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A CONSTRAINT ON THE RANDOM PACKING OF DISKS

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Abstract

This paper addresses random packing of equal-sized disks in a manner such that no disk has a gap on its circumference large enough to accommodate an extra touching neighbour. This structure generalises the deterministic packing models discussed in classical geometry (Coxeter (1961), Hilbert and Cohn-Vossen (1952)). Relationships with the dual mosaic formed by joining the centres of touching disks are established. Constraints on the neighbourhood of disks and on the packing density are established.

TESSELLATIONS; MOSAICS

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1. A disk ensemble

Consider, as in Cowan (1984), an ensemble of equal-sized disks packed together in a random way. Specifically let us assume that there exists a 'full' ensemble on $\mathbb{R}^2$ which is statistically homogeneous. A disk D within the ensemble is *full* if there is no space on its circumference to accommodate an extra neighbour. (A 'neighbour' is one that touches D.) An ensemble is full if all disks are full.

Cowan (1984) studies a model for the neighbourhood of a disk, D say. D has a random number K of neighbours and a random number G of 'gaps' on its circumference. A gap occurs between two adjacent neighbours of D when they do not touch each other. Consequently there are $T = K - G$ touchings amongst the K neighbours of D. In a full ensemble the pair (K, T) can take values listed in tabular form below. The impossibility of $K < 4$ and of those cases marked with a dot arises because of the *full* requirement and the fact that a neighbour takes up one sixth of the circumference.

$$(1)\qquad \begin{matrix} (4,0) & (4,1) & \cdot & \cdot & \cdots \\ (5,0) & (5,1) & (5,2) & (5,3) & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots (6,6) \end{matrix}$$

Let $\mu = \boldsymbol{E}(K)$ and $\nu = \boldsymbol{E}(T)$. We firstly show, using 'tessellation theory', that a necessary condition for a full ensemble is that (μ, ν) lie in the region $\mathscr{X}_1$ defined by

$$(2)\qquad \mathscr{X}_1 = \{(\mu, \nu): 4 \leqq \mu \leqq 6,\ 12\mu - 48 \leqq 4\nu \leqq 9\mu - 30\}.$$

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2. The dual of the ensemble

To prove this, consider the 'dual' tessellation (mosaic) formed by connecting the centres of all pairs of touching disks. Disk centres become *nodes* of this tessellation. Let τ be the intensity of the point process of nodes and, for a 'typical' node, let $b_{kt} \equiv \boldsymbol{P}\{K = k, T = t\}$ and $b_k \equiv \boldsymbol{P}\{K = k\} = \Sigma_t b_{kt}$. Let p_i be the probability that a 'typical' polygon of the tessellation has i sides ($i = 3, 4, 5$), and denote the mean number of sides by ρ. In general, there is no direct relationship between the $\{p_i\}$ and $\{b_k\}$ sequences, but their means are related:

$$\rho = \frac{2\mu}{\mu - 2}\left(1 - \frac{\phi}{\mu}\right) \tag{3}$$

where ϕ is the expected number of angles at a typical node which equal π. In the dual of the circle ensemble, $\phi = 0$, so

$$\rho = \frac{2\mu}{\mu - 2}. \tag{4}$$

Formula (3), and numerous other formulae associated with mosaics, are proved in Cowan (1978), (1980) and, by a different method, in the work of Mecke (1980). (This is more accessible in Stoyan and Mecke (1983) and Stoyan et al. (1987).) Interestingly, the definitions of parameters such as ρ, ϕ and μ differ in the two approaches because a different notion of 'typical' polygon or node is employed. Cowan defines a typical polygon (or node) as one sampled randomly from the finite number of such in a large domain, strictly speaking, the limit of this scheme as the domain expands to cover the plane. Convergence issues, something ignored in an earlier paper (Matschinski (1954)) which reported (4), are settled by ergodicity assumptions and Wiener's ergodic theorem. On the other hand, Mecke defines the typical polygon (or node) as one sampled by choosing an arbitrary point t and being 'lucky enough' to find a polygon centroid (or node) at t. His approach is made rigorous without the use of an ergodic assumption.

