Problem Set 4

Q27 Let X be a normed space and $x_0 \in X$. Show that $||x_0|| \leq C$ if and only if $|f(x_0)| \leq C$ for all $f \in S(X^*)$.

Q28 Complete P2 Q15, showing that $||T|| = ||T^*||$, i.e. the map $T \to T^*$ is an isometry, using one of the consequences of the Hahn-Banach theorem.

Q29 Show that if X is reflexive, then X^* is reflexive. What about the converse?

Q30 Prove the following facts about the sequence spaces l_p.

(a) For $1 < p < \infty$, $(l_p)^*$ is isometric with l_q, where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Conclude that l_p is reflexive.

(b) The dual $(c_0)^*$ is isometric with l_1.

(c) The dual $(l_1)^*$ is isometric with l_∞.

(d) The spaces c_0, l_1 and l_∞ are not reflexive.

Q31 Let X be a normed space.

(a) What are $X^\perp$ and $\{0\}^\perp$?

(b) If Y_1, Y_2 are closed subspaces of X such that $Y_1 \neq Y_2$, show that $Y_1^\perp \neq Y_2^\perp$. Is this also true if one or both subspaces are not closed?

Q32 Let Y be a subspace of the normed space X. Suppose $\dim X = n$ and $\dim Y = m$. Show that $\dim Y^\perp = n - m$.
Formulate this as a theorem about the solution set of a system of linear equations.

Q33 Let $T \in \mathfrak{B}(X, Y)$. Show that:

(a) $(\overline{\text{im}(T)})^\perp \subseteq \ker(T^*)$

(b) $\text{im}(T) \subseteq \ker(T^*)_\perp$.

What does the second part imply for solving $Tx = y$?

Q34 Let X be a set and $\mathcal{F}_b(X)$ be the real vector space of all bounded, real-valued functions on X. For each $f \in \mathcal{F}_b(X)$, let

$$||f|| = \sup_{x \in X} |f(x)|.$$

You may assume that $(\mathcal{F}_b(X), ||\cdot||)$ is a normed space.

(a) Show that $(\mathcal{F}_b(X), ||\cdot||)$ is complete.

(b) Suppose that (X, d) is a metric space and fix $a \in X$. For $x \in X$, define the function $f_x \colon X \to \mathbb{R}$ by:

$$f_x(t) = d(x, t) - d(a, t).$$

Show that $f_x \in \mathcal{F}_b(X)$, and that the map $x \to f_x$ is an isometry onto its image (where $\mathcal{F}_b(X)$ is given the metric induced by the norm). Conclude that X is homeomorphic to a subset of $\mathcal{F}_b(X)$, where both spaces are given the induced topologies.

(c) Discuss differences and analogies of (b) with the result that every normed space is isometric to a subspace of its double-dual.