Center at the critical level and commutative subalgebras

Alexander Molev

University of Sydney

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The subalgebra of invariants is

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \{P \in \mathrm{S}(\mathfrak{g}) \mid Y \cdot P = 0 \quad \text{for all} \quad Y \in \mathfrak{g}\}.$$

Let $n = \operatorname{rank} \mathfrak{g}$. Then

$$S(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[P_1, \ldots, P_n],$$

for certain algebraically independent invariants $P_1, \ldots, P_n$ of certain degrees $d_1, \ldots, d_n$ depending on $\mathfrak{g}$.

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We have the Chevalley isomorphism

$$\varsigma : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{S}(\mathfrak{h})^{W},$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and W is its Weyl group.

Type A

For $\mathfrak{g} = \mathfrak{gl}_N$ set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}$$

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$$\det(u + E) = u^N + C_1 u^{N-1} + \dots + C_N.$$

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Then $\quad \mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N} = \mathbb{C}[C_1, \dots, C_N] \quad$ and

$$\varsigma : \det(u + E) \mapsto (u + \lambda_1) \dots (u + \lambda_N), \qquad \lambda_i = E_{ii}.$$

We have

$$T_k = \operatorname{tr} E^k \in \mathrm{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$$

for all $k \geqslant 0$,

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The invariants C_k and T_k are related by the Newton formulas.

Types B, C and D

Define the orthogonal Lie algebra $\mathfrak{o}_N$ with $N = 2n$ and $N = 2n + 1$ and symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$ as subalgebras of $\mathfrak{gl}_N$ spanned by the elements F_{ij},

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We use the involution $i \mapsto i' = N - i + 1$ on the set $\{1, \dots, N\}$, and in the symplectic case set

$$\varepsilon_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n \\ -1 & \text{for} \quad i = n + 1, \dots, 2n. \end{cases}$$

The matrix $F = [F_{ij}]$ has the symmetry property $F + F' = 0$,

where we use the transposition on matrices defined by

$$(X')_{ij} = X_{j'i'} \qquad \text{or} \qquad (X')_{ij} = \varepsilon_i \varepsilon_j X_{j'i'}.$$

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If $\mathfrak{g} = \mathfrak{o}_{2n}$, then $C_n = \det F = (-1)^n \big(\mathrm{Pf}\, F\big)^2$ for the Pfaffian

$$\mathrm{Pf}\, F = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\,\sigma(2)'} \ldots F_{\sigma(2n-1)\,\sigma(2n)'}.$$

The subalgebra of invariants is

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \begin{cases} \mathbb{C}[C_1, \ldots, C_n] & \text{for} \quad \mathfrak{g} = \mathfrak{o}_{2n+1}, \quad \mathfrak{sp}_{2n} \\ \mathbb{C}[C_1, \ldots, C_{n-1}, \mathrm{Pf}\, F] & \text{for} \quad \mathfrak{g} = \mathfrak{o}_{2n}. \end{cases}$$

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Moreover, setting $\lambda_i = F_{ii}$ for $i = 1, \dots, n$, we have

$$\varsigma : \det(u + F) \mapsto \begin{cases} (u - \lambda_1^2) \dots (u - \lambda_n^2) & \text{if} \quad N = 2n \\ u(u - \lambda_1^2) \dots (u - \lambda_n^2) & \text{if} \quad N = 2n + 1. \end{cases}$$

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In the case $\mathfrak{g} = \mathfrak{o}_{2n}$,

$$\varsigma : \mathrm{Pf}\,F \mapsto \lambda_1\dots\lambda_n.$$

Poisson commutative subalgebras

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The symmetric algebra $\mathrm{S}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ admits

the Lie–Poisson bracket

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If $\mathfrak{g}$ is a simple Lie algebra with $n = \operatorname{rank} \mathfrak{g}$ then the subalgebra

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Problem: Extend $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ to a maximal Poisson commutative subalgebra of $\mathrm{S}(\mathfrak{g})$.

