On Stability and Continuity of Bounded Solutions of Degenerate Complex Monge-Ampère Equations over Compact Kähler Manifolds

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Abstract

We obtain a stability estimate for the degenerate complex Monge-Ampère operator which generalizes a result of Kołodziej in [12]. In particular, we obtain the optimal stability exponent and also treat the case when the right hand side is a general Borel measure satisfying certain regularity conditions. Moreover, our result holds for functions plurisubharmonic with respect to a big form, thus generalizing the Kähler form setting in [12]. Independently, we also provide more detail for the proof in [18] on continuity of the solution with respect to special big form when the right hand side is L^p-measure with $p > 1$.

1 Introduction

In this work, we generalize and strengthen Kołodziej's stability and continuity results concerning bounded solutions for complex Monge-Ampère equations, which are proved in [12] and [11] respectively (see also [13] for a nice summary). The solutions are understood in the sense of pluripotential theory, i.e. we do not impose any regularity assumption other than upper semi-continuity and boundedness. It is, however, a classic fact that the image of the Monge-Ampère operator can be well defined as a Borel measure in this setting.

The equation is considered over a closed Kähler manifold X of complex dimension $n \geqslant 2$. When $n = 1$ the manifold is a Riemann surface and Monge-Ampère operator is just the classic Laplace operator. Since the latter is linear, the corresponding problems can be dealt by more classic techniques.

Suppose ω is a real smooth closed semi-positive $(1,1)$-form over X, Ω is a positive Borel measure on X and $f \in L^p(X)$ for some $p > 1$ is non-negative, where the definition of the function space $L^p(X)$ is with respect to Ω. The equation under consideration is

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega.$$

Using $d = \partial + \bar{\partial}$ and $d^c := \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)$, we have $dd^c = \sqrt{-1}\partial\bar{\partial}$ and this convention is also frequently used in the literature.

As mentioned above, we require regularity of u much less than what is needed to make pointwise sense for the left hand side of the equation. More precisely, we look for solutions in the function class $PSH_\omega(X) \cap L^\infty(X)$, where $u \in PSH_\omega(X)$ means that $\omega_u := \omega + \sqrt{-1}\partial\bar{\partial}u$ is non-negative in the sense of distribution.

Of course, there is an obvious condition for the existence of such a solution coming from global integration over X, i.e. $\int_X \omega^n = \int_X f\Omega$. This condition follows from Stokes' theorem in the smooth case, and hence (by smooth approximation) in our case as well.

Kołodziej (cf. [11] and [12]) mainly studied the case when ω is a Kähler metric, or equivalently $[\omega]$ is a Kähler class and Ω is a smooth volume form. The existence of bounded solution in this case is achieved. In fact, even more general function class than $L^{p>1}$-function class has been treated in [11], but for our main interest, we restrict ourselves to $L^{p>1}$-functions. Furthermore, in his case, the bounded solution is always continuous as justified in [11]. So in the discussion of stability there, continuity of the solution can be assumed without any loss of generality.

In the following we state our first main result and refer to the next section for definitions of some notions appearing in the statement.

Theorem 1.1. *Let X be a compact Kähler Manifold and ω is a big form on X. Also assume Ω be a positive Borel measure on X which is dominated by capacity for L^p-functions with some constant $p > 1$. Let Q be a positive increasing function with polynomial growth that measures the domination of Ω, and the function κ be defined by*

$$\kappa(r) = C_{n,p}\Big(\int_{r^{-\frac{1}{n}}}^{\infty} y^{-1}(Q(y))^{-\frac{1}{n}}dy + \big(Q(r^{-\frac{1}{n}})\big)^{-\frac{1}{n}}\Big),$$

where C_n is a positive constant depending merely on the complex dimension n, p and the manifold (X, ω). Define the function γ by $\gamma(t) = C\kappa^{-1}(t)$, with κ^{-1} being the inverse function of κ. Consider any non-negative $L^p(\Omega)$-functions f and g satisfying $\int_X f\Omega = \int_X g\Omega = \int_X \omega^n$. Let ϕ and ψ in $PSH_\omega \cap L^\infty(X)$ satisfy ${\omega_\phi}^n = f\omega^n$ and ${\omega_\psi}^n = g\omega^n$ respectively and be normalized by $max_X\{\phi - \psi\} = max_X\{\psi - \phi\}$.

Then for any $\epsilon > 0$, there are a constant $C = C(X, \omega, \|f\|_p, \|g\|_p, \epsilon)$ and a constant t_0 depending on γ such that for any $t < t_0$ the inequality $\|f - g\|_{L^1} \leqslant \gamma(t)t^{n+\epsilon}$ implies

$$\|\phi - \psi\|_{L^\infty} \leqslant Ct.$$

This result means that Kołodziej's Stability Theorem still holds even if the background form is merely big. Moreover one can relax the smoothness assumptions on the measure to "*being dominated by capacity*", and the result is still true. In fact these generalizations are direct consequence of results from [3] and [4]. The non-trivial part is the improvement of the exponent from $n + 3$ (cf. [12]) to the optimal $n + \epsilon$ for any small positive ϵ.

As a natural application, we have the uniqueness of bounded solution for such equation. Another corollary is the following stability estimate, which provides the optimal exponent for the stability estimate.

Corollary 1.2. *In the same setting as Theorem 1.1, if Ω is indeed smooth, then there exists a constant $c = c(p, \epsilon, c_0)$, where $c_0 = max\{\|f\|_p, \|g\|_p\}$, such that*

$$\|\phi - \psi\|_\infty \leqslant c\|f - g\|_1^{\frac{1}{n+\epsilon}}. \tag{1.1}$$

Such an inequality was recently applied to prove Hölder continuity for solutions of Monge-Ampère equations with right hand side in $L^{p>1}$-spaces (see [14]). The optimal Hölder exponent in that result is yet to be found. However the bigger the exponent in the inequality (1.1) is, the better Hölder exponent one can get. Thus getting an optimal result in 1.1 is quite important. As Example 5.2 shows, the exponent obtained above is quite sharp.

In Section 6, we provide more detail for the proof of the result due to the second named author in [18] (or [19]). The argument given there is a little bit too sketchy (and more importantly, scattered in several chapters of the thesis for some reasons), which makes it hard to follow. Let's state this result with some accompanying background.

Theorem 1.3. *Let X be a closed Kähler manifold with $\dim_{\mathbb{C}} X = n \geqslant 2$. Suppose we have a holomorphic map $F : X \to \mathbb{CP}^N$ with the image $F(X)$ of the same dimension as X. Let ω_M be any Kähler form over some neighbourhood of $F(X)$ in $\mathbb{CP}^N$. Consider the following equation of Monge-Ampère type:*

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega,$$

where $\omega = F^*\omega_M$, Ω is a fixed smooth (non-degenerate) volume form over X and f is a non-negative function in $L^p(X)$ for some $p > 1$ satisfying $\int_X f\Omega = \int_X (F^*\omega_M)^n$. Then we have the following:

(1) (A priori Estimate) If u is a weak solution in $PSH_\omega(X) \cap L^\infty(X)$ of the equation with the normalization $sup_X u = 0$, then there is a constant C such that $\|u\|_{L^\infty} \leqslant C\|f\|^n_{L^p}$ where C only depends on F, (X, ω) and p.

(2) (Existence of Bounded Solution) There exists a bounded (weak) solution for this equation.

(3) (Continuity and Uniqueness of Bounded Solution) If F is locally birational, any bounded solution is actually the unique continuous solution.

