Problem Set 1

Q1 Let (X, d) be a metric space. Show that the axioms of a metric imply $d(x, y) \geq 0$ for all $x, y \in X$.

Q2 Let (X, d) be a metric space. Show that below function $d' \colon X \times X \to \mathbb{R}$ defines a metric on X.

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Q3 Why is $\mathbb{Q}^n$ with the induced metric from Euclidean space $\mathbb{R}^n$ not complete?

Q4 Prove Proposition 1.18: If a sequence of continuous functions of metric spaces converges uniformly to a function, then the limiting function is also continuous.

Q5 Let $X = \mathbb{R}^2$ and denote d the Euclidean metric. Let $\mathbf{0} = (0, 0)$, and define

$$d_0(x, y) = \begin{cases} d(x, y) & \text{if } x \text{ and } y \text{ lie on the same ray from the origin;} \\ d(x, \mathbf{0}) + d(\mathbf{0}, y) & \text{otherwise.} \end{cases}$$

Show that d_0 is a metric on X. (This is called the SNCF metric, with Paris at the origin. . .)

Q6 Let (X, d) be a metric space. Consider the function $f \colon [0, \infty) \to [0, \infty)$ having the following properties:

(a) f is non-decreasing, i.e. $f(a) \leq f(b)$ if $0 \leq a \leq b$;
(b) $f(x) = 0$ if and only if $x = 0$;
(c) $f(a + b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$.

For $x, y \in X$, define $d_f(x, y) = f(d(x, y))$. Show that d_f is a metric on X.
Moreover, show that the following functions have the above three properties:
$f(t) = kt$, where $k > 0$; $f(t) = t^\alpha$, where $0 < \alpha \leq 1$; and $f(t) = \frac{t}{t+1}$.

Q7 Let p be a prime number. Define the *p–adic absolute value function* $| \cdot |_p$ on $\mathbb{Q}$ by setting $|x|_p = 0$ when $x = 0$ and $|x|_p = p^{-k}$ when $x = p^k \cdot \frac{m}{n}$, where m and n are non-zero integers which are not divisible by p.
Show that for $x, y \in \mathbb{Q}$,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

and that $d(x, y) = |x - y|_p$ defines a metric on $\mathbb{Q}$. This is called the *p–adic metric.*
In fact, $d(x, z) \leq \max\{d(x, y), d(y, z)\}$, which is stronger than the triangle inequality. A metric satisfying this condition is called an *ultrametric.*

Q8 Let

$$d(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right|$$

for $n, m \in \mathbb{N}$. Then d is a metric.

(a) Let $E \subset \mathbb{N}$ be the set of positive even numbers. Find $\operatorname{diam}(E)$ and $\operatorname{diam}(\mathbb{N} \setminus E)$ in $(\mathbb{N}, d)$.
(b) For a fixed $n \in \mathbb{N}$, find all elements of $B_{\frac{1}{2n}}(n)$ and $B_{\frac{1}{2n}}(2n)$.