DETECTING EIGENVALUES IN A FOURTH-ORDER NLS EQUATION WITH A NON-REGULAR MASLOV BOX

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ABSTRACT. We use the Maslov index to study the eigenvalue problem arising from the linearisation about a standing wave in a fourth-order cubic nonlinear Schrödinger equation. We use a homotopy argument to develop a lower bound for the number of purely real unstable eigenvalues, as well as a Vakhitov-Kolokolov type stability criterion. The interesting aspects of this problem as an application of the Maslov index are the instances of non-regular conjugate points, with degenerate crossing forms of both zero and nonzero rank encountered. We handle these degeneracies with the method of partial signatures as developed by Giambò, Piccione and Portaluri.

CONTENTS

1. INTRODUCTION

The fourth-order cubic nonlinear Schrödinger (NLS) equation

$$i\Psi_t = -\frac{\beta_4}{24}\Psi_{xxxx} + \frac{\beta_2}{2}\Psi_{xx} - \gamma|\Psi|^2\Psi. \tag{1.1}$$

models the propagation of pulses in media with Kerr nonlinearity that are subject to both quartic and quadratic dispersion [KH94,ABK94,BGBK21,TABRdS19]. Here Ψ is the slowly varying complex envelope of the pulse, and β_2, β_4, and γ are real coefficients.

Solutions to (1.1) of the form $\Psi(x,t) = e^{i\beta t}\phi(x)$, $\beta \in \mathbb{R}$, are called *standing wave* solutions. Following the convention of [BGBK21], when the wave profile ϕ is a homoclinic orbit of the associated standing wave equation (given in (1.4)), we will call Ψ a *soliton solution* of (1.1). Karlsson and Höök [KH94] discovered an exact analytic family of soliton solutions to (1.1) with a squared hyperbolic secant profile. Akhmediev, Buryak and Karlsson [ABK94] observed oscillatory behaviour in the tails of solitons for certain values of β. Akhemdiev and Buryak [BA95] showed the existence of bound states of two-solitons (i.e. double-hump pulses ϕ) in the same parameter regime, and derived a stability criterion by analysing

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the dependence of the associated Hamiltonian on the energy. Karpman and Shagalov [Kar96, KS97, KS00] considered the extension of (1.1) to higher-order nonlinearities and multiple space dimensions. All of these works considered the case $\beta_4 < 0$ and $\beta_2 < 0$.

More recently, (1.1) has been the focus of a number of studies following the experimental discovery of *pure quartic solitons* (PQSs) in silicon photonic crystal waveguides [BRdSS+16]. These solitons exist through a balance of negative quartic dispersion and the nonlinear Kerr effect, for which $\beta_2 = 0$ and $\beta_4 < 0$. They have attracted much attention for their potential applicability to ultrafast lasers due to their favourable energy-width scaling [BRdSHE17, TABRdS19]. Following the discovery of PQSs, Tam *et al.* [TABRdS19] numerically investigated their existence and spectral stability. They also showed [TABRdS18, TABRdS20] that PQSs and solitons of the classical second-order NLS equation, for which $\beta_4 = 0$, are in fact part of a broader continuous family of soliton solutions to (1.1) for nonpositive dispersion coefficients β_4 and β_2.

Extending the work of Tam *et al.*, Bandara *et al.* [BGBK21] used a dynamical systems approach to find infinite families of multi-hump soliton solutions to (1.1) for $\beta_4 \neq 0$ and $\beta_2 \neq 0$. To do so, they identified solitons of (1.1) as orbits of the stationary state equation satisfied by the wave profile that are homoclinic to the origin. As a consequence of the stationary state equation being Hamiltonian, fourth-order and having two reversible symmetries, they explain that infinitely many homoclinic solutions are created when the origin transitions from a real saddle (having only purely real eigenvalues) to a saddle focus (having complex conjugate eigenvalues) as a parameter is changed. This holds provided there exists a symmetric homoclinic oribt at the point of transition (see also [CT93]). In parameter regimes where this spectral behaviour occurs, they use continuation techniques to numerically compute these homoclinic orbits, which are characterised as heteroclinic cycles between the origin and periodic orbits in the zero energy level (zero set of the Hamiltonian). Depending on the symmetry properties of the periodic orbits and the types of connections from the origin to them, the orbits are organised into infinite families accordingly. They then use numerical simulations to investigate the stability of the waves computed. They found that while many of the multi-pulse solutions were unstable, some were only weakly unstable, and therefore possibly observable in experiments over a number of dispersion lengths.

A more rigorous stability analysis was undertaken by Natali and Pastor [NP15]. They proved the orbital stability of an exact solution to the nondimensionalised equivalent of (1.1) (see (1.2)). This solution represents the family of exact solutions to (1.1) found by Karlsson and Höök in [KH94]. As observed in [NP15] (also [TABRdS20, §II]), this solution exists only for a fixed value of the frequency parameter, and is *not* part of a continuous family of solutions in that parameter. The failure of the existence of such a family renders the classical results of Grillakis, Shatah and Strauss [GSS87, GSS90] inadmissible since [GSS87, Assumption 2] does not hold in this instance.

Under certain assumptions, Parker and Aceves [PA21] proved the existence and orbital stability of a single-hump solitary wave (not the exact analytical soution of Karlsson and Höök). For any such solitary wave, they determined the existence of an associated family of multi-hump solitons, which they proved to be unstable by showing the associated linearised operator has a positive real eigenvalue. The main results of [PA21] are formulated under a number of hypotheses which will not be required in our analysis.

In this paper, we further develop the spectral stability theory for *arbitrary* single and multi-hump soliton solutions to (1.1). Our results may be applied to the infinite families of multipulse solitons numerically computed in [BGBK21]. Our goal is to determine the existence of positive real eigenvalues for the linearised operator associated with any soliton solution to (1.1). We do not require Hypothesis 2, the first part of Hypothesis 3 or Hypothesis 4 of [PA21]. The main tool of our analysis is a topological invariant from symplectic geometry known as the *Maslov index*. It is a signed count of the intersections of a path in the manifold of Lagrangian subspaces of a symplectic vector space with a certain codimension-one set, the *train* of a fixed reference plane.

Our main results are as follows. In Theorem 1.2, we provide a lower bound for the number of positive real eigenvalues associated with soliton solutions to the nondimensionalised equivalent of (1.1). The bound is given in terms of the *Morse indices* (here, the number of positive eigenvalues) of two related selfadjoint operators, as well as a certain correction term which represents a contribution to the Maslov index from a *non-regular* crossing. This includes, as a corollary, the Jones-Grillakis instability theorem, which gives sufficient conditions on the aforementioned terms for the existence of a positive, real eigenvalue. We also provide a complete proof of the *Vakhitov-Kolokolov* (VK) stability criterion (see Theorem 1.5), where spectral (in)stability is determined by the sign of a certain integral. This includes, as a special case, the stability result of [PA21]. An advantage of our analysis is in the interpretation of P and Q, afforded by the Maslov index, as the number of *conjugate points* for each of the operators L_+ and L_-. All of the required data is therefore encoded at $\lambda = 0$. As highlighted in [BJ22], numerically this is a desirable feature that a calculation with the Evans function [AGJ90] does not possess. In light of this, an alternate form of (1.17), which may be more useful for numerical computations, is given in Remark 5.5. Our results are formulated under two genericity conditions (Hypotheses 4.2 and 4.3) the removal of which will be the subject of future work.

The key feature of the eigenvalue problem herein that allows us to make use of the Maslov index is the infinitesimally symplectic structure of the eigenvalue equations which preserve Lagrangian planes. The stable and unstable subspaces of the asymptotic system give rise to two-parameter families of Lagrangian planes, the stable and unstable bundles. Their nontrivial intersection at a common $x \in \mathbb{R}$ encodes (real) eigenvalues. By exploiting homotopy invariance of the Maslov index, we will detect positive real eigenvalues by instead analysing the intersections of the unstable bundle at $\lambda = 0$ with the train of the stable subspace of the asymptotic system.

The Maslov index has been used to study the spectrum of homoclinic orbits in a number of works [Jon88,BJ95,Cor19,CH14,HLS18,BCJ+18,How23,How21]. In these cases, the Maslov index is used to detect purely real unstable eigenvalues. If monotonicity in the spectral parameter holds, as is often the case in selfadjoint problems [HLS18,BCJ+18,How23], then it is possible to give an exact count of these eigenvalues in terms of a related Lagrangian path for which the spectral parameter is zero. Howard, Latushkin and Sukhtayev [HLS18] proved the equality of the Morse and Maslov indices for Schrödinger operators on the line, where the symmetric matrix-valued potential approaches constant endstates. They apply their results to analyse the stability of nonlinear waves in various reaction-diffusion systems. Jones [Jon88] and Bose and Jones [BJ95] used the Maslov index to study the stability of homoclinic orbits in the NLS equation and a gradient reaction-diffusion system respectively. Chen and Hu [CH14] proved a stability result for standing pulses in a doubly-diffusive FitzHugh-Nagumo equation. Beck *et al.* [BCJ+18] proved the instability of pulses in gradient reaction-diffusion systems, generalising the instability result for pulses in scalar reaction-diffusion equations (see [KP13, §2.3.3]). Cornwell and Jones [Cor19,CJ18] used the Maslov index to analyse the stability of travelling waves in skew-gradient systems. Despite the eigenvalue equations not having a Hamiltonian structure, for a nonstandard symplectic form they preserve Lagrangian planes. They proved the stability of a particular travelling pulse in a doubly diffusive FitzHugh-Nagumo system by showing the Maslov index to be zero in the travelling wave co-ordinate z at $\lambda = 0$, despite lacking monotonicity in z.

A notable feature of the current problem is the occurrence of non-regular crossings, i.e. nontransversal intersections of the Lagrangian path with the train. We find instances where the *crossing form* is either identically zero, or degenerate with nonzero rank. In particular, the crossing form associated with the zero eigenvalue of the linearised operator (i.e. the conjugate point at the top left corner of the *Maslov box*) is identically zero crossing in the λ direction. This is a feature of eigenvalue problems of the form (1.12); see, for example, [CCLM23]. In addition, crossings in the x direction (when $\lambda = 0$) have a degenerate crossing form which is not identically zero. This phenomenon appears to be the result of the eigenvalue equations being fourth order, and has been encountered in [How21, How23]. In those papers, Howard and co-authors use a formulation of the Maslov index based on the spectral flow of a family of unitary matrices. Nonetheless, a degenerate crossing form can be still be observed in the spatial variable (see, for example, [How23, §6] and [How21, §5.2]). The issue is circumvented due to the crossing form being semidefinite in a neighbourhood of the crossing. By contrast, this semidefiniteness property does not hold in our case. In addition, it is unclear how to apply *Hörmander's index* (see [How21]), as was done for the fourth-order problem on the line in [How23]. The complication is the requirement of a basis of vectors for the unstable bundle (along $\lambda = 0$) at $x = +\infty$. At this point, the bundle intersects the stable subspace in a one-dimensional subspace because $\lambda = 0$ is a simple eigenvalue. It is unclear how to determine this subspace. In this paper, we use the approach of [GPP04b, GPP04a] to locally compute the Maslov index via the *partial signatures* of an associated family of symmetric bilinear forms. This allows us to handle non-regular crossings without perturbative arguments, as in [RS93]. This will involve the use of *higher-order crossing forms*, which generalise the (first-order) crossing form defined in [RS93].

Recently in [CCLM23], a similar lower bound to that in Theorem 1.2 was derived for an eigenvalue problem of the form of (1.12) on a compact interval, where $L_\pm$ are Schrödinger operators. There, the "correction term" $\mathfrak{c}$ was computed via an analysis of the *eigenvalue curves*, offering a geometric interpretation of the corresponding term in the lower bound of [KP13, Theorem 7.1.16]. The fact that the spatial domain is the entire real line renders a similar calculation in the present setting intractable.