The proof of part (1) appears in [18], Sections 2 and 3 (Pages 5–12), or in [19], Sections 4.2, 5.3, 6.1 and 6.2 (Pages 144–146, 166–169 and 173–187). The proof of (2) appears in [18], Section 4 (Page 13), or in [19], Sections 4.3, 5.3, 6.1 and 6.2 (Pages 146, 166–169 and 173–187). Here we give a more detailed proof of (3) which is discussed in [18], Section 5 (Pages 13–15), or [19], Section 7.3 (Pages 194–199).

The Monge-Ampère equation in similar setting has been studied extensively in the recent years (see [1], [7] and [8]). In particular, the a priori estimate was also obtained independently in [8] (even for more general big forms), and later generalized to more singular right hand side in [7]. As for the continuity of the solution, in despite of serious efforts, the situation is still a little bit unclear in the most general case. It is not known whether continuity holds when ω is a general big form with continuous (even smooth) potentials. This problem has attracted a lot of interest recently, and in fact this is the main motivation to present a more detailed proof of the continuity in the situation above, which contains the case with the most interest.

We wish to point out that the methods used in the proof of the stability Theorem 1.1 are independent of the regularity of solutions. So theoretically, the solutions might be discontinuous in general, but uniformly close to each other if f and g are close in L^1-norm. Needlessly to say, this could be quite a strange situation. Thus our results strongly support (but in no way have justified) the common belief that continuity would indeed hold in general.

The applications of our results could go in several directions. The semi-positive case is particularly interesting in geometry, since it appears very naturally in the study of algebraic manifolds of general type (or big line bundles in general, see e.g. [17]). In the mean time, the degeneration of the measure on the right hand side might be useful in complex dynamics and pluripotential theory. Complex dynamics often deals with such singular measures and it is very helpful to obtain any kind of regularity for the potential of such measures. The same question arises in pluripotential theory for the study of extremal functions.

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2 Preliminaries

Throughout this note we shall work on a closed Kähler manifold X with $dim_{\mathbb{C}} X = n \geqslant 2$. We equip X with a big form ω, where "*big*" is defined below.

Definition 2.1. *A smooth d-closed form ω is called big if it is pointwise semi-positive and the induced volume has a positive total integral, i.e. $\int_X \omega^n > 0$.*

It is also possible to define bigness for currents with bounded potentials (see [7]). Here we restrict to the smooth case.

We shall use the methods in pluripotential theory introduced by Bedford and Taylor (cf. [2]) and adjust them a little to the manifold case according to the description by Kołodziej (cf. [13]). The most important tool is relative capacity defined below.

Definition 2.2. *For a Borel subset K of X, we define its relative capacity with respect to ω by*

$$Cap_\omega(K) := sup\{\int_K (\omega + \partial\bar{\partial}\rho)^n |\ \rho \in PSH_\omega(X),\ 0 \leqslant \rho \leqslant 1\}.$$

Note that Kołodziej originally defined the relative capacity with respect to a Kähler form ([12]). The obvious generalization to more general background form setting has appeared in [8] and [18].

We study the Monge-Ampère equation with singular measure on the right hand side. Namely, we assume that Ω is a Borel measure instead of a smooth volume form. Then we need some restriction, since weak solutions for such an equation might not be bounded anymore (for example, if Ω is the Dirac delta measure at some point). Worse yet, there are measures for which the existence of solutions of any kind (bounded or not) is not clear so far. Therefore we impose some seemingly natural conditions on Ω that guarantee boundedness of the solutions.

Definition 2.3. *We say that a Borel measure is* **dominated by capacity** *for L^p functions if there exist constants $\alpha > 0$ and $\chi > 0$, such that for any compact $K \subset X$ and any non-negative $f \in L^p(\Omega)$ with $p > 1$, one has for some constant C independent of K that*

$$\Omega(K) \leqslant C \cdot Cap_\omega(K)^{1+\alpha}, \quad \int_K f\Omega \leqslant C \cdot Cap_\omega(K)^{1+\chi}.$$

A very similar notion, where only the first inequality is imposed, has been introduced in [8]. Both are variations of the so-called condition (A) introduced by S. Kołodziej in [11]. These conditions, which actually are stronger than condition (A), force boundedness for the solutions u of

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega.$$

See [11] for the case ω is Kähler, and [8] for the case ω is merely big.

Let's say a few words about the second inequality. When Ω is a smooth volume form, it is known (again see [11] and [8]) that the first condition is satisfied for every $\alpha > 0$. Hence by an elementary application of the Hölder inequality the second inequality also holds for every $\chi > 0$. Hölder inequality indeed implies, regardless of the smoothness of Ω, that the second inequality is a consequence of the first one provided p is big enough (if $\frac{(1+\alpha)(p-1)}{p} > 1$). In any case, one has to impose some condition, since a priori $f\Omega$ can be a lot more singular than Ω.

Note that, as in [12] Lemma 2.2 or [13] Lemma 6.5, the exponent $\chi > 0$ is used to construct the *admissible* function Q, measuring the domination by capacity, which in our situation has growth like $t^{n\chi}$, and so the function

$$\kappa(r) = C_{n,p}\Big(\int_{r^{-\frac{1}{n}}}^{\infty} y^{-1}(Q(y))^{-\frac{1}{n}}dy + \big(Q(r^{-\frac{1}{n}})\big)^{-\frac{1}{n}}\Big) \approx r^{\frac{\chi}{n}},$$

and its inverse $\gamma(t) \approx t^{\frac{n}{\chi}}$. When the volume form Ω is smooth, one can take arbitrary $\chi > 0$ (of course the larger, the better). Thus one can produce a function $\gamma(t)$ with growth like $t^{\epsilon}, \forall \epsilon > 0$ near 0. When χ is bounded from above, [1] then one can take $\gamma(t) \approx t^{\frac{n}{\chi}}$.

In order to avoid too much technicalities in the proof we shall work with the assumption that χ can be taken to be arbitrarily large. At the end (see Remark 5.1), we will explain how to modify the argument for some fixed χ and obtain the stability exponent in general as well.

The next theorem is quoted from [12], which allows us to estimate the capacity of sub-level sets of plurisubharmonic functions.

Theorem 2.4. *Suppose $\phi, \psi \in PSH_\omega(X)$ and let ϕ satisfy $0 \leqslant \phi \leqslant C$, then for $s < C + 1$, we have*

$$Cap_\omega(\{\psi + 2s < \phi\}) \leqslant \left(\frac{C+1}{s}\right)^n \int_{\{\psi+s<\phi\}} (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n.$$

The following proposition is also useful for us.

Proposition 2.5. *Let $\phi, \psi \in PSH_\omega(X)$ satisfy the inequalities $0 \leqslant \phi \leqslant a, 0 \leqslant \psi \leqslant a$. Then for any constants $m, n, t > 0$ we have*

$$Cap_\omega(\{\psi + (m+n)t < \phi\}) \leqslant \left(\frac{a+1}{nt}\right)^n \int_{\{\psi+mt<\phi\}} (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n.$$

Proof. Note that $nt \geqslant a + 1$ yields that $\{\psi + (m+n)t < \phi\}$ is empty because of the additional assumption on ψ. If $nt < a + 1$, then for any function $\rho \in PSH_\omega(X), -1 \leqslant \rho \leqslant 0$ we get the chain of sets

$$\{\psi + (m+n)t < \phi\} \subset \{\psi + mt < (1 - \frac{nt}{a+1})\phi + \frac{nt}{a+1}\rho\} \subset \{\psi + mt < \phi\}$$

and the proof is the same as the one of Theorem 2.4 (cf. [12]). The argument goes through for any function ρ as above, and so one can get the conclusion for relative capacity from definition.

