^{m} P_{0\,a},$$

because $E_0 S = S E_0$ and $P_{0\,a}$ commutes with E_b for $b \neq a$.

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The sum of the permutation operators $P_{0\,a}$ commutes with S (the Schur–Weyl duality). Applying the trace $\mathrm{tr}_{1,\dots,m}$ and using its cyclic property we get 0. □

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$$C(u) = \mathrm{cdet} \begin{bmatrix} u + E_{11} & E_{12} & \dots & E_{1N} \\ E_{21} & u + E_{22} - 1 & \dots & E_{2N} \\ \vdots & \vdots & & \vdots \\ E_{N1} & E_{N2} & \dots & u + E_{NN} - N + 1 \end{bmatrix}.$$

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All coefficients of the polynomial $C(u)$ are Casimir elements.

Indeed, observe that by the Key Lemma

$$\Big(1 - \frac{P_{ab}}{b-a}\Big)(u + E_a - a + 1)(u + E_b - b + 1)$$

$$= (u + E_b - b + 1)(u + E_a - a + 1)\Big(1 - \frac{P_{ab}}{b-a}\Big).$$

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Hence, the fusion formula for $A^{(N)}$ gives

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It remains to note that $\mathrm{tr}_{1,\dots,N}\,A^{(N)}=1$.

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For instance, for $m = 2$ we get

$$\mathrm{tr}_{1,2} P_{12}\, E_1\, E_2 = \mathrm{tr}_{1,2}\, E_2 P_{12}\, E_2 = \mathrm{tr}\, E^2$$

because $\mathrm{tr}_1\, P_{12} = 1$.

The Newton identity

Theorem [Perelomov–Popov, 1966].

We have the identity

$$1+\sum_{m=0}^{\infty}\frac{(-1)^m\,\mathrm{tr}\,E^m}{(u-N+1)^{m+1}}=\frac{C(u+1)}{C(u)}.$$

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Proof. Verify

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Hence,

$$C(u+1) - C(u) = N \operatorname{tr}_{1,\dots,N} A^{(N)} (u + E_1) \dots (u + E_{N-1} - N + 2)$$

$$= N \operatorname{tr}_{1,\dots,N} A^{(N)} C(u) (u + E_N - N + 1)^{-1}.$$

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Given an N-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_N)$, the corresponding irreducible highest weight representation $L(\lambda)$ of the Lie algebra $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ (the highest vector) such that

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$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant N.$$

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The mapping $z \mapsto \chi(z)$ defines an algebra isomorphism

$$\chi : \mathrm{Z}(\mathfrak{gl}_N) \to \mathbb{C}\,[l_1, \dots, l_N]^{\mathfrak{S}_N}$$

known as the Harish-Chandra isomorphism.

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which is the projection of the $\mathfrak{h}$-centralizer $\mathrm{U}(\mathfrak{gl}_N)^{\mathfrak{h}}$ with respect

to the direct sum decomposition

$$\mathrm{U}(\mathfrak{gl}_N)^{\mathfrak{h}} = \mathrm{U}(\mathfrak{h}) \oplus \Big(\mathrm{U}(\mathfrak{gl}_N)^{\mathfrak{h}} \cap \mathrm{U}(\mathfrak{gl}_N)\mathfrak{n}_+\Big).$$

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By the Newton formula, the Harish-Chandra images of the

Gelfand invariants are found by

$$1 + \sum_{m=0}^{\infty} \frac{(-1)^m \, \chi(\operatorname{tr} E^m)}{(u - N + 1)^{m+1}} = \prod_{i=1}^{N} \frac{u + l_i + 1}{u + l_i}.$$

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We have

$$\mathrm{Z}(\mathfrak{g}) = \mathbb{C}[P_1, \dots, P_n],$$

for certain algebraically independent invariants $P_1, \dots, P_n$

whose degrees $d_1, \dots, d_n$ are the exponents of $\mathfrak{g}$ increased by 1.

