Assignment 4

Due at the beginning of the lecture on Monday, 25 October 2010.

For all questions, you should not need to cite any literature or prove auxiliary facts. They follow from facts given in the lectures or stuff you've learned in previous courses.

Q1 Let (X, Σ) be a measurable space and ν be a σ–finite signed measure on Σ. Apply the Lebesgue-Radon-Nikodym theorem to show that there is a Σ–measurable function $h\colon X \to \mathbb{R}$ such that $|h(x)| = 1$ ν–almost everywhere and for all $A \in \Sigma$,

$$\nu(A) = \int_A h \, d|\nu|.$$

What can you say about the uniqueness of such a function h?

Q2 Suppose (X, Σ, μ) is a measure space with $\mu(X) < \infty$. For measurable functions $f, g\colon X \to \mathbb{C}$, define

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu.$$

(a) Show that d defines a metric on the set of equivalence classes of complex-valued, measurable functions on X, where two functions are considered equivalent if and only if they are equal almost everywhere.

(b) Show that $[f_n] \to [f]$ with respect to this metric if and only if $f_n \to f$ in measure.

(This indeed is Question 42 from the problem sets.)

Q3 Let $X = [0, 1] \subset \mathbb{R}$. Consider the measure spaces $(X, \mathcal{L}, m)$, where $\mathcal{L}$ is the σ–algebra of all Lebesgue measurable subsets of X and m is Lebesgue measure, and $(X, \mathcal{P}([0, 1]), \mu)$, where μ is counting measure. Let $D = \{(x, x) \mid x \in X\} \subset X \times X$. Show that the following integrals are well-defined and evaluate them:

(a) $\int \chi_D \, d(m \times \mu)$

(b) $\int \int \chi_D \, dm \, d\mu$

(c) $\int \int \chi_D \, d\mu \, dm$