The mosaic of interest in this paper has additional structure, because ν tells us, for a typical node, the mean number of triangles of the mosaic that contribute an angle to the node. We exploit this fact as follows.

Let B_r be a circular domain of radius r centred at the origin. Within B_r, let $C(B_r)$ be the number of nodes, $C_{kt}(B_r)$ the number of nodes having $K = k$ and $T = t$, $N(B_r)$ the number of polygons and $N_j(B_r)$ the number of polygons with j sides. Except for some effects near the edge of B_r, effects which are addressed rigorously in Cowan (1978), (1980), we have, by simple counting,

$$\sum_k \sum_t tC_{kt}(B_r) \doteq 3N_3(B_r). \tag{5}$$

Dividing throughout (5) by πr^2, the area of B_r, and applying an ergodic assumption, we can prove that (a) edge effects in (5) become asymptotically negligible, (b) $N_3(B_r)/N(B_r)$ converges with probability 1 to a constant which provides our definition of p_3, (c) $C_{kt}(B_r)/C(B_r)$ converges similarly to a constant which gives us our definition of b_{kt}, and (d) $N(B_r)/\pi r^2$ has similar convergence to the average area of the typical polygon. The average area is, from the tessellation theory of Cowan and Mecke, $\frac{1}{2}\tau(\mu-2)$. So after division by πr^2, the left-hand side of (5) equals

$$\frac{C(B_r)}{\pi r^2}\sum_k\sum_t \frac{tC_{kt}(B_r)}{C(B_r)} \to \tau\sum_k\sum_t tb_{kt} = \tau\nu$$

whilst the right-hand side is

$$\frac{3N_3(B_r)}{N(B_r)}\frac{N(B_r)}{\pi r^2} \to \tfrac{3}{2}p_3\tau(\mu-2).$$

Therefore $2\nu = 3p_3(\mu-2)$. Linking this with (4) and the fact that $\Sigma\, p_j = 1$, we have the complete distribution

(6)
$$p_3 = \frac{2\nu}{3(\mu-2)},\qquad p_4 = \frac{9\mu-4\nu-30}{3(\mu-2)},\qquad p_5 = \frac{2(12-3\mu+\nu)}{3(\mu-2)}.$$

Since each p must lie in [0, 1], we find that (μ, ν) must be in $\mathscr{X}_1$.

The foregoing argument can be repeated to show, in general, that

(7)
$$2\nu_j = jp_j(\mu-2)$$

where ν_j is, for a typical node, the mean number of j-sided polygons that contribute an angle to the node. (Hence $\nu = \nu_3$.) Interestingly, (μ, ν) determines ν_4 and ν_5 via (6) and (7) as follows:

$$\nu_4 = \frac{2(9\mu-4\nu-30)}{3},\qquad \nu_5 = \frac{5(12-3\mu+\nu)}{3}.$$

This means that, for the gaps on the circumference of a disk D, we know the proportions which are like Figures 1a or 1b.

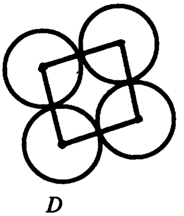

Fig. 1a

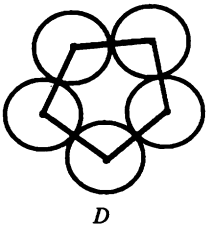

Fig. 1b

3. Further constraints on (μ, ν)

It turns out, however, that we can improve upon $\mathscr{X}_1$ by a very simple argument. The probability mass for the pair (K, T) is concentrated on the seven points of (1). For any given $\mu \in [4, 6]$, one can ask how to distribute the probability mass to maximise (or minimise) ν. This ignores the issue of whether a given probability mass distribution is geometrically or topologically feasible. Nevertheless, one can easily show that (μ, ν) must be in $\mathscr{X}_2$ defined below:

$$\mathscr{X}_2 = \{(\mu, \nu): \nu \geqq 0, \mu \geqq 4, 5\mu \geqq 2\nu + 18, 6\mu \leqq 30 + \nu\}.$$