Brauer algebra $\mathcal{B}_m(\omega)$

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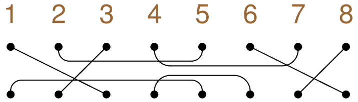
Multiplication of m-diagrams ($m = 8$):



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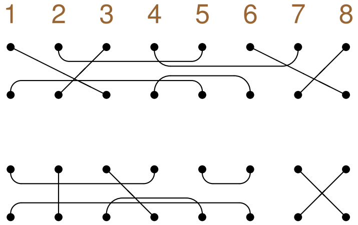

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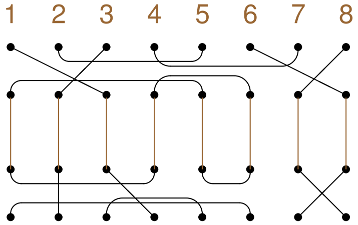

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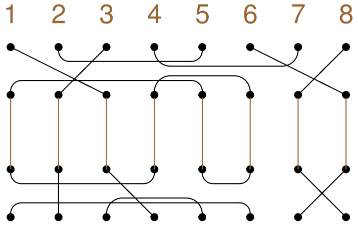
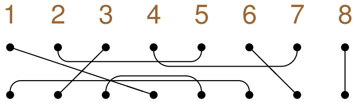


$= \omega$



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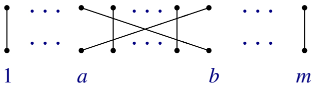

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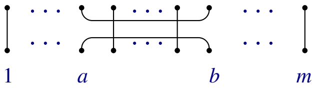

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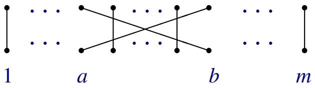

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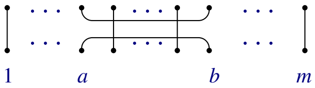

The symmetrizer in $\mathcal{B}_m(\omega)$ is the idempotent $s^{(m)}$ such that

$$s_{ab}\, s^{(m)} = s^{(m)}\, s_{ab} = s^{(m)} \qquad \text{and} \qquad g_{ab}\, s^{(m)} = s^{(m)}\, g_{ab} = 0.$$

Explicitly,

$$s^{(m)} = \frac{1}{m!} \sum_{r=0}^{\lfloor m/2 \rfloor} (-1)^r \binom{\omega/2+m-2}{r}^{-1} \sum_{d \in \mathcal{D}^{(r)}} d,$$

where $\mathcal{D}^{(r)} \subset \mathcal{B}_m(\omega)$ denotes the set of diagrams which have exactly r horizontal edges in the top.

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where the products are in the lexicographic order.

Brauer–Schur–Weyl duality

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There are commuting actions of the classical groups

in types B, C or D and the Brauer algebra with a specialized

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and

$$\big(\mathcal{B}_m(-N), Sp_N\big) \qquad \text{with} \qquad N = 2n.$$

Action in tensors

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In the case $\mathfrak{g} = \mathfrak{o}_N$ set $\omega = N$. The generators of $\mathcal{B}_m(N)$ act in the tensor space

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where $i' = N - i + 1$ and

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

In the case $\mathfrak{g} = \mathfrak{sp}_N$ with $N = 2n$ set $\omega = -N$. The generators of $\mathcal{B}_m(-N)$ act in the tensor space $(\mathbb{C}^N)^{\otimes m}$ by

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In both cases denote by $S^{(m)}$ the image of the symmetrizer $s^{(m)}$ under the action in tensors,

$$S^{(m)} \in \underbrace{\operatorname{End} \mathbb{C}^N \otimes \ldots \otimes \operatorname{End} \mathbb{C}^N}_{m}.$$

Explicitly, in the orthogonal case

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1}\right),$$

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Remark. $S^{(n+1)} = 0$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$. Consider $\gamma_m(-2n)\, S^{(m)}$,

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2}, \qquad \omega = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N \\ -2n & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

Lie algebras $\mathfrak{o}_N$ and $\mathfrak{sp}_N$

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respectively.

Introduce the $N \times N$ matrix $F = [F_{ij}]$

$$F = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}(\mathfrak{g}).$$

Theorem. For any $s \in \mathcal{B}_m(\omega)$ with $\omega = \pm N$

and $u_1, \dots, u_m \in \mathbb{C}$ the element

$$\mathrm{tr}_{1,\dots,m}\, S\,(u_1 + F_1)\dots(u_m + F_m)$$

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A version of the Newton identity also holds.

