SCALAR CURVATURE BEHAVIOR FOR FINITE TIME SINGULARITY OF KÄHLER-RICCI FLOW

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1. Introduction

Ricci flow, since the debut in the famous original work [4] by R. Hamilton, has been one of the major driving forces for the development of Geometric Analysis in the past decades. Its astonishing power is best demonstrated by the breakthrough in solving Poincaré Conjecture and Geometrization Program. For this amazing story, we refer to [1], [7], [10] and the references therein. Meanwhile, Kähler-Ricci flow, which is Ricci flow with initial metric being Kähler, has shown some of its own characters coming from the natural relation with complex Monge-Ampère equation and many interesting Algebraic Geometric objects. G. Tian's Program, as described in [14] or [15], has illustrated the direction to further improve people's understanding in many classic topics of great importance by Kähler-Ricci flow, for example, the Minimal Model Program in Algebraic Geometry.

In the current work, we give some very general discussion on Kähler-Ricci flows over closed manifolds. The closed manifold under consideration is denoted by X with $dim_{\mathbb{C}}X = n$. The computation would be done for the following version of Kähler-Ricci flow,

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t,\ \widetilde{\omega}_0 = \omega_0. \tag{1.1}$$

where ω_0 is any Kähler metric on X. The special feature of this version as shown in [15] and fully discussed in [17] is rather superficial for this work.

The short time existence of the flow is known from either R. Hamilton's general existence result on Ricci flow in [4] or the fact that Kähler-Ricci flow is indeed parabolic when considered as a flow in a properly chosen infinite dimensional space.

In sight of the optimal existence result for Kähler-Ricci flow as in [2] or [15], we know the classic solution of (1.1) exists exactly as long as the cohomology class $[\widetilde{\omega}_t]$ from formal computation remains to be Kähler. The actual meaning will be explained later.

Inevitably, it comes down to analyzing the behavior of the t-slice metric solution when time t approaches the (possibly infinite) singular time from cohomology consideration. In this work, we focus on the case when the flow singularity happens at some finite time. Now we state the main results.

Theorem 1.1. *The Kähler-Ricci flow (1.1) either exists for all time, or the scalar curvature blows up at some finite time (of singularity), i.e.*

$$sup_{X\times[0,T)}|R(\widetilde{\omega}_t)| = +\infty$$

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where T is the finite time of singularity and $R(\cdot)$ is the scalar curvature for the corresponding metric.

The possible blow-up of scalar curvature would be from above because of the known lower bound of scalar curvature (as in [13], for example). Let's also point out that the statements of this theorem and the next theorem still hold for the other two common versions of Kähler-Ricci flow with great individual interests,

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t), \ \widetilde{\omega}_0 = \omega_0,$$

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) + \widetilde{\omega}_t, \ \widetilde{\omega}_0 = \omega_0.$$

This can be justified by simple rescaling of time and metric to transform these flows from one to another.

Remark 1.2. Some fundamental results regarding the finite time blow-up of Ricci flow are known for quite a while. More precisely, it's known that curvature operator blows up from R. Hamilton's work [5] and Ricci curvature blows up from N. Sesum's work [11]. The later one also gives the blow-up of scalar curvature in real dimension 3 case.

The next result, which provides some control of the blow-up rate, needs an extra assumption described in the following with natural background from Algebraic Geometry. The motivation is the semi-ampleness of the cohomology limit at the singular time. It is of quite some interest in Algebraic Geometry as explained in [15], for example.

Still denote the finite singular time by T and use $[\omega_T]$ to represent the cohomology limit of the flow as $t \to T$ whose meaning would be very clear from the discussion in Section 2. We assume the existence of a holomorphic map

$$F : X \longrightarrow Y$$

where Y is an analytic variety smooth near the image $F(X)$ and there is a Kähler metric, $\omega_{_M}$ in a neighborhood of $F(X)$ such that $[\omega_T] = [F^*\omega_{_M}]$.

The most natural way to come up with such a picture is to actually generate a map F from the class $[\omega_T]$. Of course, this would force some conditions (of Algebraic Geometry flavor) on $[\omega_T]$. Let's point out that it would be the case when X is an algebraic manifold and the initial class $[\omega_0] \in H^{1,1}(X, \mathbb{C}) \cap H^2(X; \mathbb{Q})$ by the classic Rationality Theorem (as stated in [8]).

Theorem 1.3. *In the above setting, for the Kähler-Ricci flow (1.1) with finite time singularity at T,*

$$R(\widetilde{\omega}_t) \leqslant \frac{C}{(T-t)^2}$$

where C is a positive constant depending on the specific flow.

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2. Proof of Theorem 1.1

The proof of such a result is usually by contradiction. Let's assume **the scalar curvature stays uniformly bounded along the flow (1.1) with finite time singularity at** T. One then makes use of J. Song and G. Tian's computation for parabolic Schwarz Lemma (as in [13]) and some basic computations on the Kähler-Ricci flow to get some uniform control of the flow metric. The contradiction then comes from the general result on the existence of Kähler-Ricci flow and the numerical characterization of Kähler cone for closed Kähler manifolds by J. Demailly and M. Paun. The rest of this section contains the detailed argument. Before that, we briefly introduce the standard machinery to reduce the Kähler-Ricci flow to the level of scalar function flow as in [15] and give some explanations to some statements in Introduction.