Thus we can say that (μ, ν) must lie in $\mathscr{X}_1 \cap \mathscr{X}_2 = \mathscr{X}$, say:

$$\mathscr{X} = \{(\mu, \nu): 4 \leqq \mu \leqq 6, 6\mu - 24 \leqq 2\nu \leqq 5\mu - 18\}. \tag{8}$$

4. Density of disks

As mentioned earlier, it is known from tessellation theory that $\boldsymbol{E}(A)$, the average area of a typical polygon, is given by

$$\boldsymbol{E}(A) = \frac{2}{\tau(\mu - 2)}. \tag{9}$$

Thus $\boldsymbol{E}(A)$ depends upon τ, defined earlier as the intensity of the point process of disk centres. Thus τ measures the density of disks in the ensemble. From (9),

$$\tau = \frac{2}{\boldsymbol{E}(A)(\mu - 2)}.$$

Let $\boldsymbol{E}(A \mid j)$ be the conditional expectation of a typical polygon's area, given that it has j sides. Clearly $\boldsymbol{E}(A \mid 3) = (\sqrt{3}/4)d^2$, where d is the disk diameter. Also $(\sqrt{3}/2)d^2 < \boldsymbol{E}(A \mid 4) \leqq d^2$. It can be shown that

$$1.69518d^2 = \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{11}}{4}\right) d^2 < \boldsymbol{E}(A \mid 5) \leqq \frac{5d^2}{4} \tan\left(\frac{3\pi}{10}\right) = 1.72048d^2,$$

the upper bound corresponding to the regular pentagon whilst the lower bound corresponds to the equilateral pentagon with angles 120°, 93.56°, 120°, 103.22° and 103.22° in order (see Figure 1b). Since $\boldsymbol{E}(A) = \Sigma\, p_j \boldsymbol{E}(A \mid j)$, one can say that

$$\frac{8}{(\mu - 2)d^2\left(\sqrt{3}p_3 + 4p_4 + 5\tan\left(\frac{3\pi}{10}\right) p_5\right)} \leqq \tau < \frac{8}{(\mu - 2)d^2(\sqrt{3}p_3 + 2\sqrt{3}p_4 + (2\sqrt{3} + \sqrt{11})\, p_5)} \tag{10}$$

with the $<$ being replaced by $\leqq$ when $p_4 = p_5 = 0$. Since each p_j is a function of μ and ν, (10) provides bounds on τ for any given (μ, ν). For example, $\mu = \nu = 6$ implies $p_3 = 1$, so both bounds and hence τ equal $2/\sqrt{3}d^2 = 1.1547/d^2$. This is the densest packing. A contender for the least dense packing is provided by the case $(\mu, \nu) = (4, 0)$. Here $p_4 = 1$, and $1/d^2 \leqq \tau < 2/\sqrt{3}d^2$, the lower bound being realisable by the ensemble whose dual is a mosaic of squares.

I do not know if there exists a *realisable* full ensemble with density lower than $1/d^2$, but we can use (10) to suggest that there may be such ensembles. One can pose two questions. (a) For which values of $(\mu, \nu) \in \mathscr{X}$ is the lower bound in (10) less than $1/d^2$? (b) For which $(\mu, \nu) \in \mathscr{X}$ is the lower bound minimal for given d? The answer to (a) is: those $(\mu, \nu) \in \mathscr{X}$ such that

$$\mu < 4 + \frac{2\sqrt{3} - 16 + 10\tan(3\pi/10)}{6(5\tan(3\pi/10) - 6)}\nu = 4 + 0.23206\nu.$$

To answer (b) one can easily show that $(\mu, \nu) = (4, 1)$ is optimal, whereupon $p_1 = p_2 = p_3 = \frac{1}{3}$ and the least lower bound of (10) is

$$\frac{12}{d^2(4 + \sqrt{3} + 5\tan(3\pi/10))} = \frac{0.951327}{d^2}.$$

To achieve this lower bound, however, an equal mix of triangles, squares and *regular* pentagons is required in the mosaic. It is easily seen that no such mosaic is realisable, since a node at the vertex of a regular pentagon must combine the angle 108° with combinations of 60°, 90° and 108° to total 360°, an impossible task. It seems to be an open question whether τ can *in fact* be less than $1/d^2$. We have shown that $\tau > 0.951327/d^2$. Put another way, the proportion of space occupied by disks in a full ensemble must exceed $3\pi/(4 + \sqrt{3} + 5\tan(3\pi/10)) = 0.74717$.