1.1. **Statement of main results.** We will work with the following nondimensionalised version of (1.1) corresponding to the case of nonzero quartic dispersion ($\beta_4 \neq 0$) and positive Kerr nonlinearity ($\gamma > 0$):

$$i\psi_t = -\sigma_4\psi_{xxxx} + \sigma_2\psi_{xx} - |\psi|^2\psi, \tag{1.2}$$

where $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $\sigma_4 = \operatorname{sign}\beta_4$ and $\sigma_2 = \operatorname{sign}\beta_2$. (For the transformations used to obtain (1.2) from (1.1) for $\beta_4 \neq 0, \beta_2 \neq 0$, we refer the reader to [BGBK21, Table 1].) We will treat the case when the quartic dispersion coefficient is negative, i.e. $\sigma_4 = -1$, and we assume that $\beta_2 \neq 0$, hence $\sigma_2 \in \{\pm 1\}$. Our focus will be to determine the spectral stability of standing wave solutions

$$\psi(x,t) = e^{i\beta t}\phi(x), \qquad \phi(x) \in \mathbb{R}, \tag{1.3}$$

to (1.2), subject to perturbations in $L^2(\mathbb{R};\mathbb{C})$. Note that the wave profile ϕ satisfies the *standing wave equation*

$$\phi'''' + \sigma_2\phi'' + \beta\phi - \phi^3 = 0, \tag{1.4}$$

as seen upon substituting (1.3) into (1.2). Using the change of variables

$$\phi_1 = \phi'' + \sigma_2\phi, \qquad \phi_2 = \phi, \qquad \phi_3 = \phi', \qquad \phi_4 = \phi''', \tag{1.5}$$

we may write (1.4) as the first order Hamiltonian system

$$\begin{pmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \\ \phi_4' \end{pmatrix} = \begin{pmatrix} \phi_4 + \sigma_2\phi_3 \\ \phi_3 \\ \phi_1 - \sigma_2\phi_2 \\ -\sigma_2\phi_1 + \phi_2 - \beta\phi_2 + \phi_2^3 \end{pmatrix}. \tag{1.6}$$

Motivated by the families of homoclinic orbits discovered in [BGBK21], we consider orbits of (1.6) that are homoclinic to the origin, which correspond to soliton solutions of (1.2). We will assume that the origin in (1.6) is hyperbolic. Noting that the eigenvalues of the linearisation about the origin are given by

$$\mu^2 = \frac{1}{2}\left(-\sigma_2 \pm \sqrt{1-4\beta}\right) \tag{1.7}$$

(where we used that $\sigma_2^2 = 1$), hyperbolicity holds provided

$$\begin{cases} \beta > 0 & \text{if} \quad \sigma_2 = -1 \\ \beta > \frac{1}{4} & \text{if} \quad \sigma_2 = 1. \end{cases} \tag{1.8}$$

In the first part of (1.8), we additionally require

$$\beta \neq \frac{1}{4} \quad \text{if} \quad \sigma_2 = -1. \tag{1.9}$$

Linearising (1.2) by substituting the complex-valued perturbation

$$\psi(x,t) = \left[\phi(x) + \varepsilon\left(u(x) + iv(x)\right)e^{\lambda t}\right]e^{i\beta x} \tag{1.10}$$

for $u, v \in L^2(\mathbb{R};\mathbb{R})$ into (1.2), collecting $O(\varepsilon)$ terms and separating into real and imaginary parts leads to the following linearised dynamics in u and v:

$$\begin{aligned} -u'''' - \sigma_2 u'' - \beta u + 3\phi^2 u &= \lambda v \\ -v'''' - \sigma_2 v'' - \beta v + \phi^2 v &= -\lambda u. \end{aligned} \tag{1.11}$$

We can write (1.11) as the spectral problem

$$N\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.12}$$

where N is the linear operator

$$N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \qquad \begin{cases} L_- = -\partial_x^4 - \sigma_2\partial_x^2 - \beta + \phi^2, \\ L_+ = -\partial_x^4 - \sigma_2\partial_x^2 - \beta + 3\phi^2, \end{cases} \tag{1.13}$$

with

$$\mathrm{dom}(N) = H^4(\mathbb{R}) \times H^4(\mathbb{R}), \qquad \mathrm{dom}(L_\pm) = H^4(\mathbb{R}). \tag{1.14}$$

Our goal is to determine whether the spectrum of the unbounded and densely defined linear operator N intersects the open right half plane. Because N is Hamiltonian, its spectrum has four-fold symmetry in $\mathbb{C}$, and instability follows from any part of the spectrum lying off the imaginary axis. We will show in Section 2 that the essential spectrum of N is confined to the imaginary axis. Regarding the point spectrum, it is a requirement of the Maslov index that the eigenvalue parameter be real (the detection of complex eigenvalues via the Maslov index remains an open problem). Our task therefore is to detect *positive real eigenvalues* $\lambda \in \mathrm{Spec}(N) \cap \mathbb{R}^+$. We will give a lower bound for the count of these eigenvalues in terms of the Morse indices of the operators $L_\pm$, which are selfadjoint with the domain in (1.14) (see, for example, [Wei87]). The Morse indices of $L_\pm$ are only well-defined if their essential spectra are confined to the negative half line, and we show in Section 2 that this is indeed the case under the assumptions (1.8)–(1.9).

We point out here that the equation $L_-\phi = 0$ is just (1.4), and, differentiating (1.4) with respect to x, we have $L_+\phi_x = 0$. Thus

$$0 \in \operatorname{Spec}(L_-) \cap \operatorname{Spec}(L_+), \tag{1.15}$$

where $\phi \in \ker(L_-)$ and $\phi_x \in \ker(L_+)$. We will assume these are the only functions lying in the kernel.

Hypothesis 1.1. $\dim\ker(L_+) = \dim\ker(L_-) = 1$, where $\ker(L_-) = \operatorname{span}\{\phi\}$ and $\ker(L_+) = \operatorname{span}\{\phi_x\}$.

Notice that when $\lambda = 0$, the eigenvalue equations (1.12) decouple into the two independent equations $L_-v = 0$ and $L_+u = 0$, so that $\ker(N) = \ker(L_+) \oplus \ker(L_-)$. Hypothesis 1.1 therefore implies that $\ker(N) = \operatorname{span}\{(\phi_x, 0)^\top, (0, \phi)^\top\}$.

Let us denote

$$\begin{aligned} P &:= \#\{\text{positive eigenvalues of } L_+\}, \\ Q &:= \#\{\text{positive eigenvalues of } L_-\}, \\ n_+(N) &:= \#\{\text{positive real eigenvalues of } N\}, \end{aligned}$$

and define the quantities

$$\mathcal{I}_1 := \int_{-\infty}^{\infty} \phi_x \, \widehat{v} \, dx, \qquad \mathcal{I}_2 := \int_{-\infty}^{\infty} \phi \, \widehat{u} \, dx, \tag{1.16}$$

where $\widehat{v}$ is any solution in $H^4(\mathbb{R})$ to $-L_-v = \phi_x$ and $\widehat{u}$ is any solution in $H^4(\mathbb{R})$ to $L_+u = \phi$. Under Hypothesis 1.1 and the conditions (1.8)–(1.9), as well as two genericity conditions Hypotheses 4.2 and 4.3 which will be given in Section 4, our main result is the following:

Theorem 1.2. *Suppose $\mathcal{I}_1, \mathcal{I}_2 \neq 0$. The number of positive, real eigenvalues of the operator N satisfies*

$$n_+(N) \geq |P - Q - \mathfrak{c}|, \tag{1.17}$$

where $\mathfrak{c}$ is computed via

$$\mathfrak{c} = \begin{cases} 1 & \mathcal{I}_1 > 0, \mathcal{I}_2 < 0, \\ 0 & \mathcal{I}_1\mathcal{I}_2 > 0, \\ -1 & \mathcal{I}_1 < 0, \mathcal{I}_2 > 0. \end{cases} \tag{1.18}$$

Remark 1.3. The equations $-L_-v = \phi'$ and $L_+u = \phi$ each satisfy a solvability condition that guarantees the existence of solutions $\widehat{u}$ and $\widehat{v}$. In the case that either $\mathcal{I}_1$ or $\mathcal{I}_2$ vanishes, an extra calculation is needed to compute the correction term $\mathfrak{c}$ (the definition of which is given in (3.56)); for details, see Remark 5.7). Finally, our theorem will also hold in the case of any integer power-law nonlinearity in (1.2), i.e. in the case of standing wave solutions to

$$i\psi_t = -\sigma_4\psi_{xxxx} + \sigma_2\psi_{xx} - f(|\psi|^2)\psi, \qquad f(|\psi|^2) = |\psi|^{2p}, \;\; p \in \mathbb{Z}^+. \tag{1.19}$$

(See Remark 4.9.) However, with the standing wave solutions of [BGBK21] in mind, we have stated our results for the cubic case.

The following *Jones-Grillakis* instability theorem [Jon88, Gri88, KP13] is an immediate consequence of Theorem 1.2.

Corollary 1.4. *Standing waves for which $P - Q \neq -1, 0, 1$ are unstable.*

In this work we do not require existence of standing waves; rather, we prove that if a standing wave exists with the spectral properties of L_+ and L_- stated, then its linearised operator N satisfies Theorem 1.2.

We also have the following *Vakhitov-Kolokolov* type criterion [VK73, Pel11].

Theorem 1.5. *Suppose $P = 1$ and $Q = 0$. The standing wave $\widehat{\psi}$ is spectrally unstable if $\mathcal{I}_2 > 0$ and is spectrally stable if $\mathcal{I}_2 < 0$.*

Remark 1.6. If there exists a C^1 family of solutions $\beta \to \partial_\beta \phi(x;\beta) \in H^4(\mathbb{R})$ to the standing wave equation, then $\widehat{u} = \partial_\beta \phi(x;\beta)$ and the integral $\mathcal{I}_2$ is precisely that appearing in the classical Vakhitov-Kolokolov criterion for standing waves in the usual (second-order) NLS equation (see [Pel11, §4.2]), i.e.

$$\mathcal{I}_2 = \frac{1}{2}\frac{\partial}{\partial \beta}\int_{-\infty}^{\infty} \phi^2 dx. \tag{1.20}$$

The paper is organised as follows. In Section 2 we write down the first order system associated with (1.12), compute the essential spectra of the operators L,L_+ and N, and define the stable and unstable bundles, the main objects of our analysis. In Section 3 we provide some background material on the Maslov index before setting up the homotopy argument that will lead to the proof of the lower bound in Theorem 1.2. In Section 4 we use the Maslov index to prove that the Morse index of each of the operators L_- and L_+ is equal to the associated number of conjugate points. In Section 5 we prove Theorems 1.2 and 1.5.

2. Set-up

We first compute the essential spectra of the operators $L_\pm, N$. Using the change of variables

$$\begin{aligned} u_1 &= u'' + \sigma_2 u, & u_2 &= u, & u_3 &= u', & u_4 &= u''', \\ v_1 &= v'' + \sigma_2 v, & v_2 &= -v, & v_3 &= -v', & v_4 &= v''', \end{aligned} \tag{2.1}$$

we convert (1.11) to the (infinitesimally symplectic) first order system

$$\begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 \end{pmatrix}' = \left(\begin{array}{cccc|cccc} & & & & \sigma_2 & 0 & 1 & 0\\ & 0 & & & 0 & -\sigma_2 & 0 & 1\\ & & & & 1 & 0 & 0 & 0\\ & & & & 0 & 1 & 0 & 0\\ \hline 1 & 0 & -\sigma_2 & 0 & & & & \\ 0 & -1 & 0 & -\sigma_2 & & 0 & & \\ -\sigma_2 & 0 & \alpha(x) & \lambda & & & & \\ 0 & -\sigma_2 & \lambda & \eta(x) & & & & \end{array}\right) \begin{pmatrix} u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4 \end{pmatrix}, \tag{2.2}$$

where

$$\alpha(x) := 3\phi(x)^2 - \beta + 1, \qquad \eta(x) := -\phi(x)^2 + \beta - 1.$$

Setting

$$B = \begin{pmatrix} \sigma_2 & 0 & 1 & 0\\ 0 & -\sigma_2 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad C(x;\lambda) = \begin{pmatrix} 1 & 0 & -\sigma_2 & 0\\ 0 & -1 & 0 & -\sigma_2\\ -\sigma_2 & 0 & \alpha(x) & \lambda\\ 0 & -\sigma_2 & \lambda & \eta(x) \end{pmatrix},$$

we can write (2.2) as

$$\mathbf{w}_x = A(x;\lambda)\mathbf{w}, \tag{2.3}$$

where

$$\mathbf{w} = (u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)^\top, \qquad A(x;\lambda) = \begin{pmatrix} 0 & B\\ C(x;\lambda) & 0 \end{pmatrix}. \tag{2.4}$$

The asymptotic system for (2.2) is given by

$$\mathbf{w}_x = A_\infty(\lambda)\mathbf{w}, \tag{2.5}$$

where

$$A_\infty(\lambda) := \lim_{x\to\pm\infty} A(x,\lambda).$$

(The endstates as $x \to \pm\infty$ are the same because ϕ is homoclinic to the origin.) It now follows from [KP13, Theorem 3.1.11] that the essential spectrum of N is given by the set of $\lambda \in \mathbb{C}$ for which the matrix $A_\infty(\lambda)$ has a purely imaginary eigenvalue. A short calculation shows that

$$\mathrm{Spec}_{\mathrm{ess}}(N) = \{\lambda \in \mathbb{C} : \lambda^2 = -\left(-k^4 + \sigma_2 k^2 - \beta\right)^2 \text{ for some } k \in \mathbb{R}\} \subseteq i\mathbb{R}. \tag{2.6}$$

Notice we require $\beta \neq 0$ in order to have $0 \notin \mathrm{Spec}_{\mathrm{ess}}(N)$.