Define $\omega_t := \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ where $[\omega_\infty] = K_X$, the canonical class of X. In practice, one chooses

$$\omega_\infty = -\mathrm{Ric}(\Omega) := \sqrt{-1}\partial\bar\partial \log \frac{\Omega}{Vol_E}$$

for some smooth volume form Ω over X where Vol_E is the Euclidean volume form with respect to local holomorphic coordinate system of X. Obviously the choice of coordinates won't affect the form, $-\mathrm{Ric}(\Omega)$.

Formally, it is clear that $[\omega_t] = [\widetilde{\omega}_t]$. Now set $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar\partial u$, and then one has the following parabolic evolution equation for u over space-time,

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar\partial u)^n}{\Omega} - u, \ u(\cdot, 0) = 0, \tag{2.1}$$

which is equivalent to the original Kähler-Ricci flow (1.1).

It is time to quote the following optimal existence result of Kähler-Ricci flow (as in [2] and [15]) mentioned in Introduction.

Proposition 2.1. (1.1) (or (2.1) equivalently) exists as long as $[\omega_t]$ remains Kähler, i.e. the solution is for the time interval $[0, T)$ where

$$T = sup\{t | \, [\omega_t] \text{ is Kähler.}\}.$$

The appearance of finite time singularity means $[\omega_T]$, which is the cohomology limit mentioned in Introduction, is on the boundary of the (open) Kähler cone of X in $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$, and thus no longer Kähler. Clearly it is "numerically effective" using the natural generalization of the notion from Algebraic Geometry. From now on, we focus on those flows existing only for some finite interval $[0, T)$.

The C's below might stand for different positive constants at places. In case that this might cause unnecessary confusion, lower indices are added to tell them apart. These constants might well depend on the specific flow, for example, the finite singular time T.

The argument is divided into the following three steps.

• **Step 1. Volume Uniform Bound**

With the uniform bound on scalar curvature in $[0,T)$, we can easily derive the uniform control on the volume form along the flow, using the following evolution equation of volume form,

$$\begin{aligned}\frac{\partial \widetilde{\omega}_t^n}{\partial t} &= n\frac{\partial \widetilde{\omega}_t}{\partial t}\wedge \widetilde{\omega}_t^{n-1}\\ &= n(-\mathrm{Ric}(\widetilde{\omega}_t)-\widetilde{\omega}_t)\wedge \widetilde{\omega}_t^{n-1}\\ &= (-R-n)\widetilde{\omega}_t^n.\end{aligned}$$

Since $\widetilde{\omega}_t^n = e^{\frac{\partial u}{\partial t}+u}\Omega$, this actually tells that

$$|\frac{\partial u}{\partial t}+u|\leqslant C.$$

Remark 2.2. Instead of the assumption on scalar curvature, one can also directly assume positive lower bound for the volume form or equivalently, $\frac{\partial u}{\partial t}\geqslant -C$ since we are considering the finite time singularity case [1]. This simple observation actually brings up a very intuitive analytic understanding of Theorem 1.1, i.e. the flow (2.1) can be stopped at some finite time only because the term in log is tending to 0, i.e. no uniform lower bound.

• Step 2. Metric Estimate

We begin with the inequality from parabolic Schwarz Lemma. Throughout this note, the Laplacian Δ without lower index, is always with respect to the changing metric along the flow, $\widetilde{\omega}_t$.

Set $\phi = \langle \widetilde{\omega}_t, \omega_0\rangle$ which is obviously positive for $t\in[0,T)$. Using computation for (1.1) in [13], one has

$$(\frac{\partial}{\partial t}-\Delta)\mathrm{log}\phi \leqslant C_1\phi+1, \tag{2.2}$$

where C_1 is a positive constant depending on the bisectional curvature of ω_0. It's quite irrelevant here that ω_0 is the initial metric for the Kähler-Ricci flow. In fact, it doesn't even have to be a metric over X which is an interesting part of this computation as indicated in [13], which is useful for the proof of Theorem 1.3.

Applying Maximum Principle to (2.1) gives $u\leqslant C$. Taking t-derivative for (2.1) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle \widetilde{\omega}_t, \omega_0-\omega_\infty\rangle - \frac{\partial u}{\partial t},$$

where $\langle\cdot,\cdot\rangle$ means taking trace of the right term with respect to the left (metric) term. This equation can be reformulated into the following two equations,

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) &= \Delta\left(e^t\frac{\partial u}{\partial t}\right) - \langle \widetilde{\omega}_t, \omega_0-\omega_\infty\rangle,\\ \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right) &= \Delta\left(\frac{\partial u}{\partial t}+u\right) - n + \langle \widetilde{\omega}_t, \omega_\infty\rangle.\end{aligned}$$

Their difference gives

$$\frac{\partial}{\partial t}\left((e^t-1)\frac{\partial u}{\partial t}-u\right) = \Delta\left((e^t-1)\frac{\partial u}{\partial t}-u\right) + n - \langle \widetilde{\omega}_t, \omega_0\rangle. \tag{2.3}$$

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[1]The upper bounds for u and $\frac{\partial u}{\partial t}$ are available in general from later discussion.

