Problem Set 5

Q35 Show that a compact subset of a normed space is closed and bounded.

Q36 Let Y be a closed, proper subspace of the normed space X.
Show that for every $\varepsilon > 0$, there is a point $x \in S(X)$ such that $\text{dist}(x, Y) \geq 1 - \varepsilon$.

Q37 Show that the (closed) unit ball in l_1^n is compact.

Q38 Deduce from Lemma 4.8 that a Banach space cannot have a countably infinite algebraic basis; i.e. if $X = \text{span}\{b_k | k \in \mathbb{N}\}$ and $[b_k | k \in \mathbb{N}]$ is linearly independent, then X is incomplete. (The notation $[\ldots]$ indicates that a basis is not a set, but rather a system of vectors.)

Q39 Find normed spaces that are

(a) algebraically isomorphic, but not topologically isomorphic;
(b) topologically isomorphic, but not isometrically isomorphic.

Q40 How is the Banach–Mazur distance between X and Y related to the Banach–Mazur distance between X^* and Y^*?

Q41 Let X and Y be finite dimensional normed spaces over the same field and with the same dimension. Show that the Banach–Mazur distance between X and Y, $d(X, Y)$, is attained.