The essential spectra of the operators $L_\pm$ is computed similarly. The first order systems associated with the eigenvalue equations for each of the operators L_+ and L_- will be given in Section 4 (see (4.3) and (4.6)). It follows from a similar calculation on the asymptotic matrices associated with those systems that

$$\mathrm{Spec}_{\mathrm{ess}}(L_\pm) = \{\lambda \in \mathbb{R} : \lambda = -k^4 + \sigma_2 k^2 - \beta \text{ for some } k \in \mathbb{R}\}. \tag{2.7}$$

Given its biquadratic structure, if the equation in (2.7) has no real roots for k then the essential spectra of L_+ and L_- will be confined to the negative half line. The equation in (2.7) has no real roots for k if and only if the associated discriminant is positive, i.e.

$$16\beta^3 - 8\beta^2 + \beta = \beta(4\beta - 1)^2 > 0, \tag{2.8}$$

and, in addition, we have either

$$-8\sigma_2 > 0 \qquad \text{or} \qquad 4\beta - 1 > 0. \tag{2.9}$$

(See [Ree22], and note we have used that $\sigma_2^2 = 1$). Both (2.8) and (2.9) are satisfied for the values of β given in (1.8), (1.9). For these values of β we therefore have

$$\mathrm{Spec}_{\mathrm{ess}}(L_\pm) = \begin{cases} (-\infty, -\beta) & \sigma_2 = -1, \\ (-\infty, -\beta - \frac{1}{4}] & \sigma_2 = 1, \end{cases} \tag{2.10}$$

so that $\mathrm{Spec}_{\mathrm{ess}}(L_\pm) \subset \mathbb{R}^-$. In addition to hyperbolicity of the asymptotic matrices for the L_+ and L_- eigenvalue problems, the values of β given in (1.8), (1.9) will actually guarantee that those asymptotic matrices have an equal number of eigenvalues with positive and negative real part.

Note that the assumptions (1.8) actually guarantee that the matrix $A_\infty(\lambda)$ is hyperbolic, with an equal number of eigenvalues with positive and negative real part. Precisely, the eight eigenvalues are

$$\pm\frac{\sqrt{-\sigma_2 \pm \sqrt{1 - 4\beta \pm 4\lambda i}}}{\sqrt{2}}. \tag{2.11}$$

We denote the corresponding stable and unstable subspaces (i.e. the eigenspaces associated with eigenvalues with negative and positive real part) by $\mathbb{S}(\lambda)$ and $\mathbb{U}(\lambda)$ respectively.

Next, since $\mathrm{Spec}_{\mathrm{ess}}(N) \subset i\mathbb{R}\backslash\{0\}$, the operator $N - \lambda I$ of (1.12)–(1.14) is Fredholm for $\lambda \in \mathbb{R}$, and it follows from [San02, §3.3] that the densely-defined closed linear operator

$$T(\lambda) : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \qquad T(\lambda)u := \frac{du}{dx} - A(\cdot;\lambda)u,$$

associated with (2.3) is also Fredholm. By [San02, Theorem 3.2, Remark 3.3], (2.2) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. That is, for each fixed $\lambda \in \mathbb{R}$, on each of the intervals $\mathbb{R}^+$ and $\mathbb{R}^-$ the set of solutions to (2.2) is the direct sum of two subspaces, where one subspace consists solely of solutions that decay (exponentially) backwards in x, and the other of solutions that decay forwards in x. By flowing these subspaces under (2.2), each of these families can be extended to all of $\mathbb{R}$. This leads us to consider the spaces

$$\begin{aligned}\mathbb{E}^u(x,\lambda) &:= \{\xi \in \mathbb{R}^8 : \xi = \mathbf{w}(x;\lambda),\ \mathbf{w} \text{ solves (2.2) and } \mathbf{w}(x;\lambda) \to 0 \text{ as } x \to -\infty\},\\ \mathbb{E}^s(x,\lambda) &:= \{\xi \in \mathbb{R}^8 : \xi = \mathbf{w}(x;\lambda),\ \mathbf{w} \text{ solves (2.2) and } \mathbf{w}(x;\lambda) \to 0 \text{ as } x \to +\infty\},\end{aligned} \tag{2.12}$$

corresponding to the evaluation at $x \in \mathbb{R}$ of the spaces of solutions to (2.2) that decay (exponentially) as $x \to +\infty$ and as $x \to -\infty$, respectively. Following [AGJ90, Cor19], we call these sets the *unstable* and *stable bundles* respectively. For each $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, if we consider $\mathbb{U}(\lambda), \mathbb{S}(\lambda), \mathbb{E}^u(x,\lambda), \mathbb{E}^s(x,\lambda)$ as points in the Grassmannian of four-dimensional subspaces of $\mathbb{R}^8$,

$$\mathrm{Gr}_4(\mathbb{R}^8) = \{V \subset \mathbb{R}^8 : \dim V = 4\},$$

which (following [Fur04, HLS17]) we equip with the metric $d(V,U) = \|P_V - P_U\|$, where P_V is the orthogonal projection onto V and $\|\cdot\|$ is any matrix norm, then we have that

$$\lim_{x\to-\infty} \mathbb{E}^u(x,\lambda) = \mathbb{U}(\lambda), \qquad \lim_{x\to+\infty} \mathbb{E}^s(x,\lambda) = \mathbb{S}(\lambda). \tag{2.13}$$

That is, the orthogonal projections onto $\mathbb{E}^u(x,\lambda)$ and $\mathbb{E}^s(x,\lambda)$ converge to those on $\mathbb{U}(\lambda)$ and $\mathbb{S}(\lambda)$ as $x \to -\infty$ and $x \to \infty$, respectively. This is given in [PSS97, Corollary 2].

The important feature of the system (2.2) that makes it amenable to the Maslov index is that the coefficient matrix $A(x;\lambda)$ is infinitesimally symplectic, i.e.

$$A(x;\lambda)^T J + JA(x;\lambda) = 0, \tag{2.14}$$

which follows from the symmetry of B and $C(x;\lambda)$. This is the motivation for the choice of substitutions (2.1). Consequently, (2.2) induces a flow on the manifold of *Lagrangian planes*. In particular, the stable and unstable bundles of (2.2) are Lagrangian planes of $\mathbb{R}^8$ for all x and all λ. In addition we have that λ_0 is an eigenvalue of N if and only if for any (and hence all) $x \in \mathbb{R}$ we have

$$\mathbb{E}^u(x,\lambda_0) \cap \mathbb{E}^s(x,\lambda_0) \neq \{0\}.$$

In this case we in fact have

$$\dim \mathbb{E}^u(x,\lambda_0) \cap \mathbb{E}^s(x,\lambda_0) = \dim\ker(N - \lambda_0 I). \tag{2.15}$$

By exploiting homotopy invariance of the Maslov index, we can determine the existence of such intersections by instead analysing the evolution of the unstable bundle $\mathbb{E}^u(x,\lambda_0)$ when $\lambda_0 = 0$. This is explained in Section 3.

3. A symplectic approach to the eigenvalue problem

In this section, we give some background material on the Maslov index before describing the homotopy argument that leads to the lower bound of Theorem 1.2. Our definition of the Maslov index follows [GPP04a, GPP04b], which involves computing the *spectral flow* (the net change in the number of nonnegative eigenvalues) of a smooth curve of symmetric matrices. We begin by discussing a general framework for such a computation.

3.1. **Preliminaries: spectral flow and the partial signatures.**

We follow the discussion in [GPP04b, §2.1-2.2]. In what follows, V is a subspace of $\mathbb{R}^{2n}$ and $\mathcal{S}(V)$ is the vector space of symmetric linear operators (matrices) $T : V \to V$. Consider a smooth curve $t \mapsto L(t) \in \mathcal{S}(V)$, which has an isolated singularity at $t = t_0$, i.e. $\det L(t_0) = 0$ and $\det L(t) \neq 0$ for $0 < |t - t_0| < \varepsilon$. The following is a method to compute the jump in the number of nonnegative eigenvalues of L as t passes through t_0.

A *root function* for $L(t)$ at $t = t_0$ is a smooth map $q : [t_0 - \varepsilon, t_0 + \varepsilon] \to V$, $\varepsilon > 0$, such that $q(t_0) \in \ker(L(t_0))$. The *order* of q, $\mathrm{ord}(q)$, is the order of zero at $t = t_0$ of the map $t \mapsto L(t)q(t)$, i.e. the smallest positive integer k such that $\frac{d^k}{dt^k}\left(L(t)q(t)\right)\big|_{t=t_0} \neq 0$. With these notions we can define a sequence of spaces W_k and bilinear forms $B_k : W_k \times W_k \to \mathbb{R}$ for $k \geq 1$ as follows:

$$W_k := \{q_0 \in V : \text{there exists a root function } q \text{ with } \mathrm{ord}(q) \geq k \text{ and } q(t_0) = q_0\}, \tag{3.1}$$

$$B_k(q_0, r_0) := \frac{1}{k!}\left\langle \frac{d^k}{dt^k}\left(L(t)q(t)\right)\big|_{t=t_0}, r_0\right\rangle_{\mathbb{R}^{2n}}, \qquad q_0, r_0 \in W_k, \tag{3.2}$$

where q in (3.2) is any root function with $\mathrm{ord}(q) \geq k$ and $q(t_0) = q_0$. It follows from the definition that $W_1 \subseteq \ker(L(t_0))$. It is proven in [GPP04b, Proposition 2.4] that B_k is symmetric and independent of the choice of q, and therefore well-defined. Moreover, from [GPP04b, Proposition 2.4, Corollary 2.10] we have

$$W_{k+1} \subseteq W_k \quad \text{for all } k \geq 1, \qquad \text{and} \qquad W_{k+1} = \ker B_k. \tag{3.3}$$

Notice that if B_k is nondegenerate for some k, then $W_j = \{0\}$ for all $j > k$.

The spaces W_k can be characterised as follows. Define $L_k := \frac{1}{k!}\frac{d^k}{dt^k}L(t)|_{t=t_0}$. A *generalised Jordan chain* of length $k+1$ starting at q_0 for $L(t)$ at $t = t_0$ is a sequence of nonzero vectors $\{q_0, q_1, ..., q_k\}$, $q_i \in V$ satisfying the system of $k+1$ equations

$$\begin{aligned} &L_0 q_0 = 0,\\ &L_1 q_0 + L_0 q_1 = 0,\\ &L_2 q_0 + L_1 q_1 + L_0 q_2 = 0,\\ &\vdots\\ &\sum_{j=0}^{k} L_{k-j} q_j = 0. \end{aligned} \tag{3.4}$$

Such a chain is called *maximal* if it cannot be extended to a chain of length $k+2$, i.e. there is no solution q_{k+1} to

$$L_{k+1}q_0 + L_k q_1 + \cdots + L_1 q_k + L_0 q_{k+1} = 0. \tag{3.5}$$

For any generalised Jordan chain $\{q_0, \ldots, q_k\}$ (not necessarily maximal), the function $q(t) := \sum_{j=0}^{k}(t - t_0)^j q_j$ is a root function with $\mathrm{ord}(q) \geq k+1$ and $q(t_0) = q_0$, since

$$\frac{d^i}{dt^i}\left(L(t)q(t)\right)|_{t=t_0} = \sum_{j=0}^{i}\binom{i}{j} L^{(i-j)}(t_0)q^{(j)}(t_0) = i!\sum_{j=0}^{i} L_{i-j}q_j = 0 \qquad \text{for all } i = 0, 1, \ldots, k.$$

Here we used that $q^{(j)}(t_0) = j!q_j$ and $L^{(i-j)}(t_0) = (i-j)!L_{i-j}$ in the second equality, and (3.4) in the third equality. Conversely, any root function q with $\mathrm{ord}(q) \geq k+1$ gives a generalised Jordan chain of length (at least) $k+1$ via $q_i := \frac{1}{i!}q^{(i)}(t_0)$. This shows that:

$$\begin{aligned} W_{k+1} = \{q_0 \in V :\ \exists \text{ a generalised Jordan chain of length } k+1,\\ \text{starting at } q_0, \text{ for } L(t) \text{ at } t = t_0\}. \end{aligned} \tag{3.6}$$

Moreover, the root function q associated with any $q_0 \in W_{k+1}$ has $\mathrm{ord}(q) = k+1$ if and only if the associated Jordan chain $\{q_0, ..., q_k\}$ is maximal. Maximality of the chain holds if and only if

$$L_{k+1}q_0 + L_k q_1 + \cdots + L_1 q_k \notin \mathrm{Ran}(L_0) = \ker(L_0)^{\perp}. \tag{3.7}$$

Notice that (3.6) shows that $\ker L_0 \subseteq W_1$, since any $q_0 \in \ker L_0$ is a generalised Jordan chain of length one for $L(t)$. From our ealier observation this implies

$$W_1 = \ker L(t_0). \tag{3.8}$$

For any generalised Jordan chain $\{q_0, \dots, q_k\}$, the bilinear form B_{k+1} is given by

$$B_{k+1}(q_0, r_0) = \sum_{j=0}^{k} \langle L_{k+1-j} q_j, r_0 \rangle_{\mathbb{R}^{2n}} , \qquad q_0, r_0 \in W_{k+1}, \tag{3.9}$$

as can be seen from substituting the root function $q(t) = \sum_{j=0}^{k} (t - t_0)^j q_j$ into (3.2).