By Maximum Principle, this gives

$$(e^t-1)\frac{\partial u}{\partial t}-u-nt \leqslant C,$$

which, together with the upper bound of u and local bound for $\frac{\partial u}{\partial t}$ near $t=0$, would provide

$$\frac{\partial u}{\partial t} \leqslant C.$$

The upper bounds on $\frac{\partial u}{\partial t}$ and u together with $|\frac{\partial u}{\partial t}+u| \leqslant C$ from volume control give the uniform (lower) bounds on $\frac{\partial u}{\partial t}$ and u.

Multiplying (2.3) by a large enough constant $C_2 > C_1+1$ and combining it with (2.2), one arrives at

$$(\frac{\partial}{\partial t}-\Delta)\left(\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t}-u\right)\right) \leqslant nC_2+1-(C_2-C_1)\phi \tag{2.4}$$
$$\leqslant C-\phi.$$

Now we apply Maximum Principle for the term $\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t}-u\right)$. Considering the place where it achieves maximum value, one has

$$\phi \leqslant C,$$

and so by the bounds of u and $\frac{\partial u}{\partial t}$,

$$\log\phi + C_2\left((e^t-1)\frac{\partial u}{\partial t}-u\right) \leqslant C.$$

Hence we conclude $\phi = \langle\widetilde{\omega}_t, \omega_0\rangle \leqslant C$ using again the bounds on $\frac{\partial u}{\partial t}$ and u. This trace bound, together with volume bound $\widetilde{\omega}_t^n \leqslant C\omega_0^n$, provide the uniform bound of $\widetilde{\omega}_t$ as metric, i.e.

$$C^{-1}\omega_0 \leqslant \widetilde{\omega}_t \leqslant C\omega_0.$$

The easiest way to see this is to diagonalize $\widetilde{\omega}_t$ with respect to ω_0 and deduce the uniform control of the eigenvalues from the above trace and volume bounds.

• **Step 3. Contradiction**

The metric (lower) bound makes sure that for any fixed analytic variety in X, the integral of the proper power of $\widetilde{\omega}_t$ is bounded away from 0, and so the limiting class $[\omega_T]$ would have positive intersection with any analytic variety by taking the cohomology limit. Thus by Theorem 4.1 in [3], we conclude that $[\omega_T]$ is actually Kähler. This contradicts with the assumption of finite time singularity at T in sight of Proposition 2.1.

Hence we have finished the proof of Theorem 1.1.

Remark 2.3. In sight of this numerical characterization of Kähler cone for any general closed Kähler manifold by J. Demailly and M. Paun, the blow-up of curvature operator or Ricci curvature in closed Kähler manifold case is fairly obvious. The situation of scalar curvature is the first non-trivial statement.

Also, if X is an algebraic manifold, then we can apply the more classic characterization of ampleness by S. Kleiman as in [6] to draw the contradiction (see [9] for the complete story in Algebraic Geometry).

3. Proof of Theorem 1.3

It would be more satisfying to gain some control for the blow-up of scalar curvature at finite time of singularity. Of course, this is also important for further analysis of the singularity. That is what we are going to do in this section by mainly following the argument in [18]. It is also divided into three steps.

• **Step 1. 0-th Order Estimates**

$u \leqslant C$ is directly from (2.1). Recall that t-derivative of (2.1) is

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle\widetilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \frac{\partial u}{\partial t},$$

which has the following variations,

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) = \Delta\left(e^t\frac{\partial u}{\partial t}\right) - \langle\widetilde{\omega}_t, \omega_0 - \omega_\infty\rangle,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) = \Delta\left(\frac{\partial u}{\partial t} + u\right) - n + \langle\widetilde{\omega}_t, \omega_\infty\rangle. \tag{3.1}$$

A proper linear combination of these equations provides the following "finite time version" of the second equation above [2],

$$\frac{\partial}{\partial t}\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1-e^{t-T})\frac{\partial u}{\partial t} + u\right) - n + \langle\widetilde{\omega}_t, \omega_T\rangle.$$

As before, the difference of the original two equations gives

$$\frac{\partial}{\partial t}\left((1-e^{t})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1-e^{t})\frac{\partial u}{\partial t} + u\right) - n + \langle\widetilde{\omega}_t, \omega_0\rangle,$$

which implies the "essential decreasing" of metric potential along the flow, i.e.

$$\frac{\partial u}{\partial t} \leqslant \frac{nt + C}{e^t - 1}.$$

Notice that this estimate only depends on the initial value of u and its upper bound along the flow. It is uniform away from the initial time.

Another t-derivative gives

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2}\right) + e^{-t}\langle\widetilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \frac{\partial^2 u}{\partial t^2} - |\frac{\partial\widetilde{\omega}_t}{\partial t}|^2_{\widetilde{\omega}_t}.$$

Take summation with the one time t-derivative to arrive at

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - |\frac{\partial\widetilde{\omega}_t}{\partial t}|^2_{\widetilde{\omega}_t},$$

which gives

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \leqslant Ce^{-t}.$$

This implies the "essential decreasing" of volume form along the flow, i.e.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) \leqslant Ce^{-t}.$$

[2] By taking $e^{-\infty} = 0$, this is exactly (3.1).

