Problem Set 5

Q1 Let X be a set. Given any collection σ of subsets of X, the collection

$$\{U_1 \cap \ldots \cap U_k \mid U_i \in \sigma, k \in \mathbb{N}\} \cup \{\emptyset, X\}$$

defines a basis for a topology on X. Moreover, this is the smallest topology on X containing σ.

Q2 In the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences with the box topology, the sequence

$$x_m = \left(\frac{1}{m!}e^{-mn}\right)_{n=1}^{\infty}$$

does not converge to the zero sequence as $m \to \infty$.

Q3 The map $f\colon \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ given by $f(x) = (x, x, x, \ldots)$ is not continuous at $x = 0$ if $\mathbb{R}^{\mathbb{N}}$ is given the box topology.

Q4 Let (X_α, τ_α), $\alpha \in A$, be a family of topological spaces and $X = \prod_{\alpha \in A} X_\alpha$ with the product topology. Show that the sequence $(y_n)_{n=1}^{\infty} \subseteq X$ converges to the limit $y \in X$ if and only if for each $\alpha \in A$, $p_\alpha(y_n)$ converges to $p_\alpha(y)$.

Q5 Let X be a normed space.

(a) Show that $(x_n)_{n=1}^{\infty} \subseteq X$ converges in the weak topology to $x \in X$ if and only if

$$f(x_n) \to f(x)$$

for each $f \in X^*$. Decide whether a weak limit (if it exists) is unique.

(b) Show that $(f_n)_{n=1}^{\infty} \subseteq X^*$ converges in the weak-star topology to $f \in X^*$ if and only if

$$f_n(x) \to f(x)$$

for each $x \in X$. Decide whether a weak-star limit (if it exists) is unique.

Q6 Recall that if $X = c_0$, then $X^* \cong l_1$ and $X^{**} \cong l_\infty$. Let $e_n = (\delta_{kn})_{k=1}^{\infty}$ in either space.

(a) Show that $(e_n)_{n=1}^{\infty}$ converges weakly in X to zero, but that it doesn't converge (strongly) in X.

(b) Show that $(e_n)_{n=1}^{\infty}$ converges in the weak-star topology on X^* to zero, but that it doesn't converge weakly or strongly in X^*.

(c) Show that $(\sum_{m=n}^{\infty} e_m)_{n=1}^{\infty}$ converges in the weak-star topology on X^{**} to zero, but it doesn't converge weakly or strongly in X^{**}.