Centers of vertex algebras

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A series of the form

$$c(z) = \sum_{n \in \mathbb{Z}} c_n z^{-n-1} \in \operatorname{End} V[[z, z^{-1}]]$$

is called a field, if for any $v \in V$ there exists an integer $N \geqslant 0$

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is called a field, if for any $v \in V$ there exists an integer $N \geqslant 0$ such that $c_n v = 0$ for all $n \geqslant N$.

Equivalently, the series $c(z)v$ contains finitely many negative powers of z for any $v \in V$.

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the translation T is an operator $T : V \to V$ and

the state-field correspondence Y is a linear map

$$Y : V \to \mathrm{End}\, V[[z, z^{-1}]]$$

such that the image of any element $a \in V$ is a field, $Y : a \mapsto a(z)$,

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \qquad a_{(n)} \in \mathrm{End}\, V.$$

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- $\big[T, a(z)\big] = \partial_z a(z)$ for each $a \in V$,
- for any states $a, b \in V$ there exists a nonnegative integer N such that $(z-w)^N [a(z), b(w)] = 0$.

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The commutator is given by

$$\left[a_{(m)}, b_{(k)}\right] = \sum_{n \geqslant 0} \binom{m}{n} \left(a_{(n)}\, b\right)_{(m+k-n)}.$$

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Proposition. The center $\mathfrak{z}(V)$ of any vertex algebra V is a unital commutative associative algebra with respect to the product $ab := a_{(-1)}b$.

Affine vertex algebras

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$$V_\kappa(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{U}(\widehat{\mathfrak{g}})\big(\mathfrak{g}[t] + \mathbb{C}(K - \kappa)\big).$$

We view $V_\kappa(\mathfrak{g})$ as a $\widehat{\mathfrak{g}}$-module. It is called the vacuum module of level κ.

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The state-field correspondence Y is defined as follows. First,

$$Y(X[-1], z) = \sum_{r\in\mathbb{Z}} X[r] z^{-r-1} =: X(z).$$

For any $r_i \geqslant 0$ we have

$$Y(X_1[-r_1 - 1] \dots X_m[-r_m - 1], z) = \frac{1}{r_1! \dots r_m!} :\partial_z^{r_1} X_1(z) \dots \partial_z^{r_m} X_m(z):,$$

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where

$$a(z)_+ = \sum_{r<0} a_{(r)} z^{-r-1} \qquad \text{and} \qquad a(z)_- = \sum_{r \geqslant 0} a_{(r)} z^{-r-1}.$$

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$\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $\mathrm{U}(t^{-1}\mathfrak{g}[t^{-1}])$.

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There exist elements $S_1, \dots, S_n \in \mathrm{U}(t^{-1}\mathfrak{g}[t^{-1}])$ such that

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$\mathfrak{z}(\widehat{\mathfrak{g}})$ is called the Feigin–Frenkel center associated with $\mathfrak{g}$.

Detailed exposition: [E. Frenkel, 2007].

Example. $\mathfrak{g} = \mathfrak{gl}_N$.

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Set $\tau = -\partial_t$ and consider the $N \times N$ matrix $\tau + E[-1]$ given by

$$\tau + E[-1] = \begin{bmatrix} \tau + E_{11}[-1] & E_{12}[-1] & \dots & E_{1N}[-1] \\ E_{21}[-1] & \tau + E_{22}[-1] & \dots & E_{2N}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ E_{N1}[-1] & E_{N2}[-1] & \dots & \tau + E_{NN}[-1] \end{bmatrix}.$$

Theorem [Chervov–Talalaev, 2006, Chervov–M. 2009].

The coefficients $\phi_1, \dots, \phi_N$ of the polynomial

$$\mathrm{cdet}\big(\tau + E[-1]\big) = \tau^N + \phi_1\,\tau^{N-1} + \cdots + \phi_{N-1}\,\tau + \phi_N$$

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Example. For $N = 2$

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with

$$\phi_1 = E_{11}[-1] + E_{22}[-1],$$

$$\phi_2 = E_{11}[-1]\, E_{22}[-1] - E_{21}[-1]\, E_{12}[-1] + E_{22}[-2].$$

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Set

$$E[-1] = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij}[-1] \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big).$$

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and let $H^{(m)}$ and $A^{(m)}$ denote the symmetrizer and anti-symmetrizer in

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$$\mathrm{tr}_{1,\dots,m}\, A^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big)$$

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belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$.

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Moreover, each family

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Extension to types B_n, C_n, D_n and G_2:

[M. 2013], [M.–Ragoucy–Rozhkovskaya, 2016].

Quantum vacuum modules

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$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)}\, u^{-r}$$

and

$$t_{ij}^{+}(u) = \delta_{ij} - \sum_{r=1}^{\infty} t_{ij}^{(-r)}\, u^{r-1}.$$

The defining relations are

$$R(u-v)\,T_1(u)\,T_2(v)=T_2(v)\,T_1(u)\,R(u-v),$$

$$R(u-v)\,T_1^+(u)\,T_2^+(v)=T_2^+(v)\,T_1^+(u)\,R(u-v),$$

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where the coefficients of powers of u, v belong to

$$\mathrm{End}\,\mathbb{C}^N \otimes \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{DY}(\mathfrak{gl}_N)$$

and

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \qquad \text{and} \qquad T^+(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}^+(u).$$

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We also use the normalized R-matrix

$$\overline{R}(u) = g(u)\, R(u),$$

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$$R(u) = 1 - P\,u^{-1},$$

where P is the permutation operator in $\mathbb{C}^N \otimes \mathbb{C}^N$.