One also has $\frac{\partial}{\partial t}\left(e^t \frac{\partial u}{\partial t}\right) \leqslant C$ which induces

$$\frac{\partial u}{\partial t} \leqslant (Ct + C)e^{-t}.$$

After plugging in $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$, the metric flow equation (1.1) becomes,

$$\mathrm{Ric}(\widetilde{\omega}_t) = -\sqrt{-1}\partial\bar{\partial}\left(u + \frac{\partial u}{\partial t}\right) - \omega_\infty.$$

Taking trace with respect to $\widetilde{\omega}_t$ for the equation above and using the trivial identity $n = \langle \widetilde{\omega}_t, \omega_t + \sqrt{-1}\partial\bar{\partial}u \rangle$, we have

$$R = -\Delta\left(u + \frac{\partial u}{\partial t}\right) - \langle \widetilde{\omega}_t, \omega_\infty \rangle = e^{-t}\langle \widetilde{\omega}_t, \omega_0 - \omega_\infty \rangle - \Delta\left(\frac{\partial u}{\partial t}\right) - n,$$

where R denotes the scalar curvature of $\widetilde{\omega}_t$. In sight of (3.1), we also have

$$R = -n - \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right),$$

and so the estimate got for $\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right)$ before is nothing but the well known lower bound for scalar curvature.

Recall that we focus on the smooth solution of Kähler-Ricci flow in $X \times [0, T)$ with finite time singularity at T.

Remark 3.1. For Step 1, we only need that the smooth limiting background form $\omega_T \geqslant 0$. It is indeed equivalent to assume $[\omega_T]$ has a smooth non-negative representative and presumably weaker than the class being "semi-ample", i.e. the existence of a map F described before that statement of Theorem 1.3 in Introduction.

Recall the following equation derived before

$$\frac{\partial}{\partial t}\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u\right) - n + \langle \widetilde{\omega}_t, \omega_T \rangle. \tag{3.2}$$

With $\omega_T \geqslant 0$, by Maximum Principle, one has

$$(1 - e^{t-T})\frac{\partial u}{\partial t} + u \geqslant -C.$$

Together with the known upper bounds, we conclude

$$|(1 - e^{t-T})\frac{\partial u}{\partial t} + u| \leqslant C.$$

• Step 2. Parabolic Schwarz Estimate

Use the set-up as in [13] for the map F described before the statement of Theorem 1.3. Let $\varphi = \langle \widetilde{\omega}_t, F^*\omega_{_M} \rangle$ which is clearly non-negative, then one has, over $X \times [0, T)$,

$$(\frac{\partial}{\partial t} - \Delta)\varphi \leqslant \varphi + C\varphi^2 - H,$$

where C is related to the bisectional curvature bound of $\omega_{_M}$ near $F(X)$ and $H \geqslant 0$ is described as follows. Using normal coordinates locally over X and Y, with indices i, j and α, β respectively, $\varphi = |F_i^\alpha|^2$ and $H = |F_{ij}^\alpha|^2$ with summations for all indices. Notice that the normal coordinates over X is changing along the flow with the metric. Using this inequality, one has

$$(\frac{\partial}{\partial t}-\Delta)\log\varphi \leqslant C\varphi+1.$$

Remark 3.2. For application, the map F is generated by (some multiple of) the class $[\omega_T]$ with Y being some projective space $\mathbb{CP}^N$, and so ω_T is $F^*\omega$ where ω is (some multiple of) Fubini-Study metric over Y.

Define

$$v := (1-e^{t-T})\frac{\partial u}{\partial t}+u$$

and we know $|v| \leqslant C$ for the previous step. We also have (3.2),

$$(\frac{\partial}{\partial t}-\Delta)v = -n+\langle\widetilde{\omega}_t,\omega_T\rangle = -n+\varphi.$$

After taking a large enough positive constant A, the following inequality is true,

$$(\frac{\partial}{\partial t}-\Delta)(\log\varphi - Av) \leqslant -\varphi+C.$$

Since v is bounded, Maximum Principle can be used to deduce $\varphi \leqslant C$, i.e.

$$\langle\widetilde{\omega}_t,\omega_T\rangle \leqslant C.$$

- **Step 3. Gradient and Laplacian Estimates**

In this part, we derive gradient and Laplacian estimates for v. Recall that

$$(\frac{\partial}{\partial t}-\Delta)v = -n+\varphi,\ \ \varphi=\langle\widetilde{\omega}_t,\omega_T\rangle.$$

Standard computation (as in [13]) gives:

$$(\frac{\partial}{\partial t}-\Delta)(|\nabla v|^2) = |\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar{\nabla}v|^2 + 2\mathrm{Re}(\nabla\varphi,\nabla v),$$

$$(\frac{\partial}{\partial t}-\Delta)(\Delta v) = \Delta v + (\mathrm{Ric}(\widetilde{\omega}_t),\sqrt{-1}\partial\bar{\partial}v)+\Delta\varphi.$$

Again, all the ∇, Δ and $(\cdot,\cdot)$ are with respect to $\widetilde{\omega}_t$ and $\nabla\bar{\nabla}v$ is just $\partial\bar{\partial}v$.