□

In Section 6 we shall work with (locally) birational mappings. Although these are fairly standard objects, we feel that it is worth giving the definitions as well as to show some illuminating examples.

Definition 2.6. *A meromorphic mapping $F : X \dashrightarrow Y$ between two complex varieties X and Y is called birational if it has an inverse (in the sense of meromorphic map) such that $F^{-1} : Y \dashrightarrow X$ is also meromorphic.*

[1] We assume it is a fixed constant depending on the measure μ.

A typical example of such a mapping is as follows. If X carries a big line bundle L, then the Iitaka Fibration Theorem (cf. [16]) states that the linear series corresponding to L^m generate (for sufficiently large $m \in \mathbb{N}$) a meromorphic morphism into $\mathbb{CP}^N$ which is birational onto its image. If moreover L is semi-ample then the mapping is holomorphic, i.e. the map is defined over X.

Definition 2.7. *A meromorphic mapping $F : X \dashrightarrow Y$ is called locally birational if for every small enough neighbourhood U of any point on $F(X)$, each component of $F^{-1}(U)$ is birational to U.*

The next example is a classic double-point, which shows that these two notions are indeed different.

Example 2.8. *Consider the following map*

$$F : \mathbb{C} \ni t \to \big(t^2 - 1, t(t^2 - 1)\big) \in \mathbb{C}^2.$$

The image $F(\mathbb{C})$ sits in the variety $\{(z_1, z_2) \in \mathbb{C}^2 | z_1^2 + z_1^3 = z_2^2\}$. Observe that F is a bijection onto its image, except for the points 1 and -1 being mapped to $(0, 0)$. Then for any small enough neighborhood U of $(0, 0)$, its pre-image is disconnected and the connected components are not birational to U. However this map is clearly birational.

Recall that Theorem 1.3 is stated for locally birational holomorphic mapping, and in sight of the discussion on Pages 124–128 in [16] on algebraic fiber space, locally birational is not such a restrictive assumption at all.

For further introduction regarding pluripotential theory on Kähler manifolds we refer to [13]. A good reference for the geometric part is [16].

3 Stability for Non-degenerate Monge-Ampère Equations

We begin with stating Kołodziej's original stability theorem (cf. [12], Theorem 4.1). Note however that in the Kähler case we know that the weak solutions are actually continuous (cf. Section 2.4 in [11]).

Theorem 3.1. *Let ω be a Kähler form on a compact manifold X and A be a fixed positive constant. Then for any non-negative L^p-functions f and g with $p > 1$ satisfying $\int_X f\omega^n = \int_X g\omega^n = \int_X \omega^n$ and $\|f\|_p, \|g\|_p < A$, let ϕ and ψ in $PSH_\omega(X) \cap L^\infty(X)$ satisfy ${\omega_\phi}^n = f\omega^n$ and ${\omega_\psi}^n = g\omega^n$ respectively and be normalized by $max_X\{\phi - \psi\} = max_X\{\psi - \phi\}$. Then there exists $t_0 > 0$ depending on γ [2] such that for every $t < t_0$ if $\|f - g\|_{L^1} \leqslant \gamma(t)t^{n+3}, t < t_0$, then*

$$\|\phi - \psi\|_{L^\infty} \leqslant Ct,$$

for some C depending on $\gamma, \omega, X,$ and *A.*

[2] γ is defined as in the statement of Theorem 1.1.

Now one gets the following corollary (cf. [12], Corollary 4.4):

Corollary 3.2. *For any $\epsilon > 0$, there exists $c = c(\epsilon, p, c_0)$ with c_0 being the upper bound for L^p-norms of f and g such that*

$$\|\phi - \psi\|_\infty \leqslant c\|f - g\|_1^{\frac{1}{n+3+\epsilon}}$$

provided ϕ and ψ are normalized as in the above theorem.

Before proceeding further, we observe a small improvement on the stability exponent in the above corollary.

Note that in the definition of set $G = \{f < (1 - t^2)g\}$ in line 2 on Page 679 in [12], one can change t^2 to $\frac{t}{b}$ for a sufficiently large constant b, and the same argument still goes through except in the last step, one has to change the set E_4 in line 5 on Page 680 in [12] to E_s for some constant s depending only on b. Hence using Proposition 2.5, one can get rid of the constant in the term $\gamma(t)t^n$ which is affected by b. So $\|f - g\|_1 \leqslant \gamma(t)t^{n+2}$ implies $\|\phi - \psi\|_\infty \leqslant Ct$. In particular, the estimate in Corollary 3.2 holds with exponent $\frac{1}{n+2+\epsilon}$.

4 Adjustment to Degenerate Case

Now we begin to adjust Kołodziej's argument in [12] for the situation in Theorem 1.1. The argument (with the exponent $n+2$) can be repeated line-by-line except for two issues. One has to justify Comparison Principle in this setting and the inequality in line 4 on Page 679 in [12] for the case of merely bounded ω-plurisubharmonic functions. In the following, we treat them one by one.

4.1 Comparison Principle

In [3], the authors constructed decreasing smooth approximation for bounded functions plurisubharmonic with respect to a Kähler metric. Using this, they were able to prove Comparison Principle for any bounded functions plurisubharmonic with respect to a Kähler form.

Though the version we want would be for some background form $\omega \geqslant 0$, it would follow from their version of Comparison Principle because we can perturb ω by $\epsilon\omega_0$ with $\omega_0 > 0$ and any constant $\epsilon > 0$ [3]. Functions plurisubharmonic with respect to ω would still be plurisubharmonic with respect to $\omega + \epsilon\omega_0$. Letting $\epsilon \to 0$ in the conclusion of their version of Comparison Principle, we can conclude the following result, which deals with the first issue of running through Kołodziej's argument.

Theorem 4.1. *For $\phi, \psi \in PSH_\omega(X) \cap L^\infty(X)$, where X is a closed Kähler manifold and $\omega \geqslant 0$ is a big form over X, one has*

$$\int_{\{\phi<\psi\}} (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n \leqslant \int_{\{\phi<\psi\}} (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n.$$

Clearly, we only need $\omega \geqslant 0$ for this theorem in general.

[3] X being Kähler guarantees the existence of ω_0.

4.2 Inequalities for Mixed Measures

The first observation is that although Kołodziej (as in [12]) considered equations of the form

$$\omega_\psi^n = f\omega^n, \ \ \omega_\phi^n = g\omega^n,$$

the volume form ω^n would play no significant role in the proof. The essential step is to justify the following inequality.

For ϕ and ψ continuous ω-plurisubharminic functions with f and g being integrable functions on X, suppose (locally) we have

$$\omega_\psi^n \geqslant f\omega^n, \ \ \omega_\phi^n \geqslant g\omega^n,$$

then the following inequalities for mixed measures hold for any $k \in \{0, 1, \cdots, n\}$,

$$\omega_\psi^k \wedge \omega_\phi^{n-k} \geqslant f^{\frac{k}{n}} g^{\frac{n-k}{n}} \omega^n.$$

Now we need to generalize the above result for more general measures and moreover for merely bounded functions ϕ and ψ. The following theorem is essentially taken from [4].