Proof of the theorem relies on the matrix form of the defining relations for $\mathrm{U}(\mathfrak{g})$:

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where both sides are regarded as elements of the algebra $\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{g})$ and

$$F_1 = \sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes F_{ij}, \qquad F_2 = \sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes F_{ij}.$$

Theorem. For $\mathfrak{g} = \mathfrak{o}_N$ the image of the Casimir element

$$\gamma_{2k}(N)\,\mathrm{tr}\, S^{(2k)}\,(F_1 - k)\ldots(F_{2k} + k - 1)$$

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Theorem. For $\mathfrak{g}_N = \mathfrak{sp}_{2n}$ the image of the Casimir element

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where $l_i = F_{ii} + n - i + 1$ for $i = 1, \dots, n$.

More constructions of Casimir elements for the Lie algebras

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The Harish-Chandra images $\chi(\mathbb{S}_\lambda)$ are

the shifted Schur polynomials.

Affine Kac–Moody algebras

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Define an invariant bilinear form on a simple Lie algebra $\mathfrak{g}$,

$$\langle X, Y \rangle = \frac{1}{2h^\vee} \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y),$$

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where $h^\vee$ is the dual Coxeter number.

For the classical types, $\langle X, Y\rangle = \mathrm{const}\cdot\mathrm{tr}\,X\,Y$,

$$h^\vee = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{sl}_N, \qquad \mathrm{const} = 1 \\ N-2 & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N, \qquad \mathrm{const} = \frac{1}{2} \\ n+1 & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}, \qquad \mathrm{const} = 1. \end{cases}$$

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$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\, K,$$

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Problem: What are Casimir elements for $\widehat{\mathfrak{g}}$?

The universal enveloping algebra at the critical level $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ is the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the ideal generated by $K + h^\vee$.

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By [Kac 1974], the canonical quadratic Casimir element belongs to a completion $\widetilde{\mathrm{U}}_{-h^\vee}(\widehat{\mathfrak{g}})$ of $\mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})$ with respect to the left ideals I_m, $m \geqslant 0$, generated by $t^m\mathfrak{g}[t]$.

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- Extension to Lie superalgebras.
- Extension to quantum affine algebras.

Example: $\mathfrak{g} = \mathfrak{gl}_N$. Defining relations for $\mathrm{U}(\widehat{\mathfrak{gl}}_N)$:

$$E_{ij}[r]\,E_{kl}[s] - E_{kl}[s]\,E_{ij}[r]$$

$$= \delta_{kj}\,E_{i\,l}[r+s] - \delta_{i\,l}\,E_{kj}[r+s] + r\,\delta_{r,-s}\left(\delta_{kj}\,\delta_{i\,l} - \frac{\delta_{ij}\,\delta_{kl}}{N}\right)K.$$

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For all $r \in \mathbb{Z}$ the sums

$$\sum_{i=1}^{N} E_{ii}[r]$$

are Casimir elements.

For $r \in \mathbb{Z}$ set

$$C_r = \sum_{i,j=1}^{N} \Big(\sum_{s<0} E_{ij}[s]\, E_{ji}[r-s] + \sum_{s\geqslant 0} E_{ji}[r-s]\, E_{ij}[s] \Big).$$

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All C_r are Casimir elements at the critical level, they belong to the completed universal enveloping algebra $\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$.

Introduce the (formal) Laurent series

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Given two Laurent series $a(z)$ and $b(z)$,

their normally ordered product is defined by

$$: a(z)\, b(z) : \ = a(z)_+\, b(z) + b(z)\, a(z)_-.$$

Note

$$\sum_{r\in\mathbb{Z}} C_r\, z^{-r-2} = \sum_{i,j=1}^{N} \Big(E_{ij}(z)_+ E_{ji}(z) + E_{ji}(z) E_{ij}(z)_- \Big).$$

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Hence, all coefficients of the series

$$\mathrm{tr}\, :E(z)^2: \; = \sum_{i,j=1}^{N} :E_{ij}(z) E_{ji}(z):$$

are Casimir elements.