Define

$$\Psi := \frac{|\nabla v|^2}{C-v}.$$

Since v is bounded, one can easily make sure the denominator is positive, bounded and also away from 0. We have the following computation,

$$\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\Psi = (\frac{\partial}{\partial t}-\Delta)\left(\frac{|\nabla v|^2}{C-v}\right)\\
&= \frac{1}{C-v}\cdot\frac{\partial}{\partial t}(|\nabla v|^2)+\frac{|\nabla v|^2}{(C-v)^2}\cdot\frac{\partial v}{\partial t} - \left(\frac{(|\nabla v|^2)_{\bar{i}}}{C-v}+\frac{v_{\bar{i}}|\nabla v|^2}{(C-v)^2}\right)_i\\
&= \frac{|\nabla v|^2}{(C-v)^2}\cdot(\frac{\partial}{\partial t}-\Delta)v + \frac{1}{C-v}\cdot(\frac{\partial}{\partial t}-\Delta)(|\nabla v|^2) - \frac{v_i\cdot(|\nabla v|^2)_{\bar{i}}}{(C-v)^2} - v_{\bar{i}}\cdot\left(\frac{|\nabla v|^2}{(C-v)^2}\right)_i\\
&= \frac{|\nabla v|^2}{(C-v)^2}\cdot(\frac{\partial}{\partial t}-\Delta)v + \frac{1}{C-v}\cdot(\frac{\partial}{\partial t}-\Delta)(|\nabla v|^2) - \frac{2\mathrm{Re}(\nabla v,\nabla|\nabla v|^2)}{(C-v)^2} - \frac{2|\nabla v|^4}{(C-v)^3}.
\end{aligned}$$

Plug in the equalities from before and rewrite the differential equality for Ψ as follows,

$$\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\Psi \\
&=\frac{(-n+\varphi)|\nabla v|^2}{(C-v)^2}+\frac{|\nabla v|^2-|\nabla\nabla v|^2-|\nabla\bar{\nabla}v|^2}{C-v}+\frac{2\mathrm{Re}(\nabla\varphi,\nabla v)}{C-v} \\
&-\frac{2\mathrm{Re}(\nabla v,\nabla|\nabla v|^2)}{(C-v)^2}-\frac{2|\nabla v|^4}{(C-v)^3}.
\end{aligned}\tag{3.3}$$

We also need the following computations,

$$\begin{aligned}
|(\nabla v,\nabla|\nabla v|^2)| &= |v_i(v_j v_{\bar{j}})_{\bar{i}}| \\
&= |v_i v_{\bar{j}} v_{j\bar{i}} + v_i v_j v_{\bar{j}\bar{i}}| \\
&\leqslant |\nabla v|^2(|\nabla\nabla v|+|\nabla\bar{\nabla}v|) \\
&\leqslant \sqrt{2}|\nabla v|^2(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2)^{\frac{1}{2}},
\end{aligned}$$

$$\nabla\Psi=\nabla\left(\frac{|\nabla v|^2}{C-v}\right)=\frac{\nabla(|\nabla v|^2)}{C-v}+\frac{|\nabla v|^2\nabla v}{(C-v)^2}.$$

Together with the bounds for φ and $C-v$, we can have the following computation with ϵ being a small positive constant which might be different from place to place,

$$\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)\Psi \\
&\leqslant C|\nabla v|^2+\epsilon\cdot|\nabla\varphi|^2-C(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2) \\
&-(2-\epsilon)\mathrm{Re}\left(\nabla\Psi,\frac{\nabla v}{C-v}\right)-\epsilon\cdot\frac{\mathrm{Re}(\nabla v,\nabla|\nabla v|^2)}{(C-v)^2}-\epsilon\cdot\frac{|\nabla v|^4}{(C-v)^3} \\
&\leqslant C|\nabla v|^2+\epsilon\cdot|\nabla\varphi|^2-C(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2) \\
&-(2-\epsilon)\mathrm{Re}\left(\nabla\Psi,\frac{\nabla v}{C-v}\right)+\epsilon\cdot(|\nabla\nabla v|^2+|\nabla\bar{\nabla}v|^2)-\epsilon\cdot|\nabla v|^4 \\
&\leqslant C|\nabla v|^2+\epsilon\cdot|\nabla\varphi|^2-(2-\epsilon)\mathrm{Re}\left(\nabla\Psi,\frac{\nabla v}{C-v}\right)-\epsilon\cdot|\nabla v|^4.
\end{aligned}$$

We need a few more calculations to set up Maximum Principle argument. Recall that $\varphi=\langle\widetilde{\omega}_t,\omega_T\rangle$ and,

$$(\frac{\partial}{\partial t}-\Delta)\varphi\leqslant\varphi+C\varphi^2-H.$$

With the description of φ and H before and the estimate for φ from Step 2, i.e. $\varphi\leqslant C$, we can conclude as in [13] that

$$H\geqslant C|\nabla\varphi|^2.$$

Now one arrives at

$$(\frac{\partial}{\partial t}-\Delta)\varphi\leqslant C-C|\nabla\varphi|^2.\tag{3.4}$$

We also have the following inequality,

$$|\left(\nabla\varphi,\frac{\nabla v}{C-v}\right)|\leqslant\epsilon\cdot|\nabla\varphi|^2+C\cdot|\nabla v|^2.\tag{3.5}$$