If the chain $\{q_0, \dots, q_k\}$ is not maximal (i.e. it can be extended to $\{q_0, \dots, q_{k+1}\}$ where q_{k+1} solves (3.5)), then for all $i = 0, \dots, k$ and any $r_0 \in W_{i+1}$, we have

$$B_{i+1}(q_0, r_0) = \sum_{j=0}^{i} \langle L_{i+1-j} q_j, r_0 \rangle_{\mathbb{R}^{2n}} = - \langle L_0 q_{i+1}, r_0 \rangle_{\mathbb{R}^{2n}} = - \langle q_{i+1}, L_0 r_0 \rangle_{\mathbb{R}^{2n}} = 0.$$

Here, the second equality follows for $i = 0, \dots, k-1$ from (3.4) and for $i = k$ from (3.5), and we used that $r_0 \in W_{i+1} \subseteq \ker L(t_0)$. On the other hand, if the Jordan chains associated with $q_0, r_0 \in W_{k+1}$ are both of length $k+1$ and maximal, then $B_{k+1}(q_0, r_0)$ is nondegenerate. This follows from the symmetry of B_{k+1} and (3.7).

The family of bilinear forms $\{B_k\}_k$ can be used to compute the jump in the number of nonnegative eigenvalues of $L(t)$ as t increases through t_0. The following is taken from [GPP04b, Proposition 2.9]. We denote by $n_+(S), n_-(S), n_+^0(S), n_-^0(S)$ respectively the number of positive, negative, nonnegative and nonpositive eigenvalues (squares) of the symmetric matrix (symmetric bilinear form) S. For the bilinear forms defined in (3.2), the integers

$$n_-(B_k), \quad n_+(B_k), \quad n_+(B_k) - n_-(B_k), \tag{3.10}$$

for $k \geq 1$ are called, respectively, the *kth partial negative index*, the *kth partial positive index* and the *kth partial signature* of $L(t)$ at $t = t_0$. The integers in (3.10) are collectively referred to as the *partial signatures* of the curve of symmetric matrices $L(t)$ at $t = t_0$.

Proposition 3.1. *Suppose* $[t_0 - \varepsilon, t_0 + \varepsilon] \mapsto L(t) \in \mathcal{S}(V)$ *is a smooth curve of symmetric matrices with an isolated singularity at* $t = t_0$*,* $\{\lambda_i(t)\}$ *are the smooth curves of eigenvalues of* $L(t)$*, and the associated spaces* W_k *and bilinear forms* B_k *are as in* (3.1)*,*(3.2)*. For all nonconstant* $\lambda_i(t)$ *vanishing at* $t = t_0$*, assume the zero of* $\lambda_i(t)$ *at* $t = t_0$ *is of finite order, and that for each eigenvalue* $\lambda_i(t)$*, there exists a smooth family of unit eigenvectors* $u_i(t)$*, where the* u_i *are pairwise orthogonal for each* t*. Then the following hold:*

(i) $W_k = \operatorname{span}\{u_i(t_0) : i \in \{1, \dots, n\}$ *is such that* $\lambda_i^{(j)}(t_0) = 0$ *for all* $j < k\}$*;*

(ii) if $q \in W_k$ *is an eigenvector of* $\lambda_i(t_0)$*, where* $\lambda_i^{(j)}(t_0) = 0$ *for all* $j < k$*, then* $B_k(q, r) = \frac{1}{k!} \lambda_i^{(k)}(t_0) \langle q, r \rangle$ *for all* $r \in W_k$*;*

(iii) $$n_+^0(L(t_0 + \varepsilon)) - n_+^0(L(t_0)) = - \sum_{k \geq 1} n_-(B_k),$$

$$n_+^0(L(t_0)) - n_+^0(L(t_0 - \varepsilon)) = \sum_{k \geq 1} \left(n_-(B_{2k}) + n_+(B_{2k-1}) \right),$$

$$n_+^0(L(t_0 + \varepsilon)) - n_+^0(L(t_0 - \varepsilon)) = \sum_{k \geq 1} \left(n_+(B_{2k-1}) - n_-(B_{2k-1}) \right),$$

where each of the sums on the right hand side of the previous three equations have a finite number of nonzero terms.

Note that the negative index $n_-(B_k)$ (resp. the positive index $n_+(B_k)$) is the number of i's in $\{1, \dots, n\}$ such that $\lambda_i(t)$ has a zero of order k at $t = t_0$ and whose kth derivative is negative (resp. positive) at $t = t_0$. Note as well that to obtain the formulas in (ii), we have manipulated the corresponding formulas in [GPP04b, Proposition 2.9] using the following formula from [GPP04b, Corollary 2.11]:

$$\sum_{k\geq 1} (n_+(B_k) + n_-(B_k)) = \dim\ker(L(t_0)). \tag{3.11}$$

For some illustrative examples involving computation of the spaces W_k, the forms B_k and the behaviour the eigenvalues $\lambda_i(t)$ in some simple cases when $V = \mathbb{R}^2$ and $L(t) \in \mathbb{R}^{2\times 2}$, see [GPP04b, Examples 2.8, 2.12].

3.2. **The Maslov index.** In this section we follow the discussions in [Arn67, RS93, GPP04b]. Consider $\mathbb{R}^{2n}$ equipped with the symplectic form

$$\omega(u, v) = \langle Ju, v\rangle_{\mathbb{R}^{2n}}, \qquad J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}. \tag{3.12}$$

A *Lagrangian subspace* of $\mathbb{R}^{2n}$ is one that is n dimensional and upon which the symplectic form vanishes. We denote the Grassmannian of all Lagrangian subspaces of $\mathbb{R}^{2n}$ by

$$\mathcal{L}(n) := \{\Lambda \subset \mathbb{R}^{2n} : \dim\Lambda = n, \omega(u, v) = 0 \ \forall\ u, v \in \Lambda\}. \tag{3.13}$$

A *frame* for a Lagrangian subspace Λ of $\mathbb{R}^{2n}$ is a $2n \times n$ matrix whose columns span Λ. Such a frame has the form

$$\begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \text{where } X^\top Y = Y^\top X, \qquad X, Y \in \mathbb{R}^{n\times n}. \tag{3.14}$$

The symmetry of $X^\top Y$ follows from the vanishing of (3.12). Such a frame is not unique; right multiplication by an invertible matrix will yield a different frame for the same space. In particular, if X is invertible then an equivalent frame is

$$\begin{pmatrix} I \\ YX^{-1} \end{pmatrix}, \qquad \text{where } \left(YX^{-1}\right)^\top = YX^{-1}. \tag{3.15}$$

Arnol'd [Arn67] defined a Maslov index for non-closed curves as follows. Any fixed $V \in \mathcal{L}(n)$ gives rise to a decomposition of $\mathcal{L}(n)$ via $\mathcal{L}(n) = \bigcup_{k=0}^{n} \mathcal{T}_k(V)$, where each stratum $\mathcal{T}_k(V) := \{W \in \mathcal{L}(n) : \dim(W \cap V) = k\}$ has codimension $k(k+1)/2$. The *train* $\mathcal{T}(V)$ of V is the set of all Lagrangian planes that intersect V nontrivially, i.e. $\mathcal{T}(V) := \bigcup_{k=1}^{n} \mathcal{T}_k(V)$. From the fundamental lemma of [Arn67], $\mathcal{T}_1(V)$ is two-sidedly imbedded in $\mathcal{L}(n)$, that is, there exists a continuous vector field on $\mathcal{L}(n)$ that is everywhere transverse to $\mathcal{T}_1(V)$. Such a vector field therefore defines a 'positive' and a 'negative' side of $\mathcal{T}_1(V)$. For any continuous curve $\Lambda : [a, b] \to \Lambda(n)$ with endpoints lying off the train and which intersects $\mathcal{T}(V)$ only in $\mathcal{T}_1(V)$, its *Maslov index* is given by $\nu_+ - \nu_-$, where ν_+ (ν_-) is the number of points of passage of Λ from the negative to the positive side (from the positive to the negative side) of $\mathcal{T}_1(V)$. Robbin and Salamon [RS93] gave a definition in terms of *crossing forms*, which is based on an identification of the tangent space of $\mathcal{L}(n)$ with the space of quadratic forms. Their definition required neither transversality at the endpoints nor of intersections only with $\mathcal{T}_1(V)$. However, nondegeneracy of the quadratic crossing form is required; this is equivalent to the path having only transversal intersections with $\mathcal{T}(V)$. They extended the definition to *all* continuous Lagrangian paths (i.e. those for which the crossing form is degenerate) via homotopy invariance (see Proposition 3.3).

Giambò, Piccione and Portaluri [GPP04b, GPP04a] gave a formula for the Maslov index of an analytic Lagrangian path having isolated possibly nontransversal intersections with $\mathcal{T}(V)$. This is given below. In doing so, they did away with the nondegeneracy assumption of [RS93] (at least for analytic paths). To the analytic Lagrangian path they associate a locally-defined smooth curve of symmetric bilinear forms, the spectral flow of which is shown to locally compute the Maslov index.

Suppose $\Lambda : [a,b] \to \mathcal{L}(n)$ is an analytic path of Lagrangian subspaces, and let $V \in \mathcal{L}(n)$ be fixed. Suppose further that $t = t_0$ is an isolated *crossing*, that is, $\Lambda(t_0) \cap V \neq \{0\}$, and choose any $W \in \mathcal{L}(n)$ which is transverse to both $\Lambda(t_0)$ and V. By continuity, W is transversal to $\Lambda(t)$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\varepsilon > 0$ small enough, and there exists a smooth, unique family of matrices $R(t)$, viewed as operators from V into W, such that $\Lambda(t)$ is the graph of $R(t)$ for each t, i.e. $\Lambda(t) = \operatorname{graph}(R(t)) = \{q + R(t)q : q \in V\}$. This allows one to define a smooth curve of bilinear forms

$$[t_0 - \varepsilon, t_0 + \varepsilon] \ni t \mapsto \omega(R(t)\cdot, \cdot)|_{V\times V} \tag{3.16}$$

on V, which are symmetric for each t on account of $\Lambda(t)$ being Lagrangian. Indeed, for all $u, v \in V$ we have

$$\begin{aligned} \omega(R(t)u, v) &= \omega(u + R(t)u, v) \\ &= \omega(u + R(t)u, v + R(t)v) - \omega(u + R(t)u, R(t)v) \\ &= -\omega(u, R(t)v) = \omega(R(t)v, u). \end{aligned} \tag{3.17}$$

Moreover, we have

$$(\ker R(t)) \cap V = \ker\left(\omega(R(t)\cdot, \cdot)|_{V\times V}\right) = \Lambda(t) \cap V, \tag{3.18}$$

and from our assumptions the right hand side is nontrivial precisely when $t = t_0$. In this way we see that any crossing of the path Λ with the train $\mathcal{T}(V)$ corresponds to an isolated singularity of the locally-defined form (3.16).

Denote by $\pi_1(\mathcal{L}(n))$ the fundamental groupoid of $\mathcal{L}(n)$, i.e. the set of (fixed-endpoint) homotopy classes $[\Lambda]$ of paths $\Lambda : [a,b] \to \mathcal{L}(n)$, equipped with the partial operation $[\Lambda]\cdot[\xi] = [\Lambda * \xi]$, where $*$ is the concatenation of two paths $\Lambda, \xi : [a,b] \to \mathcal{L}(n)$, which is only defined if $\Lambda(b) = \xi(a)$. For all $V \in \mathcal{L}(n)$, it is proven in [GPP04b, Corollary 3.5] that there is a unique integer-valued homomorphism $\mu(\cdot; V)$ on $\pi_1(\mathcal{L}(n))$[1] such that the following holds. With our earlier choice of W, i.e. such that $W \in \mathcal{T}_0(V) \cap \mathcal{T}(\Lambda(t_0))$, if $\Lambda : [a,b] \to \mathcal{T}_0(W)$ then $\mu([\Lambda]; V)$ is given by the spectral flow of the family of forms (3.16) defined over $[a,b]$. Note (3.16) is well-defined over the entire interval in this case because $\Lambda : [a,b] \to \mathcal{T}_0(W)$. The Maslov index of *any* continuous path Λ is then defined to be $\mu([\Lambda]; V)$, and the authors prove in [GPP04b, Proposition 3.11], using Proposition 3.1, that it is computable via the partial signatures of (3.16) at each isolated crossing with $\mathcal{T}(V)$. For our purposes, it will suffice to use the latter computational tool as our definition of the Maslov index.