Similarly, all coefficients of the series

$$\mathrm{tr} : E(z)^3 : \; = \sum_{i,j,k=1}^{N} : E_{ij}(z)\, E_{jk}(z)\, E_{ki}(z) :$$

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Correction term: all coefficients of the series

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Invariants of the vacuum module

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The vacuum module at the critical level is the $\widehat{\mathfrak{g}}$-module

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The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of $\mathfrak{g}[t]$-invariants

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\,v = 0\}.$$

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The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of $\mathfrak{g}[t]$-invariants

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\,v = 0\}.$$

Note $V(\mathfrak{g}) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ as a vector space.

Invariants of the vacuum module

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Hence, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

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Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a Segal–Sugawara vector.

Theorem (Feigin–Frenkel, 1992, Frenkel, 2007).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

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Explicit constructions of such sets and a new proof of

the theorem for the classical types A, B, C, D:

[Chervov–Talalaev, 2006, Chervov–M., 2009, M. 2013].

Example: $\mathfrak{g} = \mathfrak{gl}_N$.

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Set $\tau = -d/dt$ and consider the $N \times N$ matrix

$$\tau + E[-1] = \begin{bmatrix} \tau + E_{11}[-1] & E_{12}[-1] & \dots & E_{1N}[-1] \\ E_{21}[-1] & \tau + E_{22}[-1] & \dots & E_{2N}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ E_{N1}[-1] & E_{N2}[-1] & \dots & \tau + E_{NN}[-1] \end{bmatrix}.$$

The coefficients $\phi_1, \dots, \phi_N$ of the polynomial

$$\operatorname{cdet}\big(\tau + E[-1]\big) = \tau^N + \phi_1\,\tau^{N-1} + \dots + \phi_{N-1}\,\tau + \phi_N$$

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For $N = 2$

$$\begin{aligned} \mathrm{cdet}\big(\tau + E[-1]\big) &= \big(\tau + E_{11}[-1]\big)\big(\tau + E_{22}[-1]\big) - E_{21}[-1]E_{12}[-1] \\ &= \tau^2 + \phi_1\, \tau + \phi_2 \end{aligned}$$

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with

$$\phi_1 = E_{11}[-1] + E_{22}[-1],$$

$$\phi_2 = E_{11}[-1]\, E_{22}[-1] - E_{21}[-1]\, E_{12}[-1] + E_{22}[-2].$$

To get another family of Segal–Sugawara vectors, expand

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The following are Segal–Sugawara vectors for $\mathfrak{gl}_N$:

$$\operatorname{tr} E[-1], \qquad \operatorname{tr} E[-1]^2, \qquad \operatorname{tr} E[-1]^3, \qquad \operatorname{tr} E[-1]^4 - \operatorname{tr} E[-2]^2.$$

The corresponding central elements in $\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$ are recovered by the state-field correspondence map Y which takes elements of the vacuum module $V(\mathfrak{gl}_N)$ to Laurent series in z;

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By definition,

$$Y : E_{ij}[-1] \mapsto E_{ij}(z) = \sum_{r \in \mathbb{Z}} E_{ij}[r]\, z^{-r-1}.$$

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are topological generators of the center of $\widetilde{\mathrm{U}}_{-N}(\widehat{\mathfrak{gl}}_N)$.

Remark. The theorem holds in the same form for any complete set of Segal–Sugawara vectors.

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$$\operatorname{gr} \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \cong \mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$$

which yields a $\mathfrak{g}[t]$-module structure on the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \cong \mathrm{S}\big(\mathfrak{g}[t, t^{-1}]/\mathfrak{g}[t]\big)$.

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$$P_{(r)} = T^r P\big(X_1[-1], \dots, X_d[-1]\big), \qquad r \geqslant 0,$$

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Theorem (Raïs–Tauvel 1992, Beilinson–Drinfeld 1997).

If $P_1, \dots, P_n$ are algebraically independent generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then the elements $P_{1,(r)}, \dots, P_{n,(r)}$ with $r \geqslant 0$ are algebraically independent generators of $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

Explicit generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$. Type A

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and recall its elements $H^{(m)}$ and $A^{(m)}$.