Now we look at the function $\Psi+\varphi$. By choosing $\epsilon>0$ small enough in the above computation, which also affects the choices of C's, we have

$$(\frac{\partial}{\partial t}-\Delta)(\Psi+\varphi)\leqslant C+C|\nabla v|^2-\epsilon\cdot|\nabla v|^4-(2-\epsilon)\mathrm{Re}\left(\nabla(\Psi+\varphi),\frac{\nabla v}{C-v}\right).$$

At the maximum value point of $\Psi+\varphi$, which is either at the initial time or not, we see $|\nabla v|^2$ can not be too large. It's then easy to conclude the upper bound for $\Psi+\varphi$, and so for Ψ. Hence we have bounded the gradient, i.e.

$$|\nabla v|\leqslant C.$$

Now we want to do similar thing for the Laplacian, Δv. Define

$$\Phi:=\frac{C-\Delta v}{C-v}.$$

Similar computation as before gives the following

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)\Phi&=(\frac{\partial}{\partial t}-\Delta)\left(\frac{C-\Delta v}{C-v}\right)\\&=-\frac{1}{C-v}\cdot(\frac{\partial}{\partial t}-\Delta)\Delta v+\frac{C-\Delta v}{(C-v)^2}\cdot(\frac{\partial}{\partial t}-\Delta)v+\frac{2\mathrm{Re}(\nabla v,\nabla\Delta v)}{(C-v)^2}\\&\quad-\frac{2|\nabla v|^2(C-\Delta v)}{(C-v)^3}\\&=-\frac{1}{C-v}\cdot\big(\Delta v+(\mathrm{Ric}(\widetilde{\omega}_t),\sqrt{-1}\partial\bar{\partial}v)+\Delta\varphi\big)+\frac{C-\Delta v}{C-v}\cdot(-n+\varphi)\\&\quad+\frac{2\mathrm{Re}(\nabla v,\nabla\Delta v)}{(C-v)^2}-\frac{2|\nabla v|^2(C-\Delta v)}{(C-v)^3}.\end{aligned}\tag{3.6}$$

We also have

$$\nabla\left(\frac{C-\Delta v}{C-v}\right)=\frac{(C-\Delta v)\nabla v}{(C-v)^2}-\frac{\nabla\Delta v}{C-v}.$$

Recall that it is already known $(0\leqslant)\varphi\leqslant C$. The following inequality follows from standard computation as in [13] and has actually been used to derive the inequality for the parabolic Schwarz estimate,

$$\Delta\varphi\geqslant\big(\mathrm{Ric}(\widetilde{\omega}_t),\omega_T\big)+H-C\varphi^2.$$

As $H\geqslant 0$ and $0\leqslant\varphi\leqslant C$, we have

$$(\mathrm{Ric}(\widetilde{\omega}_t),\sqrt{-1}\partial\bar{\partial}v)+\Delta\varphi\geqslant(\mathrm{Ric}(\widetilde{\omega}_t),\sqrt{-1}\partial\bar{\partial}v+\omega_T)-C.\tag{3.7}$$

Recall that we are considering the finite time singularity case $T < \infty$, $v = (1-e^{t-T})\frac{\partial u}{\partial t} + u$ and $\omega_T = \omega_\infty + e^{-T}(\omega_0 - \omega_\infty)$. The following is then obvious,

$$\begin{aligned}
\mathrm{Ric}(\widetilde{\omega}_t) &= -\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t} + u\right) - \omega_\infty \\
&= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T}\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t}\right) + e^{-T}(\omega_0 - \omega_\infty) \\
&= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T}\left(\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t}\right) - e^{-t}(\omega_0 - \omega_\infty)\right) \\
&= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T}\frac{\partial\widetilde{\omega}_t}{\partial t} \\
&= -\sqrt{-1}\partial\bar{\partial}v - \omega_T - e^{t-T}\big(-\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t\big),
\end{aligned}$$

This gives

$$(1-e^{t-T})\mathrm{Ric}(\widetilde{\omega}_t) = -\sqrt{-1}\partial\bar{\partial}v - \omega_T + e^{t-T}\widetilde{\omega}_t,$$

and so we have the two equations below,

$$\mathrm{Ric}(\widetilde{\omega}_t) = -\frac{\sqrt{-1}\partial\bar{\partial}v + \omega_T}{1-e^{t-T}} + \frac{e^{t-T}\widetilde{\omega}_t}{1-e^{t-T}},$$

$$(1-e^{t-T})R = -\Delta v - \langle\widetilde{\omega}_t, \omega_T\rangle + ne^{t-T}.$$

As $R \geqslant -C$ and $\langle\widetilde{\omega}_t, \omega_T\rangle \geqslant 0$, we have $\Delta v \leqslant C$. So the numerator of Φ, $C - \Delta v$, is positive for large enough C.

