RICCI FLOW ON QUASIPROJECTIVE MANIFOLDS

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Abstract. We consider the Kähler-Ricci flow on complete finite-volume metrics that live on the complement of a divisor in a compact Kähler manifold $\overline{X}$. Assuming certain spatial asymptotics on the initial metric, we compute the singularity time in terms of cohomological data on $\overline{X}$. We also give a sufficient condition for the singularity, if there is one, to be type-II.

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1. Introduction

In this paper we study the Ricci flow on certain finite-volume complete Kähler metrics that live on complements of divisors in compact Kähler manifolds. Our motivation, which we now describe, comes from the general goal of understanding singularities in Ricci flow.

It is known, since Hamilton's first Ricci flow paper [13], that singularities in a Ricci flow on a manifold M arise from curvature blowup. The nature of the blowup is important in the analysis of the singularity. We let Rm denote sectional curvatures. If T_{sing} is a first singularity time then the singularity is said to be *type-I* if there is a constant $C < \infty$ so

Date: July 20, 2009.
2000 *Mathematics Subject Classification.* 53C44,32Q15.
The research of the first author was partially supported by NSF grant DMS-0604829.

that $|\operatorname{Rm}|(m,t) \leq \frac{C}{T_{\text{sing}}-t}$ for all $m \in M$ and $t < T_{\text{sing}}$. Otherwise, the singularity is said to be *type-II.*

In Ricci flow, the natural scaling is that *time* $\sim$ *distance*^{2}. Since *curvature* $\sim$ *distance*$^{-2}$, a naive dimensional analysis would suggest that all singularities are type-I. However, this is not the case. The first type-II singularity was found on a noncompact surface which is diffeomorphic to $\mathbb{R}^2$, but whose initial metric $g(0)$ describes a hyperbolic cusp capped off by a ball. The singular time is $T_{\text{sing}} = \frac{1}{4\pi}\operatorname{Vol}(\mathbb{R}^2, g(0))$. At any time $t < T_{\text{sing}}$, the volume is $\operatorname{Vol}(\mathbb{R}^2, g(0)) - 4\pi t$. Hence as $t \to T_{\text{sing}}$, there is no volume left. The geometric behavior as the time t approaches T_{sing} is as follows. For $t < T_{\text{sing}}$, one can divide the surface into an inner region I_t and an outer region O_t. As one goes out the end, the metric on O_t has asymptotically constant negative curvature $k(t)$, with $k(t)$ remaining bounded as $t \to T_{\text{sing}}$. However, as $t \to T_{\text{sing}}$, the outer region disappears and the inner region I_t dominates. The curvature on I_t goes to infinity pointwise as $t \to T_{\text{sing}}$ and its geometry approaches a ray, in the pointed Gromov-Hausdorff sense. After a parabolic rescaling to normalize the curvature, I_t approaches a special Ricci flow solution, the cigar soliton, as one approaches the singularity time. For these results and more, we refer to papers by Daskalopoulos-del Pino-Hamilton-Sesum [7, 8, 9, 10] and references therein.

The goal of this paper is to extend some of these two-dimensional results to higher dimensions. A starting point in the two-dimensional analysis is the use of isothermal coordinates on $\mathbb{R}^2$, in order to write the Ricci flow equation as a scalar equation for the conformal factor. This method clearly does not work in higher dimensions, so we must take another approach. Our approach is based on the observation that $\mathbb{R}^2$, with a finite-volume asymptotically hyperbolic metric, can be considered as $S^2 - \text{pt}$, with a metric on S^2 which, in local coordinates near pt, approaches the Poincaré metric $\frac{4dzd\overline{z}}{|z|^2\log^2(|z|^{-2})}$. This is an example of a quasiprojective manifold, meaning the complement $X = \overline{X} - D$ of an effective divisor D with simple normal crossings in a compact Kähler manifold $\overline{X}$. Another simple example of a quasiprojective manifold comes from taking the product $X = (S^2 - \text{pt}) \times (S^2 - \text{pt})$ of the previous manifold with itself. Then $X = \overline{X} - D$, where $\overline{X} = S^2 \times S^2$ and $D = (S^2 \times \text{pt}) \cup (\text{pt} \times S^2)$.

In what follows, we will speak equivalently of a Kähler metric or a Kähler form. Let $\omega_X(0)$ be a complete Kähler metric with bounded sectional curvature on a complex manifold X. It is known that there is some $\epsilon > 0$ so that there is a Ricci flow solution on the time interval $[0, \epsilon]$ with initial metric $\omega_X(0)$, complete time slices and uniformly bounded sectional curvature [23]. It is easy to see that the time-t metric $g(t)$ is Kähler with respect to the initial (and fixed) complex structure, so it makes sense to talk about the ensuing Kähler-Ricci flow [1].

By definition, the singularity time T_{sing} is the supremum of the numbers $T > 0$ with the property that there is a Ricci flow solution $\omega_X(t)$ with the given value at $t = 0$, defined for $t \in [0, T]$, having complete time slices and uniformly bounded sectional curvature on the time interval $[0, T]$. Note that T_{sing} could be infinity, which corresponds to not having a singularity.

In order to state the main result, we introduce some terminology. Given a compact Kähler manifold $\overline{X}$ of complex dimension n, we write $[K_{\overline{X}}] \in \mathrm{H}^{(1,1)}(\overline{X}; \mathbb{R}) \cap \mathrm{H}^2(\overline{X}; \mathbb{Z})$ for the first

Chern class of the canonical line bundle $K_{\overline{X}} = \Lambda^{n,0}\overline{X}$. Note that $[K_{\overline{X}}]$ is the negative of the first Chern class of the holomorphic tangent bundle, so $[K_{\overline{X}}] = -c_1(\overline{X})$.

For us, a divisor $D = \sum_i D_i$ in $\overline{X}$ is a formal sum of complex submanifolds of $\overline{X}$ with complex codimension one. There is a corresponding class $[D] \in \mathrm{H}^{(1,1)}(\overline{X};\mathbb{R}) \cap \mathrm{H}^2(\overline{X};\mathbb{Z})$, whose Poincaré dual $*[D] \in \mathrm{H}_{2n-2}(\overline{X};\mathbb{Z})$ is the sum of the pushforwards of the fundamental classes of the D_i's. Hereafter we will assume that D has normal crossings.

Recall that a class $c \in \mathrm{H}^{(1,1)}(\overline{X};\mathbb{R})$ is a *Kähler class* if there is a closed positive form $\omega \in \Omega^{(1,1)}(\overline{X})$ whose de Rham cohomology class is c. In such a case, we write $c > 0$.

The main theorem of the paper concerns a Kähler-Ricci flow solution on $X = \overline{X} - D$ whose initial metric is a finite-volume Kähler metric $\omega_X(0)$ with "superstandard" spatial asymptotics. This notion, which will be made precise in Definition 8.7, roughly means that the metric at infinity can be decomposed into families of products of hyperbolic cusp metrics. (An example of superstandard spatial asymptotics is the product metric on $X = (S^2 - \mathrm{pt}) \times (S^2 - \mathrm{pt})$ from before.) One motivation for considering such asymptotics is that they arise for the finite-volume Kähler-Einstein metric on X that exists when $[K_{\overline{X}} + D] > 0$ [16, 22, 24, 27, 28].

If $\omega_X(0)$ has superstandard spatial asymptotics then in terms of the inclusion $X \subset \overline{X}$, we can extend $\omega_X(0)$ by zero to get a closed $(1,1)$-current on $\overline{X}$; see Theorem 6.6. There is a corresponding cohomology class $[\omega_X(0)] \in \mathrm{H}^{(1,1)}(\overline{X};\mathbb{R})$.

The goal now is to express properties of the Kähler-Ricci flow on X in terms of cohomological data. We remark that it may not be immediately clear which cohomology group is the relevant one. For example, one may think that it should be some sort of cohomology of X. However, it turns out that what's relevant is the cohomology of the compactification $\overline{X}$. (As a precedent, the cohomology of the compactification is also key to the previously-mentioned work on finite-volume Kähler-Einstein manifolds.) We show that we can effectively compute T_{sing} from cohomological data on $\overline{X}$. We also give a sufficient condition to ensure a type-II singularity.

Theorem 1.1. *Suppose that $\omega_X(0)$ is a Kähler metric on X with superstandard spatial asymptotics.*
1. The singularity time T_{sing} of the ensuing (unnormalized) Kähler-Ricci flow equals the supremum of the numbers $T > 0$ so that $[\omega_X(0)] + 2\pi T[K_{\overline{X}} + D] \in \mathrm{H}^{(1,1)}(\overline{X};\mathbb{R})$ is a Kähler class on $\overline{X}$.
2. If $D \neq \emptyset$, $T_{\mathrm{sing}} < \infty$ and $[\omega_X(0)] + 2\pi T_{\mathrm{sing}}[K_{\overline{X}} + D]$ vanishes in $\mathrm{H}^{(1,1)}(\overline{X};\mathbb{R})$ then there is a type-II singularity at time T_{sing}.

When X has one complex dimension, Theorem 1.1 recovers some of the surface results mentioned before; see Example 9.4.

In the course of proving Theorem 1.1 we obtain some results about Kähler-Ricci flow that are valid for a wider class of initial metrics. We now describe some of these results, in order of decreasing generality.

In Theorem 4.1 we characterize the singularity time for a normalized Kähler-Ricci flow on any complex manifold X, whose initial metric $\omega_X(0)$ is complete with bounded curvature.

For $T \geq 0$, put $\omega_T = -\mathrm{Ric}(\omega_X(0)) + e^{-T}(\omega_X(0) + \mathrm{Ric}(\omega_X(0)))$. Theorem 4.1 says that the singularity time T_{sing} equals the supremum of the numbers $T > 0$ with the property that there is some $F_T \in C^\infty(X)$ so that

- $\omega_T + \sqrt{-1}\partial\overline{\partial}F_T$ is a Kähler metric on X which is biLipschitz to $\omega_X(0)$, and
- For each $k \geq 0$, the k-th covariant derivatives of F_T (with respect to the initial metric $\omega_X(0)$) are uniformly bounded on X.

Theorem 4.1 is an extension of [26, Proposition 1.1] by Tian and Zhang, which dealt with the case when X is compact. The interest of Theorem 4.1 is that the issue of computing T_{sing} is reduced to a flow-independent question on X.

The next main result, Theorem 5.1, concerns long-time convergence. Under the assumption that the initial metric $\omega_X(0)$ satisfies $-\mathrm{Ric}(\omega_X(0)) + \sqrt{-1}\partial\overline{\partial}f > \epsilon\omega_X(0)$ for some $\epsilon > 0$ and some smooth function f with bounded covariant derivatives, we show that the normalized Kähler-Ricci flow (3.2) exists forever and that its time slices converge smoothly to a complete Kähler-Einstein metric on X with Einstein constant -1. This is an extension of [4, Theorem 1.1] by Chau.

The next goal is to characterize the singularity time in cohomological terms. To do so, we specialize to initial metrics on a quasiprojective manifold $X = \overline{X} - D$ that satisfy "standard" spatial asymptotics. In Theorem 7.1 we show that this property is shared by the time slices of the ensuing normalized Kähler-Ricci flow. We can then extend $\omega_X(t)$ by zero to define a closed $(1,1)$ current on $\overline{X}$ and a corresponding cohomology class $[\omega_X(t)] \in \mathrm{H}^{(1,1)}(\overline{X};\mathbb{R})$. We prove that $[\omega_X(t)]$ equals $e^{-t}[\omega_X(0)] + 2\pi(1-e^{-t})[K_{\overline{X}} + D]$. In Theorem 6.6 we show that if the Kähler-Ricci flow on X, with initial metric $\omega_X(0)$, extends to time T then $[\omega_X(T)]$ is a Kähler class on $\overline{X}$. The proof uses a characterization of Kähler classes that is due to Demailly-Paun [12].

In Theorem 8.14 we further specialize to initial metrics on $X = \overline{X} - D$ that satisfy "superstandard" spatial asymptotics. We show that this property is again shared by the time slices of the ensuing Kähler-Ricci flow. In Theorem 8.16 we show that if $[\omega_X(t)]$ happens to be a Kähler class on $\overline{X}$ then $\omega_X(t)$ can be written as $\omega_t + \sqrt{-1}\partial\overline{\partial}F_t$ for an appropriate $F_t \in C^\infty(X) \cap L^\infty(X)$. Along with Theorem 4.1, this proves the first part of Theorem 1.1.

The proof of the second part of Theorem 1.1 is by contradiction. Suppose that the singularity is type-I. By a result of Naber [20] (which is based on Perelman's work [21]), there is a spacetime sequence (x_i, t_i) with $t_i \to T_{\mathrm{sing}}$ so that after rescaling by $\frac{1}{T_{\mathrm{sing}} - t_i}$, the corresponding pointed Ricci flow solutions converge to a κ-noncollapsed gradient shrinking soliton Y with uniformly bounded curvature. Since $D \neq \emptyset$, the manifold Y is noncompact. By our assumption on the limit of the Kähler class, Y has finite volume. This leads to a contradiction. Therefore, the singularity must be type-II.

We mention some open problems. The first problem is to understand what kind of rescaling limits can arise from type-II singularities as above. A general construction of Hamilton gives a rescaling limit which is an eternal solution, i.e. which exists for $t \in \mathbb{R}$ [6, Proposition 8.17]. The question is whether it must be a gradient steady soliton, as is the case in one complex dimension, where one gets the cigar soliton. Another question is which gradient steady solitons can occur as rescaling limits.

A second problem is to work out a spatial asymptotic expansion for the metric $\omega_X(t)$, assuming some precise spatial asymptotics for $\omega_X(0)$. The analogous question for a Kähler-Einstein metric on $X = \overline{X} - D$, which exists when $[K_{\overline{X}} + D] > 0$, was addressed in [22, 28].

We thank Lei Ni for a helpful comment.

2. Conventions

Given a Kähler manifold X of complex dimension n, the Kähler form is a real $(1,1)$-form ω which can be expressed in holomorphic normal coordinates at a point p by $\omega(p) = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz^i \wedge d\overline{z}^i$. The Kähler form of the Poincaré metric is given on the upper half plane $H = \{w \in \mathbb{C} : \Im(w) > 0\}$ by $\frac{\sqrt{-1}}{2} \frac{dw \wedge d\overline{w}}{(\Im(w))^2}$. This is the pullback of the Kähler form

$$2\sqrt{-1} \frac{dz \wedge d\overline{z}}{|z|^2 \log^2(|z|^{-2})} = -\sqrt{-1}\partial\bar{\partial} \log \left(|z|^2 \log^2(|z|^{-2}) \right) \tag{2.1}$$

on $\mathbb{C}^* = \mathbb{C} - \{0\}$, under the map $z = e^{\sqrt{-1}w}$.