Theorem 4.2. *Suppose the non-negative Borel measure Ω is dominated by capacity, and let ϕ and ψ be two bounded ω-plurisubharminc functions on a Kähler manifold. If the following inequalities hold*

$$\omega_\psi^n \geqslant f\Omega, \ \ \omega_\phi^n \geqslant g\Omega,$$

for some $f, g \in L^p(\Omega),\ p > 1$, then $\forall k \in \{0, 1, \cdots, n\}$,

$$\omega_\psi^k \wedge \omega_\phi^{n-k} \geqslant f^{\frac{k}{n}} g^{\frac{n-k}{n}} \Omega.$$

In [12] (Lemma 1.2), this result was proved under the assumption that both ϕ and ψ are continuous and $\Omega = \omega^n$. The result is clearly local, and it can be rephrased in the setting of a ball in $\mathbb{C}^n$. Then the argument makes use of approximation for which a solution for the Dirichlet problem with boundary data.

Since we deal with merely bounded functions, one cannot expect continuity on the boundary of the ball in general. Fortunately, as observed in [4], we can line-by-line follow the approximation argument from [12] whenever the measure on the right hand side is the Lebesgue measure. Indeed, approximation at the boundary will not converge uniformly towards discontinuous boundary data, but the sequence of approximation solutions is still decreasing. This implies convergence with respect to capacity from a classic result in [2], which is enough for the argument to go through. In the case when ω^n is changed to a general measure dominated by capacity one can not rely only on the argument from [12]. Meanwhile domination by capacity would force the measure Ω to vanish on pluripolar sets, hence one can use Theorem 1.3 in [4] to draw the conclusion. We refer to [4] for more detail.

5 Improvement on Stability Exponent

In this section, we improve the exponent from Kołodziej's Stability Theorem, i.e. Theorem 3.1. The strategy is to iterate the original argument, defining at each step a new function ρ (cf. line 14, Page 678 in [12]) and use the previous step to get estimates for $\|\rho-\psi\|_\infty$, which in turn can be used to choose the new set E (cf. line 1, Page 679 in [12]) in a "better" way. To the authors' knowledge such an iteration process is quite original and should be of some interest by itself.

To begin with, we fix a small constant $\epsilon > 0$. The argument is divided into the following three parts as follows.

The first part is the original argument quoted before with the improvement mentioned after Corollary 3.2, which is the starting point. In the sequel, the original argument will be often mentioned as Step 1.

The second part is the description of the iteration. Since the first step differs slightly from all the later ones, we give a detailed description of it below and then illustrate how to proceed further.

The mechanism is based on the fact that $\|f-g\|_1 \leqslant \gamma(t)t^\beta$ [4] would yield $\int_{\{\psi+kat<\phi\}}(\omega+\sqrt{-1}\partial\bar{\partial}\psi)^n \leqslant c_0t^n$ for some constants k and c_0. In the following, c_i's denote constants independent of the relevant quantities.

Then applying Proposition 2.5, we have a constant $k_1 > 0$ depending only on c_0 such that the set $\{\psi + ((k+k_1)a+2)\,t < \phi\}$ is empty (cf. Page 680 in [12]), and so $\|\phi-\psi\|_\infty \leqslant ((k+k_1)a+2)\,t$. Here a is the L^∞-bound of the solution.

Now we try to find β as small as possible for which this implication holds with uniform control on c_0 and larger k if needed. Note that from now on instead of ω^n, we use the measure Ω. It follows from the discussion above that Step 1 is not affected by that.

Assume $||f-g||_1 \leqslant \gamma(t)t^\beta$ with $t<1$, for some β to be chosen later. Then if $l = t^{\frac{\beta}{n+2}}$ with $\beta < n+2$, we have $\|f-g\|_1 \leqslant \gamma(l)l^{n+2}$, and from Step 1 we know that

$$\int_{E_2} g\Omega \leqslant \gamma(l)l^n, \tag{5.1}$$

where, as in Kołodziej's original argument, we denote $E_k := \{\psi < \phi - kat\}$. Hence

$$\int_{E_2} g\Omega \leqslant c_1 t^{\frac{\beta n}{n+2}}, \quad t \leqslant t_0 \tag{5.2}$$

recalling that $\gamma(t)$ is bounded and decreases to 0, as $t \searrow 0$.

Now we will find such a $\beta < n+2$. δ is a small positive constant to be fixed later. Consider the following "new" function, comparing with the function g_1 in Kołodziej's proof,

$$g_1(z) = \begin{cases} (1+\frac{t^\delta}{2})g(z), & z \in E_2 \\ c_2 g(z), & z \in X \setminus E_2, \end{cases}$$

[4] In the improved original proof as Step 1, $\beta = n+2$.

where $0 \leqslant c_2 \leqslant 1$ is chosen such that $\int_X g_1 \Omega = 1$. The constant $\frac{1}{2}$ is taken to assure that the integral over E_2 is less than 1. Note that in despite of the fact that the case t being small is of the main interest, when δ is also small the quantity t^δ cannot be controlled by a constant less than 1.

As in Step 1 we find a solution ρ to the equation $\omega_\rho^n = g_1 \omega^n, max_X \rho = 0$. Also, $\rho \geqslant -a$ for $\|g_1\|_p < 3A$ and renormalize ρ by adding a constant so that $max_X(\psi - \rho) = max_X(\rho - \psi)$, which can be done in a uniformly controlled way.

Now by Step 1, we have

$$\begin{aligned}
\|\rho - \psi\|_\infty &\leqslant c_3 \|g - g_1\|_1^{\frac{1}{n+2+\epsilon}} \\
&= c_3 \left(\int_{E_2} |g - g_1| \omega^n + \int_{X \setminus E_2} |g - g_1| \omega^n \right)^{\frac{1}{n+2+\epsilon}} \\
&= c_3 \left(\frac{t^\delta}{2} \int_{E_2} g \omega^n + (1 - c_2) \int_{X \setminus E_2} g \omega^n \right)^{\frac{1}{n+2+\epsilon}} \\
&= c_3 \left(\frac{t^\delta}{2} \int_{E_2} g \omega^n + \int_{X \setminus E_2} g \omega^n - \int_X g_1 \omega^n + (1 + \frac{t^\delta}{2}) \int_{E_2} g \omega^n \right)^{\frac{1}{n+2+\epsilon}} \\
&= c_3 \left(t^\delta \int_{E_2} g \omega^n \right)^{\frac{1}{n+2+\epsilon}} \leqslant c_4 t^{\frac{\delta + \frac{n\beta}{n+2}}{n+2+\epsilon}}.
\end{aligned}$$

If δ is sufficiently small and $\beta > n$ the last exponent is less than 1 [5] and we define $\alpha = 1 - \frac{\delta + \frac{\beta n}{n+2}}{n+2+\epsilon}$.

For $s = \frac{2c_4}{a} + 2$, we obtain the following chain of sets,

$$\begin{aligned}
E_s &= \{\psi + sat < \phi\} \\
&= \{(1 - \frac{1}{2}t^\alpha)(\psi + sat) < (1 - \frac{1}{2}t^\alpha)\phi\} \\
&\subset E := \{\psi < (1 - \frac{1}{2}t^\alpha)\phi + \frac{1}{2}t^\alpha \rho + \frac{1}{2}c_4 t - sat(1 - \frac{1}{2}t^\alpha)\} \\
&\subset \{\psi < (1 - \frac{1}{2}t^\alpha)\phi + \frac{1}{2}t^\alpha \psi + c_4 t - sat(1 - \frac{1}{2}t^\alpha)\} \\
&= \{\psi + \left(s - \frac{c_4}{a(1 - \frac{1}{2}t^\alpha)} \right) at < \phi\} \subset E_2,
\end{aligned} \tag{5.3}$$

where the term $\frac{1}{2}$, as before, is introduced in order to estimate the term $1 - \frac{1}{2}t^\alpha$ from below.