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$$\mathrm{tr}_{1,\dots,m}\, A^{(m)}\big(\tau + E[-1]_1\big)\dots\big(\tau + E[-1]_m\big)$$

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The defining relations can be written in the form

$$E[r]_1 \, E[s]_2 - E[s]_2 \, E[r]_1$$

$$= \big(E[r+s]_1 - E[r+s]_2\big) \, P_{12} + r \, \delta_{r,-s} \, \big(1 - N \, P_{12}\big).$$

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The elements ψ_{ma} and θ_{ma} are expressed in terms of the ϕ_{ma} through the MacMahon Master Theorem and the Newton identities, respectively.

The coefficients of the column-determinant are related to the ϕ_{ma} through the relation

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \mathrm{tr}_{1,\dots,N} A^{(N)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_N\big).$$

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This follows from the property

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implied by the fact that $\tau + E[-1]$ is a Manin matrix.

Types B, C and D

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Recall the symmetrizers associated with $\mathfrak{o}_N$ and $\mathfrak{sp}_{2n}$:

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1}\Big),$$

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Also,

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2}, \qquad \omega = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N \\ -2n & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

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Combine into a matrix

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In addition, in the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Pfaffian

$$\mathrm{Pf}\,F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma \cdot F_{\sigma(1)\,\sigma(2)'}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

belongs to $\mathfrak{z}(\widehat{\mathfrak{o}}_{2n})$.

Moreover, $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}$ is a complete set of

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$\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n-2\,2n-2}, \phi'_n$ is a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n}$, where $\phi'_n = \mathrm{Pf}\, F[-1]$.

Affine Harish-Chandra isomorphism

For a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ consider the

Harish-Chandra homomorphism

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the projection modulo the left ideal generated by $t^{-1}\mathfrak{n}_-[t^{-1}]$.

The restriction to $\mathfrak{z}(\widehat{\mathfrak{g}})$ yields the Harish-Chandra isomorphism

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where $\mathcal{W}({}^L\mathfrak{g})$ is the classical $\mathcal{W}$-algebra associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$ [Feigin and Frenkel, 1992].

Example $\mathfrak{g} = \mathfrak{gl}_N$. Set $\mu_i[r] = E_{i\,i}[r]$. We have

$$\mathfrak{f} : \mathrm{cdet}\big(\tau + E[-1]\big) \mapsto \big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big).$$

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Define the elements $\mathcal{E}_1, \dots, \mathcal{E}_N$ by the Miura transformation

$$\big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = \tau^N + \mathcal{E}_1 \tau^{N-1} + \dots + \mathcal{E}_N.$$

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Explicitly,

$$\mathcal{E}_m = e_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big)\, 1$$

is the noncommutative elementary symmetric function,

$$e_m(x_1, \dots, x_p) = \sum_{i_1 > \dots > i_m} x_{i_1} \dots x_{i_m}.$$

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$$+ 2\,\mu_1[-2] + \mu_2[-2],$$

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For the Lie algebra $\mathfrak{g} = \mathfrak{o}_{2n}$ the image is

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The Harish-Chandra image of the Pfaffian

$$\operatorname{Pf} F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\,\sigma(2)'}[-1] \dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

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$$\mathrm{Pf}\, F[-1] \mapsto \big(\mu_1[-1] - T\big) \dots \big(\mu_n[-1] - T\big)\, 1.$$

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Miura transformation for $\mathfrak{o}_{2n+1}$ [Drinfeld–Sokolov 1985]:

$$\big(\tau - \mu_1[-1]\big)\ldots\big(\tau - \mu_n[-1]\big)\,\tau\,\big(\tau + \mu_n[-1]\big)\ldots\big(\tau + \mu_1[-1]\big)$$

$$= \tau^{2n+1} + \mathcal{E}_2\,\tau^{2n-1} + \mathcal{E}_3\,\tau^{2n-2} + \cdots + \mathcal{E}_{2n+1}.$$

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the V_i are the screening operators.