In the same fashion as (3.2), we define the *kth-order crossing form* by

$$\mathfrak{m}^{(k)}_{t_0}(\Lambda, V)(q_0, r_0) = \frac{d^k}{dt^k}\,\omega\big(R(t)q(t), r_0\big)\Big|_{t=t_0}, \qquad q_0, r_0 \in W_k, \tag{3.19}$$

where q is a *root function* for (3.16) at $t = t_0$ with $\operatorname{ord}(q) \geq k$, i.e. a smooth map $q : [t_0 - \varepsilon, t_0 + \varepsilon] \to V$ such that $q(t_0) \in \ker JR(t_0) = \ker R(t_0)$ and $\dfrac{d^i}{dt^i} JR(t)q(t)|_{t=t_0} = 0$ for $i = 1, \dots, k-1$, and

$$\begin{aligned} W_k = \{q_0 \in V :\ &\exists \text{ a generalised Jordan chain of length } k, \\ &\text{starting at } q_0, \text{ for the curve of bilinear forms in (3.16) at } t = t_0\}. \end{aligned} \tag{3.20}$$

(For more details on these terms, see Section 3.1.) We will mostly work with the associated quadratic form

$$\mathfrak{m}^{(k)}_{t_0}(\Lambda, V)(q_0) := \mathfrak{m}^{(k)}_{t_0}(\Lambda, V)(q_0, q_0) \qquad q_0 \in W_k. \tag{3.21}$$

For notational convenience we will sometimes drop the subscript zero for the functions in W_k; it will be clear from the context whether q denotes a root function or a fixed vector in V. In the case that $k = 1$, we will drop the superscript and write $\mathfrak{m}_{t_0}(\Lambda, V)$. Following [RS93],

[1] i.e. a map $\mu : \pi_1(\mathcal{L}(n)) \to \mathbb{Z}$ such that $\mu([\Lambda]*[\xi]) = \mu([\Lambda]) + \mu([\xi])$ for all $[\Lambda], [\xi] \in \pi(\mathcal{L}(n))$ with $\Lambda(b) = \xi(a)$

a crossing $t = t_0$ will be called *regular* if $\mathfrak{m}_{t_0}$ is nondegenerate; otherwise, $t = t_0$ will be called *non-regular*. Denoting by $n_+(B)$ and $n_-(B)$ the number of positive, respectively negative, squares of the quadratic form B, we define the sequence of *partial signatures* of (3.19) (as in (3.10)):

$$n_-(\mathfrak{m}_{t_0}^{(k)}), \quad n_+(\mathfrak{m}_{t_0}^{(k)}), \quad \mathrm{sign}(\mathfrak{m}_{t_0}^{(k)}) = n_+(\mathfrak{m}_{t_0}^{(k)}) - n_-(\mathfrak{m}_{t_0}^{(k)}).$$

It is proven in [GPP04b, Lemma 3.10] that these integers are independent of the choice of W and are therefore well-defined. The Maslov index of the Lagrangian path Λ is then given as follows, as in [GPP04b, Proposition 3.11].

Definition 3.2. *Suppose $\Lambda : [a, b] \to \mathcal{L}(n)$ is an analytic path of Lagrangian subspaces, whose intersections with $\mathcal{T}(V)$ are isolated. Its* Maslov index *is given by*

$$\mathrm{Mas}(\Lambda, V; [a, b]) = -\sum_{k\geq 1} n_-\left(\mathfrak{m}_a^{(k)}\right) + \sum_{t_0\in(a,b)} \left(\sum_{k\geq 1} \mathrm{sign}\left(\mathfrak{m}_{t_0}^{(2k-1)}\right)\right) + \sum_{k\geq 1}\left(n_+\left(\mathfrak{m}_b^{(2k-1)}\right) + n_-\left(\mathfrak{m}_b^{(2k)}\right)\right), \tag{3.22}$$

where the right hand side has a finite number of nonzero terms.

Notice that at all interior crossings $t_0 \in (a, b)$, only the signatures of the crossing forms of odd order contribute; at the initial point the negative indices of crossing forms of all order contribute; while at the final point, the negative indices of the forms of even order and the positive indices of the forms of odd order contribute. From (3.11), we have that

$$\sum_{k\geq 1}\left(n_+(\mathfrak{m}_{t_0}^{(k)}) + n_-(\mathfrak{m}_{t_0}^{(k)})\right) = \dim \Lambda(t_0) \cap V, \tag{3.23}$$

so that by taking sufficiently many higher order crossing forms, a crossing t_0 will always contribute $\dim \Lambda(t_0) \cap V$ summands (the signs of which may offset each other) to the Maslov index.

We point out that Definition 3.2 includes, as a special case, the definition given by Robbin and Salamon [RS93] in the case that all crossings are regular. To see this, we compute $\mathfrak{m}_{t_0}$ from (3.19):

$$\mathfrak{m}_{t_0}(\Lambda, V)(q_0) = \frac{d}{dt}\omega\left(R(t)q_0, q_0\right)\big|_{t=t_0}, \qquad q_0 \in \Lambda(t_0) \cap V, \tag{3.24}$$

where we used the symmetry of $JR(t_0)$, and (3.8), (3.18) to obtain $W_1 = \Lambda(t_0) \cap V$. If $\mathfrak{m}_{t_0}$ is nondegenerate, it follows from (3.3) that $W_2 = \{0\}$ and therefore $W_k = \{0\}$ for $k \geq 3$. Thus the forms $\mathfrak{m}_{t_0}^{(k)}$ are trivial for $k \geq 2$, and from (3.23) we have $n_+(\mathfrak{m}_{t_0}) + n_-(\mathfrak{m}_{t_0}) = \dim \Lambda(t_0) \cap V$. Thus, the Maslov index of a path $\Lambda : [a, b] \to \mathcal{L}(n)$ with only regular crossings is given by

$$\mathrm{Mas}(\Lambda, V; [a, b]) = -n_-\left(\mathfrak{m}_a\right) + \sum_{t_0\in(a,b)} \mathrm{sign}\left(\mathfrak{m}_{t_0}\right) + n_+\left(\mathfrak{m}_b\right), \tag{3.25}$$

just as in [RS93, §2].

Two special cases will be important in our analysis. The first is the instance of a non-regular crossing $t_0 = a$ at the initial point of the path $\Lambda : [a, b] \to \mathcal{L}(n)$, for which the first-order crossing form is identically zero and the second-order crossing form is nondegenerate. Then

$$n_+(\mathfrak{m}_a^{(2)}) + n_-(\mathfrak{m}_a^{(2)}) = \dim \Lambda(t_0) \cap V, \tag{3.26}$$

and from Definition 3.2 we see that, for $\varepsilon > 0$ small enough,

$$\operatorname{Mas}(\Lambda, V; [a, a+\varepsilon]) = -n_-(\mathfrak{m}_a^{(2)}), \tag{3.27}$$

just as in [CCLM23, Proposition 4.15] and [DJ11, Proposition 3.10]. Note that in this case, we have $W_2 = (\ker R(t_0)) \cap V = \ker\big(\dot R(t_0)\big) \cap V = \Lambda(t_0) \cap V$, and the second-order crossing form (3.19) is given by (where dot denotes d/dt)

$$\begin{aligned}
\mathfrak{m}_{t_0}^{(2)}(\Lambda, V)(q_0) &= \frac{d^2}{dt^2}\,\omega\big(R(t)q(t), q_0\big)\Big|_{t=t_0}, \\
&= \omega\big(\ddot R(t)q_0, q_0\big) + \omega\big(\dot R(t_0)\dot q(t_0), q_0\big) + \omega\big(R(t_0)\ddot q(t_0), q_0\big), \\
&= \omega\big(\ddot R(t)q_0, q_0\big),
\end{aligned} \tag{3.28}$$

for $q_0 \in \Lambda(t_0) \cap V$, where we used the symmetry of $JR(t_0)$ and $J\dot R(t_0)$.

The second special case is the instance of a non-regular interior crossing $t_0 \in (a, b)$ for which the first-order form is degenerate with nonzero rank, the second-order form is identically zero, and the third-order form is nondegenerate. Then

$$n_+(\mathfrak{m}_{t_0}^{(1)}) + n_-(\mathfrak{m}_{t_0}^{(1)}) + n_+(\mathfrak{m}_{t_0}^{(3)}) + n_-(\mathfrak{m}_{t_0}^{(3)}) = \dim \Lambda(t_0) \cap V, \tag{3.29}$$

and if t_0 is the only crossing in $[t_0 - \varepsilon, t_0 + \varepsilon]$, its contribution to the Maslov index is

$$\operatorname{Mas}(\Lambda, V; [t_0 - \varepsilon, t_0 + \varepsilon]) = \operatorname{sign} \mathfrak{m}_{t_0}^{(1)} + \operatorname{sign} \mathfrak{m}_{t_0}^{(3)}. \tag{3.30}$$

We summarise the important properties of the Maslov index for the current analysis in the following proposition, as in [GPP04b, Lemma 3.8] (see also [RS93, Theorem 2.3]).

Proposition 3.3. *The Maslov index enjoys*

(1) *(Homotopy invariance.) If two paths $\Lambda_1, \Lambda_2 : [a, b] \longrightarrow \mathcal{L}(n)$ are homotopic with fixed endpoints, then*

$$\operatorname{Mas}(\Lambda_1(t), V; [a, b]) = \operatorname{Mas}(\Lambda_2(t), \Lambda_0; [a, b]). \tag{3.31}$$

(2) *(Additivity under concatenation.) For $\Lambda(t) : [a, c] \longrightarrow \mathcal{L}(n)$ and $a < b < c$,*

$$\operatorname{Mas}(\Lambda(t), V; [a, c]) = \operatorname{Mas}(\Lambda(t), V; [a, b]) + \operatorname{Mas}(\Lambda(t), V; [b, c]). \tag{3.32}$$

(3) *(Symplectic additivity.) Identify the Cartesian product $\mathcal{L}(n) \times \mathcal{L}(n)$ as a submanifold of $\mathcal{L}(2n)$. If $\Lambda = \Lambda_1 \oplus \Lambda_2 : [a, b] \to \mathcal{L}(2n)$ where $\Lambda_1, \Lambda_2 : [a, b] \to \mathcal{L}(n)$, and $V = V_1 \oplus V_2$ where $V_1, V_2 \in \mathcal{L}(n)$, then*

$$\operatorname{Mas}(\Lambda(t), V; [a, b]) = \operatorname{Mas}(\Lambda_1(t), V_1; [a, b]) + \operatorname{Mas}(\Lambda_2(t), V_2; [a, b]). \tag{3.33}$$

(4) *(Zero property.) If $\Lambda : [a, b] \longrightarrow \mathcal{T}_k(V)$ for any fixed integer k, then*

$$\operatorname{Mas}(\Lambda(t), V; [a, b]) = 0. \tag{3.34}$$

Suppose now that we have a pair of Lagrangian paths $(\Lambda_1, \Lambda_2) : [a, b] \to \mathcal{L}(n) \times \mathcal{L}(n)$, or a *Lagrangian pair*. Using the symplectic additivity property of Proposition 3.3, it is possible to define the Maslov index of such an object (as in [GPP04b, RS93, Fur04]), where crossings are values $t_0 \in [a, b]$ such that $\Lambda_1(t_0) \cap \Lambda_2(t_0) \neq \{0\}$. Precisely, one realises the Lagrangian pair as the path $\Lambda_1 \oplus \Lambda_2$ in the doubled space $\mathbb{R}^{4n}$ equipped with the symplectic form $\Omega = \omega \times (-\omega)$, where

$$\Omega((u_1, u_2), (v_1, v_2)) = \omega(u_1, v_1) - \omega(u_2, v_2), \qquad u_1, u_2, v_1, v_2 \in \mathbb{R}^{2n}. \tag{3.35}$$

Crossings of the pair then correspond to intersections of the path $\Lambda_1 \oplus \Lambda_2 : [a, b] \to \mathbb{R}^{4n}$ with the diagonal subspace $\triangle = \{(x, x) : x \in \mathbb{R}^{2n}\} \subset \mathbb{R}^{4n}$. The resultant Maslov index,

$$\operatorname{Mas}(\Lambda_1, \Lambda_2; [a, b]) \coloneqq \operatorname{Mas}(\Lambda_1 \oplus \Lambda_2, \triangle; [a, b]), \tag{3.36}$$

is thus a signed count of the intersections of Λ_1 and Λ_2 which, loosely speaking, measures the winding of Λ_1 *relative to* Λ_2.