Consider the "new" set

$$G := \{f < \left(1 - \frac{t^{\alpha + 3\delta}}{8n2^{\frac{n-1}{n}}} \right) g\}.$$

[5] If $\beta < n$, we are already done.

In sight of $h(t) = (1+\frac{t^\delta}{2})^{\frac{1}{n}} - 1 - \frac{1}{4n2^{\frac{n-1}{n}}}t^{2\delta}$ increasing in $[0,1]$ and hence being non-negative there, we conclude as in Step 1 that on $E\setminus G$,

$$\begin{aligned}
(\omega_{\frac{1}{2}t^\alpha\rho+(1-\frac{1}{2}t^\alpha)\phi})^n &\geqslant \left((1-\frac{1}{2}t^\alpha)(1-\frac{t^{\alpha+3\delta}}{8n2^{\frac{n-1}{n}}})^{\frac{1}{n}} + (1+\frac{t^\delta}{2})^{\frac{1}{n}}\frac{1}{2}t^\alpha\right)^n g\Omega \\
&\geqslant \left((1-\frac{1}{2}t^\alpha)(1-\frac{t^{\alpha+3\delta}}{8n2^{\frac{n-1}{n}}}) + (1+\frac{1}{4n2^{\frac{n-1}{n}}}t^{2\delta})\frac{1}{2}t^\alpha\right)^n g\Omega \\
&\geqslant (1+\frac{t^{\alpha+2\delta}}{16n2^{\frac{n-1}{n}}})g\Omega.
\end{aligned} \tag{5.4}$$

As in Step 1, on G we have

$$\frac{t^{\alpha+3\delta}}{8n2^{\frac{n-1}{n}}}\int_G g\Omega \leqslant \int_G (g-f)\Omega \leqslant \gamma(t)t^\beta, \tag{5.5}$$

so using (5.4), (5.5) and Comparison Principle, we obtain

$$\left(1+\frac{t^{\alpha+2\delta}}{16n2^{\frac{n-1}{n}}}\right)\int_{E\setminus G} g\Omega \leqslant \int_E \omega^n_{(1-t^\alpha)\phi+t^\alpha\rho} \leqslant \int_E g\Omega \leqslant \int_{E\setminus G} g\Omega + c_5\gamma(t)t^{\beta-\alpha-3\delta}. \tag{5.6}$$

Finally, we obtain

$$\int_{E\setminus G} g\Omega \leqslant c_6\gamma(t)t^{\beta-2\alpha-5\delta},$$

$$\int_{E_s} g\Omega - 8n2^{\frac{n-1}{n}}t^{\beta-\alpha-3\delta} \leqslant \int_{E_s\setminus G} g\Omega \leqslant \int_{E\setminus G} g\Omega.$$

Combine them to arrive at

$$\int_{E_s} g\Omega \leqslant c_7\gamma(t)t^{\beta-2\alpha-5\delta}.$$

If $\beta - 2\alpha - 5\delta = n$, we can proceed as in Step 1 to get $\max(\phi-\psi) = \max(\psi-\phi) \leqslant ((s+s_1)a+2)\,t$ for some s_1 depending only on c_7 and $\|\phi-\psi\|_\infty \leqslant C(\epsilon)||f-g||_1^{\frac{1}{\beta+\epsilon}}$. Moreover,

$$\beta\left(1+\frac{\frac{2n}{n+2}}{n+2+\epsilon}\right) = n+2+5\delta - \frac{2\delta}{n+2+\epsilon}.$$

It is clear that if δ is sufficiently small, β is smaller than $n+2$. Hence we get an improvement.

Now in the third and last part we iterate the argument.

Consider $\|f-g\|_1 \leqslant \gamma(t)t^{\beta_{k+1}}$. As before, for $l = t^{\frac{\beta_{k+1}}{\beta_k}}$, $\int_{E_r} g\Omega \leqslant Ct^{\frac{n\beta_{k+1}}{\beta_k}}$, comparing with (5.1), where r is chosen so that we can use the estimate on appropriate sublevel set from the previous step.

Choosing δ_{k+1} small enough and repeating the above argument, one gets

$$\beta_{k+1} = n + 2\alpha_{k+1} + 5\delta_{k+1}.$$

where $\alpha_{k+1} = 1 - \frac{\delta_{k+1} + \frac{\beta_{k+1} n}{n+2}}{n+2+\epsilon}$. This yields

$$\beta_{k+1}\left(1 + \frac{2n}{\beta_k(\beta_k + \epsilon)}\right) = n + 2 + 5\delta_{k+1} - \frac{2\delta_{k+1}}{\beta_k + \epsilon}. \tag{5.7}$$

Choosing $\{\delta_k\}$ to be a sequence of sufficiently small constants decreasing to 0, one can obtain that $\{\beta_k\}$ is convergent as $n \geqslant 2$. Suppose A is the limit of the sequence $\{\beta_k\}$, one gets

$$A\left(1 + \frac{2n}{A(A+\epsilon)}\right) = n + 2$$

which implies

$$A = \frac{n + 2 - \epsilon + \sqrt{(n-2+\epsilon)^2 + 8\epsilon}}{2}.$$

Clearly, when $\epsilon \to 0^+$, $A \to n$, so β_k can be arbitrarily close to n for k big enough if we take small enough ϵ.

Hence we have proved Corollary 1.2.

Remark 5.1. *In the case when the measure Ω is dominated by capacity for $L^{p>1}$ functions but the constant χ is fixed, one can construct $Q(t)$ and afterwards $\kappa(t), \gamma(t)$ in such a way that $\gamma(t) \approx t^{\frac{n}{\chi}}$. Then one can use the same iteration technique as above with the exception that inequality (5.2) should be improved to*

$$\int_{E_2} g\Omega \leqslant Ct^{\frac{n\beta}{\chi(n+2)} + \frac{\beta n}{n+2}},$$

where the factor $t^{\frac{n\beta}{\chi(n+2)}}$ comes from the estimate of γ. The recurrence (5.7) now reads

$$\beta_{k+1}\left(1 + \frac{2n(1+\frac{1}{\chi})}{\beta_k(\beta_k + \frac{n}{\chi})}\right) = n + 2 + 5\delta_{k+1} - \frac{2\delta_{k+1}}{\beta_k + \frac{n}{\chi}}. \tag{5.8}$$

Again this is a convergent sequence and it can be seen that

$$\lim_{k\to\infty} \beta_k = n.$$

Hence the stability estimate in this case reads

$$\|\phi - \psi\|_\infty \leqslant c(\epsilon, c_0, X, \mu)\|f - g\|_{L^1(d\mu)}^{\frac{1}{n+\frac{n}{\chi}+\epsilon}}. \tag{5.9}$$

The following example shows that the exponent obtained in our corollary is fairly sharp.