Now we can continue the estimation (3.7) as follows.

$$\begin{aligned}
&\big(\mathrm{Ric}(\widetilde{\omega}_t), \sqrt{-1}\partial\bar{\partial}v\big) + \Delta\varphi \\
&\geqslant \big(\mathrm{Ric}(\widetilde{\omega}_t), \sqrt{-1}\partial\bar{\partial}v + \omega_T\big) - C \\
&= \left(-\frac{\sqrt{-1}\partial\bar{\partial}v + \omega_T}{1-e^{t-T}} + \frac{e^{t-T}\widetilde{\omega}_t}{1-e^{t-T}}, \sqrt{-1}\partial\bar{\partial}v + \omega_T\right) - C \\
&= -\frac{|\sqrt{-1}\partial\bar{\partial}v + \omega_T|^2}{1-e^{t-T}} + \frac{e^{t-T}(\Delta v + \langle\widetilde{\omega}_t, \omega_T\rangle)}{1-e^{t-T}} - C.
\end{aligned}$$

As $C^{-1}(T-t) \leqslant 1 - e^{t-T} \leqslant C(T-t)$ for $t \in [0, T)$, using $\Delta v = \Delta v - C + C$ and $0 \leqslant \langle\widetilde{\omega}_t, \omega_T\rangle \leqslant C$, we have

$$\begin{aligned}
&\big(\mathrm{Ric}(\widetilde{\omega}_t), \sqrt{-1}\partial\bar{\partial}v\big) + \Delta\varphi \\
&= -\frac{|\sqrt{-1}\partial\bar{\partial}v + \omega_T|^2}{1-e^{t-T}} + \frac{e^{t-T}(\Delta v + \langle\widetilde{\omega}_t, \omega_T\rangle)}{1-e^{t-T}} - C \\
&\geqslant -\frac{1}{T-t}\big((1+\epsilon)|\sqrt{-1}\partial\bar{\partial}v|^2 + C|\omega_T|^2\big) - \frac{C}{T-t}(C - \Delta v) - \frac{C}{T-t} \\
&\geqslant -\frac{1+\epsilon}{T-t}|\sqrt{-1}\partial\bar{\partial}v|^2 - \frac{C}{T-t}(C - \Delta v) - \frac{C}{T-t},
\end{aligned}$$

where $|\omega_T|^2 \leqslant C$, which follows from $0 \leqslant \langle\widetilde{\omega}_t, \omega_T\rangle \leqslant C$, is applied in the last step.

Now we can continue the computation for Φ, (3.6), as follows.

$$(\frac{\partial}{\partial t} - \Delta)\Phi \leqslant \frac{C}{T-t} + \frac{C}{T-t}\cdot(C - \Delta v) + \frac{(1+\epsilon)|\nabla\bar{\nabla}v|^2}{(T-t)(C-v)} - 2\mathrm{Re}\left(\nabla\Phi, \frac{\nabla v}{C-v}\right).$$

Using $\Phi = \frac{C-\Delta v}{C-v} \geqslant C(C-\Delta v)$, one arrives at

$$(\frac{\partial}{\partial t}-\Delta)((T-t)\Phi) \leqslant C+C\cdot(C-\Delta v)+\frac{(1+\epsilon)|\nabla\bar{\nabla}v|^2}{C-v}-2\mathrm{Re}\left(\nabla\big((T-t)\Phi\big), \frac{\nabla v}{C-v}\right).$$

In sight of (3.4) and (3.5), we have

$$(\frac{\partial}{\partial t}-\Delta)\varphi \leqslant C - 4\mathrm{Re}\left(\nabla\varphi, \frac{\nabla v}{C-v}\right) + C|\nabla v|^2.$$

Also, (3.3) can be rewritten as

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)\Psi \leqslant & \frac{(-n+\varphi)|\nabla v|^2}{(C-v)^2} + \frac{|\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar{\nabla}v|^2}{C-v} \\ & + 2\mathrm{Re}\left(\nabla\varphi, \frac{\nabla v}{C-v}\right) - 2\mathrm{Re}\left(\Psi, \frac{\nabla v}{C-v}\right).\end{aligned}$$

Using the known bound for $|\nabla v|$ and choosing proper $0<\epsilon<1$, we have

$$\begin{aligned}&(\frac{\partial}{\partial t}-\Delta)\big((T-t)\Phi+2\Psi+2\varphi\big) \\ &\leqslant C + C\cdot(C-\Delta v) - 2\mathrm{Re}\left(\nabla\big((T-t)\Phi+2\Psi+2\varphi\big), \frac{\nabla v}{C-v}\right) - C|\nabla\bar{\nabla}v|^2 \\ &\leqslant C + C\cdot(C-\Delta v) - 2\mathrm{Re}\left(\nabla\big((T-t)\Phi+2\Psi+2\varphi\big), \frac{\nabla v}{C-v}\right) - C(C-\Delta v)^2\end{aligned}$$

where $|\nabla\bar{\nabla}v|^2 \geqslant C(\Delta v)^2 \geqslant C(C-\Delta v)^2 - C$ is applied in the last step.

Now we apply Maximum Principle. At maximum value point of the function $(T-t)\Phi+2\Psi+2\varphi$, we have $C-\Delta v \leqslant C_1$. Using the known bounds on Ψ and φ, we arrive at

$$(T-t)\Phi+2\Psi+2\varphi \leqslant C,$$

and so

$$\Phi \leqslant \frac{C}{T-t}, \text{ i.e. } \Delta v \geqslant -\frac{C}{T-t}.$$

Finally since $(1-e^{t-T})R = -\Delta v - \langle\widetilde{\omega}_t, \omega_T\rangle + ne^{t-T}$, we conclude that

$$R \leqslant \frac{C}{(T-t)^2}$$

and finish the proof of Theorem 1.3.