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$$X_i \mapsto X_i \, z^{-1} + \mu(X_i),$$

where z is a variable:

$$P\big(X_1 \, z^{-1} + \mu(X_1), \ldots, X_l \, z^{-1} + \mu(X_l)\big)$$
$$= P^{(0)} \, z^{-d} + \cdots + P^{(d-1)} \, z^{-1} + P^{(d)}.$$

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Denote by $\overline{\mathcal{A}}_\mu$ the subalgebra of $\mathrm{S}(\mathfrak{g})$ generated by all elements $P^{(i)}$ associated with all invariants $P \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.

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- If μ is a regular semi-simple element of $\mathfrak{g}^* \cong \mathfrak{g}$,

  then the elements

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  so that $\overline{\mathcal{A}}_\mu$ has the maximal possible transcendence

  degree $(\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g})/2$.

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For any regular $\mu \in \mathfrak{g}^*$ the elements $P_k^{(i)}$

are free generators of $\overline{\mathcal{A}}_\mu$.

Example. For $\mathfrak{g} = \mathfrak{gl}_N$ set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}, \qquad \mu = \begin{bmatrix} \mu_{11} & \dots & \mu_{1N} \\ \vdots & & \vdots \\ \mu_{N1} & \dots & \mu_{NN} \end{bmatrix}$$

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and write

$$\det\left(u + \mu + E\, z^{-1}\right) = \sum_{0 \leqslant i \leqslant k \leqslant N} C_k^{(i)}\, z^{-k+i}\, u^{N-k}.$$

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and write

$$\det\big(u + \mu + E\, z^{-1}\big) = \sum_{0 \leqslant i \leqslant k \leqslant N} C_k^{(i)}\, z^{-k+i}\, u^{N-k}.$$

The elements $C_k^{(i)}$ with $k = 1, \dots, N$ and $i = 0, 1, \dots, k-1$ are algebraically independent generators of $\overline{\mathcal{A}}_\mu$ for regular μ.

Also write

$$\operatorname{tr}\left(\mu + E z^{-1}\right)^k = \sum_{i=0}^{k} T_k^{(i)}\, z^{-k+i}.$$

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All elements $T_k^{(i)}$ belong to the subalgebra $\overline{\mathcal{A}}_\mu$ of $\mathrm{S}(\mathfrak{gl}_N)$.

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We would like to find a (maximal) commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{g})$ (together with its free generators) such that $\mathrm{gr}\,\mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.

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M. Nazarov and G. Olshanski, 1996:

$\mathcal{A}_\mu$ is produced for classical types, μ regular semi-simple.

Explicit free generators of $\mathcal{A}_\mu$ for $\mathfrak{g} = \mathfrak{gl}_N$:

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The solution uses the Feigin–Frenkel center associated with $\widehat{\mathfrak{g}}$.

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The value $\kappa = -h^\vee$ corresponds to the critical level.

Feigin–Frenkel center

Consider the left ideal $\mathrm{I} = \mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})\,\mathfrak{g}[t]$ and let

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The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the associative algebra defined as the quotient

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathrm{Norm}\,\mathrm{I}/\mathrm{I}.$$

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Hence, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

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Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a Segal–Sugawara vector.

Theorem (Feigin–Frenkel, 1992).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

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There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

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We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

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If S is a Segal–Sugawara vector of degree d, set

$$\rho(S) = S^{(0)} z^{-d} + \cdots + S^{(d-1)} z^{-1} + S^{(d)}.$$

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- If $S_1, \dots, S_n$ is a complete set of Segal–Sugawara vectors of the respective degrees $d_1, \dots, d_n$ then the elements

$$S_k^{(i)}, \qquad k = 1, \dots, n, \quad i = 0, 1, \dots, d_k - 1,$$

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Conjecture (*loc. cit.*) The last claim holds for any $\mu \in \mathfrak{g}^*$.

Explicit construction of $\mathcal{A}_\mu$

Use complete sets of Segal–Sugawara vectors $S_1, \dots, S_n$

produced in A. Chervov and D. Talalaev, 2006,

and also A. Chervov and A. M., 2009 (in type A)

and A. M., 2013 (types B, C and D).