5. Discussion

It is necessary to make two technical remarks. First, we have utilised some examples where the ensemble is highly regular, deterministic in character rather than stochastic. By convention, we incorporate such structures into the framework of a statistically homogeneous process by randomly offsetting the basic repeating unit from the origin. In particular, one ensures that the origin is uniformly distributed within the area of the repeating unit. For example, the densest mosaic comprising only equilateral triangles is made stationary by ensuring that the origin is uniformly distributed within one of the triangles. Such processes are not ergodic in the sense stated in Cowan's tessellation theory, yet all of the conclusions of that theory remain valid for these non-ergodic mosaics. This follows because spatial averages in these regular mosaics tend to the same non-random limits as their ergodic counterparts.

Secondly, we mention other non-ergodic cases, where it may appear that Mecke's method can still be used when the ergodic methods of this paper fail. There is, however, a 'cost' in Mecke's interpretation of the basic formulae of tessellation theory in non-ergodic cases, as the following example shows.

Consider an ensemble which is, with probability $\frac{1}{2}$, the most dense ensemble whose dual is the triangular lattice and, with probability $\frac{1}{2}$, the ensemble whose dual is the square lattice (common d in both cases). This process is not ergodic. Given the former model, $\mu = 6$ and $\rho = 3$ whilst given the latter, $\mu = 4$ and $\rho = 4$. In each case, the basic formula (4) holds. Yet one is tempted to say that, unconditionally, $\mu = 5$ and $\rho = 3.5$. A consequence of this reasonable statement is a violation of (4). So, in which sense is (4) valid in the non-ergodic situation?

This apparent paradox is resolved by recognising that, given one is 'lucky enough' to have a node at a chosen observation site t the chances that the former process was employed is $2/(2+\sqrt{3})$, due to Bayes' theorem and the higher intensity of nodes in the former case. Similarly, given a polygon centroid at t, the chance that the triangular model was employed is $4/(4+\sqrt{3})$. Thus the true 'Mecke' $\mu = 4(3+\sqrt{3})/(2+\sqrt{3}) = 5.0718$ whilst the true 'Mecke' $\rho = 4(3+\sqrt{3})/(4+\sqrt{3}) = 3.3022$. Formula (4) is valid with these values, but at some cost to the intuition.

Future work will study the extent to which our methods apply to full ensembles with more than one size of disk. Then, we expect interesting questions on both *maximum* and minimum packing density to arise.

We conclude by noting that the (μ, ν) values in the 'local' models analysed in Cowan (1984) do *not* lie in $\mathscr{X}$. This confirms the worries expressed in that paper that the models applied locally do not extend to the whole ensemble.

References

COWAN, R. (1978) The use of the ergodic theorems in random geometry. *Suppl. Adv. Appl. Prob.* **10**, 47–57.

COWAN, R. (1980) Properties of ergodic random mosaic processes. *Math. Nachr.* **97**, 89–102.

COWAN, R. (1984) A model for the random packing of disks in the neighbourhood of one disk. *SIAM J. Appl. Math.* **44**, 839–853.

COXETER, H. S. M. (1961) *Introduction to Geometry*. Wiley, New York.

HILBERT, D. AND COHN-VOSSEN, S. (1952) *Geometry and the Imagination*. Chelsea, New York.

MATSCHINSKI, M. (1954) Considérations statistiques sur les polygones et les polyèdres. *Publ. Inst. Statist. Univ. Paris* **3**, 179–201.

MECKE, J. (1980) Palm methods for stationary random mosaics. In *Combinational Principles in Stochastic Geometry*, ed. R. V. Ambartzumian, pp. 124–132, Armenian Academy of Sciences, Erevan.

STOYAN, D. AND MECKE, J. (1983) *Stochastische Geometrie*. Akademie-Verlag, Berlin.

STOYAN, D., KENDALL, W. S. AND MECKE, J. (1987) *Stochastic Geometry and Its Applications*. Wiley, New York; Akademie-Verlag, Berlin.