Example 5.2. *Fix appropriate positive constants B, D such that $D < B$ and $2^{2\alpha}B < \log 2 + D$ for some fixed $\alpha \in (0,1)$. We have the function*

$$\widehat{\rho}(z) := \begin{cases} B\|z\|^{2\alpha}, & ||z|| \leqslant 1 \\ max\{B\|z\|^{2\alpha}, \log\|z\| + D\}, & 1 \leqslant \|z\| \leqslant 2 \\ \log\|z\| + D, & \|z\| \geqslant 2 \end{cases}$$

is plurisubharmonic in $\mathbb{C}^n$ and of logarithmic growth. One can smooth out $\widehat{\rho}$ so that the new function ρ is again of logarithmic growth, radially symmetric, smooth away from the origin and $\rho(z) = B\|z\|^{2\alpha}$ for $\|z\| \leqslant \frac{3}{4}$.

Via the standard inclusion

$$\mathbb{C}^n \ni z \longrightarrow [1 : z] \in \mathbb{CP}^n,$$

one identifies $\rho(z)$ with

$$\overline{\rho}([z_0 : z_1 : \cdots : z_n]) := \rho\left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right) - \frac{1}{2}\log\left(1 + \frac{||z||^2}{|z_0|^2}\right) \in PSH(\mathbb{CP}^n, \omega_{FS}),$$

where ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^n$, and the values of $\overline{\rho}$ on the hypersurface $\{z_0 = 0\}$ are understood as limits of values of $\overline{\rho}$ when z_0 approaches 0. It is clear that $\omega_{\overline{\rho}}^n = (dd^c\rho)^n$ in the chart $z_0 \neq 0$ and in fact one can ignore what happens on the hypersurface at infinity.

Now for a vector $h \in \mathbb{C}^n$ (with small length) one can define $\rho_h(z) := \rho(z+h)$ and similarly the corresponding $\overline{\rho}_h$. Note that when $\|h\| \to 0, \overline{\rho}_h \rightrightarrows \overline{\rho}$. One also has

$$B\|h\|^{2\alpha} \leqslant \|\overline{\rho}_h - \overline{\rho}\|_\infty. \tag{5.10}$$

The Monge-Ampère measures of $\overline{\rho}$ and $\overline{\rho}_h$ are smooth except at the origin and $-h$ respectively, and belong to $L^p(\omega_{FS}^n)$, for some $p > 1$ depending on α.

Clearly $\int_{\mathbb{CP}^n} |\omega_{\overline{\rho}}^n - \omega_{\overline{\rho}_h}^n| = \int_{\mathbb{C}^n} |(dd^c\rho)^n - (dd^c\rho_h)^n|$. To estimate the term on the right hand side, we divide $\mathbb{C}^n$ into three pieces to estimate the total integral:

$$\int_{\mathbb{C}^n} |(dd^c\rho)^n - (dd^c\rho_h)^n| = \int_{\{\|z\| \leqslant 2\|h\|\}} |\cdot| + \int_{\{2\|h\| < \|z\| \leqslant \frac{1}{2}\}} |\cdot| + \int_{\{\|z\| > \frac{1}{2}\}} |\cdot|.$$

Using the fact that ρ and ρ_h are smooth functions in a neighbourhood of $\{\|z\| > \frac{1}{2}\}$, one can easily estimate the last term by $C_0\|h\|$ for a constant C_0 independent of h. For the first two terms, we have $(dd^c\rho)^n = B^n\|z\|^{2n(\alpha-1)}$ and $(dd^c\rho_h)^n = B^n\|z+h\|^{2n(\alpha-1)}$, where the standard Euclidean measure is omitted.

Now for the first term, we use a computation in [15].

$$\begin{aligned} \int_{\{\|z\| \leqslant 2\|h\|\}} |(dd^c\rho)^n - (dd^c\rho_h)^n| &= B^n \int_{\{\|z\| \leqslant 2\|h\|\}} |\|z\|^{2n(\alpha-1)} - \|z+h\|^{2n(\alpha-1)}| \\ &\leqslant 2B^n \int_{\{\|z\| \leqslant 3\|h\|\}} \|z\|^{2n(\alpha-1)} = C_1\|h\|^{2n\alpha}. \end{aligned}$$

For the second term, we have

$$\begin{aligned}\int_{\{2\|h\|\leqslant\|z\|\leqslant\frac{1}{2}\}} |(dd^c\rho)^n-(dd^c\rho_h)^n| &= B^n\int_{\{2\|h\|\leqslant\|z\|\leqslant\frac{1}{2}\}} |\|z\|^{2n(\alpha-1)}-\|z+h\|^{2n(\alpha-1)}| \\ &\leqslant B^n\int_{\{2\|h\|\leqslant\|z\|\}}\int_0^1 |\nabla\|z+th\|^{2n(\alpha-1)}|\cdot\|h\|dt \\ &\leqslant C_2\|h\|\int_{\{\|h\|\leqslant\|z\|\}}\|z\|^{2n(\alpha-1)-1}\leqslant C_3||h||^{2n\alpha},\end{aligned}$$

provided $\alpha<\frac{1}{2n}$ *so that the integral is finite.*

Finally we conclude for small $\|h\|$,

$$\int_{\mathbb{P}^n}|\omega_\rho^n-\omega_{\rho_h}^n|\leqslant C_1\|h\|^{2n\alpha}+C_3\|h\|\leqslant C_4\|h\|^{2n\alpha}. \tag{5.11}$$

Suppose that we have a stability estimate $\|\phi-\psi\|_\infty\leqslant C_5\|f-g\|_1^{\frac{1}{m}}$. *Then combining with (5.10) and (5.11), one gets*

$$\|h\|^{2\alpha}\leqslant C_6(\|h\|^{2n\alpha})^{\frac{1}{m}},\alpha\in(0,\frac{1}{2n}).$$

As $\|h\|\to 0$, *this can hold only if* $m\geqslant n$. *Corollary 1.2 gets us as close to* n *as possible. It remains interesting to see whether* n *itself is allowed as the exponent.*

Remark 5.3. *In [8], they showed a stability estimate of another type. In the setting as above with* Ω *being* ω^n,

$$\|\phi-\psi\|_\infty\leqslant c(\epsilon,c_0,\omega)\|\phi-\psi\|_{L^2(\omega^n)}^{\frac{2}{nq+2+\epsilon}} \tag{5.12}$$

where c_0 *is a constant that controls* L^p*-norms of Monge-Ampère measures of* ϕ *and* ψ. *Using the same reasoning as in [8], one can show more generally that*

$$\|\phi-\psi\|_\infty\leqslant c(\epsilon,c_0,\omega)\|\phi-\psi\|_{L^s(\omega^n)}^{\frac{s}{nq+s+\epsilon}},\forall s>0. \tag{5.13}$$

Using the same example and similar estimates one can show that this exponent is also sharp, provided that $p<2$ *and* $s>\frac{2np}{2-p}$ [6]. *It is, however, very likely that these exponents are sharp in general.*

6 Continuity of Solutions for Degeneration from Locally Birational Map

In this section we give more detail for the proof of the continuity statement in Theorem 1.3. Recall that in our setting there exists a holomorphic mapping

[6] The reason for these restrictions is that the second integral we estimate as in the example would be divergent otherwise.

$F : X \to \mathbb{CP}^N$ such that $\omega = F^*\omega_M$ with ω_M being a Kähler form in the projective space. Note that F by assumption is locally birational.

Consider the image $Y = F(X)$. By the Proper Mapping Theorem Y is a (singular in general) subvariety in $\mathbb{CP}^N$. It is also clear that Y is irreducible and a locally irreducible variety where the latter follows from the local birationality assumption.