For $\mathfrak{g} = \mathfrak{gl}_N$ set

$$E = \begin{bmatrix} E_{11} & \dots & E_{1N} \\ \vdots & & \vdots \\ E_{N1} & \dots & E_{NN} \end{bmatrix}, \qquad \mu = \begin{bmatrix} \mu_{11} & \dots & \mu_{1N} \\ \vdots & & \vdots \\ \mu_{N1} & \dots & \mu_{NN} \end{bmatrix}.$$

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Write

$$\mathrm{cdet}\big(-\partial_z + \mu + E\, z^{-1}\big) = \sum_{0\leqslant i\leqslant k\leqslant N} \widehat{C}_k^{(i)}\, z^{-k+i}\, \partial_z^{N-k}$$

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and

$$\operatorname{tr}\left(-\partial_z + \mu + E z^{-1}\right)^k 1 = \sum_{i=0}^{k} \widehat{T}_k^{(i)}\, z^{-k+i}.$$

Theorem. For any μ all elements $\widehat{C}_k^{(i)}$ and $\widehat{T}_k^{(i)}$ belong to the commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{gl}_N)$.

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If μ is regular, then the elements of each of these families with $k = 1, \dots, N$ and $i = 0, 1, \dots, k-1$ are algebraically independent generators of $\mathcal{A}_\mu$.

Examples. We get the following algebraically independent generators of the algebra $\mathcal{A}_\mu$ for regular μ:

for $\mathfrak{gl}_2 : \qquad \operatorname{tr} E, \quad \operatorname{tr} \mu E, \quad \operatorname{tr} E^2$

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$$2\operatorname{tr}\mu^2 E^2 + \operatorname{tr}(\mu E)^2,\quad \operatorname{tr} E^3,\quad \operatorname{tr}\mu E^3,\quad \operatorname{tr} E^4.$$

Types B, C and D

The symmetric group $\mathfrak{S}_m$ acts on the tensor space

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$$P_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(m-b)}.$$

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in the orthogonal case, and

$$Q_{ab} = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j \, 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}$$

in the symplectic case, where $i' = N - i + 1$.

Define the respective symmetrizer as the operator

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \Big(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1} \Big),$$

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Set

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2}, \qquad \omega = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N \\ -2n & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

Combine the generators of $\mathfrak{g} = \mathfrak{o}_N, \quad \mathfrak{sp}_N$ into the matrix

$$F = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}(\mathfrak{g}).$$

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For any $\mu \in \mathfrak{g}^*$ write

$$\gamma_m(\omega) \operatorname{tr} S^{(m)} (-\partial_z + \mu_1 + F_1 z^{-1}) \ldots (-\partial_z + \mu_m + F_m z^{-1}) \, 1 = \sum_{i=0}^{m} L_m^{(i)} \, z^{-m+i}.$$

In the case of $\mathfrak{o}_{2n}$ consider the Pfaffian

$$\begin{aligned}
&\mathrm{Pf}\,(\mu + F z^{-1}) \\
&\quad = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn}\sigma \cdot (\mu + F z^{-1})_{\sigma(1)\,\sigma(2)'} \dots (\mu + F z^{-1})_{\sigma(2n-1)\,\sigma(2n)'} \\
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Theorem. For any $\mu \in \mathfrak{g}^*$ all elements $L_m^{(i)}$

(together with the $P^{(i)}$ in type D)

belong to the commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{g})$.

Theorem. Suppose $\mu \in \mathfrak{g}^*$ is regular.

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In types B and C the elements $L_m^{(0)}, \dots, L_m^{(m-1)}$ with $m = 2, 4, \dots, 2n$ are algebraically independent generators of the maximal commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{o}_{2n+1})$ and $\mathrm{U}(\mathfrak{sp}_{2n})$.

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In type D the elements $L_m^{(0)}, \dots, L_m^{(m-1)}$ with $m = 2, 4, \dots, 2n-2$ and $P^{(0)}, \dots, P^{(n-1)}$ are algebraically independent generators of the maximal commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{o}_{2n})$.

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