The right hand side of (3.36) is computed with Definition 3.2. To that end, using Ω as the symplectic form in (3.19) for the path $\Lambda_1 \oplus \Lambda_2$, we define the *kth-order relative crossing form* of the Lagrangian pair (Λ_1, Λ_2) to be the quadratic form

$$\mathfrak{m}^{(k)}_{t_0}(\Lambda_1, \Lambda_2)(q) := \mathfrak{m}^{(k)}_{t_0}(\Lambda_1, \Lambda_2(t_0))(q) - \mathfrak{m}^{(k)}_{t_0}(\Lambda_2, \Lambda_1(t_0))(q), \quad q \in W^k, \tag{3.37}$$

where $W_k \subseteq \Lambda_1(t_0) \cap \Lambda_2(t_0)$. Using these forms in Definition 3.2 thus allows us to compute the Maslov index of the pair $(\Lambda_1, \Lambda_2) : [a, b] \to \mathcal{L}(n) \times \mathcal{L}(n)$. In the case that $\Lambda_2 = V$ is constant, the Maslov index of the pair reduces to the Maslov index of the single path Λ_1, with respect to the reference plane V.

The Maslov index is invariant for Lagrangian pairs that are *stratum homotopic*. This result will be needed in our analysis, and we give a proof below. The result for single paths can be found in [RS93, Theorem 2.4]. Suppose the pairs $(\Lambda_1, \Lambda_2) : [a, b] \to \mathcal{L}(n)$ and $(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) : [a, b] \to \mathcal{L}(n)$ are stratum homotpic, i.e. there exist continuous mappings $H_1, H_2 : [0, 1] \times [a, b] \to \mathcal{L}(n)$ such that

$$\begin{aligned} H_1(0, \cdot) &= \Lambda_1(\cdot), & H_2(0, \cdot) &= \Lambda_2(\cdot) \\ H_1(1, \cdot) &= \widetilde{\Lambda}_1(\cdot), & H_2(1, \cdot) &= \widetilde{\Lambda}_2(\cdot), \end{aligned}$$

for which $\dim(H_1(s, a) \cap H_2(s, a))$ and $\dim(H_1(s, b) \cap H_2(s, b))$ are constant with respect to $s \in [0, 1]$. (The name "stratum homotopy" derives from the fact that

$$H_1(s, a) \oplus H_2(s, a) \in \mathcal{T}_{k_1}(\triangle), \qquad H_1(s, b) \oplus H_2(s, b) \in \mathcal{T}_{k_2}(\triangle),$$

for all $s \in [0, 1]$ and fixed integers k_1, k_2.) Then we have:

Lemma 3.4.

$$\mathrm{Mas}(\Lambda_1, \Lambda_2; [a, b]) = \mathrm{Mas}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2; [a, b]). \tag{3.38}$$

Proof. Consider the continuous mapping $H = H_1 \oplus H_2 : [0, 1] \times [a, b] \to \mathcal{L}(n) \times \mathcal{L}(n)$. By continuity of H and homotopy invariance (i.e. property (3) of Proposition 3.3), we have

$$\begin{aligned} &\mathrm{Mas}(H(0, \cdot), \triangle; [a, b]) + \mathrm{Mas}(H(\cdot, b), \triangle; [0, 1]) \\ &\qquad - \mathrm{Mas}(H(1, \cdot), \triangle; [a, b]) - \mathrm{Mas}(H(\cdot, a), \triangle; [0, 1]) = 0. \end{aligned} \tag{3.39}$$

Using (3.36) we have

$$\mathrm{Mas}(H(0, \cdot), \triangle; [a, b]) = \mathrm{Mas}(\Lambda_1, \Lambda_2; [a, b]), \quad \mathrm{Mas}(H(1, \cdot), \triangle; [a, b]) = \mathrm{Mas}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2; [a, b]).$$

By assumption $\dim(H(\cdot, a) \cap \triangle) = \dim(H_1(\cdot, a) \cap H_2(\cdot, a))$ and $\dim(H(\cdot, b) \cap \triangle) = \dim(H_1(\cdot, b) \cap H_2(\cdot, b))$ are constant, so by property (4) of Proposition 3.3 the Maslov indices of the second and fourth terms in (3.39) are zero. Equation (3.38) follows. □

For a Lagrangian pair, when the first-order form $\mathfrak{m}_{t_0}(\Lambda_1, \Lambda_2)$ of (3.37) at $t = a$ is identically zero, and the second order form $\mathfrak{m}^{(2)}_a(\Lambda_1, \Lambda_2)$ is nondegenerate, equation (3.27) becomes

$$\mathrm{Mas}(\Lambda_1, \Lambda_2; [a, a + \varepsilon]) = -n_-(\mathfrak{m}^{(2)}_a(\Lambda_1, \Lambda_2)). \tag{3.40}$$

This formula will be needed in our application to the eigenvalue problem (1.12). In particular, the crossing corresponding to the zero eigenvalue of the operator N is not regular in the λ direction, and the conditions for (3.40) are met under the assumption that $\mathcal{I}_1, \mathcal{I}_2 \neq 0$.

We will call a crossing $t = t_0$ *positive* if

$$\sum_{k\geq 1}\left(n_+(\mathfrak{m}_{t_0}^{(2k-1)})\right) = \dim\Lambda(t_0)\cap V, \tag{3.41}$$

and *negative* if

$$\sum_{k\geq 1}\left(n_-(\mathfrak{m}_{t_0}^{(2k-1)})\right) = \dim\Lambda(t_0)\cap V. \tag{3.42}$$

In light of Definition 3.2, if t_0 is a positive interior crossing, or a positive crossing at the final point $t_0 = b$, then it contributes $\dim\Lambda(t_0)\cap V$ to the Maslov index. Similarly, if t_0 is a negative interior crossing, or a negative crossing at the initial point $t_0 = a$, then its contribution is $-\dim\Lambda(t_0)\cap V$. Note, however, that with this convention, the final crossing $t_0 = b$ may still contribute $\dim\Lambda(b)\cap V$ if it is not positive, and the initial point $t_0 = a$ may still contribute $-\dim\Lambda(a)\cap V$ if it is not negative.

3.3. **Lagrangian pairs and the Maslov box.** We first discuss the regularity and Lagrangian property of the stable and unstable bundles. Recall $\mathbb{E}^s(x,\lambda)$ and $\mathbb{E}^u(x,\lambda)$ defined in (2.12) for $x\in\mathbb{R}$ and $\lambda\in\mathbb{R}$. We extend $\mathbb{E}^s$ to $x=+\infty$ and $\mathbb{E}^u$ to $x=-\infty$ by setting

$$\mathbb{E}^s(+\infty,\lambda) := \mathbb{S}(\lambda), \qquad \mathbb{E}^u(-\infty,\lambda) := \mathbb{U}(\lambda). \tag{3.43}$$

Thus by (2.13), $\mathbb{E}^s$ and $\mathbb{E}^u$ are continuous on $(-\infty,\infty]\times\mathbb{R}$ and $[-\infty,\infty)\times\mathbb{R}$ respectively. Furthermore, since the right hand side of (2.2) is analytic in λ and x, it follows that the solution spaces $\mathbb{E}^s$ and $\mathbb{E}^u$ are analytic on $(x,\lambda)\in\mathbb{R}\times\mathbb{R}$ (note that $x=\pm\infty$ is excluded). We remark here that the mapping

$$\lambda\mapsto\lim_{x\to\infty}\mathbb{E}^u(x;\lambda) \tag{3.44}$$

is discontinuous at eigenvalues $\lambda\in\operatorname{Spec}(N)$. Indeed, if $\lambda\notin\operatorname{Spec}(N)$, then $\lim_{x\to\infty}\mathbb{E}^u(x;\lambda) = \mathbb{U}(\lambda)$ (again as points on the Grassmannian $\operatorname{Gr}_4(\mathbb{R}^8)$), while if $\lambda\in\operatorname{Spec}(N)$ is an eigenvalue then $\lim_{x\to\infty}\mathbb{E}^u(x;\lambda)\cap\mathbb{S}(\lambda)\neq\{0\}$. Now since $\mathbb{U}(\lambda)\cap\mathbb{S}(\lambda)=\{0\}$ i.e. $\mathbb{U}(\lambda)\in\mathcal{T}_0(\mathbb{S}(\lambda))$, and $\mathcal{T}_0(\mathbb{S}(\lambda))$ is an open subset of $\mathcal{L}(n)$ with boundary $\mathcal{T}(\mathbb{S}(\lambda))$, it follows that $\mathbb{U}(\lambda)$ is bounded away from $\mathcal{T}(\mathbb{S}(\lambda))$. For more details see the Appendix in [HLS18].

Remark 3.5. The Maslov index is defined for Lagrangian paths over compact intervals. Following [HLS18] we will sometimes compactify $\mathbb{R}$ via the change of variables

$$x = \ln\left(\frac{1+\tau}{1-\tau}\right), \qquad \tau\in[-1,1]. \tag{3.45}$$

(Similar transformations are used in [BCJ+18, AGJ90].) Notationally we will use a hat to indicate such a change has been made, for example,

$$\widehat{\mathbb{E}}^{s,u}(\tau,\cdot) := \mathbb{E}^{s,u}\left(\ln\left(\frac{1+\tau}{1-\tau}\right),\cdot\right), \quad \tau\in[-1,1]. \tag{3.46}$$

In this case, (3.43) implies that $\widehat{\mathbb{E}}^u(-1,\lambda)=\mathbb{U}(\lambda)$ and $\widehat{\mathbb{E}}^s(1,\lambda)=\mathbb{S}(\lambda)$.

Lemma 3.6. *The spaces $\mathbb{E}^u(x;\lambda)$ and $\mathbb{E}^s(x;\lambda)$ are Lagrangian subspaces of $\mathbb{R}^8$ for all $x\in[-\infty,\infty]$ and $\lambda\in\mathbb{R}$.*

Proof. First, recall that $\dim\mathbb{U}(\lambda)=\dim\mathbb{S}(\lambda)=4$ (we showed in (3.43) that $A_\infty(\lambda)$ is hyperbolic with four eigenvalues of positive real part and four of negative real part.) It follows from the continuity of $\mathbb{E}^u$ on $[-\infty,\infty)\times\mathbb{R}$ that $\dim\mathbb{E}^u(x,\lambda)=4$ for all $(x,\lambda)\in[-\infty,\infty)\times\mathbb{R}$. A similar argument shows $\dim\mathbb{E}^s(x,\lambda)=4$ for $(x,\lambda)\in(-\infty,\infty]\times\mathbb{R}$.