Recall that an upper semi-continuous function u on a singular variety W is called weakly plurisubharmonic if for every holomorphic disk $f : \Delta \to W$, the function $f^*u := u \circ f$ is a subharmonic function (see [9]). Theorem 5.31 in that paper states (in a much more general situation of Stein spaces) that any such function u can be extended locally to a usual plurisubharmonic function in the ambient space, i.e. for every $x \in Y$ there exists a small Euclidean ball B in $\mathbb{CP}^N$, centered at x and a function $v \in PSH(B)$, such that $v|_{B\cap Y} = u$.

The continuity is proved by contradiction argument. Suppose ϕ is a discontinuous solution of the Monge-Ampère equation under study. Since we already know that ϕ is bounded, we can also assume it is positive by adding a uniform constant.

Define $d := \sup(\phi - \phi_*) > 0$, where ϕ_* denotes the lower semicontinuous regularization of ϕ. Note that the supremum is attained, and in the closed set

$$E := \{\phi - \phi_* = d\},$$

there exists a point x_0 such that $\phi(x_0) = min_E\phi$.

By assumption there exist analytic sets $Z \subset X$ and $W \subset Y = F(X)$ such that $F|_{X\setminus Z} \to Y \setminus W$ is a biholomorphism and moreover $S := \{\omega^n = 0\} \subset Z$.

There are two cases for x_0 in S or not. In the case of $x_0 \notin S$, ω is strictly positive in a small ball centered at x_0 and repeating the argument from Section 2.4 in [11], we obtain a contradiction. So from now on we assume that $x_0 \in S$.

Consider $F(x_0) = z$ and take a neighborhood U of z in Y, such that each component of its pre-image is birational to it. Choose the one, $\overline{U}$, containing x_0. For the rest of the argument we restrict ourselves to

$$F : \overline{U} \to U.$$

Consider the push-forward of ϕ on U defined below

$$(F_*\phi)(z) := \begin{cases} \phi(w), & z \in U \setminus W, w \in \overline{U} \setminus Z, F(w) = z \\ \limsup_{\zeta \in \overline{U}\setminus Z, F(\zeta)\to z} \phi(\zeta), & otherwise \end{cases}$$

and a local potential η for the Kähler form ω_M on $U \cap \mathbb{CP}^N$. Eventually, we are going to choose η properly, but at this moment, that is not necessary.

The following lemma is important.

Lemma 6.1. *$\eta + F_*\phi$ is weakly plurisubharmonic on U.*

Proof. Weak plurisubharmonicity is a local property, so it is enough to check it in a small neighborhood of any point in U.

For regular points of U, this is evident for the (open) part which is biholomorphic to the (open) part in $\overline{U}$ because biholomorphism preserves plurisubharmonicity. Then the plurisubharmonicity for the regular part would follow from the semi-continuity of the function and the classic unique extension result for plurisubharmonic functions through subvarieties (see, for example, [6], Chapter I, (5.24) Theorem).

However at singular points of U one might a priori run into trouble as the example of a double point shows. Indeed, take the double point variety as in Example 2.8. Fix any subhamonic function w on $\mathbb{C}$ satisfying $w(-1) >> w(1)$. Now the value of the pushforward at $(0, 0)$ equals $w(-1) = max\{w(-1), w(1)\}$. If this pushforward were weakly subharmonic then on a small disk centered at 1 the function

$$\tilde{w}(t) := \begin{cases} w(t), & t \neq 1 \\ w(-1), & t = 1 \end{cases}$$

would be subharmonic itself. But $\tilde{w}$ does not satisfy the sub-mean value inequality at 1. Hence the push-forward of a subharmonic function w on $\mathbb{C}$ cannot be weakly subharmonic on the image if $w(1) \neq w(-1)$. The assumption of local birationality is mainly forced to rule out situations like this.

Observe that local birationality forces the analytic set Y to be locally irreducible. Then our lemma would follow from a classic theorem (see [5], Theorem 1.7) stating that on a locally irreducible variety Y, for a locally bounded plurisubharmonic function w defined on $\operatorname{Reg} Y$, i.e. the regular part of Y, the extension via limsup procedure $w(z) := \limsup_{\zeta \to z, \zeta \in \operatorname{Reg} Y} w(\zeta)$ is indeed a weak plurisubharmonic function.

A more direct argument can also be found in [19], Pages 194–197, where one goes through the definition of weak plurisubharmonic function quoted before using desingularization.

□

Remark 6.2. *In fact, the key idea is that the local birationality assumption guarantees that the pre-image of each point in $F(X)$ is a connect variety (from topological consideration) which could be just a point, when restricted to the component $\overline{U}$. Then along that variety, $\eta + \phi$ is plurisubharmonic and so has to be a constant. This is essentially why one has this natural push-forward construction preserving plurisubharmonicity.*

Now ω_M is the Kähler metric which defines ω, i.e. $\omega = F^*\omega_M$. We need to choose a good η, the local potential of ω_M near $z = F(x_0)$ in $\mathbb{CP}^N$.

We proceed exactly as in [11]. In a local coordinate ball B'' centered at z, choose a local potential ρ which is clearly strictly plurisubharmonic and smooth.

It can be expanded as

$$\begin{aligned}\rho(z+h) &= \rho(z) + 2\Re\left(\sum_{j=1}^{n} a_j h_j + \sum_{j,k=1}^{n} b_{jk} h_j h_k\right) + \sum_{j,k=1}^{n} c_{j\bar{k}} h_j \bar{h}_k + o(|h|^2)\\ &= \Re\left(P(h)\right) + H(h) + o(|h|^2),\end{aligned}$$

where h is the coordinate system, P is a complex polynomial in h and H is the complex Hessian at z.

Exactly as in [11], Lemma 2.3.1, $\eta := \rho - \Re P(\cdot - z)$ is also a local potential for ω_M, with the additional property that η has a strict local minimum at z using that at this point that H is strictly positive definite. This means that for a smaller ball, which after possible shrinking we still denote by B'', $\inf_{\partial B''} \eta > \eta(z) + b''$ for some positive constant b''. The ball B'' can be any ball centered at z. By adding a constant if necessary one can further assume that $\eta(z) > 0$.

Now by using the extension result in [9], we have an even smaller Euclidean ball B' in B'' centered at z and a function $\psi \in PSH(B')$, such that

$$\psi|_{U\cap B'} = \eta + F_*\phi.$$

On a neighborhood of a slightly smaller ball B (avoiding the boundary for convolution), ψ can be approximated by a sequence of smooth plurisubharmonic functions ψ_j decreasing towards it. This can be achieved using classic convolution construction (see, for example, [6], Chapter I, (5.5) Theorem). And one still has that $\inf_{\partial B} \eta > \eta(z) + b$ for some constant $b > 0$ from our choice of η because the η chosen before has its value growing from its minimum at the center.

Now we pull back the ball and the approximation functions to X. Let $V := F^{-1}(B \cap U)$ and $u_j := F^*(\psi_j)$, which are defined only on small neighborhood of x_0, V and still continuous plurisubharmonic functions on V decreasing towards $u := F^*\eta + \phi$. [7]

Note that V would no longer be a Euclidean domain anymore, i.e. it can not be contained in $\mathbb{C}^n$. Nevertheless $F^*\eta$ is a global potential of ω on this set from construction. This is the essential difference between this case and Kołodziej's Kähler case.

Next we prove a lemma which is essentially contained in [11], Section 2.4, for the sake of completeness.