Let L be a holomorphic line bundle over X with Hermitian metric h_L. If σ is a section of L then we write $|\sigma|_L^2 = h_L(\sigma, \sigma)$. There is a unique connection ∇^L which is compatible with both the Hermitian metric h_L and the holomorphic structure on L. Let $F(h_L) \in \Omega^2(M)$ be the curvature form of ∇^L. The de Rham cohomology class of $\frac{\sqrt{-1}}{2\pi} F(h_L)$ equals $c_1(L) \in \mathrm{Im}(\mathrm{H}^2(X; \mathbb{Z}) \to \mathrm{H}^2(X; \mathbb{R}))$. If σ is a local holomorphic section of L then $F(h_L) = -\partial\overline{\partial} \log |\sigma|_{h_L}^2$. If $K_X = \Lambda^{n,0} X$ is the canonical bundle of X then we will write $[K_X]$ for $c_1(K_X) = -c_1(X)$.

The Ricci form is

$$\mathrm{Ric} = -\sqrt{-1} F(h_{K_X}) = \sqrt{-1}\partial\overline{\partial} \log |\sigma|_{K_X}^2 = -\sqrt{-1}\partial\overline{\partial} \log \det(g_{i\overline{j}}), \tag{2.2}$$

where σ is locally $dz^1 \wedge \ldots \wedge dz^n$. Then $[\mathrm{Ric}] = -2\pi c_1(K_X) = 2\pi c_1(X) \in \mathrm{H}^2(X; \mathbb{R})$. For the Poincaré metric on $\mathbb{C}^*$, $|\sigma|_{K_X}^2 = |z|^2 \log^2(|z|^{-2})$, so $\mathrm{Ric}(\omega) = -\omega$.

3. The potential flow

We consider Ricci flow on a connected complex manifold X of complex dimension n, which may be non-compact. Suppose that ω_0 is a smooth complete Kähler metric on X with bounded curvature. The unnormalized Kähler-Ricci flow equation is

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t), \ \widetilde{\omega}_0 = \omega_0, \tag{3.1}$$

while for us the normalized Kähler-Ricci flow equation is

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t, \ \widetilde{\omega}_0 = \omega_0. \tag{3.2}$$

(Note that the right-hand side of (3.1) differs by a factor of two from the usual Ricci flow equation $\frac{dg}{dt} = -2\,\mathrm{Ric}$.) One can pass between solutions of (3.1) and (3.2) by rescaling the metric and reparametrizing time, so there is no essential difference between the two equations. Theorem 1.1 is stated for the unnormalized equation (3.1) but in the rest of the paper, we use the normalized equation (3.2). The reason is that the Poincaré metric is a static solution of (3.2); this fact will be convenient in some constructions.

There is some $T > 0$ so that there is a solution of (3.2) on the time interval $[0, T]$, having complete time slices and uniformly bounded curvature on $[0, T]$ [23]. Furthermore, such a solution is unique on $[0, T]$ [5].

As is standard in Kähler-Ricci flow, we reduce (3.2) to a scalar equation. To do so, note that if we have a solution of (3.2) then after passing to de Rham cohomology, we get an ordinary differential equation in $\mathrm{H}^2(X; \mathbb{R})$:

$$\begin{aligned} \frac{d}{dt}[\widetilde{\omega}_t] &= -[\mathrm{Ric}(\widetilde{\omega}_t)] - [\widetilde{\omega}_t] \\ &= -[\mathrm{Ric}(\omega_0)] - [\widetilde{\omega}_t]. \end{aligned} \tag{3.3}$$

The solution to (3.3) is $[\widetilde{\omega}_t] = -[\mathrm{Ric}(\omega_0)] + e^{-t}\left([\omega_0] + [\mathrm{Ric}(\omega_0)]\right)$. This suggests putting

$$\omega_t = -\mathrm{Ric}(\omega_0) + e^{-t}\left(\omega_0 + \mathrm{Ric}(\omega_0)\right) \tag{3.4}$$

and making an ansatz for a solution of (3.2) of the form $\omega_t + \sqrt{-1}\partial\overline{\partial}u$ for some scalar function.

Consider the equation

$$\frac{\partial u}{\partial t} = \log \frac{\left(\omega_t + \sqrt{-1}\partial\bar{\partial}u\right)^n}{\omega_0^n} - u, \;\; u(0, \cdot) = 0. \tag{3.5}$$

It is implicit that we only consider solutions u of (3.5) on time intervals so that $\omega_t + \sqrt{-1}\partial\bar{\partial}u > 0$. Note that a solution of (3.5) automatically has $\frac{\partial u}{\partial t}(0, \cdot) = 0$.

Lemma 3.6. *Suppose that there is a solution to (3.2) on a time interval* $[0, T]$*, with complete time slices and uniformly bounded curvature. Then there is a smooth solution for* u *in (3.5) on the time interval* $[0, T]$ *so that*
1. For each $t \in [0, T]$*,* $\omega_t + \sqrt{-1}\partial\bar{\partial}u$ *is a Kähler metric which is biLipschitz equivalent to* ω_0*.*
2. For each k*, the* k*-th covariant derivatives of* u *(with respect to the initial metric* ω_0*) are uniformly bounded.*

Also, $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$*.*

Conversely, suppose that there is a smooth solution to (3.5) on a time interval $[0, T]$ *so that*
1. For each $t \in [0, T]$*,* $\omega_t + \sqrt{-1}\partial\bar{\partial}u$ *is a Kähler metric which is biLipschitz equivalent to* ω_0*.*
2. For each k*, the* k*-th covariant derivatives of* u *(with respect to the initial metric* ω_0*) are uniformly bounded.*

Then $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ *is a solution to (3.2) on* $[0, T]$*, with complete time slices and uniformly bounded curvature.*

Proof. Suppose that we have a solution to (3.2) on a time interval $[0, T]$, with complete time slices and uniformly bounded curvature. Put

$$u(t) = \int_0^t e^{s-t} \log \frac{\widetilde{\omega}_s^n}{\omega_0^n} \, ds, \tag{3.7}$$

so that

$$\frac{\partial u}{\partial t} = \log \frac{\widetilde{\omega}_t^n}{\omega_0^n} - u. \tag{3.8}$$

Then for each k, the k-th covariant derivatives of u (with respect to the initial metric ω_0) are uniformly bounded. Also,

$$\frac{\partial}{\partial t} \left(\widetilde{\omega}_t - \omega_t - \sqrt{-1}\partial\bar{\partial} u \right) = - \left(\widetilde{\omega}_t - \omega_t - \sqrt{-1}\partial\bar{\partial} u \right). \tag{3.9}$$

As $\widetilde{\omega}_t - \omega_t - \sqrt{-1}\partial\bar{\partial} u$ vanishes at $t = 0$, it follows that $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial} u$ for all t. Thus u satisfies (3.5).

Conversely, suppose that we have a smooth solution to (3.5) on a time interval $[0, T]$ so that each $\omega_t + \sqrt{-1}\partial\bar{\partial} u$ is a Kähler metric which is biLipschitz equivalent to ω_0, and for each k, the k-th covariant derivatives of u (with respect to the initial metric ω_0) are uniformly bounded. Putting $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial} u$ gives a solution to (3.2) on $[0, T]$. Because $\widetilde{\omega}_t$ is biLipschitz equivalent to ω_0, each time slice is complete. From the derivative bounds on u, the curvature of $\widetilde{\omega}_t$ is uniformly bounded on $[0, T]$. $\square$

Remark 3.10. In view of the uniqueness of $\widetilde{\omega}$ on $[0, T]$, the uniqueness of u on $[0, T]$ is an immediate consequence, since u must satisfy the equation (3.8) and hence be given by (3.7).

4. Existence result

In this section we characterize the first singularity time for a Kähler-Ricci flow solution on a general complex manifold. Recall the definition of ω_t from (3.4).

Theorem 4.1. *Suppose that ω_0 is a complete Kähler metric on a complex manifold X, with bounded curvature.*

Let T_1 be the supremum (possibly infinite) of the numbers T' so that there is a smooth solution for u in (3.5) on the time interval $[0, T']$ such that

1. *For each $t \in [0, T']$, $\omega_t + \sqrt{-1}\partial\bar{\partial} u$ is a Kähler metric which is biLipschitz equivalent to ω_0 and*
2. *For each k, the k-th covariant derivatives of u (with respect to the initial metric ω_0) are uniformly bounded on $[0, T']$.*

Let T_2 be the supremum (possibly infinite) of the numbers T for which there is a function $F_T \in C^\infty(X)$ such that

3. *$\omega_T + \sqrt{-1}\partial\bar{\partial} F_T$ is a Kähler metric which is biLipschitz equivalent to ω_0 and*
4. *For each k, the k-th covariant derivatives of F_T (with respect to the initial metric ω_0) are uniformly bounded.*

Then $T_1 = T_2$.

Proof. If there is a solution for u in (3.5) on a time interval $[0, T']$ satisfying (1) and (2) then we can take $F_{T'} = u(T')$ to show that $T_2 \geq T_1$. Thus it suffices to show that $T_1 \geq T_2$. That is, we need to show that if we can find a function F_T satisfying (3) and (4) then we can solve u in (3.5) on the time interval $[0, T)$ so that for each $T' \in [0, T)$, the restriction of the solution to $[0, T']$ satisfies (1) and (2).

We know that there is a solution u for short time satisfying (1) and (2). Suppose initially that we have a solution on some time interval $[0, \widehat{T})$, with $\widehat{T} < T$, so that (1) and (2) are satisfied on subintervals $[0, T'] \subset [0, \widehat{T})$. Our goal is to derive uniform estimates for the solution u and $\widetilde{\omega}_t$, i.e. to show that there are positive numbers $C > 1$ and $\{A_k\}_{k=0}^{\infty}$ so that for all $t \in [0, \widehat{T})$ and $k \geq 0$, $\sup_{x \in X} |\nabla^k u|(t, x) \leqslant A_k$ and $C^{-1}\omega_0 \leqslant \widetilde{\omega}_t \leqslant C\omega_0$.

Note: in what follows, C always stands for a positive constant, which might be different from place to place.

We now give some equations derived from (3.5), as in [26]. All of the inner products and Laplacians are computed using the metric $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. We also use the fact that

$$\triangle u = \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\sqrt{-1}\partial\bar{\partial}u\right), \tag{4.2}$$

where Δ stands for Laplacian operator with respect to the flow metric $\widetilde{\omega}_t$.

First, the t-derivative of (3.5) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\operatorname{Tr}\left(\widetilde{\omega}_t^{-1}(\omega_0 + \operatorname{Ric}(\omega_0))\right) - \frac{\partial u}{\partial t}. \tag{4.3}$$

This implies that

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) = \Delta\left(e^t\frac{\partial u}{\partial t}\right) - \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}(\omega_0 + \operatorname{Ric}(\omega_0))\right). \tag{4.4}$$

Also, since

$$\begin{aligned} n &= \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\widetilde{\omega}_t\right) = \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}(\omega_t + \sqrt{-1}\partial\bar{\partial}u)\right) \\ &= \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}(-\operatorname{Ric}(\omega_0) + e^{-t}(\omega_0 + \operatorname{Ric}(\omega_0)))\right) + \triangle u, \end{aligned} \tag{4.5}$$

we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) = \Delta\left(\frac{\partial u}{\partial t} + u\right) - n - \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\operatorname{Ric}(\omega_0)\right). \tag{4.6}$$

A linear combination of (4.4) and (4.6) gives that for any $T > 0$,

$$\frac{\partial}{\partial t}\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u\right) - n + \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\omega_T\right). \tag{4.7}$$

(Equation (4.6) can be viewed as the limiting case of equation (4.7) when $T \to \infty$.)

Next, the t-derivative of (4.3) gives

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) =&\Delta\left(\frac{\partial^2 u}{\partial t^2}\right) - \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\frac{\partial \widetilde{\omega}_t}{\partial t}\widetilde{\omega}_t^{-1}\sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}\right) + \\ &+ e^{-t}\operatorname{Tr}\left(\widetilde{\omega}_t^{-1}(\omega_0 + \operatorname{Ric}(\omega_0))\right) + e^{-t}\operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\frac{\partial \widetilde{\omega}_t}{\partial t}\widetilde{\omega}_t^{-1}(\omega_0 + \operatorname{Ric}(\omega_0))\right) + \\ &- \frac{\partial^2 u}{\partial t^2}.\end{aligned} \tag{4.8}$$

As

$$- \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\frac{\partial \widetilde{\omega}_t}{\partial t}\widetilde{\omega}_t^{-1}\left[\sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t} - e^{-t}(\omega_0 + \operatorname{Ric}(\omega_0))\right]\right) = - \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}\frac{\partial \widetilde{\omega}_t}{\partial t}\right)^2 = - \left|\frac{\partial \widetilde{\omega}_t}{\partial t}\right|^2, \tag{4.9}$$

we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2}\right) + e^{-t}\operatorname{Tr}\left(\widetilde{\omega}_t^{-1}(\omega_0 + \operatorname{Ric}(\omega_0))\right) - \frac{\partial^2 u}{\partial t^2} - \left|\frac{\partial \widetilde{\omega}_t}{\partial t}\right|^2. \tag{4.10}$$

Its summation with (4.3) is

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) - \left|\frac{\partial \widetilde{\omega}_t}{\partial t}\right|^2. \tag{4.11}$$

By assumption the curvature is uniformly bounded on compact intervals of $[0, \widehat{T})$. Hence we can apply the maximum principle freely on such intervals. Applying it to (4.11) gives

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \leqslant Ce^{-t}, \tag{4.12}$$

where $C = \sup_{x\in X}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right)(0, x)$. From (3.8),

$$\widetilde{\omega}_t^n = e^{\frac{\partial u}{\partial t} + u}\omega_0^n. \tag{4.13}$$

Hence (4.12) indicates the "essential decreasing" of the volume form along the flow, i.e.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) \leqslant Ce^{-t}. \tag{4.14}$$

Equivalently, $\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) \leqslant C$, so

$$\frac{\partial u}{\partial t} \leqslant Cte^{-t}, \tag{4.15}$$

which implies that

$$u \leqslant C. \tag{4.16}$$

To get a lower bound on u, we use (4.7). We have a smooth bounded function F_T so that $\omega_T + \sqrt{-1}\partial\bar{\partial}F_T$ is a Kähler metric. Then (4.7) can be reformulated as

$$\begin{aligned}\frac{\partial}{\partial t}\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u - F_T\right) =&\Delta\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u - F_T\right) - n + \\ &+ \operatorname{Tr}\left(\widetilde{\omega}_t^{-1}(\omega_T + \sqrt{-1}\partial\bar{\partial}F_T)\right).\end{aligned} \tag{4.17}$$

The maximum principle gives

$$(1 - e^{t-T}) \frac{\partial u}{\partial t} + u - F_T + nt \geq - \sup F_T. \tag{4.18}$$

Equations (4.15) and (4.18) imply a uniform lower bound for u on $[0, \widehat{T})$, of the form

$$u \geq -Cte^{-t}(1 - e^{t-T}) + F_T - \sup F_T - nt. \tag{4.19}$$

Also, equations (4.16) and (4.18) imply a uniform lower bound for $\frac{\partial u}{\partial t}$ on $[0, \widehat{T})$, of the form

$$\frac{\partial u}{\partial t} \geq \frac{-C + F_T - \sup F_T - nt}{1 - e^{t-T}}. \tag{4.20}$$

So far we have obtained 0-th order estimates for u and $\frac{\partial u}{\partial t}$. In order to get uniform higher order estimates (for u) on $[0, \widehat{T})$, we modify the background form ω_t to make it a Kähler form.