Next, for $x\in\mathbb{R}$, let $\mathbf{w}_1(x;\lambda),\mathbf{w}_2(x;\lambda)\in\mathbb{E}^u(x;\lambda)$. We have:

$\omega(\mathbf{w}_1(x;\lambda),\mathbf{w}_2(x;\lambda)) = \langle J\mathbf{w}_1(x;\lambda),\mathbf{w}_2(x;\lambda)\rangle,$

$$
\begin{aligned}
&= \int_{-\infty}^{x} \frac{d}{ds}\langle J\mathbf{w}_1(s;\lambda), \mathbf{w}_2(s;\lambda)\rangle ds,\\
&= \int_{-\infty}^{x} \langle JA(s;\lambda)\mathbf{w}_1(s;\lambda), \mathbf{w}_2(s;\lambda)\rangle + \langle J\mathbf{w}_1(s;\lambda), A(s;\lambda)\mathbf{w}_2(s;\lambda)\rangle ds,\\
&= \int_{-\infty}^{x} \left\langle \left(A(s;\lambda)^\top J + JA(s;\lambda)\right)\mathbf{w}_1(s;\lambda), \mathbf{w}_2(s;\lambda)\right\rangle ds,\\
&= 0,
\end{aligned}
$$

where we used (2.14), i.e. that $A(x;\lambda)$ is infinitesimally symplectic. The proof for $\mathbb{E}^s(x;\lambda)$ is similar, but the integral is taken over $[x,\infty)$. We have shown that $\mathbb{E}^u$ and $\mathbb{E}^s$ are Lagrangian on $\mathbb{R}\times\mathbb{R}$. That this property extends to $x=\pm\infty$ follows the closedness of $\mathcal{L}(n)$ as a submanifold of the Grassmannian of n-dimensional subspaces of $\mathbb{R}^{2n}$. (Note this latter property follows from the continuity of the symplectic form ω.) □

We are now ready to give the homotopy argument that leads to the lower bound of Theorem 1.2. Consider the following path of Lagrangian pairs

$$
\Gamma \ni (x,\lambda) \mapsto (\mathbb{E}^u(x,\lambda), \mathbb{E}^s(\ell,\lambda)) \in \mathcal{L}(4)\times\mathcal{L}(4), \tag{3.47}
$$

where $\ell \gg 1$ needs to be chosen large enough so that

$$
\mathbb{U}(\lambda)\cap\mathbb{E}^s(x,\lambda) = \{0\} \quad \text{for all} \quad x \geq \ell \tag{3.48}
$$

(see Remark 3.7). Here $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$, where the Γ_i are the contours

$$
\begin{aligned}
&\Gamma_1 : x\in[-\infty,\ell],\;\; \lambda = 0, \quad \Gamma_3 : x\in[-\infty,\ell],\;\; \lambda=\lambda_\infty,\\
&\Gamma_2 : x=\ell,\;\; \lambda\in[0,\lambda_\infty], \quad\; \Gamma_4 : x=-\infty,\;\; \lambda=\lambda\in[0,\lambda_\infty],
\end{aligned} \tag{3.49}
$$

in the λx-plane (see Fig. 1). The set Γ has been referred to by some as the *Maslov box* [HLS18, Cor19], although the associated homotopy argument (outlined below) can be seen in as far back as the works of Bott [Bot56], Edwards [Edw64], Arnol'd [Arn67] and Duistermaat [Dui76]. Notice that along Γ_1 and Γ_3, the second entry $\mathbb{E}^s(\ell,\lambda)$ of (3.47) is fixed. The Maslov index of (3.47) along these pieces thus reduces to the Maslov index for a single path with respect to a fixed reference plane. Along Γ_2 and Γ_4, however, we have a genuine Lagrangian pair.

Crossings of (3.47) are thus points $(x,\lambda)\in\Gamma$ such that

$$
\mathbb{E}^u(x,\lambda)\cap\mathbb{E}^s(\ell,\lambda) \neq \{0\}.
$$

Recalling that λ is an eigenvalue of N if and only if $\mathbb{E}^u(x,\lambda)\cap\mathbb{E}^s(x,\lambda)\neq\{0\}$ for all $x\in\mathbb{R}$, it follows that the λ-values of the crossings along Γ_2 are exactly the eigenvalues of N. In particular, because $0\in\mathrm{Spec}(N)$ there will be a crossing at $(x,\lambda)=(0,\ell)$. From Hypothesis 1.1 we have $\ker(L_-)=\mathrm{span}\{\phi\}$ and $\ker(L_+)=\mathrm{span}\{\phi\}$. The corresponding solutions of (2.2),

$$
\boldsymbol{\phi}(x) := \begin{pmatrix} 0\\ \phi''(x)+\sigma_2\phi(x)\\ 0\\ -\phi(x)\\ 0\\ -\phi'(x)\\ 0\\ \phi'''(x)\end{pmatrix}, \qquad \boldsymbol{\varphi}(x) := \begin{pmatrix} \phi'''(x)+\sigma_2\phi'(x)\\ 0\\ \phi'(x)\\ 0\\ \phi''(x)\\ 0\\ \phi''''(x)\\ 0\end{pmatrix}, \tag{3.50}
$$

(obtained from (2.1) with $v=\phi$ and $u=\phi'$ respectively) will therefore satisfy $\boldsymbol{\phi}(x),\boldsymbol{\varphi}(x)\in\mathbb{E}^u(x;0)\cap\mathbb{E}^s(x;0)$ for all $x\in\mathbb{R}$.

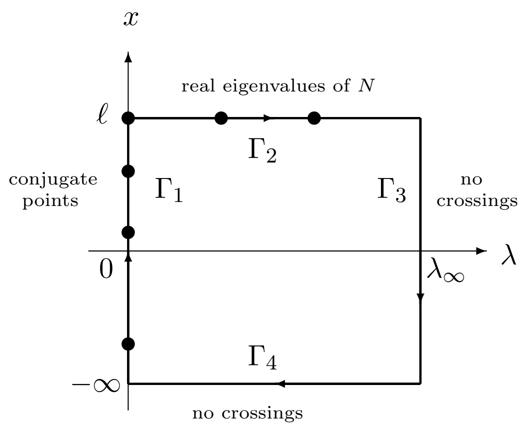

Figure 1. Maslov box in the λx-plane, with edges oriented in a clockwise fashion. The crossing at the top left corner $(0, \ell)$ corresponds to the zero eigenvalue of N. Noting that $\lambda \in \mathbb{R}$ is a spectral parameter, and therefore lives on the real axis in $\mathbb{C}$, it is natural to place λ on the horizontal axis.

Remark 3.7. That the path (3.44) is discontinuous in λ prohibits taking Γ_2 to be at $x = +\infty$. Taking Γ_2 to be at $x = \ell$ for ℓ large enough avoids this issue. Chen and Hu [CH07] showed that by taking ℓ large enough so that (3.48) holds, the Maslov index of (3.47) along Γ_1 is independent of the choice of ℓ. For more details, see [CH07, Cor19].

Crossings along Γ_1, i.e. points $(x, \lambda) = (x_0, 0)$ such that

$$\mathbb{E}^u(x_0, 0) \cap \mathbb{E}^s(\ell, 0) \neq \{0\}, \tag{3.51}$$

are called *conjugate points*. Recall that when $\lambda = 0$ the eigenvalue equations (1.12) decouple into two independent equations for the operators L_+ and L_-. Similarly, when $\lambda = 0$ the first order system (2.2) decouples into two independent systems for the u and v variables. In Section 4 the eigenvalue problems for the operators L_+ and L_- will be written as first order systems; the stable and unstable bundles for the L_+ system will be denoted by $\mathbb{E}^s_+(x, \lambda)$ and $\mathbb{E}^u_+(x, \lambda)$, respectively, while the stable and unstable bundles for the L_- system will be denoted by $\mathbb{E}^{s,u}_-(x, \lambda)$. For the system (2.2), as a result of the decoupling at $\lambda = 0$ we have

$$\mathbb{E}^u(x, 0) = \mathbb{E}^u_+(x, 0) \oplus \mathbb{E}^u_-(x, 0) \quad \text{and} \quad \mathbb{E}^s(x, 0) = \mathbb{E}^s_+(x, 0) \oplus \mathbb{E}^s_-(x, 0), \tag{3.52}$$

so that

$$\begin{aligned} &\{x \in \mathbb{R} : \mathbb{E}^u(x, 0) \cap \mathbb{E}^s(\ell, 0) \neq \{0\}\} = \\ &\qquad \{x \in \mathbb{R} : \mathbb{E}^u_+(x, 0) \cap \mathbb{E}^s_+(\ell, 0) \neq \{0\}\} \cup \{x \in \mathbb{R} : \mathbb{E}^u_-(x, 0) \cap \mathbb{E}^s_-(\ell, 0) \neq \{0\}\}. \end{aligned} \tag{3.53}$$

The precise notion of the direct sums in (3.52) will be given in Section 5. When dealing with conjugate points, we will show in Section 4 that it suffices to use the stable subspace $\mathbb{S}(0)$ (instead of $\mathbb{E}^s(\ell, 0)$) as the reference plane to do computations. That $\mathbb{S}(0) = \mathbb{S}_+(0) \oplus \mathbb{S}_-(0)$, where $\mathbb{S}_\pm(0)$ is the stable subspace of the asymptotic first order system for the eigenvalue problem for $L_\pm$, leads to the following classification of conjugate points.

Definition 3.8. *An L_+* conjugate point *is a point $(x, \lambda) = (x_0, 0)$ such that $\mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0) \neq \{0\}$. An L_-* conjugate point *is similarly defined via $\mathbb{E}^u_-(x_0, 0) \cap \mathbb{S}_-(0) \neq \{0\}$.*

Since the solid rectangle $[-\infty, \ell] \times [0, \lambda_\infty]$ is contractible and the map (3.47) is continuous, the image of the boundary of the rectangle in $\mathcal{L}(4) \times \mathcal{L}(4)$ is homotopic to a fixed point.

From homotopy invariance (Proposition 3.3), it follows that

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,\cdot),\mathbb{E}^s(\cdot,\cdot);\Gamma)=0. \tag{3.54}$$

By additivity under concatenation, we can decompose the left hand side into the contributions coming from the constituent sides of the Maslov box, i.e.

$$\begin{aligned}\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell])+\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\lambda_\infty])\\ -\operatorname{Mas}(\mathbb{E}^u(\cdot,\lambda_\infty),\mathbb{E}^s(\ell,\lambda_\infty);[-\infty,\ell])-\operatorname{Mas}(\mathbb{E}^u(-\infty,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\lambda_\infty])=0.\end{aligned} \tag{3.55}$$

Note we have included minus signs for the last two terms in order to be consistent with the clockwise orientation of the Maslov box (see Fig. 1). We will show in Section 5 that in fact these last two Maslov indices are zero. A distinguished quantity will be the contribution to the Maslov index of the conjugate point $(x,\lambda)=(\ell,0)$ at the top left corner of the Maslov box,

$$\mathfrak{c}:=\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[\ell-\varepsilon,\ell])+\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\varepsilon]), \tag{3.56}$$

where $\varepsilon>0$ is small. This is because the crossing $(\ell,0)$ is non-regular in λ, and hence higher order crossing forms are needed to compute the second term in (3.56). It follows once more from additivity under concatenation that

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell-\varepsilon])+\mathfrak{c}+\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[\varepsilon,\lambda_\infty])=0. \tag{3.57}$$

We will compute the first term of (3.57) by counting L_+ and L_- conjugate points. By bounding the third term, computing $\mathfrak{c}$ and rearranging, we will arrive at the statement of Theorem 1.2. Before doing so, we turn our attention to the computation of the Morse indices of L_+ and L_- via the Maslov index.

4. Spectral counts for the operators L_+ and L_-

In this section we focus on the spectral problems for the operators L_+ and L_-. Specifically, for each operator we prove that the Morse index is equal to the number of conjugate points on $\mathbb{R}$. Proposition 4.1 is proven under two genericity conditions which will be formulated later on.

Proposition 4.1. *Assume Hypotheses 4.2 and 4.3. The number of positive eigenvalues of L_+ is equal to the number of L_+-conjugate points on $\mathbb{R}$ (up to multiplicity),*

$$P=\sum_{x\in\mathbb{R}}\dim\left(\mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0)\right). \tag{4.1}$$

A similar assertion holds for L_-.

We will prove the proposition in a series of lemmas, focusing on the L_+ operator; the spectral count for L_- follows similarly with minor adjustments. Many of the ideas here have already been discussed in §3, and so in the interest of expediency we present only the main arguments. In what follows, we use a subscript + or − to indicate that objects pertain to the eigenvalue problem for L_+ or L_-.

The eigenvalue equation for L_+,

$$-u''''-\sigma_2u''-\beta u+3\phi^2u=\lambda u,\qquad u\in H^4(\mathbb{R}), \tag{4.2}$$

can be reduced to the following first order system via the u substitutions in (2.1),

$$\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}'=\begin{pmatrix}0&0&\sigma_2&1\\0&0&1&0\\1&-\sigma_2&0&0\\-\sigma_2&\alpha(x)-\lambda&0&0\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}. \tag{4.3}$$

where $\alpha(x) = 3\phi(x)^2 - \beta + 1$. Similar to (2.2), we write this system as

$$\mathbf{u}_x = A_+(x,\lambda)\mathbf{u}, \tag{4.4}$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)^\top$ and

$$A_+(x,\lambda) = \begin{pmatrix} 0 & B_+ \\ C_+(x,\lambda) & 0 \end{pmatrix}, \quad B_+ = \begin{pmatrix} \sigma_2 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_+(x,\lambda) = \begin{pmatrix} 1 & -\sigma_2 \\ -\sigma_2 & \alpha(x)-\lambda \end{pmatrix}.$$

Likewise, the eigenvalue equation for L_-,

$$-v'''' - \sigma_2 v'' - \beta v + \phi^2 v = \lambda v, \qquad v \in H^4(\mathbb{R}), \tag{4.5}$$

can be reduced to the following first order system via the v substitutions in (2.1),

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & -\sigma_2 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & -\sigma_2 & 0 & 0 \\ -\sigma_2 & \eta(x)+\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}. \tag{4.6}$$

where $\eta(x) = -\phi(x)^2 + \beta - 1$. We write this as

$$\mathbf{v}_x = A_-(x,\lambda)\mathbf{v}, \tag{4.7}$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)^\top$ and

$$A_-(x,\lambda) = \begin{pmatrix} 0 & B_- \\ C_-(x,\lambda) & 0 \end{pmatrix}, \quad B_- = \begin{pmatrix} -\sigma_2 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_-(x,\lambda) = \begin{pmatrix} -1 & -\sigma_2 \\ -\sigma_2 & \eta(x)+\lambda \end{pmatrix}.$$

The coefficient matrices $A_\pm(x,\lambda)$ are infinitesimally symplectic, satisfying equation (2.14). In order to be consistent with (2.2) at $\lambda = 0$, we have used the same substitutions (2.1) to reduce (4.2) and (4.5) to (4.3) and (4.6) respectively. Notice that λ appears with a different sign in (4.3) and (4.6), due to the subsitutions for u_2 and u_3 in (2.1) having different signs to the corresponding substitutions for v_2 and v_3. This will be the reason for the difference in sign of the Maslov indices in Lemma 4.4.