Lemma 6.3. *There exist $a_0 > 0, t > 1$ such that the sets*

$$W(j,c) := \{tu + d - a_0 + c < u_j\}$$

are non-empty and relatively compact in V for every constant c belonging to an interval which does not depend on $j > j_0$.

[7] There is no need to worry about the boundary issue from convolution construction in U because all we need is a smooth decreasing approximation for $F^*\eta + \phi$ for a neighborhood centered at x_0 from pulling back a neighborhood of $z = F(x_0)$ in $\mathbb{CP}^N$. There is some related discussion in [19], Pages 182–183.

Proof. Define $E(0) := \{u - u_* = d\} \cap \overline{V} = E \cap \overline{V}$, and also the sets $E(a) := E := \{u - u_* \geqslant d - a\} \cap \overline{V}$. They are all closed and $E(a)$ decreases towards $E(0)$ as $a \searrow 0$.

Define $c(a) := \phi(x_0) - \min_{E_a} \phi$. We have that $\limsup_{a \to 0^+} c(a) \leqslant 0$ because otherwise we would get a contradiction from the definition of d. Hence we can have

$$c(a) < \frac{1}{3}b \quad \text{for} \quad 0 < a < a_0 < min(\frac{1}{3}b, d).$$

Let $A := u(x_0)$. Note that $A > d$ since the potential is greater than 0 at x_0, and ϕ, a function positive everywhere, has to be greater than d at x_0. One can choose $t > 1$, such that it satisfies

$$(t-1)(A-d) < a_0 < (t-1)(A - d + \frac{2}{3}b). \tag{6.1}$$

Now if $y \in \partial V \cap E(a_0)$, one gets

$$u_*(y) \geqslant \eta(F(x_0)) + b + F^* F_* \phi(x_0) \geqslant A - d + \frac{2}{3}b.$$

Hence $u(y) \leqslant u_*(y) + d < tu_*(y) + d - a_0$. Note that this inequality still holds in a neighbourhood of $\partial V \cap E(a_0)$. Taking another neighborhood relatively compact in the first and applying Hartogs type of argument, one obtains

$$u_j < tu(y) + d - a_0, \quad \forall j > j_1.$$

For the rest part of ∂V, the same inequality holds if we take big enough j_1 and the proof is even simpler, since $u - u_*$ is less than $d - a_0$ there. This proves the relative compactness of $W(j, c)$ in V.

Note that from the left part of (6.1), one has $(t-1)u_*(x_0) < a_0$, and so

$$tu_*(x_0) < u(x_0) - d - a_1 + a_0 < u_j(x_0) - d - a_1 + a_0$$

for some constant $a_1 > 0$. This implies that the sets $W(j, c)$ for $c \in (0, a_1)$ contain some points near x_0, and so they are non-empty.

The proof of the lemma is thus finished.

□

Now we are going to apply the version of Lemma 2.3.1 from [11] for our case. There is quite something to take care of because we are no longer considering a Euclidean domain. It can be seen that the original argument in [11] can be carried through line by line in sight of the following observations (as pointed out in [19], Chapter 5).

1. The classic definition of relative capacity for Euclidean domains can be generalized naturally to the current situation, preserving a lot of properties. There are plenty of references on this topic, for example, [13].

2. There is no need to involve relative extremal function even in Kołodziej's original proof. When drawing conclusion on relative capacity, one can instead go through the estimation for any admissible plurisubharmonic function in Definition 2.2. This idea also appears in the proof of Proposition 2.5.

3. Comparison Principle can still be applied as discussed in Section 4.

Running through Kołodziej's argument, one can bound the relative capacity, $Cap(W(j, a_1), V)$ from below by a uniform positive constant.

On the other hand, $W(j, a_1) \subset \{u + (d - a_0 + a_1) < u_j\}$, and so there is contradiction from the fact that the decreasing sequence $\{u_j\}$ actually converges towards u with respect to capacity.

Hence one concludes that ϕ is continuous.

Remark 6.4. *As we have seen the argument cannot be applied in the case of a birational map. The local birational assumption is needed to allow the push-forward construction. Fortunately, this assumption holds for most cases with geometric interest as pointed out in Section 2.*

7 Final Remarks

Complex Monge-Ampère equations are of great interest in various aspects of mathematics. In [19], the following version of the Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = e^u \Omega$$

is studied. Of course the degenerate case as in the setting of Theorem 1.3 is the main focus.

Using the argument in [12], we observe that the main result in this work would also apply there. More precisely, one has the following theorem. [8]

Theorem 7.1. *Let ω be a big form and u_1 and u_2 be ω-plurisubharmonic solutions for the following Monge-Ampère equations:*

$$\omega_{u_1}^n = e^{u_1}\Omega_1, \quad \omega_{u_2}^n = e^{u_2}\Omega_2,$$

where Ω_1 and Ω_2 are smooth volume forms. Then for any $\epsilon > 0$, there exist positive constants t_0 and C depending only on ϵ, (X, ω) and $L^{p>1}$-norms of Ω_1 and Ω_2, such that if

$$\int_X |\Omega_1 - \Omega_2| \leqslant \gamma(t)t^{n+\epsilon},$$

then one concludes

$$\|u_1 - u_2\|_\infty \leqslant Ct$$

for $t < t_0$.

[8] This theorem can be stated in a more general form as in [12].

Proof. Since Comparison Principle with respect to a big form is available and by Theorem 1.1, we have stability with exponent $n+\epsilon$ the proof is entirely the same as in Theorem 5.2 in [12].

$\square$

The following problems are related to the results in [14] and [8], stating that when ω is a Kähler form on a compact Kähler manifold, the solutions of

$$\omega_\phi^n = f\omega^n, \quad f \in L^p(\omega^n) \text{ for } p > 1,$$

are Hölder continuous. In general the Hölder exponent depends on the manifold X, n and p ([14]). Under the additional assumption that X is *homogeneous*, i.e. the automorphism group $\mathrm{Aut}(X)$ acts transitively the exponent is independent of X and is not less that $\frac{2}{nq+2}, q = \frac{p}{p-1}$ (see [8]). One can further ask the questions below of various interest.

1. Is the solution continuous when ω is semi-positive and big in general? If this is the case, can one expect Hölder continuity?
2. Does the Hölder exponent on a general manifold really depend on the manifold? In the corresponding result for the flat case in [10], the Hölder exponent is uniform and independent of the domain. Moreover the proof in [14] strongly depends on a regularization procedure for ω-plurisubharmonic functions, which consists of patching local regularizations, and this is the point where the geometry of the manifold influences the exponent. In particular are there other regularization procedures of a more global nature that are not so affected by the local geometry?
3. Is the exponent for the homogeneous case sharp? Note that for the flat case in [10] there is also a gap between the exponent given there $\frac{2}{qn+1}$ and the exponent $\frac{2}{qn}$, for which an example is shown.
4. It is interesting to compare the stability results we have and the one in [8]. In particular, is the stability exponent in [8] sharp in general?
5. It would be very interesting to achieve Hölder continuity to more singular measures. One possible application of such a result would be a criterion for Hölder continuity of the Siciak Extremal Function of certain compact sets in $\mathbb{C}^n$ (see [13] for more discussion). Such a property is very important from pluripotential theory point of view. So one has to study the equilibrium measure of the compact sets. The problem is that such measures are singular with respect to the Lebesgue measure, while [14] and [8] rely strongly on the smoothness of ω^n. However, as argument here shows, some argument can be adjusted to singular measures as well.

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