First, for any $t \in [0, \widehat{T})$, one has

$$\omega_t = \left(\frac{e^{-t} - e^{-T}}{1 - e^{-T}} \right) \omega_0 + \left(\frac{1 - e^{-t}}{1 - e^{-T}} \right) \omega_T. \tag{4.21}$$

Putting

$$\widehat{\omega}_t = \left(\frac{e^{-t} - e^{-T}}{1 - e^{-T}} \right) \omega_0 + \left(\frac{1 - e^{-t}}{1 - e^{-T}} \right) (\omega_T + \sqrt{-1} \partial \bar{\partial} F_T) \tag{4.22}$$

gives a Kähler form which, in fact, is uniformly Kähler for $t \in [0, \widehat{T})$. If we put

$$v = u - \left(\frac{1 - e^{-t}}{1 - e^{-T}} \right) F_T \tag{4.23}$$

then from (4.21), (4.22) and (4.23),

$$\widetilde{\omega}_t = \omega_t + \sqrt{-1} \partial \bar{\partial} u = \widehat{\omega}_t + \sqrt{-1} \partial \bar{\partial} v. \tag{4.24}$$

The evolution equation for v is

$$\frac{\partial v}{\partial t} = \log \frac{(\widehat{\omega}_t + \sqrt{-1} \partial \bar{\partial} v)^n}{\omega_0^n} - v - \frac{F_T}{1 - e^{-T}}, \ \ v(0, \cdot) = 0. \tag{4.25}$$

The initial value for $\frac{\partial v}{\partial t}$ is no longer zero, but this will not cause any problems. The extra term $\frac{F_T}{1 - e^T}$ in the right-hand side of (4.25) is also well controlled. The point is that equation (4.25) is phrased in terms of a background metric $\widehat{\omega}_t$ which is uniformly Kähler on $[0, \widehat{T})$. The 0-th order bounds on u and $\frac{\partial u}{\partial t}$ imply 0-th order bounds on v and $\frac{\partial v}{\partial t}$.

In the following, we sketch the argument to obtain the higher order estimates, which is fairly standard.

We begin with an estimate on $\triangle_{\widehat{\omega}} v$. We use computations in [29] to derive an inequality for solutions of (4.25). This inequality is closely related to to [1, (1.5)], [26, (2.3)]) and [27, (15)].

Lemma 4.26. *Given a solution of (4.25), there is an inequality of the form*

$$
\begin{aligned}
(4.27)\qquad e^{Cv}(\Delta - \frac{\partial}{\partial t})\big(e^{-Cv}(n+\Delta_{\widehat{\omega}_t} v)\big) \geq & \Delta_{\widehat{\omega}_t}(\log\frac{\Omega}{\widehat{\omega}_t^n}) - n^2 \inf_{i\neq j} R_{i\bar{i}j\bar{j},t} - n + \\
& (C\frac{\partial v}{\partial t} - C)(n+\Delta_{\widehat{\omega}} v) + \\
& (C + \inf_{i\neq j} R_{i\bar{i}j\bar{j},t}) e^{-\frac{\frac{\partial v}{\partial t} + v + \log\frac{\Omega}{\widehat{\omega}_t^n}}{n-1}} (n+\Delta_{\widehat{\omega}} v)^{\frac{n}{n-1}},
\end{aligned}
$$

where

$$(4.28)\qquad n + \Delta_{\widehat{\omega}_t} v = \mathrm{Tr}\left(\widehat{\omega}_t^{-1}(\widehat{\omega}_t + \sqrt{-1}\partial\bar{\partial} v)\right) = \mathrm{Tr}\left(\widehat{\omega}_t^{-1}\widetilde{\omega}_t\right) > 0,$$

$$(4.29)\qquad \Omega = e^{\frac{F_T}{1-e^{-T}}}\omega_0^n$$

and C is a constant that depends on t, ω_0 and F_T.

Proof. As in [29, Section 2], suppose that ϕ is a smooth solution of

$$(4.30)\qquad (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^f \omega^n,$$

where ω is a Kähler metric on a Kähler manifold X and f is a smooth function. We have the following inequality at any point $p \in X$:

$$
\begin{aligned}
(4.31)\qquad e^{C\phi}\Delta_{\widetilde{\omega}}\big(e^{-C\phi}(n+\Delta_\omega \phi)\big) \geq & \Delta_\omega f - n^2 \inf_{i\neq j} R_{i\bar{i}j\bar{j}} - Cn(n+\Delta\phi) + \\
& (C + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}) e^{-\frac{F}{n-1}} (n+\Delta\phi)^{\frac{n}{n-1}}.
\end{aligned}
$$

Here $R_{i\bar{i}j\bar{j}}$ comes from the curvature tensor for the metric ω, written in terms of any unitary frame $\{e_i\}_{i=1}^n$, and C is any (fixed) positive constant such that $C + \inf_{i\neq j} R_{i\bar{i}j\bar{j}} > 0$ at p. We emphasize that this inequality is pointwise and the "inf" is taken at the point p.

Now we consider the flow. Equation (4.25) can be reformulated as

$$(4.32)\qquad (\widehat{\omega}_t + \sqrt{-1}\partial\bar{\partial} v)^n = e^{\frac{\partial v}{\partial t} + v + \log\frac{\Omega}{\widehat{\omega}_t^n}}\widehat{\omega}_t^n.$$

From (4.31),

$$
\begin{aligned}
(4.33)\qquad e^{Cv}\Delta_{\widehat{\omega}_t}\big(e^{-Cv}(n+\Delta_{\widehat{\omega}_t} v)\big) \geq & \Delta_{\widehat{\omega}_t}\left(\frac{\partial v}{\partial t} + v + \log\frac{\Omega}{\widehat{\omega}_t^n}\right) - n^2 \inf_{i\neq j} R_{i\bar{i}j\bar{j}} - \\
& Cn(n+\Delta_{\widehat{\omega}_t} v) + \\
& (C + \inf_{i\neq j} R_{i\bar{i}j\bar{j}}) e^{-\frac{\frac{\partial v}{\partial t} + v + \log\frac{\Omega}{\widehat{\omega}_t^n}}{n-1}} (n+\Delta_{\widehat{\omega}_t} v)^{\frac{n}{n-1}},
\end{aligned}
$$

where $R_{i\bar{i}j\bar{j}}$ is computed using the metric $\widehat{\omega}_t$ and C is a positive constant such that $C + \inf_{i\neq j} R_{i\bar{i}j\bar{j}} > 0$.

Next, one computes that

$$
\begin{aligned}
(4.34)\quad e^{Cv}\left(-\frac{\partial}{\partial t}\right)\left(e^{-Cv}(n+\Delta_{\widehat{\omega}_t}v)\right) =& C\frac{\partial v}{\partial t}(n+\Delta_{\widehat{\omega}_t}v)-\frac{\partial}{\partial t}(n+\Delta_{\widehat{\omega}_t}v)\\
=& C\frac{\partial v}{\partial t}(n+\Delta_{\widehat{\omega}_t}v)-\frac{\partial}{\partial t}\operatorname{Tr}\left(\widehat{\omega}_t^{-1}\sqrt{-1}\partial\bar{\partial}\frac{\partial v}{\partial t}\right)\\
=& C\frac{\partial v}{\partial t}(n+\Delta_{\widehat{\omega}_t}v)+\operatorname{Tr}\left(\widehat{\omega}_t^{-1}\frac{\partial\widehat{\omega}_t}{\partial t}\widehat{\omega}_t^{-1}\sqrt{-1}\partial\bar{\partial}\frac{\partial v}{\partial t}\right)-\\
& \Delta_{\widehat{\omega}_t}(\frac{\partial v}{\partial t}).
\end{aligned}
$$

Adding (4.33) and (4.34) gives

$$
\begin{aligned}
(4.35)\quad e^{Cv}\left(\Delta_{\widehat{\omega}_t}-\frac{\partial}{\partial t}\right)\left(e^{-Cv}(n+\Delta_{\widehat{\omega}_t}v)\right) \geq& \Delta_{\widehat{\omega}_t}(\log\frac{\Omega}{\widehat{\omega}_t^n})-n^2\inf_{i\neq j}R_{i\bar{i}j\bar{j}}-n+\\
& (-Cn+C\frac{\partial v}{\partial t}+1)(n+\Delta_{\widehat{\omega}_t}v)+\\
& \operatorname{Tr}\left(\widehat{\omega}_t^{-1}\frac{\partial\widehat{\omega}_t}{\partial t}\widehat{\omega}_t^{-1}\sqrt{-1}\partial\bar{\partial}\frac{\partial v}{\partial t}\right)+\\
& (C+\inf_{i\neq j}R_{i\bar{i}j\bar{j}})e^{-\frac{\frac{\partial v}{\partial t}+v+\log\frac{\Omega}{\widehat{\omega}_t^n}}{n-1}}(n+\Delta_{\widehat{\omega}_t}v)^{\frac{n}{n-1}}.
\end{aligned}
$$

Since $\frac{\partial\widehat{\omega}_t}{\partial t}$ is relatively bounded with respect to $\widehat{\omega}_t$, one has

$$
(4.36)\quad \operatorname{Tr}\left(\widehat{\omega}_t^{-1}\frac{\partial\widehat{\omega}_t}{\partial t}\widehat{\omega}_t^{-1}\sqrt{-1}\partial\bar{\partial}\frac{\partial v}{\partial t}\right)\geq -C(n+\Delta_{\widehat{\omega}}v)-C
$$

for some $C>0$. Hence after a redefinition of C, we have

$$
\begin{aligned}
(4.37)\quad e^{Cv}\left(\Delta_{\widehat{\omega}_t}-\frac{\partial}{\partial t}\right)\left(e^{-Cv}(n+\Delta_{\widehat{\omega}}v)\right) \geq& \Delta_{\widehat{\omega}_t}(\log\frac{\Omega}{\widehat{\omega}_t^n})-n^2\inf_{i\neq j}R_{i\bar{i}j\bar{j}}-n+\\
& (C\frac{\partial v}{\partial t}-C)(n+\Delta_{\widehat{\omega}}v)+\\
& (C+\inf_{i\neq j}R_{i\bar{i}j\bar{j}})e^{-\frac{\frac{\partial v}{\partial t}+v+\log\frac{\Omega}{\widehat{\omega}_t^n}}{n-1}}(n+\Delta_{\widehat{\omega}}v)^{\frac{n}{n-1}}.
\end{aligned}
$$

This proves the lemma. $\square$

Using Lemma 4.26 and the 0-th order bounds on v and $\frac{\partial v}{\partial t}$, along with the uniform control on $\widehat{\omega}_t$ as a metric, we conclude that there is an estimate of the form

$$
\begin{aligned}
(4.38)\quad \left(\Delta_{\widehat{\omega}_t}-\frac{\partial}{\partial t}\right)\left(e^{-Cv}(n+\Delta_{\widehat{\omega}_t}v)\right) &\geq -C+(C\frac{\partial v}{\partial t}-C)(n+\Delta_{\widehat{\omega}_t}v)+C(n+\Delta_{\widehat{\omega}_t}v)^{\frac{n}{n-1}}\\
&\geq -C-C(n+\Delta_{\widehat{\omega}_t}v)+C(n+\Delta_{\widehat{\omega}_t}v)^{\frac{n}{n-1}}.
\end{aligned}
$$

The maximum principle now gives an *a priori* upper bound for $e^{-Cv}(n+\Delta_{\widehat{\omega}_t}v)$, and hence also for $(n+\Delta_{\widehat{\omega}_t}v)$.

This Laplacian upper bound gives a trace upper bound on $\widetilde{\omega}_t$, relative to $\widehat{\omega}_t$, from (4.24). There is also a determinant lower bound on $\widetilde{\omega}_t$ relative to $\widehat{\omega}_t$, coming from (4.32), where we use the lower bounds on v and $\frac{\partial v}{\partial t}$. This gives a uniform bound on $\widetilde{\omega}_t$, relative to $\widehat{\omega}_t$.

Next, we look at third order estimates. Following [1] and [29], we consider the expression $S = \widetilde{g}^{i\bar{j}}\widetilde{g}^{k\bar{l}}\widetilde{g}^{\lambda\bar{\eta}} v_{i\bar{l}\lambda} v_{\bar{j}k\bar{\eta}}$, where $\widetilde{g}_{i\bar{j}}$ is the metric tensor corresponding to $\widetilde{\omega}_t$. As in [4, Section 5.3], there are estimates of the form

$$(\Delta_{\widehat{\omega}_t} - \frac{\partial}{\partial t})S \geq -C \cdot S - C, \tag{4.39}$$

$$(\Delta_{\widehat{\omega}_t} - \frac{\partial}{\partial t})\Delta_{\widehat{\omega}_t} v \geq C \cdot S - C.$$

Choosing $A > 0$ large enough, as in [1, (1.25)] and [4, (5.17)], there is an estimate of the form

$$(\Delta_{\widehat{\omega}_t} - \frac{\partial}{\partial t})(S + A\Delta_{\widehat{\omega}_t} v) \geq C \cdot S - C. \tag{4.40}$$

Applying the maximum principle and using the uniform control on $\Delta_{\widehat{\omega}_t} v$, we obtain an *a priori* upper bound on S. This provides a spatial $C^{2,\alpha}$-bound for v and a C^{α}-bound for the metric coefficients of $\widetilde{\omega}_t$.

One can then obtain further derivative bounds (cf. [4, Section 5.4]) and apply parabolic Schauder estimates (cf. [4, Section 5.5]). In this way, one obtains *a priori* estimates on all of the derivatives of v. These imply the desired derivative estimates on u.

To summarize, we assumed that we have a solution of (3.5) on a time interval $[0, \widehat{T})$, with $\widehat{T} < T$, so that conditions (1) and (2) are satisfied on compact subintervals of $[0, \widehat{T})$. Then we have shown that there are numbers $C > 1$ and $\{A_k\}_{k=0}^{\infty}$ so that for all $t \in [0, \widehat{T})$ and $x \in X$, we have $|\nabla^k u|(t, x) \leq A_k$ and $C^{-1}\omega_0 \leqslant \widetilde{\omega}_t \leqslant C\omega_0$.