4. Further Remarks

Here we want to indicate how these results would sit in the big picture. There are several closely related results also worth mentioning. Some remarks below should give people some idea about the essential difference between finite and infinite time singularities for Kähler-Ricci flow.

- In [12], following Perelman's idea, N. Sesum and G. Tian have proved that for X with $c_1(X)>0$, for any initial Kähler metric ω such that $[\omega]=c_1(X)$, the Kähler-Ricci flow

$$\frac{\partial\widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) + \widetilde{\omega}_t$$

has uniformly bounded scalar curvature and diameter for $t \in [0, \infty)$. Notice that this is not finite time singularity, and so does not bring trouble to our comments after Theorem 1.1. Using simply rescaling of time and metric, one can see for our flow (1.1) with $[\omega_0] = c_1(X) > 0$,

$$R(\widetilde{\omega}_t) \leqslant \frac{C}{T-t}$$

for $t \in [0, T)$ where the finite singular time $T = \log 2$, which is a better control than Theorem 1.3 in this special case. So the conclusion of Theorem 1.3, though fairly general, should not be optimal for many special cases of interests.

- For the infinite time singularity case, the scalar curvature would be uniformly bounded for all time if the infinite time limiting class, $[\omega_\infty]$, provides a holomorphic fiber bundle structure for X, i.e. the map F as in our setting is a smooth (holomorphic) bundle map. This is actually proved in [13] if one restricts to smooth collapsing case.
- Still for the infinite time singularity case, the scalar curvature would also be bounded if the limiting class is "semi-ample and big", i.e. the (possibly singular) image of the map F is of the same dimension as X, which is usually called global volume non-collapsing case. This result is proved in [18]. The more recent work of Yuguang Zhang, [16], has given a nice application.
- A major character of Kähler-Ricci flow is the cohomology information in a finite dimensional cohomology space. It provides natural expectation of the behavior for flow metric, though up to this moment, most of the behavior remains very difficult to justify. Meanwhile, scalar curvature also provides very condensed information about metric, and so it is reasonable to conjecture close relation between cohomology data of Kähler-Ricci flow and behavior of scalar curvature. That is exactly what we have achieved in this note. It gives us hope that the cohomology data would indeed provide good prediction of Kähler-Ricci flow.

References

[1] Cao, Huaidong; Zhu, Xiping: A complete proof of the Poincaré and Geometrization Conjectures - application of the Hamilton-Perelman theory of the Ricci flow. Asian J. Math. 10 (2006), no. 2, 165–492.

[2] Cascini, Paolo; La Nave, Gabriele: Kähler-Ricci flow and the minimal model program for projective varieties. ArXiv:math/0603064 (math.AG).

[3] Demailly, Jean-Pierre; Paun, Mihai: Numerical characterization of the Kähler cone of a compact Kähler manifold. Ann. of Math. 159 (2004), 1247–1274.

[4] Hamilton, Richard S.: Three-manifolds with positive Ricci curvature. J. Differential Geom. 17 (1982), no. 2, 255-306.

[5] Hamilton, Richard S.: The formation of singularities in the Ricci flow. Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.

[6] Kleiman, Steven L.: Toward a numerical theory of ampleness. Ann. of Math. (2) 84 (1966), 293–344.

[7] Kleiner, Bruce; Lott, John: Notes on Perelman's papers. Geometry and Topology 12, 2587–2855.

[8] Kollar, Janos; Mori, Shigefumi: Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Tranlated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998, vii+254 pp.

[9] Lazarsfeld, Robert: Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern

Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004. xviii+387 pp.

[10] Morgan, John; Tian, Gang: Ricci flow and the Poincarè Conjecture. Clay Mathematics Insititute.

[11] Sesum, Natasa: Curvature tensor under the Ricci flow. Amer. J. Math. 127 (2005), no. 6, 1315–1324.

[12] Sesum, Natasa; Tian, Gang: Bounding scalar curvature and diamater along the Kähler Ricci flow (after Perelman). J. Inst. Math. Jussieu 7 (2008), no. 3, 575–587. 53C44.

[13] Song, Jian; Tian, Gang: The Kähler-Ricci flow on surfaces of positive Kodaira dimension. Invent. Math. 170 (2007), no. 3, 609–653.

[14] Tian, Gang: New results and problems on Kähler-Ricci flow. To appear in Asterisque.

[15] Tian, Gang; Zhang, Zhou: On the Kähler-Ricci flow on projective manifolds of general type. Chinese Annals of Mathematics - Series B, Volume 27, Number 2, 179–192.

[16] Zhang, Yuguang: Miyaoka-Yau inequality for minimal projective manifolds of general type. ArXiv:0812.0462 (math.DG) (math.AG). Accepted by Proceedings of AMS.

[17] Zhang, Zhou: Degenerate Monge-Ampere equations over projective manifolds. Mathematics Ph. D Thesis at MIT.

[18] Zhang, Zhou: Scalar Curvature Bound for Kähler-Ricci Flows over Minimal Manifolds of General Type. International Mathematics Research Notices 2009; doi: 10.1093/imrn/rnp073.

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