We also use the normalized R-matrix

$$\overline{R}(u) = g(u)\,R(u),$$

where

$$g(u) = 1 + \sum_{i=1}^{\infty} g_i\,u^{-i}, \qquad g_i \in \mathbb{C},$$

is uniquely determined by the relation

$$g(u+N) = g(u)\,(1-u^{-2}).$$

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Proposition. The assignments

$$E_{ij}[r-1] \mapsto \bar{t}_{ij}^{(r)}, \qquad E_{ij}[-r] \mapsto \bar{t}_{ij}^{(-r)} \qquad \text{and} \qquad K \mapsto \overline{C}$$

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with $r \geqslant 1$ define an algebra isomorphism

$$\mathrm{U}(\widehat{\mathfrak{gl}}_N) \to \mathrm{gr}\,\mathrm{DY}(\mathfrak{gl}_N).$$

The vacuum module $\mathcal{V}_c(\mathfrak{gl}_N)$ at the level $c \in \mathbb{C}$ over $\mathrm{DY}(\mathfrak{gl}_N)$ is

$$\mathcal{V}_c(\mathfrak{gl}_N) = \mathrm{DY}(\mathfrak{gl}_N)/\mathrm{DY}(\mathfrak{gl}_N)\langle C - c,\ t_{ij}^{(r)} \,|\, r \geqslant 1\rangle.$$

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Let $\widehat{\mathcal{V}}$ denote the completion of $\mathcal{V}_{-N}(\mathfrak{gl}_N)$ by the descending filtration defined by setting the degree of $t_{ij}^{(-r)}$ to be r.

By the proposition, $\operatorname{gr} \mathrm{Y}^{+}(\mathfrak{gl}_N) \cong \mathrm{U}\big(t^{-1}\mathfrak{gl}_N[t^{-1}]\big)$ so that $\widehat{\mathcal{V}}$ is a quantization of the vacuum module at the critical level over $\widehat{\mathfrak{gl}}_N$.

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Introduce the subspace of invariants by

$$\mathfrak{z}(\widehat{\mathcal{V}}) = \{v \in \widehat{\mathcal{V}} \mid t_{ij}(u)\,v = \delta_{ij}\,v\},$$

so that any element of $\mathfrak{z}(\widehat{\mathcal{V}})$ is annihilated by all $t_{ij}^{(r)}$ with $r \geqslant 1$.

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the product is in the lexicographical order on the pairs (a, b).

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For a standard μ-tableau $\mathcal{U}$ with entries in $\{1, \dots, m\}$ introduce the contents $c_a = c_a(\mathcal{U})$ for $a = 1, \dots, m$ so that $c_a = j - i$ if a occupies the box (i, j) in $\mathcal{U}$.

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The symmetric group $\mathfrak{S}_m$ acts by permuting the tensor factors in $(\mathbb{C}^N)^{\otimes m}$. Denote by $\mathcal{E}_{\mathcal{U}}$ the image of $e_{\mathcal{U}}$ under this action.

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$$R(u_1,\dots,u_m)\big|_{u_1=c_1}\big|_{u_2=c_2}\dots\big|_{u_m=c_m} = h(\mu)\,\mathcal{E}_{\mathcal{U}},$$

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$$\mathbb{T}^{+}_{\mu}(u) = \mathrm{tr}_{1,\dots,m}\,\mathcal{E}_{\mathcal{U}}\,T^{+}_{1}(u+c_1)\dots T^{+}_{m}(u+c_m).$$

This is a power series in u independent of $\mathcal{U}$, whose coefficients are elements of the completed vacuum module $\widehat{\mathcal{V}}$.

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Theorem [Okounkov, 1996]. The quantum immanants $\mathbb{S}_\mu$ form a basis of the center of $\mathrm{U}(\mathfrak{gl}_N)$.

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Hence all coefficients of the polynomial

$$\mathrm{tr}_{1,\dots,m} A^{(m)} T_1^+(u) \dots T_m^+(u-m+1)$$

$$= \mathrm{tr}_{1,\dots,m} A^{(m)} T_1^+(u) e^{-\partial_u} \dots T_m^+(u) e^{-\partial_u} \cdot e^{m\partial_u}$$

belong to $\mathfrak{z}(\widehat{\mathcal{V}})$.

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The element

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we recover the differential operator in $\tau = -\partial_t$:

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$$= \phi_{m\,0}\, \tau^m + \phi_{m\,1}\, \tau^{m-1} + \dots + \phi_{m\,m}.$$

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A similar calculation works for the row-diagram $\mu = (m)$, but no Segal–Sugawara vectors are known for arbitrary μ.