Let $\{t_i\}_{i=1}^{\infty}$ be a sequence in $[0, \widehat{T})$ with $\lim_{i\to\infty} t_i = \widehat{T}$. We can extract a subsequence of $\{u(t_i, \cdot)\}_{i=1}^{\infty}$ that converges in the pointed C^{∞}-topology to some $u_{\widehat{T}}(\cdot) \in C^{\infty}(X)$. Now $u_{\widehat{T}}$ is uniformly bounded on X, along with its covariant derivatives (with respect to ω_0). From (4.13), it follows that $\omega_{\widehat{T}} + \sqrt{-1}\partial\bar{\partial} u_{\infty}$ is Kähler and biLipschitz to ω_0, with bounded curvature. Hence we can solve the equation in (3.5) to get a solution U on a time interval $[\widehat{T}, \widehat{T} + \epsilon)$ with initial condition $U(\widehat{T}) = u_{\widehat{T}}$. One then shows that the solutions $u(\cdot)$, on $[0, \widehat{T})$, and $U(\cdot)$, on $[\widehat{T}, \widehat{T} + \epsilon)$, join to form a smooth solution of (3.5) on $[0, \widehat{T} + \epsilon)$, which satisfies conditions (1) and (2) on compact subintervals. It follows that there is a solution of (3.5) on $[0, T)$ so that for each $T' \in [0, T)$, the restriction of the solution to $[0, T']$ satisfies conditions (1) and (2). This finishes the proof. □

5. Long-time convergence

In this section we prove a long-time convergence result for the normalized Kähler-Ricci flow equation, under the assumption that the initial metric satisfies an inequality of the form $-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f > \epsilon\omega_0$ for some bounded function f and some positive constant ϵ. We show that the solution smoothly approaches a complete Kähler-Einstein metric, having Einstein constant -1. This result can be seen as a generalization of results in [1], [26] concerning Kähler-Ricci flow solutions on compact manifolds. It is also a generalization of [4, Theorem 1.1], which proves the same conclusion under the assumption that $-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f = \omega_0$.

Theorem 5.1. *1. Suppose that ω_0 is a complete Kähler metric on a complex manifold X, with bounded curvature, such that $-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f \geq 0$ for some smooth function f with bounded k-th covariant derivatives (with respect to ω_0) for each $k \geq 0$. Then the flow (3.2) (or equivalently (3.5)) exists forever.*
2. Suppose in addition that $-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f > \epsilon\omega_0$ for some $\epsilon > 0$. Then the flow (3.2) (or equivalently (3.5)) converges smoothly to a complete Kähler-Einstein metric with Einstein constant -1.

Proof. Suppose first that $-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f \geq 0$. Then

$$\omega_t + \left(1 - e^{-t}\right) \sqrt{-1}\partial\bar{\partial} f = e^{-t}\omega_0 + \left(1 - e^{-t}\right) \left(-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f\right) \geq e^{-t}\omega_0. \tag{5.2}$$

From Theorem 4.1, the flow (3.2) exists forever.

Now suppose that $-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f > \epsilon\omega_0$. To prove the long-time convergence, we need estimates that are uniform in time. The upper bounds on u and $\frac{\partial u}{\partial t}$ from (4.16) and (4.15) are uniform for all time. For the lower bound, we use the following variation on (4.6):

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u - f\right) = \Delta\left(\frac{\partial u}{\partial t} + u - f\right) - n + \mathrm{Tr}\left(\widetilde{\omega}_t^{-1}(-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f)\right). \tag{5.3}$$

Now

$$\begin{aligned} \mathrm{Tr}\left(\widetilde{\omega}_t^{-1}(-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f)\right) &\geq n \cdot \left(\frac{(-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f)^n}{\widetilde{\omega}_t^n}\right)^{\frac{1}{n}} \\ &\geq n \cdot \left(\frac{(\epsilon\omega_0)^n}{\widetilde{\omega}_t^n}\right)^{\frac{1}{n}} \\ &= n\epsilon\, e^{-\frac{1}{n}\left(\frac{\partial u}{\partial t} + u\right)}, \end{aligned} \tag{5.4}$$

so (5.3) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u - f\right) \geq \Delta\left(\frac{\partial u}{\partial t} + u - f\right) - n + n\epsilon\, e^{-\frac{f}{n}}\, e^{-\frac{1}{n}\left(\frac{\partial u}{\partial t} + u - f\right)}. \tag{5.5}$$

Putting $Y(t) = \inf_{x \in X}\left(\frac{\partial u}{\partial t} + u - f\right)(x,t)$, we can apply the maximum principle to (5.5) to conclude that $Y(t)$ is bounded below by the solution $c(t)$ to the ordinary differential equation

$$\frac{dc}{dt} = -n + n\epsilon\, e^{-\frac{\sup f}{n}}\, e^{-\frac{c}{n}} \tag{5.6}$$

with initial condition $c(0) = Y(0)$. It follows that there is a lower bound

$$\frac{\partial u}{\partial t} + u \geq -C \tag{5.7}$$

which is uniform in t. When combined with the upper bounds on u and $\frac{\partial u}{\partial t}$, equation (5.7) provides uniform lower bounds for u and $\frac{\partial u}{\partial t}$.

As in the proof of Theorem 4.1, we now transform the flow equation in order to prove the higher order estimates. Putting

$$\widehat{\omega}_t = e^{-t}\omega_0 + (1-e^{-t})(-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}f), \tag{5.8}$$

we have

$$\widehat{\omega}_t \geq \left(e^{-t} + \epsilon(1-e^{-t})\right)\omega_0, \tag{5.9}$$

so the family $\{\widehat{\omega}_t\}_{t\geq 0}$ is uniformly Kähler. Next, putting

$$w = u - (1-e^{-t})f, \tag{5.10}$$

we can write

$$\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u = \widehat{\omega}_t + \sqrt{-1}\partial\bar{\partial}w. \tag{5.11}$$

Then the flow equation (3.5) becomes

$$\frac{\partial w}{\partial t} = \log\frac{(\widehat{\omega}_t + \sqrt{-1}\partial\bar{\partial}w)^n}{\omega_0^n} - w - f,\ w(0,\cdot) = 0. \tag{5.12}$$

We can use this equation to find higher order estimates on u, as in the proof of Theorem 4.1. Note that the background metric $\widehat{\omega}_t$ is uniformly Kähler and from (5.8), it is uniformly bounded above. Hence the higher order estimates will also be uniform in time. So we have achieved uniform estimates on u for all time.

We now justify the convergence. Using the uniform bounds that we have obtained so far, equation (4.3) implies an inequality of the form

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) \geq \Delta\left(e^t\frac{\partial u}{\partial t}\right) - C. \tag{5.13}$$

From the maximum principle, $e^t\frac{\partial u}{\partial t} + Ct \geq 0$ and so

$$\frac{\partial u}{\partial t} \geq -Cte^{-t}. \tag{5.14}$$

Combining (4.15) and (5.14), we conclude that $\lim_{t\to\infty} u(x,t) = u_\infty(x)$ for some function u_∞ on X. Using the uniform higher order bounds on $u(t)$, one sees that there is uniform C^K-convergence of $u(t)$ toward u_∞ for any $K > 0$. Taking the limit of (3.2) as $t\to\infty$ shows that the limiting Kähler metric $\omega_\infty = -\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}u_\infty$ satisfies $\mathrm{Ric}(\omega_\infty) = -\omega_\infty$. Also, ω_∞ is biLipschitz equivalent to ω_0; see the discussion after (4.38) and note that

$$\omega_\infty = (-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}f) + \sqrt{-1}\partial\bar{\partial}(u_\infty - f). \tag{5.15}$$

Thus ω_∞ is complete. □

For an example of Theorem 5.1, see Example 6.18.

6. Standard spatial asymptotics

In this section we begin to specify the spatial asymptotics that we want to consider. The goal is to come up with the widest class of spatial asymptotics which is preserved by the Ricci flow, and for which we can prove something nontrivial. We introduce the notion of "standard" spatial asymptotics for a Kähler metric ω_X on a quasiprojective manifold $X = \overline{X} - D$. Assuming standard spatial asymptotics, we prove some properties of the extension of ω_X by zero to $\overline{X}$. As an example of standard spatial asymptotics, in the case when $[K_{\overline{X}} + D] > 0$, we show how to recover the Kähler-Einstein metric on X [16, 24, 27] using the Kähler-Ricci flow, via Theorem 5.1 or [4, Theorem 1.1].

In the next section we will show that the property of having standard spatial asymptotics is preserved by Ricci flow. In Section 8 we will consider a refinement called "superstandard" spatial asymptotics.

Suppose that $\overline{X}$ is a compact n-dimensional complex manifold and D is an effective divisor with simple normal crossings. Put $X = \overline{X} - D$. Let $D = \sum_{i=1}^{k} D_i$ be the decomposition of D into its irreducible components. Let L_i be the holomorphic line bundle on $\overline{X}$ corresponding to D_i. Put $L_D = \bigotimes_{i=1}^{k} L_i$.

Let h_{L_i} be a Hermitian metric on L_i. There is a holomorphic section σ_i of L_i whose zero-set is D_i, unique up to multiplication by a nonzero complex number. The section σ_i is nondegenerate along D_i [11, Theorem II.(6.6)]. That is, the restriction of the bundle map $\nabla^{L_i}\sigma_i : T\overline{X} \to L_i$ to D_i factors through an isomorphism $T\overline{X}/TD_i \to L_i\big|_{D_i}$.

Given a multi-index $I = (i_1, \ldots, i_m)$, put $D_I = \bigcap_{j=1}^{m} D_{i_j}$. We write $|I| = m$. Put $D_I^{int} = D_I - \bigcup_{I' : |I'| > m} D_{I'}$. Then D_I^{int} is a smooth complex manifold of complex dimension $n - m$, possibly noncompact.

Let Δ denote the open unit ball in $\mathbb{C}$. Put $\Delta^* = \Delta - \{0\}$ and $H = \{z \in \mathbb{C} : \Im(z) > 0\}$. There is a holomorphic covering map $\pi : H \to \Delta^*$ given by $\pi(z) = e^{iz}$. Suppose that $\overline{x} \in D_I^{int}$. After permutation of indices, we can assume that $\overline{x} \in (D_1 \cap D_2 \cap \ldots \cap D_m) - (D_{m+1} \cup\ D_{m+2} \cup \ldots \cup D_k)$. We write 0 for $(0, \ldots, 0) \in \Delta^n$. Then there is a neighborhood U of $\overline{x}$ in $\overline{X}$ and a biholomorphic map $F_{\overline{x}} : \Delta^n \to U$ so that

(1) For $k > m$, $U \cap D_k = \emptyset$.
(2) $F_{\overline{x}}(0) = \overline{x}$.
(3) For $1 \leq k \leq m$, $F_{\overline{x}}(\Delta^{k-1} \times \{0\} \times \Delta^{n-m}) = U \cap D_k$.
(4) For $1 \leq k \leq m$, $\parallel \sigma_i(F_{\overline{x}}(z)) \parallel_{L_i}^2 = h_i |z^i|^2$ for some positive function $h_i \in C^\infty(\Delta^n)$.

In particular, $F_{\overline{x}}\left((\Delta^*)^m \times \Delta^{n-m}\right) = U \cap X$. Passing to the universal cover gives a holomorphic covering map $\widetilde{F}_{\overline{x}} : H^m \times \Delta^{n-m} \to U \cap X$.

The map $G_{\overline{x}}$ on Δ^{n-m}, given by $G_{\overline{x}}(w) = F_{\overline{x}}(0, w)$, is a biholomorphic map from Δ^{n-m} to a neighborhood of $\overline{x}$ in D_I^{int}.

Let ω_X be a Kähler metric on X. Then $\widetilde{F}_{\overline{x}}^* \omega_X$ is a Kähler metric on $H^m \times \Delta^{n-m}$ which is invariant under translation in the H^m-factor by $2\pi\mathbb{Z}^m$. Given $r \in (\mathbb{R}^+)^m$, define a biholomorphic map $\alpha_r : H^m \to H^m$ by $\alpha_r(h_1, \ldots, h_m) = (r_1 h_1, \ldots, r_m h_m)$. If Z is an auxiliary space then we will also write α_r for $(\alpha_r, \mathrm{Id}) : H^m \times Z \to H^m \times Z$.

Definition 6.1. Let $\{\omega_{D_I^{int}}\}$ be complete Kähler metrics on $\{D_I^{int}\}$. Let $\{c_i\}_{i=1}^m$ be positive numbers. Then ω_X has standard spatial asymptotics associated to $\{\omega_{D_I^{int}}\}$ and $\{c_i\}_{i=1}^m$ if for every $\overline{x} \in \overline{X}$ and every local parametrization $F_{\overline{x}}$,

$$\lim_{r\to\infty} \alpha_r^* \widetilde{F}_{\overline{x}}^* \omega_X \ = \ \sum_{i=1}^m c_i \frac{\sqrt{-1}}{2} \frac{dz^i \wedge d\overline{z}^i}{\Im(z^i)^2} + G_{\overline{x}}^* \omega_{D_I^{int}}, \tag{6.2}$$

where $\lim_{r\to\infty}$ means that $r_i \to \infty$ for each $1 \le i \le m$. The limit in (6.2) is taken in the pointed C^∞-topology around the basepoint $(\sqrt{-1}, \ldots, \sqrt{-1}) \times 0 \in H^m \times \Delta^{n-m}$.

Definition 6.3. Let $\{u_{D_I^{int}}\}$ be smooth functions on $\{D_I^{int}\}$. Then a function $u_X \in C^\infty(X)$ has standard spatial asymptotics associated to $\{u_{D_I^{int}}\}$ if for every $\overline{x} \in \overline{X}$ and every local parametrization $F_{\overline{x}}$

$$\lim_{r\to\infty} \alpha_r^* \widetilde{F}_{\overline{x}}^* u_X \ = \ G_{\overline{x}}^* u_{D_I^{int}}, \tag{6.4}$$

where $\lim_{r\to\infty}$ means that $r_i \to \infty$ for each $1 \le i \le m$. The limit in (6.2) is taken in the pointed C^∞-topology around the basepoint $(\sqrt{-1}, \ldots, \sqrt{-1}) \times 0 \in H^m \times \Delta^{n-m}$.

Remark 6.5. Note that if U is any neighborhood of $(\sqrt{-1}, \ldots, \sqrt{-1}) \in H^m$ then for some $K_U > 0$, $\bigcup_{r\ge 1} \alpha_r(U)$ contains $\{z \in \mathbb{C}^m : \Re(z^i) \in [0, 2\pi], \Im(z^i) \ge K_U \text{ for } 1 \le i \le m\}$.