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One verifies directly that

$$V_i \big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = 0.$$

Equivalently,

$$V_i : \mu_i(z) \mapsto \exp \int \big(\mu_i(z) - \mu_{i+1}(z)\big)\, dz,$$

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Affine Poisson vertex algebra $\mathcal{V}(\mathfrak{g})$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and

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$$\mathcal{V} = \mathbb{C}[X_1^{(r)}, \ldots, X_d^{(r)} \mid r = 0, 1, 2, \ldots] \qquad \text{with} \quad X_i^{(0)} = X_i,$$

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equipped with the derivation ∂,

$$\partial(X_i^{(r)}) = X_i^{(r+1)}$$

for all $i = 1, \dots, d$ and $r \geqslant 0$.

Introduce the λ-bracket on $\mathcal{V}$ as a linear map

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$$\{a_\lambda bc\} = \{a_\lambda b\}c + \{a_\lambda c\}b.$$

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The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is a Poisson vertex algebra equipped with the λ-bracket

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Motivation: Hamiltonian equations

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for $u = u(t) \in \mathcal{W}(\mathfrak{g})$ with the Hamiltonian $H \in \mathcal{W}(\mathfrak{g})$.

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De Sole, Kac and Valeri, 2013-15; Drinfeld and Sokolov, 1985.

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The Hamiltonian equation with $H = \dfrac{u^2}{2}$ is equivalent to

the KdV equation

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The Hamiltonian equation with

$$K = \frac{1}{2}\,u^3 - \frac{1}{4}\,u\,u''$$

is also equivalent to the KdV equation.

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We will work with the algebra $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

$$\partial\, E_{ij}^{(r)} - E_{ij}^{(r)}\, \partial = E_{ij}^{(r+1)}.$$

Generators of $\mathcal{W}(\mathfrak{gl}_N)$

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The invariant symmetric bilinear form on $\mathfrak{gl}_N$ is defined by

$$\langle X, Y\rangle = \operatorname{tr} XY, \qquad X, Y \in \mathfrak{gl}_N.$$

Expand the determinant with entries in $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

$$\det \begin{bmatrix} \partial + E_{11} & 1 & 0 & 0 & \dots & 0 \\ E_{21} & \partial + E_{22} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{N-1\,1} & E_{N-1\,2} & E_{N-1\,3} & \dots & \dots & 1 \\ E_{N\,1} & E_{N\,2} & E_{N\,3} & \dots & \dots & \partial + E_{N\,N} \end{bmatrix}$$

$$= \partial^N + w_1\, \partial^{N-1} + \cdots + w_N.$$

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$$= \partial^N + w_1\, \partial^{N-1} + \dots + w_N.$$

Theorem. All elements $w_1, \dots, w_N$ belong to $\mathcal{W}(\mathfrak{gl}_N)$. Moreover,

$$\mathcal{W}(\mathfrak{gl}_N) = \mathbb{C}[w_1^{(r)}, \dots, w_N^{(r)} \mid r \geqslant 0].$$

Chevalley-type theorem

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Let

$$\phi : \mathcal{V}(\mathfrak{p}) \to \mathcal{V}(\mathfrak{h})$$

denote the homomorphism of differential algebras defined on the generators as the projection $\mathfrak{p} \to \mathfrak{h}$ with the kernel $\mathfrak{n}_-$.

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The restriction of ϕ to $\mathcal{W}(\mathfrak{g})$ is injective. The embedding

$$\phi : \mathcal{W}(\mathfrak{g}) \hookrightarrow \mathcal{V}(\mathfrak{h})$$

is often called the Miura transformation.

Theorem.

The restriction of the homomorphism ϕ to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ yields an isomorphism

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The restriction of the homomorphism ϕ to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ yields an isomorphism

$$\phi : \mathcal{W}(\mathfrak{g}) \to \widetilde{\mathcal{W}}(\mathfrak{g}),$$

where $\widetilde{\mathcal{W}}(\mathfrak{g})$ is the subalgebra of $\mathcal{V}(\mathfrak{h})$ which consists of the elements annihilated by all screening operators V_i,

$$\widetilde{\mathcal{W}}(\mathfrak{g}) = \bigcap_{i=1}^{n} \ker V_i.$$