Theorem 6.6. *If ω_X has standard spatial asymptotics then*

(1) *ω_X extends by zero to an element $[\omega_X] \in \mathrm{H}^{(1,1)}(\overline{X})$.*
(2) *For any $0 \le j \le n$, ω_X^j extends to an element of $\mathrm{H}^{(j,j)}(\overline{X})$ which equals $[\omega_X]^j$.*
(3) *$\mathrm{Ric}(\omega_X)$ extends by zero to an element $[\mathrm{Ric}(\omega_X)] \in \mathrm{H}^{(1,1)}(\overline{X})$ which equals $-[K_{\overline{X}}+D]$.*
(4) *$[\omega_X]$ lies in the Kähler cone of $\overline{X}$.*

Proof. (1). We use results from [11, Chapter 3]. From Definition 6.1, the Kähler metric $F_{\overline{x}}^*\omega_X$ is uniformly biLipshitz equivalent to $2\sqrt{-1}\sum_{i=1}^m c_i \frac{dz^i \wedge d\overline{z}^i}{|z^i|^2 \log^2(|z^i|^{-2})} + G_{\overline{x}}^*\omega_{D_I^{int}}$ on $(\Delta^*)^m \times \Delta^{n-m}$. It follows that ω_X, considered as a current on $\overline{X}$, has locally finite mass in the sense of [11, Remark III.(1.15)]. From the Skoda-El Mir extension theorem [11, Theorem III.(2.3)], ω_X extends to a closed current on $\overline{X}$ and hence a class $[\omega_X] \in \mathrm{H}^{(1,1)}(\overline{X})$.
(2). The same argument as in part (1) shows that ω_X^j extends to an element $[\omega_X^j] \in \mathrm{H}^{(j,j)}(\overline{X})$. The extension of ω_X to $\overline{X}$ has no singular support. Also, ω_X^j is L^1 on $\overline{X}$, so it is plausible that $[\omega_X^j] = [\omega_X]^j$.

To see this, let $g_{\overline{X}}$ be a Kähler metric on $\overline{X}$. Let $\triangle^{(j,j)}$ denote the (nonnegative) Hodge Laplacian on $\Omega^{(j,j)}(\overline{X})$ associated to the Kähler form $\omega_{\overline{X}}$. For small $\epsilon > 0$, the Schwartz kernel $e^{-\epsilon\triangle^{(j,j)}}(x,y)$ is well approximated by $(4\pi\epsilon)^{-n}\, e^{-\frac{d_{\overline{X}}(x,y)^2}{4\epsilon}} P_{x,y}$, where $P_{x,y}$ denotes parallel transport from $\Lambda_y^{(j,j)}$ to $\Lambda_x^{(j,j)}$ along a minimal geodesic from y to x (In what follows, we can assume that x is not in the cut locus of y.)

For any $\epsilon > 0$, $e^{-\epsilon\triangle^{(1,1)}}\omega_X$ is a smooth closed form on $\overline{X}$ whose cohomology class equals $[\omega_X] \in \mathrm{H}^{(1,1)}(\overline{X};\mathbb{R})$. Hence $\left(e^{-\epsilon\triangle^{(1,1)}}\omega_X\right)^j$ represents $[\omega_X]^j$, while $e^{-\epsilon\triangle^{(j,j)}}\omega_X^j$ represents

$[\omega_X^j]$. In particular, for any smooth closed form $\alpha \in \Omega^{(n-j,n-j)}(\overline{X})$, we have

$$\int_{\overline{X}} [\omega_X^j] \wedge [\alpha] = \lim_{\epsilon \to 0} \int_{\overline{X}} \left(e^{-\epsilon \triangle^{(j,j)}} \omega_X^j \right) \wedge \alpha = \lim_{\epsilon \to 0} \int_{\overline{X}} \omega_X^j \wedge \left(e^{-\epsilon \triangle^{(n-j,n-j)}} \alpha \right) = \int_X \omega_X^j \wedge \alpha. \tag{6.7}$$

On the other hand,

$$\int_{\overline{X}} [\omega_X]^j \wedge [\alpha] = \lim_{\epsilon \to 0} \int_{\overline{X}} \left(e^{-\epsilon \triangle^{(1,1)}} \omega_X \right)^j \wedge \alpha. \tag{6.8}$$

We claim that there is L^1-convergence

$$\lim_{\epsilon \to 0} \left(e^{-\epsilon \triangle^{(1,1)}} \omega_X \right)^j = \omega_X^j. \tag{6.9}$$

If not then there is a sequence $\{O_k\}_{k=1}^\infty$ of nonempty open subsets of $\overline{X}$, with $\mathrm{diam}(O_k) \le \frac{1}{k}$, so that for each k we do not have L^1-convergence in (6.9) on O_k. After passing to a subsequence, we can assume that there is some $\overline{x} \in \overline{X}$ so that $\lim_{k \to \infty} \overline{O_k} = \{\overline{x}\}$ in the Hausdorff topology. Choose a biholomorphic map $F_{\overline{x}} : \Delta^n \to U$ as before with $G_{\overline{x}}(\Delta^{n-m}) = F_{\overline{x}}(0, \Delta^{n-m}) \subset D_I^{int}$. The standard asymptotics from Definition 6.1 control ω_X on U. In particular, as $(z^1, \ldots, z^m) \to 0$, $F_{\overline{x}}^* \omega_X$ approaches $2\sqrt{-1} \sum_{i=1}^m c_i \frac{dz^i \wedge d\overline{z}^i}{|z^i|^2 \log^2(|z^i|^{-2})} + G_{\overline{x}}^* \omega_{D_I^{int}}$. Combining with the uniform heat kernel asymptotics of $e^{-\epsilon \triangle^{(1,1)}}$ on U, one sees that there is some $K > 0$ so that for any $k \ge K$, there is L^1-convergence in (6.9) on O_k. This is a contradiction.

It follows that

$$\lim_{\epsilon \to 0} \int_{\overline{X}} \left(e^{-\epsilon \triangle^{(1,1)}} \omega_X \right)^j \wedge \alpha = \int_X \omega_X^j \wedge \alpha. \tag{6.10}$$

Thus $[\omega_X^j] = [\omega_X]^j$ in $\mathrm{H}^{(j,j)}(\overline{X}; \mathbb{R})$.
(3). The same argument as in part (1) shows that $\mathrm{Ric}(\omega_X)$ extends to an element $[\mathrm{Ric}(\omega_X)] \in \mathrm{H}^{(1,1)}(\overline{X})$. From the asymptotics in Definition 6.1, $\frac{h_{K_X}}{\prod_{i=1}^k |\sigma_i|_{L_i}^2 \log^2(|\sigma_i|_{L_i}^{-2})}$ extends to a continuous Hermitian metric $h_{K_{\overline{X}}}$ on $K_{\overline{X}}$. Now $h_{K_{\overline{X}}} \otimes \bigotimes_{i=1}^k h_{L_i}$ is a Hermitian metric on $K_{\overline{X}} \otimes L_D$ and on X, there is an equality of currents:

$$\mathrm{Ric}(\omega_X) = -\sqrt{-1} F(h_{K_X}) = -\sqrt{-1} F(h_{K_{\overline{X}}} \otimes \bigotimes_{i=1}^k h_{L_i}) + \sqrt{-1} \partial \overline{\partial} \sum_{i=1}^k \log \log^2(|\sigma_i|_{L_i}^{-2}). \tag{6.11}$$

The extension by zero of $\sqrt{-1}\partial\overline{\partial} \sum_{i=1}^k \log \log^2(|\sigma_i|_{L_i}^{-2})$ to $\overline{X}$ is a closed $(1,1)$-current which is the image under $\sqrt{-1}\partial\overline{\partial}$ of $\sum_{i=1}^k \log \log^2(|\sigma_i|_{L_i}^{-2}) \in L^1(\overline{X})$. Hence $[\sqrt{-1}\partial\overline{\partial} \sum_{i=1}^k \log \log^2(|\sigma_i|_{L_i}^{-2})]$ vanishes in $\mathrm{H}^{(1,1)}(X; \mathbb{R})$ and so $[\mathrm{Ric}(\omega_X)] = -2\pi [K_{\overline{X}} + D]$.
(4). Let $\omega_{\overline{X}}$ be an arbitrary smooth Kähler form on $\overline{X}$. From [12, Theorem 4.2], it suffices to show that $\int_Y [\omega_{\overline{X}}]^{p-j} [\omega_X]^j > 0$ for all irreducible analytic sets Y and all $0 \le j \le p$, where $\dim_{\mathbb{C}} Y = p$.

For any $\epsilon > 0$, $\int_Y [\omega_{\overline{X}}]^{p-j} [\omega_X]^j = \int_Y \omega_{\overline{X}}^{p-j} \wedge \left(e^{-\epsilon \triangle^{(j,j)}} \omega_X^j \right)$. Suppose first that $Y \not\subset D$. Then $Y - D$ is dense in Y. By taking ϵ small and using the asymptotics in Definition 6.1, it follows that $\int_Y [\omega_{\overline{X}}]^{p-j} [\omega_X]^j = \int_Y \omega_{\overline{X}}^{p-j} \wedge \omega_X^j > 0$. Now suppose that $Y \subset D_I$ and D_I is minimal with respect to this property. For $x \in D_I^{int}$, let $R_x^* : \Lambda_x^{(j,j)} \overline{X} \to \Lambda_x^{(j,j)} D_I^{int}$ be the pullback map. Using the asymptotics in Definition 6.1, if $x \in D_I^{int}$ then

(6.12)

$$\begin{aligned}\lim_{\epsilon \to 0} R_x^* \left(\left(e^{-\epsilon \triangle^{(j,j)}} \omega_X^j \right)(x) \right) &= \lim_{\epsilon \to 0} \int_{\overline{X}} R_x^* \left(e^{-\epsilon \triangle^{(j,j)}}(x,y)\, \omega_X^j(y) \right) \mathrm{dvol}_{\overline{X}}(y) \\ &= \lim_{\epsilon \to 0} \int_{\overline{X}} (4\pi\epsilon)^{-n}\, e^{-\frac{d_{\overline{X}}(x,y)^2}{4\epsilon}} R_x^* P_{x,y} \omega_X^j(y)\ \mathrm{dvol}_{\overline{X}}(y) \ = \ \omega^j_{D_I^{int}}(x).\end{aligned}$$

It follows that $\int_Y [\omega_{\overline{X}}]^{p-j} [\omega_X]^j = \int_Y \omega_{\overline{X}}^{p-j} \wedge \omega^j_{D_I^{int}} > 0$. □

Remark 6.13. Part (2) of Theorem 6.6 has a more direct proof if ω_X has superstandard spatial asymptotics in the sense of Definition 8.7 below. Part (3) of Theorem 6.6 also follows from [19, §1].

Example 6.14. Let $\omega_{\overline{X}}$ be a Kähler metric on $\overline{X}$. Given positive numbers $\{c_i\}_{i=1}^k$, define a $(1,1)$-form on X by

$$\begin{aligned}\omega_X &= \omega_{\overline{X}} - \sqrt{-1} \partial \overline{\partial} \sum_{i=1}^k c_i \log \log^2 |\sigma_i|_{L_i}^{-2} \\ &= \omega_{\overline{X}} - 2\sqrt{-1} \frac{\partial \overline{\partial} \sum_{i=1}^k c_i \log |\sigma_i|_{L_i}^{-2}}{\log |\sigma_i|_{L_i}^{-2}} + 2\sqrt{-1} \sum_{i=1}^k c_i \frac{\partial \log |\sigma_i|_{L_i}^{-2}}{\log |\sigma_i|_{L_i}^{-2}} \wedge \frac{\overline{\partial} \log |\sigma_i|_{L_i}^{-2}}{\log |\sigma_i|_{L_i}^{-2}}.\end{aligned} \tag{6.15}$$

Now $\widehat{\omega} = \omega_{\overline{X}} + 2\sqrt{-1} \sum_{i=1}^k c_i \frac{\partial \log |\sigma_i|_{L_i}^{-2}}{\log |\sigma_i|_{L_i}^{-2}} \wedge \frac{\overline{\partial} \log |\sigma_i|_{L_i}^{-2}}{\log |\sigma_i|_{L_i}^{-2}}$ is a Kähler metric on X. For any $\epsilon > 0$, if the Hermitian metrics h_{L_i} are multiplied by a sufficiently small constant then

$$-\epsilon \widehat{\omega} \leq 2\sqrt{-1} \frac{\partial \overline{\partial} \sum_{i=1}^k c_i \log |\sigma_i|_{L_i}^{-2}}{\log |\sigma_i|_{L_i}^{-2}} \leq \epsilon \widehat{\omega}. \tag{6.16}$$

Hence by rescaling the Hermitian metrics, we can achieve that ω_X defines a Kähler metric on X.

One can check that ω_X has standard spatial asymptotics. To describe $\omega_{D_I^{int}}$, suppose that $D_I = \bigcap_{j=1}^m D_{i_j}$. After permuting indices, we can assume that $D_I = D_1 \cap D_2 \cap \ldots \cap D_m$. Then

$$\omega_{D_I^{int}} = \omega_{\overline{X}} \Big|_{D_I^{int}} - \sqrt{-1} \partial \overline{\partial} \sum_{i=m+1}^k c_i \log \log^2 |\sigma_i|_{L_i}^{-2}, \tag{6.17}$$

where the last computation is performed on D_I^{int}.

Example 6.18. Suppose that $\overline{X}$ is a compact Kähler manifold, D is an effective divisor in $\overline{X}$ with simple normal crossings and $[K_{\overline{X}} + D] > 0$. We use the Kähler-Ricci flow and Theorem 5.1, or [4, Theorem 1.1], to construct the Kähler-Einstein metric on $X = \overline{X} - D$ which is known to exist from [16, 24, 27].

The first step, as in [16, 24, 27], is to construct a 0-th order approximation to the Kähler-Einstein metric using an idea of Carlson-Griffiths [3, Proposition 2.1]. Namely, since $[K_{\overline{X}} + D] > 0$, we can find a Hermitian metric $h_{K_{\overline{X}} \otimes L_D}$ on $K_{\overline{X}} \otimes L_D$ so that $\sqrt{-1} F(h_{K_{\overline{X}} \otimes L_D}) > 0$.

Fix $\omega_{\overline{X}} = \sqrt{-1}F(h_{K_{\overline{X}} \otimes L_D})$. Now perform the construction of Example 6.14 with $c_1 = c_2 = \ldots = c_k = 1$ to get a Kähler metric ω_X on X, with corresponding Hermitian metric h_{K_X} on K_X. The construction also produces a Hermitian metric h_D on L_D. This, along with $h_{K_{\overline{X}} \otimes L_D}$, gives a Hermitian metric $h_{K_{\overline{X}}}$ on $K_{\overline{X}}$. Then

$$- \operatorname{Ric}(\omega_X) + \sqrt{-1}\partial\overline{\partial} \log \frac{h_{K_X}}{h_{K_{\overline{X}}} \prod_{i=1}^k |\sigma_i|^2_{L_i} \log^2 |\sigma_i|^{-2}_{L_i}} = \omega_X \tag{6.19}$$

on X. From the standard spatial asymptotics, $\log \frac{h_{K_X}}{h_{K_{\overline{X}}} \prod_{i=1}^k |\sigma_i|^2_{L_i} \log^2 |\sigma_i|^{-2}_{L_i}}$ has bounded covariant derivatives (with respect to ω_X). We can now apply Theorem 5.1 or [4, Theorem 1.1] to ω_X.

However, to be more general, suppose that f_1 is any smooth function on X so that

(1) f_1 has bounded covariant derivatives (with respect to ω_X) and
(2) $\omega_X + \sqrt{-1}\partial\overline{\partial} f_1$ is a Kähler metric which is biLipschitz equivalent to ω_X.

Then

$$\begin{aligned} &- \operatorname{Ric}(\omega_X + \sqrt{-1}\partial\overline{\partial} f_1) + \\ &\sqrt{-1}\partial\overline{\partial} \left(\log \frac{\omega_X^n h_{K_X}}{(\omega_X + \sqrt{-1}\partial\overline{\partial} f_1)^n h_{K_{\overline{X}}} \prod_{i=1}^k |\sigma_i|^2_{L_i} \log^2 |\sigma_i|^{-2}_{L_i}} + f_1 \right) = \\ &\omega_X + \sqrt{-1}\partial\overline{\partial} f_1. \end{aligned} \tag{6.20}$$

Putting

$$f = \log \frac{\omega_X^n h_{K_X}}{(\omega_X + \sqrt{-1}\partial\overline{\partial} f_1)^n h_{K_{\overline{X}}} \prod_{i=1}^k |\sigma_i|^2_{L_i} \log^2 |\sigma_i|^{-2}_{L_i}} + f_1, \tag{6.21}$$

Theorem 5.1, or [4, Theorem 1.1], implies that the normalized Kähler-Ricci flow starting with the initial metric $\omega_0 = \omega_X + \sqrt{-1}\partial\overline{\partial} f_1$ converges to a complete Kähler-Einstein metric on X with Einstein constant -1. (Such a metric is necessarily unique.) From the evolution formulas for the volume and scalar curvature under Ricci flow, one easily shows that the Kähler-Einstein metric has finite volume. In the case of complex dimension one, we recover some of the results of [15].

7. Preservation of standard spatial asymptotics

In this section we show that the property of having standard spatial asymptotics is preserved by the Kähler-Ricci flow. We use this to give an upper bound on the singularity time T_{sing}.

Theorem 7.1. *Suppose that $\omega_X(0)$ has standard spatial asymptotics associated to $\{\omega_{D_I^{int}}(0)\}$ and $\{c_i\}_{i=1}^m$. Suppose that the normalized Kähler-Ricci flow $\omega_X(t)$, with initial Kähler form $\omega_X(0)$, exists on a maximal time interval $[0,T)$ in the sense of Theorem 4.1. Then for all $t \in [0,T)$, $\omega_X(t)$ has standard asymptotics associated to $\{\omega_{D_I^{int}}(t)\}$ and $\{1 + (c_i - 1)e^{-t}\}_{i=1}^m$, where $\omega_{D_I^{int}}(t)$ is a normalized Kähler-Ricci flow solution on D_I^{int} with initial Kähler-form $\omega_{D_I^{int}}(0)$.*

Proof. Suppose first that D is a smooth divisor C with a trivial holomorphic normal bundle. Then there is a biholomorphic map $F : \Delta \times C \to V$ to a neighborhood V of C, with F restricting to the identity map from $\{0\} \times C$ to $C \subset V$. The restriction $F\Big|_{\Delta^* \times C}$: $\Delta^* \times C \to V \cap X$ has a lift to a holomorphic covering map $\widetilde{F} : H \times C \to V \cap X$. Suppose that the conclusion of the theorem is not true. Then for some $t' \in [0, T)$ and some $\epsilon > 0$, there is a sequence $r_j \to \infty$ so that each $\alpha_{r_j}^* \widetilde{F}^* \omega_X(t')$ has distance at least ϵ from $(1+(c_1-1)e^{-t'}) \frac{\sqrt{-1}}{2} \frac{dz^1 \wedge \overline{z}^1}{\Im(z^1)^2} + \omega_C(t')$ in the pointed C^∞-topology. (We use basepoint $\{\sqrt{-1}\} \times c_0$ for some arbitrary $c_0 \in C$.)

From our assumptions, there is a uniform positive lower bound on the injectivity radius of $\widetilde{F}^* \omega_X(0)$ at $\alpha_{r_j}(\{\sqrt{-1}\} \times c_0)$ or, equivalently, of $\alpha_{r_j}^* \widetilde{F}^* \omega_X(0)$ at $\{\sqrt{-1}\} \times c_0$. By Hamilton's compactness theorem [14], after passing to a subsequence, there is a pointed limit

$$\lim_{j \to \infty} \Big(H \times C, \{\sqrt{-1}\} \times c_0, \alpha_{r_j}^* \widetilde{F}^* \omega_X(t) \Big) = \big(H \times C, \{\sqrt{-1}\} \times c_0, \omega_\infty(t) \big) \tag{7.2}$$

for some normalized Ricci flow solution $\omega_\infty(t)$ on $H \times C$ that exists for $t \in [0, T)$, with bounded curvature on compact time intervals. (Note that in taking the limit we do not have to perform diffeomorphisms.) Also by assumption, $\lim_{j \to \infty} \alpha_{r_j}^* \widetilde{F}^* \omega_X(0) = c_1 \frac{\sqrt{-1}}{2} \frac{dz^1 \wedge d\overline{z}^1}{\Im(z^1)^2} + \omega_C(0)$. From the uniqueness of Ricci flow solutions with bounded curvature on compact time intervals [5], it follows that $\omega_\infty(t') = (1 + (c_1 - 1)e^{-t'}) \frac{\sqrt{-1}}{2} \frac{dz^1 \wedge d\overline{z}^1}{\Im(z^1)^2} + \omega_C(t')$, where $\omega_C(\cdot)$ is a normalized Ricci flow solution on C. This is a contradiction, thereby proving the theorem in this special case.

We now discuss the case when D is a smooth divisor C but its holomorphic normal bundle need not be trivial. In this case we may not be able to use the covering space argument from before; for example, if V is a tubular neighborhood of C then $V - C$ may be simply connected and we cannot increase the injectivity radius by passing to a cover. On the other hand, in some sense this problem is irrelevant, since we can localize on C and parametrize a neighborhood $U \subset \overline{X}$ of $\overline{x} \in C$ by a holomorphic map $F_{\overline{x}} : \Delta^n \to \overline{X}$ with $F_{\overline{x}}(0) = \overline{x}$ and $F_{\overline{x}}(\{0\} \times \Delta^{n-1}) \subset C$. Then we can consider the pullback metric $\widetilde{F}_{\overline{x}}^* \omega_X(t)$ on the cover $H \times \Delta^{n-1}$ of $\Delta^* \times \Delta^{n-1}$ and try to run the previous argument. However, there is a new problem because the metric on $H \times \Delta^{n-1}$ is not complete, whereas we need complete metrics to apply Ricci flow compactness. Again, this problem is somewhat irrelevant, since we should be able to patch together the local parametrizations $F_{\overline{x}} : \Delta^n \to \overline{X}$ as $\overline{x}$ varies over C and thereby effectively pass to the setting of complete metrics. To do so, it is convenient to use the language of étale groupoids. We use the notion of a Ricci flow on an étale groupoid, as explained in [17, Section 5] and [18, Section 3].

Let us first reformulate the earlier setting, when the holomorphic normal bundle is trivial, in terms of étale groupoids. Let T_v denote translation in H by $v \in \mathbb{R}$. Then $\alpha_r^{-1} T_v \alpha_r = T_{r^{-1}v}$. It follows that $\alpha_r^* \widetilde{F}^* \omega_X(0)$ is invariant under translation by $2\pi r^{-1}\mathbb{Z}$. Then $\lim_{r \to \infty} \alpha_r^* \widetilde{F}_{\overline{x}}^* \omega_X(0) = \frac{\sqrt{-1}}{2} \frac{dz^1 \wedge d\overline{z}^1}{\Im(z^1)^2} + \omega_C(0)$, where the right-hand side is invariant under translation by $\lim_{r \to \infty} 2\pi r^{-1}\mathbb{Z} = \mathbb{R}_\delta$ on H. Here $\mathbb{R}_\delta$ denotes the group $\mathbb{R}$ with the discrete topology. Equivalently, the pointed limit

$$\lim_{r\to\infty}\left(X,\widetilde{F}(\alpha_r(\sqrt{-1}),c_0),\omega_X(0)\right)\cong\lim_{r\to\infty}\left(\Delta^*\times C,(\pi(\alpha_r(\sqrt{-1})),c_0),F^*\omega_X(0)\right) \tag{7.3}$$

exists as a pointed Riemannian groupoid, whose underlying étale groupoid is the cross-product groupoid $(H\times C)\rtimes\mathbb{R}_\delta$, with the Kähler form on the space of units $H\times C$ being $c_1\frac{\sqrt{-1}}{2}\frac{dz^1\wedge d\bar{z}^1}{\Im(z^1)^2}+\omega_C(0)$. Then the normalized Kähler-Ricci flow solution on the étale groupoid is given by the $\mathbb{R}_\delta$-invariant normalized Kähler-Ricci flow solution $((1+(c_1-1)e^{-t}))\frac{\sqrt{-1}}{2}\frac{dz^1\wedge d\bar{z}^1}{\Im(z^1)^2}+\omega_C(t)$ on the space of units $H\times C$.

In the case when the holomorphic normal bundle of C need not be trivial, we take $\overline{x}\in C$ and choose a local parametrization $F_{\overline{x}}:\Delta^n\to\overline{X}$ with $F_{\overline{x}}(0)=\overline{x}$ and $F_{\overline{x}}(\{0\}\times\Delta^{n-1})\subset C$. Then the pointed limit $\lim_{r\to\infty}\left(X,\widetilde{F}_{\overline{x}}(\alpha_r(\sqrt{-1}),0),\omega_X(0)\right)$ exists as a pointed Riemannian groupoid, whose underlying étale groupoid is the cross-product groupoid $(H\times C)\rtimes\mathbb{R}_\delta$, with the Kähler form on the space of units $H\times C$ being $c_1\frac{\sqrt{-1}}{2}\frac{dz^1\wedge d\bar{z}^1}{\Im(z^1)^2}+\omega_C(0)$. Again, the normalized Kähler-Ricci flow solution on the étale groupoid is given by the $\mathbb{R}_\delta$-invariant normalized Kähler-Ricci flow solution $((1+(c_1-1)e^{-t}))\frac{\sqrt{-1}}{2}\frac{dz^1\wedge d\bar{z}^1}{\Im(z^1)^2}+\omega_C(t)$ on the space of units $H\times C$, where $\omega_C(\cdot)$ is a normalized Kähler-Ricci flow solution on C.

Now the uniqueness argument of [5] extends to Ricci flow solutions on étale groupoids. Along with the compactness result for Ricci flow solutions on étale groupoids [17, Theorem 1.4], we can prove the theorem using a contradiction argument as before.

Finally, in the case of general D, suppose that $\overline{x}\in D_I^{int}$. Let $F_{\overline{x}}:\Delta^n\to U$ be the holomorphic parametrization near $\overline{x}$. Then the pointed limit $\lim_{r\to\infty}\left(X,\widetilde{F}_{\overline{x}}(\alpha_r(\sqrt{-1},\ldots,\sqrt{-1}),0),\omega_X(0)\right)$ exists as a pointed Riemannian groupoid, whose underlying étale groupoid is the cross-product groupoid $(H^m\times D_I^{int})\rtimes\mathbb{R}^m_\delta$, with the Kähler form on the space of units $H^m\times D_I^{int}$ being $\sum_{i=1}^m c_i\frac{\sqrt{-1}}{2}\frac{dz^i\wedge d\bar{z}^i}{\Im(z^i)^2}+\omega_{D_I^{int}}(0)$. The normalized Kähler-Ricci flow solution on the étale groupoid is given by the $\mathbb{R}^m_\delta$-invariant normalized Kähler-Ricci flow solution $\sum_{i=1}^k((1+(c_i-1)e^{-t}))\frac{\sqrt{-1}}{2}\frac{dz^i\wedge d\bar{z}^i}{\Im(z^i)^2}+\omega_{D_I^{int}}(t)$ on the space of units $H^m\times D_I^{int}$, where $\omega_{D_I^{int}}(t)$ is a complete normalized Kähler-Ricci flow solution on D_I^{int}. The theorem now follows from a contradiction argument as before. $\square$

Remark 7.4. It follows that under the hypotheses of Theorem 7.1, the normalized Kähler-Ricci flow exists on each D_I^{int} for a time interval of at least $[0,T)$, with bounded curvature on compact subintervals of $[0,T)$. Note in this regard that Theorem 1.1 is consistent with passing to the divisor, in the sense that $(K_{\overline{X}}+L_{D_I})\big|_{D_I}=K_{D_I}$, and if c is a Kähler class on $\overline{X}$ then its pullback to D_I is a Kähler class on D_I.

Remark 7.5. Continuing with the previous remark, the divisor D is itself a complex space in the sense of [11, Section II.5]. This suggests that one should be able to extend the results of this paper from the setting of pairs $(\overline{X},D)$ to the setting of complex spaces Y, or some class thereof. For example, a standard Kähler metric on such a complex space would consist of complete Kähler metrics on the strata Y_j-Y_{j-1} of Y having "standard" spatial asymptotics as one approaches (in Y_j) a substratum Y_k of Y_j.

Corollary 7.6. *Let $u_{D_I^{int}}(t) \in C^\infty(D_I^{int})$ be the time-t solution of (3.5) on D_I^{int}. Then the time-t solution $u_X(t) \in C^\infty(X)$ of (3.5) on X has standard spatial asymptotics associated to $\{\mathrm{const.}_I(t) + u_{D_I^{int}}(t)\}$, where $\mathrm{const.}_I(t)$ is spatially constant.*

Proof. This follows from (3.7) and Theorem 7.1. □

Corollary 7.7. *Suppose that $\omega_X(0)$ has standard spatial asymptotics associated to $\{\omega_{D_I^{int}}(0)\}$ and $\{c_i\}_{i=1}^m$. Let $T_1 = T_2$ be the parameter of Theorem 4.1.*

Let T_3 be the supremum (possibly infinite) of the numbers T' so that there is a smooth solution for u in (3.5) on the time interval $[0, T']$ such that $\omega_t + \sqrt{-1}\partial\bar{\partial}u$ is a Kähler metric with standard spatial asymptotics associated to $\{\omega_{D_I^{int}}(t)\}$ and $\{(1 + (c_i - 1)e^{-t})\}_{i=1}^k$.

Let T_4 be the supremum (possibly infinite) of the numbers T for which there is a function $F_T \in C^\infty(X)$, with standard spatial asymptotics associated to $\{u_{D_I^{int}}(T)\}$, such that $\omega_T + \sqrt{-1}\partial\bar{\partial}F_T$ is a Kähler metric .

Then $T_1 = T_2 = T_3 = T_4$.

Proof. This follows from Theorem 4.1 and Corollary 7.6. □

Corollary 7.8. *Under the hypotheses of Theorem 7.1, the maximal existence time $T_{\mathrm{sing}} \in (0, \infty]$ of the normalized Kähler-Ricci flow on X is bounded above by*

$$\sup\{t \in \mathbb{R}^+ : e^{-t}[\omega_X(0)] + 2\pi(1 - e^{-t})[K_{\overline{X}} + D] \text{ lies in the Kähler cone of } \overline{X}\}. \tag{7.9}$$

Proof. Suppose that $T' < T_{\mathrm{sing}}$. From Theorem 6.6 and the normalized Ricci flow equation,

$$\frac{d}{dt}[\omega_X(t)] = 2\pi[K_{\overline{X}} + D] - [\omega_X(t)] \tag{7.10}$$

in $\mathrm{H}^{(1,1)}(\overline{X}; \mathbb{R})$. Thus

$$[\omega_X(T')] = e^{-T'}[\omega_X(0)] + 2\pi(1 - e^{-T'})[K_{\overline{X}} + D]. \tag{7.11}$$

Also from Theorem 6.6, $[\omega_X(T')]$ is a Kähler class on $\overline{X}$. The corollary follows. □

8. Superstandard spatial asymptotics

In this section we introduce the notion of superstandard spatial asymptotics. We show that having this property is preserved under the Ricci flow. We then prove the first part of Theorem 1.1.

We first prove a lemma regarding the singular support of the $\partial\overline{\partial}$-operator applied to certain functions. In general, if $X = \overline{X} - D$ and $J \in C^\infty(X)$, let $\overline{J}$ be the extension of J by zero to $\overline{X}$. We can construct $\sqrt{-1}\partial\overline{\partial}J \in \Omega^{(1,1)}(X)$ and extend it by zero to $\overline{X}$, or we could also consider the $(1,1)$-current $\sqrt{-1}\partial\overline{\partial}\overline{J}$ on $\overline{X}$. These two currents do not have to be the same. For example, if $\overline{X} = S^2$ and $D = \mathrm{pt}$, suppose that $J \in C^\infty(S^2 - \mathrm{pt})$ equals $\log|z|$ in a neighborhood of $\mathrm{pt} = \{0\}$. The extension by zero of $\partial\overline{\partial}J$ to S^2 has no singular support and represents a nonzero class in $\mathrm{H}^2(S^2; \mathbb{R})$, whereas the current $\sqrt{-1}\partial\overline{\partial}\overline{J}$ has singular support at pt and vanishes in $\mathrm{H}^2(S^2; \mathbb{R})$.

The next lemma gives a sufficient condition on J for the two extensions to agree on $\overline{X}$.

Lemma 8.1. *Let $\omega_{\overline{X}}$ be a smooth Kähler form on $\overline{X}$. Given $J \in C^\infty(X)$, suppose that $|J(x)| = o(\log \prod_{i=1}^k |\sigma_i|_{L_i}^{-2})$ as $x \to D$. Suppose that $\sqrt{-1}\partial\overline{\partial}J$ has locally finite mass on $\overline{X}$ and there is some $C > 0$ so that when restricted to $X \subset \overline{X}$,*

$$\sqrt{-1}\partial\overline{\partial}J \geq -C\omega_{\overline{X}}. \tag{8.2}$$

If $\overline{J}$ denotes the extension of J by zero to $\overline{X}$ then the current $\sqrt{-1}\partial\overline{\partial}\overline{J}$ on $\overline{X}$ equals the extension by zero of the current $\sqrt{-1}\partial\overline{\partial}J$ on X.

Proof. The extension of J by zero to $\overline{X}$ defines an L^1-function $\overline{J}$ on $\overline{X}$ and hence a distribution on $\overline{X}$ (which in fact is independent of the extension). Thus it makes sense to talk about the current $\sqrt{-1}\partial\overline{\partial}\overline{J}$ on $\overline{X}$. From (8.2), $\sqrt{-1}\partial\overline{\partial}\overline{J}$ is measurable. (See [11, Theorem I.(5.8)] for the analogous statement about plurisubharmonic functions.)

Let $\overline{\sqrt{-1}\partial\overline{\partial}J}$ denote the extension by zero, to $\overline{X}$, of the current $\sqrt{-1}\partial\overline{\partial}J$ on X. By Theorem 6.6, $\overline{\sqrt{-1}\partial\overline{\partial}J}$ is a closed $(1,1)$-current on $\overline{X}$. Putting $T = \sqrt{-1}\partial\overline{\partial}\overline{J} - \overline{\sqrt{-1}\partial\overline{\partial}J}$ gives a closed measurable current which is supported on D and which is nonnegative by (8.2). By [11, Corollary III.(2.14)], $T = \sum_{i=1}^k c_i \delta_{D_i}$ for some nonnegative constants $\{c_i\}_{i=1}^k$. However, it is easy to show that if c_i is nonzero then J has a logarithmic singularity near D_i; see the Green-Riesz formula [11, Proposition I.(4.22a)] and [11, Example III.(6.9)]. This contradicts the assumption on J. □

To motivate the definition of superstandard spatial asymptotics, we first prove a result about the Ricci curvature of a metric with standard spatial asymptotics.

Lemma 8.3. *If ω_X has standard spatial asymptotics then we can write*

$$\mathrm{Ric}(\omega_X) = \eta'_{\overline{X}} - \sqrt{-1}\partial\overline{\partial}\left(- \sum_{i=1}^k \log\log^2 |\sigma_i|_{L_i}^{-2} + H' \right) \tag{8.4}$$

on X, where

- *$\eta'_{\overline{X}}$ is a smooth closed $(1,1)$-form on $\overline{X}$ whose de Rham cohomology class is $-[K_{\overline{X}} + D]$, and*
- *$H' \in C^\infty(X) \cap L^\infty(X)$.*

Proof. Choose a Hermitian metric $h_{K_{\overline{X}} \otimes L_D}$ on $K_{\overline{X}} \otimes L_D$. Along with Hermitian metrics $\{h_{L_i}\}_{i=1}^k$ on $\{L_i\}_{i=1}^k$, we obtain a Hermitian metric $h_{K_{\overline{X}}}$ on $K_{\overline{X}}$. Then

$$\mathrm{Ric}(\omega_X) = -\sqrt{-1}F(h_{K_{\overline{X}}\otimes L_D}) - \sqrt{-1}\partial\overline{\partial}\left(- \sum_{i=1}^k \log\log^2 |\sigma_i|_{L_i}^{-2} + \log \frac{h_{K_{\overline{X}}} \prod_{i=1}^k |\sigma_i|_{L_i}^2 \log^2 |\sigma_i|_{L_i}^{-2}}{h_{K_X}} \right) \tag{8.5}$$

on X. Put $\eta'_{\overline{X}} = -\sqrt{-1}F(h_{K_{\overline{X}}\otimes L_D})$ and $H' = \log \frac{h_{K_{\overline{X}}} \prod_{i=1}^k |\sigma_i|_{L_i}^2 \log^2 |\sigma_i|_{L_i}^{-2}}{h_{K_X}}$. By the standard spatial asymptotics, $H' \in L^\infty(X)$. The lemma follows. □

Remark 8.6. It follows from elliptic estimates that for each $k \geq 0$, the function H' has uniform bounds on its k-th covariant derivatives; see the end of the proof of Theorem 8.16.

Recall that Definition 6.1 of standard asymptotics involves some parameters $\{c_i\}_{i=1}^k$.

Definition 8.7. A Kähler metric ω_X on X has superstandard spatial asymptotics if it has standard spatial asymptotics and one can write

$$\omega_X = \eta_{\overline{X}} - \sqrt{-1}\partial\overline{\partial}\left(\sum_{i=1}^{k} c_i \log\log^2 |\sigma_i|_{L_i}^{-2} + H\right) \tag{8.8}$$

where

- $\eta_{\overline{X}}$ is a smooth closed $(1,1)$-form on $\overline{X}$,
- h_{L_i} is a Hermitian metric on the line bundle L_i and
- $H \in C^\infty(X) \cap L^\infty(X)$.

Example 8.9. If $\overline{X} = S^2$ and $D = \mathrm{pt}$, suppose that in terms of a local coordinate z near pt, the metric takes the form $\omega_X = -\sqrt{-1}\partial\overline{\partial}\left(\log\log^2 |z|^{-2} + \log\log\log^2 |z|^{-2}\right)$. Then ω_X has standard asymptotics but does not have superstandard asymptotics.

Lemma 8.10. *The property of having superstandard spatial asymptotics is independent of the choice of Hermitian metrics* $\{h_{L_i}\}_{i=1}^k$.

Proof. Given a Hermitian metric h_{L_i}, any other Hermitian metric on L_i can be written as $\phi_i h_{L_i}$ for some positive $\phi_i \in C^\infty(\overline{X})$.Then

$$\log\log^2(\phi_i^{-1}|\sigma_i|_{L_i}^{-2}) - \log\log^2 |\sigma_i|_{L_i}^{-2} = 2\log\left(1 + \frac{\log \phi_i^{-1}}{\log |\sigma_i|_{L_i}^{-2}}\right), \tag{8.11}$$

which is bounded on $\overline{X}$. The lemma follows. □

Example 8.12. Continuing with Example 6.14, one can check that ω has superstandard spatial asymptotics.

Lemma 8.13. *If* ω_X *has superstandard spatial asymptotics then* $[\omega_X] = [\eta_{\overline{X}}]$ *in* $\mathrm{H}^{(1,1)}(\overline{X})$.

Proof. Let $\omega_{\overline{X}}$ be a smooth Kähler form on $\overline{X}$. First, from (8.8) and the definition of standard asymptotics, if $\overline{C} > 0$ is sufficiently large then there is some $C > 0$ so that $\sqrt{-1}\partial\overline{\partial}(H - \overline{C}\sum_{i=1}^k c_i \log\log^2 |\sigma_i|_{L_i}^{-2}) \geq -C\omega_{\overline{X}}$ on X. Lemma 8.1 implies that the extension of $\sqrt{-1}\partial\overline{\partial}(H - \overline{C}\sum_{i=1}^k c_i \log\log^2 |\sigma_i|_{L_i}^{-2})$ by zero to $\overline{X}$ vanishes in $\mathrm{H}^{(1,1)}(\overline{X})$. It also follows from Lemma 8.1 that the extension of $-\sqrt{-1}\partial\overline{\partial}\sum_i c_i \log\log^2 |\sigma_i|_{L_i}^{-2}$ vanishes in $\mathrm{H}^{(1,1)}(\overline{X})$; see (6.15). Thus $[\omega_X] = [\eta_{\overline{X}}] \in \mathrm{H}^{(1,1)}(\overline{X})$. □

Theorem 8.14. *Suppose that* $\omega_X(0)$ *has superstandard spatial asymptotics. Suppose that the normalized Kähler-Ricci flow* $\omega_X(t)$*, with initial Kähler metric* $\omega_X(0)$*, exists on a maximal time interval* $[0,T)$ *in the sense of Theorem 4.1. Then for all* $t \in [0,T)$*,* $\omega_X(t)$ *has superstandard spatial asymptotics.*

Proof. Recall the definition of ω_t from (3.4). Applying (8.4) and (8.8) to $\omega_X(0)$, we can write

$$\begin{aligned}(8.15)\quad \omega_X(t) =& \omega_t + \sqrt{-1}\partial\overline{\partial}u(t) \\ =& -\eta'_{\overline{X}} + e^{-t}(\eta_{\overline{X}} + \eta'_{\overline{X}}) - \\ & \sqrt{-1}\partial\overline{\partial}\left(\sum_i (1+e^{-t}(c_i-1))\log\log^2|\sigma_i|^{-2}_{L_i} - H' + e^{-t}(H+H') - u(t)\right).\end{aligned}$$

From Corollary 7.6, $u(t) \in L^\infty(X)$. □

Theorem 8.16. *Suppose that $\omega_X(0)$ has superstandard spatial asymptotics. Then the maximal existence time $T \in (0,\infty]$ of the Kähler-Ricci flow on X, in the sense of Theorem 4.1, equals*

$$(8.17)\qquad \sup\{t \in \mathbb{R}^+ : e^{-t}[\omega_X(0)] + 2\pi(1-e^{-t})[K_{\overline{X}} + D] \text{ lies in the Kähler cone of } \overline{X}\}.$$

Proof. From Theorem 4.1, it suffices to show that if $e^{-t}[\omega_X(0)] + 2\pi(1-e^{-t})[K_{\overline{X}} + D]$ lies in the Kähler cone of $\overline{X}$ then there is a function $F_t \in C^\infty(X)$ such that

(1) $\omega_t + \sqrt{-1}\partial\overline{\partial}F_t$ is a Kähler metric which is biLipschitz equivalent to $\omega_X(0)$, and
(2) For each k, the k-th covariant derivatives of F_t (with respect to the initial metric $\omega_X(0)$) are uniformly bounded.

Suppose that $\omega_{\overline{X}}$ is a Kähler metric on $\overline{X}$ whose class in $\mathrm{H}^{(1,1)}(\overline{X})$ equals $e^{-t}[\omega_X(0)] + 2\pi(1-e^{-t})[K_{\overline{X}} + D]$. We construct a Kähler metric ω_X on X as in Example 8.12, using the constants $\{1 + e^{-t}(c_i - 1)\}_{i=1}^k$. We now write

$$(8.18)\qquad \omega_X = \omega_t + \sqrt{-1}\partial\overline{\partial}F$$

and show that we can solve for F. That is, we show that we can solve

$$(8.19)\qquad \sqrt{-1}\partial\overline{\partial}F = \omega_X + \mathrm{Ric}(\omega_X(0)) - e^{-t}\left(\omega_X(0) + \mathrm{Ric}(\omega_X(0))\right).$$

From Lemma 8.10, for the purposes of the proof we can assume that the Hermitian metrics h_{L_i} are the same in the construction of ω_X and in the superstandard behavior of $\omega_X(0)$. Let $\eta_{\overline{X}}$ and $\eta'_{\overline{X}}$ be the $(1,1)$-forms on $\overline{X}$ involved in the superstandard behavior of $\omega_X(0)$. From (6.15), (8.4) and (8.8), we can write

$$\begin{aligned}(8.20)\quad \omega_X + \mathrm{Ric}(\omega_X(0)) - e^{-t}\left(\omega_X(0) + \mathrm{Ric}(\omega_X(0))\right) =& \omega_{\overline{X}} + \eta'_{\overline{X}} - e^{-t}(\eta_{\overline{X}} + \eta'_{\overline{X}}) \\ & - \sqrt{-1}\partial\overline{\partial}(H' - e^{-t}(H+H')).\end{aligned}$$

From Lemma 8.13 and our assumption on $\omega_{\overline{X}}$, we know that $\omega_{\overline{X}}$ and $-\eta'_{\overline{X}} + e^{-t}(\eta_{\overline{X}} + \eta'_{\overline{X}})$ both represent the same class in $\mathrm{H}^{(1,1)}(\overline{X})$, namely $e^{-t}[\omega_X(0)] + 2\pi(1-e^{-t})[K_{\overline{X}} + D]$. Thus

$$(8.21)\qquad \omega_{\overline{X}} + \eta'_{\overline{X}} - e^{-t}(\eta_{\overline{X}} + \eta'_{\overline{X}}) = \sqrt{-1}\partial\overline{\partial}f$$

for some $f \in C^\infty(\overline{X})$.

From (8.20) and (8.21), we can solve (8.18) for some $F \in L^\infty(X)$. From (8.18), the Laplacian $\triangle_{\omega_X(0)}F = \mathrm{Tr}\left(\omega_X(0)^{-1}\sqrt{-1}\partial\overline{\partial}F\right)$ has bounded k-th covariant derivatives (with respect to $\omega_X(0)$) for each k. By elliptic regularity (where near the divisor we work on the covering spaces $H^m \times \Delta^{n-m}$, which have bounded geometry), we conclude that F also has bounded k-th covariant derivatives for each k. This proves the theorem. $\square$

This finishes the proof of the first part of Theorem 1.1. Theorem 1.1 is stated for the unnormalized Kähler-Ricci flow (3.1) instead of the normalized Kähler-Ricci flow (3.2), so one has to make the translation between the two.

9. Singularity type

In this section we give sufficient conditions for the Kähler-Ricci flow on a quasiprojective manifold to have a type-II singularity. We give examples in which this happens.

Theorem 9.1. *Suppose that $\omega_X(t)$ is a Kähler-Ricci flow solution on a quasiprojective manifold $X = \overline{X} - D$, $D \neq \emptyset$, whose initial metric $\omega_X(0)$ has superstandard spatial asymptotics. Suppose that the maximal existence time T_{sing}, in the sense of Theorem 4.1, is finite. Suppose that there is a number $C > 0$ so that for all $t \in [0, T_{\text{sing}})$, we have $\operatorname{vol}(X, g(t)) = \frac{1}{n!} \int_X \omega_X(t)^n \leq C(T_{\text{sing}} - t)^n$. Then the Ricci flow has a type-II singularity at time T_{sing}, i.e. $\limsup_{t \to T_{\text{sing}}} \left((T_{\text{sing}} - t) \sup_{x \in X} |\operatorname{Rm}(x,t)| \right) = \infty$.*

Proof. If the theorem is not true then there is some $C' > 0$ so that for all $x \in X$ and $t \in [0, T_{\text{sing}})$, we have $|\operatorname{Rm}(x,t)| \leq \frac{C'}{T_{\text{sing}} - t}$. From [20, Theorem 1.4], for any $x' \in X$ there is a sequence of times $t_i \to T_{\text{sing}}$ so that if we put $\tau_i = T_{\text{sing}} - t_i$ then the rescaled Ricci flow solutions $g_i(x,t) = \tau_i^{-1} g\left(x, T_{\text{sing}} + t\tau_i\right)$ have a pointed limit $(X, g_i, (x', -1)) \stackrel{i \to \infty}{\to} (Y, g_\infty, (y_\infty, -1))$. Here (Y, g_∞) is a complete gradient shrinking soliton with bounded curvature which is κ-noncollapsed at all scales, for some $\kappa > 0$, in the sense of Perelman [21]. (Note that there is no $\kappa > 0$ so that the initial metric $\omega_X(0)$ is κ-noncollapsed at all scales. Nevertheless, in this setting the blowup limit is κ-noncollapsed at all scales for some κ; see [20, Remark 2.2].)

From our assumption, $(Y, g(-1))$ has finite volume. However, the κ-noncollapsing now implies that Y is compact. (We thank Lei Ni for this remark.) Namely, if Y is noncompact then it contains an infinite sequence of disjoint unit balls. The κ-noncollapsing, along with the bounded curvature, implies that there is a uniform positive lower bound on the volumes of these balls. This is a contradiction.

Thus Y is compact. This implies that X is compact, which is a contradiction. The theorem follows. $\square$

Corollary 9.2. *Suppose that $\omega_X(t)$ is a Kähler-Ricci flow solution on a quasiprojective manifold $X = \overline{X} - D$, $D \neq \emptyset$, whose initial metric $\omega_X(0)$ has superstandard spatial asymptotics. If $T_{\text{sing}} < \infty$ and $\lim_{t \to T_{\text{sing}}} [\omega_X(t)] = 0$ in $\mathrm{H}^{(1,1)}(\overline{X}; \mathbb{R})$ then there is a type-II singularity at time T_{sing}.*

Proof. From the smoothness of $[\omega_X(t)]$, we can write $[\omega_X(t)] = (T_{\text{sing}} - t) R(t)$ for some smooth function $R : [0, T_{\text{sing}}] \to \mathrm{H}^{(1,1)}(\overline{X}; \mathbb{R})$. Then there is a constant $C < \infty$ so that for

$t \in [0, T_{\text{sing}})$,

$$\int_X \omega_X^n(t) = \int_{\overline{X}} [\omega_X(t)]^n \leq C(T_{\text{sing}} - t)^n. \tag{9.3}$$

The corollary follows. □

This finishes the proof of the second part of Theorem 1.1. We now give some examples, using the unnormalized Kähler-Ricci flow of Theorem 1.1.

Example 9.4. Suppose that $\overline{X} = S^2$ and $D = \text{pt}$, so $X = S^2 - \text{pt} = \mathbb{R}^2$. Let $[S^2] \in \mathrm{H}^{(1,1)}(S^2; \mathbb{R}) \cap \mathrm{H}^2(S^2; \mathbb{Z})$ denote the fundamental class in cohomology. Then $[K_{\overline{X}}] = -2[S^2]$ and $[D] = [S^2]$. From Theorem 1.1, T_{sing} is the supremum of the numbers $T > 0$ so that $[\omega_0] - 2\pi T[S^2] \in \mathrm{H}^{(1,1)}(\overline{X}; \mathbb{R})$ is a Kähler class. That is, $T_{\text{sing}} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega_X(0) = \frac{1}{2\pi} \operatorname{Vol}(\mathbb{R}^2, g(0))$. (As we are now dealing with the unnormalized Kähler-Ricci equation $\frac{d\omega}{dt} = -\operatorname{Ric}$, the singularity time given here differs by a factor of two from the result $\frac{1}{4\pi} \operatorname{Vol}(\mathbb{R}^2, g(0))$ stated in the introduction for the unnormalized Ricci flow $\frac{dg}{dt} = -2 \operatorname{Ric}$.)

As $[\omega_0] - 2\pi T_{\text{sing}}[S^2]$ vanishes in $\mathrm{H}^{(1,1)}(\overline{X}; \mathbb{R})$, we conclude that there is a type-II singularity at time T_{sing}, in agreement with the results of Daskalopoulos-del Pino-Hamilton-Sesum [7, 8, 9, 10].

Example 9.5. Taking a product of the previous example with S^2, suppose that $\overline{X} = S^2 \times S^2$ and $D = \{\text{pt}\} \times S^2$. Let $[S^2]_1, [S^2]_2 \in \mathrm{H}^{(1,1)}(S^2 \times S^2; \mathbb{R}) \cap \mathrm{H}^2(S^2 \times S^2; \mathbb{Z})$ denote the fundamental classes of the two sphere factors. Then $K_{\overline{X}} = -2[S^2]_1 - 2[S^2]_2$ and $D = [S^2]_2$. We conclude that T_{sing} is the supremum of the times T so that $\int_{[S^2]_1} \omega_X(0) - 4\pi T > 0$ and $\int_{[S^2]_2} \omega_X(0) - 2\pi T > 0$.

- If $\frac{\int_{[S^2]_1} \omega_X(0)}{4\pi} < \frac{\int_{[S^2]_2} \omega_X(0)}{2\pi}$ then $T_{\text{sing}} = \frac{\int_{[S^2]_1} \omega_X(0)}{4\pi}$. Since $[\omega_X(0)] + 2\pi T_{\text{sing}}[K_{\overline{X}} + D]$ is nonvanishing, we cannot conclude that there is a type-II singularity. In fact, if the initial metric $\omega_X(0)$ is a product metric then the first S^2-factor shrinks to a point at the singularity time before the other factor can collapse, and we have a type-I singularity.
- If $\frac{\int_{[S^2]_1} \omega_X(0)}{4\pi} = \frac{\int_{[S^2]_2} \omega_X(0)}{2\pi}$ then T_{sing} is this common value. Since $[\omega_X(0)] + 2\pi T_{\text{sing}}[K_{\overline{X}} + D] = 0$, there is a type-II singularity.
- If $\frac{\int_{[S^2]_1} \omega_X(0)}{4\pi} > \frac{\int_{[S^2]_2} \omega_X(0)}{2\pi}$ then $T_{\text{sing}} = \frac{\int_{[S^2]_2} \omega_X(0)}{2\pi}$. Since $[\omega_X(0)] + 2\pi T_{\text{sing}}[K_{\overline{X}} + D]$ is nonvanishing, we cannot conclude that there is a type-II singularity, although there is one if $\omega_X(0)$ is a product metric.

Example 9.6. Suppose that $\overline{X} = \mathbb{C}P^n$ and D consists of k copies of $\mathbb{C}P^{n-1}$ in general position. Let $[H] \in \mathrm{H}^{(1,1)}(\mathbb{C}P^n; \mathbb{R}) \cap \mathrm{H}^2(\mathbb{C}P^n; \mathbb{Z})$ be the hyperplane class. Then $[K_{\overline{X}}] = -(n+1)[H]$ and $[D] = k[H]$, so T_{sing} is the supremum of the numbers $T > 0$ so that $[\omega_0] + 2\pi(-n - 1 + k)T[H] \in \mathrm{H}^{(1,1)}(\overline{X}; \mathbb{R})$ is a Kähler class.

- If $k > n + 1$ then $T_{\text{sing}} = \infty$. In this case there is a finite-volume Kähler-Einstein metric ω_{KE} on X with Einstein constant -1 [16, 24, 27]. Theorem 5.1 says that for a wide class of initial metrics, the normalized Kähler-Ricci flow will converge to ω_{KE}.

- If $k = n+1$ then $T_{\text{sing}} = \infty$. In this case there is a complete Ricci-flat Kähler metric $\omega_{Ricci-flat}$ on X [25]. It should be possible to show that for a large class of initial metrics, the unnormalized Kähler-Ricci flow converges to a multiple of $\omega_{Ricci-flat}$.
- If $k < n+1$ then $T_{\text{sing}} = \frac{\int_{\mathbb{C}P^1 \cap X} \omega_X(0)}{2\pi(n+1-k)}$, where $\mathbb{C}P^1$ denotes a generic complex line in $\overline{X} = \mathbb{C}P^n$. If in addition $k \neq 0$ then there is a type-II singularity.

Note that when $k = 1$, there is a $U(n)$-invariant superstandard initial Kähler metric on $X = \mathbb{C}^n = \mathbb{C}P^n - \mathbb{C}P^{n-1}$. At infinity, it looks like a family of hyperbolic cusps parametrized by $\mathbb{C}P^{n-1}$. It is plausible that in this case, there is a rescaling limit at the singular time which is a $U(n)$-invariant gradient steady Kähler-Ricci soliton. Examples of the latter are in [2]. From [7, 8, 9, 10], we know that there is such a rescaling limit when $n = 1$.

References

[1] H. Cao, "Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds", Invent. Math. 81, p. 359-372 (1985)

[2] H. Cao, "Existence of gradient Kähler-Ricci solitons", in Elliptic and parabolic methods in geometry, A.K. Peters, p. 1-16 (1996)

[3] J. Carlson and P. Griffiths, "A defect relation for equidimensional holomorphic mappings between algebraic varieties", Ann. Math. 95, p. 557-584 (1972)

[4] A. Chau, "Convergence of the Kähler-Ricci flow on noncompact Kähler manifolds", J. Diff. Geom. 66, p. 211-232 (2004)

[5] B.-L. Chen and X.-P. Zhu, "Uniqueness of the Ricci flow on complete noncompact manifolds", J. Diff. Geom. 74, p. 119-154 (2006)

[6] B. Chow, P. Lu and L. Ni, Hamilton's Ricci flow, American Math. Soc., Providence (2006)

[7] P. Daskalopoulos and M. del Pino, "On a singular diffusion equation", Comm. Anal. Goem. 3, p. 523-542 (1995)

[8] P. Daskalopoulos and M. del Pino, "Type II collapsing of maximal solutions to the Ricci flow in $\mathbb{R}^2$", Ann. Inst. H. Poincaré Anal. Nonlinéaire 24, p. 851-874 (2007)

[9] P. Daskalopoulos and R. Hamilton, "Geometric estimates for the logarithmic fast diffusion equation", Comm. Anal. Geom. 12, p. 143-164 (2004)

[10] P. Daskalopoulos and N. Sesum, "On the extinction profile of solutions to fast diffusion", J. Reine Angew. Math. 622, p. 95-119 (2008)

[11] J.-P. Demailly, "Complex analytic and algebraic geometry", http://www-fourier.ujf-grenoble.fr/~demailly/books.html

[12] J.-P. Demailly and M. Paun, "Numerical characterization of the Kähler cone of a compact Kähler manifold", Ann. Math. 159, p. 1247-1274 (2004)

[13] R. Hamilton, "Three-manifolds with positive Ricci curvature", J. Diff. Geom. 17, p. 255-306 (1982)

[14] R. Hamilton, "A compactness property for solutions of the Ricci flow", Amer. J. Math. 117, p. 545-572 (1995)

[15] L. Ji, R. Mazzeo and N. Sesum, "Ricci flow on surfaces with cusps", preprint, http://arxiv.org/abs/math/0703357 (2007)

[16] R. Kobayashi, "Kähler-Einstein metric on an open algebraic manifold", Osaka J. Math. 21, p. 399-418 (1984)

[17] J. Lott, "On the long-time behavior of type-III Ricci flow solutions", Math. Ann. 339, p. 627-666 (2007)

[18] J. Lott, "Dimensional reduction and the long-time behavior of Ricci flow", preprint, http://arxiv.org/abs/0711.4063 to appear, Comm. Math. Helv.

[19] D. Mumford, "Hirzebruch's proportionality theorem in the noncompact case", Invent. Math. 42, p. 239-272 (1977)

[20] A. Naber, "Noncompact shrinking four solitons with nonnegative curvature", preprint, http://arxiv.org/abs/0710.5579 (2008)
[21] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, http://arxiv.org/abs/math/0211159 (2002)
[22] G. Schumacher, "Asymptotics of Kähler-Einstein metrics on quasi-projective manifolds and an extension theorem on holomorphic maps", Math. Ann. 311, p. 631-645 (1998)
[23] W.-X. Shi, "Deforming the metric on complete Riemannian manifolds", J. Diff. Geom. 30, p. 223-301 (1989)
[24] G. Tian and S.-T. Yau, "Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry", in Mathematical aspects of string theory (San Diego, Calif., 1986), p. 574628, Adv. Ser. Math. Phys. 1, World Sci. Publishing, Singapore (1987)
[25] G. Tian and S.-T. Yau, "Complete Kähler manifolds with zero Ricci curvature I", J. Amer. Math. Soc. 3, p. 579-609 (1990)
[26] G. Tian and Z. Zhang, "On the Kähler-Ricci flow on projective manifolds of general type", Chinese Annals of Math. - Series B 27, p. 179-192 (2006)
[27] H. Tsuji, "Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type", Math. Ann. 281, p. 123-133 (1988)
[28] D. Wu, "Higher canonical asymptotics of Kähler-Einstein metrics on quasiprojective manifolds", Comm. Anal. and Geom. 14, p. 795-845 (2006)
[29] S.-T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation I", Comm. Pure Appl. Math. 31, p. 339-411 (1978)

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