The asymptotic matrices $A_+(\lambda) := \lim_{x\to\pm\infty} A_+(x,\lambda)$ and $A_-(\lambda) := \lim_{x\to\pm\infty} A_-(x,\lambda)$ each have two eigenvalues with negative real part and two with positive real part. We denote the associated stable and unstable subspaces by $\mathbb{S}_\pm(\lambda)$ and $\mathbb{U}_\pm(\lambda)$. Reasoning as in Section 3.3, associated with each of the systems (4.3) and (4.6) are stable and unstable bundles,

$$\begin{aligned} \mathbb{E}^u_+(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{u}(x;\lambda),\ \mathbf{u} \text{ solves (4.3) and } \mathbf{u}(x;\lambda) \to 0 \text{ as } x \to -\infty\}, \\ \mathbb{E}^s_+(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{u}(x;\lambda),\ \mathbf{u} \text{ solves (4.3) and } \mathbf{u}(x;\lambda) \to 0 \text{ as } x \to +\infty\}, \\ \mathbb{E}^u_-(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{v}(x;\lambda),\ \mathbf{v} \text{ solves (4.6) and } \mathbf{v}(x;\lambda) \to 0 \text{ as } x \to -\infty\}, \\ \mathbb{E}^s_-(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{v}(x;\lambda),\ \mathbf{v} \text{ solves (4.6) and } \mathbf{v}(x;\lambda) \to 0 \text{ as } x \to +\infty\}, \end{aligned} \tag{4.8}$$

which, when considered as points on the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$, converge to the stable and unstable subspaces at $\pm\infty$ as follows,

$$\lim_{x\to-\infty} \mathbb{E}^u_+(x,\lambda) = \mathbb{U}_+(\lambda), \qquad \lim_{x\to+\infty} \mathbb{E}^s_+(x,\lambda) = \mathbb{S}_+(\lambda),$$
$$\lim_{x\to-\infty} \mathbb{E}^u_-(x,\lambda) = \mathbb{U}_-(\lambda), \qquad \lim_{x\to+\infty} \mathbb{E}^s_-(x,\lambda) = \mathbb{S}_-(\lambda).$$

That $\mathbb{E}^u_+(x,\lambda), \mathbb{E}^u_-(x,\lambda), \mathbb{E}^s_+(x,\lambda), \mathbb{E}^s_-(x,\lambda)$ are Lagrangian subspaces of $\mathbb{R}^4$, with the mappings $(x,\lambda) \mapsto \mathbb{E}^u_\pm(x,\lambda)$ being continuous on $[-\infty,\infty) \times \mathbb{R}$ and $(x,\lambda) \mapsto \mathbb{E}^{u,s}_\pm(x,\lambda)$ analytic on $\mathbb{R} \times \mathbb{R}$, follows from the same arguments as in Section 3.3. We omit the proofs.

In order to show Proposition 4.1, we need to write down frames for $\mathbb{S}_\pm(0)$ that we can do computations with. To that end, first note that the asymptotic matrices $A_\pm(0)$ satisfy $\mathrm{Spec}(A_+(0)) = \mathrm{Spec}(A_-(0)) = \{\pm\mu_1, \pm\mu_2\}$, where

$$\mu_1 = \frac{\sqrt{-\sigma_2 - \sqrt{1-4\beta}}}{\sqrt{2}}, \qquad \mu_2 = \frac{\sqrt{-\sigma_2 + \sqrt{1-4\beta}}}{\sqrt{2}}. \tag{4.9}$$

Under the assumption (1.8), we have $\mu_2 = \bar{\mu}_1$ whenever $\beta \geq 1/4$ (for both $\sigma_2 = 1$ and $\sigma_2 = -1$), and $\mu_1, \mu_2 \in \mathbb{R}$ when $\sigma_2 = -1$ and $0 < |\beta| \leq 1/4$. The corresponding eigenvectors are given by

$$\mathbf{u}_1 = \begin{pmatrix} \mu_2^2 \\ -1 \\ \mu_1 \\ \mu_1^3 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} \mu_1^2 \\ -1 \\ \mu_2 \\ \mu_2^3 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{v}_1 = \begin{pmatrix} \mu_2^2 \\ 1 \\ -\mu_1 \\ \mu_1^3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} \mu_1^2 \\ 1 \\ -\mu_2 \\ \mu_2^3 \end{pmatrix}, \tag{4.10}$$

where $\ker(A_+(0) + \mu_i) = \mathrm{span}\{\mathbf{u}_i\}$ and $\ker(A_-(0) + \mu_i) = \mathrm{span}\{\mathbf{v}_i\}$, $i = 1, 2$. Notice that the vectors $\mathbf{u}_i, \mathbf{v}_i$ for $i = 1, 2$ are complex-valued if $\beta \geq 1/4$. We collect these vectors into the columns of two frames, which we denote with 2×2 blocks P_i, M_i, $i = 1, 2$ via

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} := \begin{pmatrix} \mu_2^2 & \mu_1^2 \\ -1 & -1 \\ \mu_1 & \mu_2 \\ \mu_1^3 & \mu_2^3 \end{pmatrix}, \qquad \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} := \begin{pmatrix} \mu_2^2 & \mu_1^2 \\ 1 & 1 \\ -\mu_1 & -\mu_2 \\ \mu_1^3 & \mu_2^3 \end{pmatrix}. \tag{4.11}$$

All of the matrices P_i, M_i are invertible under (1.8) and (1.9). Right multiplying each frame in (4.11) by the inverse of its upper 2×2 block yields the following *real* frame for $\mathbb{S}_\pm(0)$,

$$\mathbf{S}_\pm = \begin{pmatrix} I \\ S_\pm \end{pmatrix}, \qquad S_\pm = \frac{1}{\sqrt{2\sqrt{\beta} - \sigma}} \begin{pmatrix} \mp 1 & \sigma - \sqrt{\beta} \\ \sigma - \sqrt{\beta} & \pm(\sqrt{\beta}\sigma + \beta - 1) \end{pmatrix}, \tag{4.12}$$

where $S_+ = P_2 P_1^{-1}$ and $S_- = M_2 M_1^{-1}$.

An important relation exists between $S_\pm$ and the blocks of the asymptotic matrix $A_\pm(0)$ that will be needed in our analysis. Define $C_\pm(x) := C_\pm(x, 0)$ and

$$\widehat{C}_+(x) := \begin{pmatrix} 0 & 0 \\ 0 & 3\phi(x)^2 \end{pmatrix}, \quad \widehat{C}_-(x) := \begin{pmatrix} 0 & 0 \\ 0 & -\phi(x)^2 \end{pmatrix}, \quad \widetilde{C}_\pm := \begin{pmatrix} \pm 1 & -\sigma_2 \\ -\sigma_2 & \mp(\beta - 1) \end{pmatrix}, \tag{4.13}$$

so that $C_\pm(x) = \widehat{C}_\pm(x) + \widetilde{C}_\pm$. Because the columns of the frames in (4.11) are eigenvectors of $A_\pm(0)$, we have

$$\begin{pmatrix} 0 & B_+ \\ \widetilde{C}_+ & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} D_+, \qquad D_+ = \mathrm{diag}\{-\mu_1, -\mu_2\}, \tag{4.14}$$

with a similar equation holding for $A_-(0)$ and the frame (M_1, M_2). That is, $B_+ P_2 = P_1 D_+$ and $\widetilde{C}_+ P_1 = P_2 D_+$. It follows that

$$\widetilde{C}_+ = P_2 D_+ P_1^{-1} = \left(P_2 P_1^{-1}\right)\left(P_1 D_+ P_2^{-1}\right)\left(P_2 P_1^{-1}\right) = S_+ B_+ S_+. \tag{4.15}$$

It can be similarly shown that

$$\widetilde{C}_- = S_- B_- S_-. \tag{4.16}$$

The first intermediate result that will be used in the proof of Proposition 4.1 is Lemma 4.4, which proves sign-definiteness of the L_+ and L_- conjugate points on Γ_1. For it, we will require two genericity conditions. For details on how the first may be removed, see Remark 4.8.

Hypothesis 4.2. For any $x_0 \in \mathbb{R}$ where $\mathbb{E}^u_\pm(x_0, 0) \cap \mathbb{S}_\pm(0) \neq \{0\}$, we assume $\phi(x_0) \neq 0$.

Denote a frame for the unstable bundle $\mathbb{E}^u_\pm(x,0)$ by

$$\mathbf{U}_\pm(x) = \begin{pmatrix} X_\pm(x) \\ Y_\pm(x) \end{pmatrix}, \qquad X_\pm(x), Y_\pm(x) \in \mathbb{R}^{2\times 2}. \tag{4.17}$$

We will assume that in the event of a one dimensional crossing on Γ_1, the intersection of the unstable bundle with the stable subspace does not perfectly align with the span of the first column of the frame $\mathbf{S}_\pm$.

Hypothesis 4.3. Suppose $\dim \mathbb{E}^u_\pm(x_0,0)\cap\mathbb{S}_\pm(0) = 1$. Then there exist vectors $k = (a,b) \in \mathbb{R}^2$ and $h = (c,d) \in \mathbb{R}^2$ so that $\mathbf{U}_\pm(x_0)h = \mathbf{S}_\pm k$. We assume that $a \neq 0$.

Lemma 4.4. *Assume Hypotheses 4.2 and 4.3. Each crossing $x = x_0 \in \mathbb{R}$ of the Lagrangian path $x \mapsto (\mathbb{E}^u_+(x,0), \mathbb{S}_+(0))$ is negative. Thus*

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0); [-\infty,\infty)) = -\sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0)\right). \tag{4.18}$$

Similarly, each crossing $x = x_0 \in \mathbb{R}$ of $x \mapsto \left(\mathbb{E}^u_-(x,0), \mathbb{S}_-(0)\right)$ is positive, and we have

$$\operatorname{Mas}(\mathbb{E}^u_-(\cdot,0), \mathbb{S}_-(0); [-\infty,\infty)) = \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_-(x,0)\cap\mathbb{S}_-(0)\right). \tag{4.19}$$

Remark 4.5. In the above lemma (and throughout), by having the domain of the Lagrangian paths $x \mapsto (\mathbb{E}^u_\pm(x,0), \mathbb{S}_\pm(0))$ as $x \in [-\infty,\infty)$, we mean that $\tau \in [-1, 1-\varepsilon]$ for the compactified path $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ for some small $\varepsilon > 0$ (see Remark 3.5). Note however that the initial point $\tau = -1$ ($x = -\infty$) is never a conjugate point because $\mathbb{U}_+(0)\cap\mathbb{S}_+(0) = \{0\}$. On the other hand, $\tau = +1$ ($x = \infty$) is always a conjugate point, because $\mathbb{E}^u_\pm(+\infty,0) \in \mathcal{T}_1(\mathbb{S}_\pm(0))$ on account of Hypothesis 1.1; nonetheless, because crossings are isolated (c.f. Lemma 4.10), we can make $\varepsilon > 0$ as small as we like.

The proof of Lemma 4.4 will focus on the L_+ problem, with the modifications needed for the L_- problem listed at the end. In order to compute the partial signatures of Definition 3.2, we will explicitly construct the matrix family $R(x)$ defining the curve of symmetric bilinear forms $\omega(R(x)\cdot,\cdot)\big|_{\mathbb{S}_+(0)\times\mathbb{S}_+(0)}$ in (3.16). This is given in Lemma 4.6. Recall that for each x near x_0, $R(x)$ is the unique matrix, when viewed as an operator from $\mathbb{S}_+(0)$ into $\mathbb{S}_+(0)^\perp$, whose graph is the Lagrangian plane $\mathbb{E}^u_+(x,0)$. For ease of presentation we will drop the subscript $+$ on the frame $\mathbf{U}_+(x)$ for the unstable bundle, which we denote by

$$\mathbf{U}(x) = \begin{pmatrix} X(x) \\ Y(x) \end{pmatrix}. \tag{4.20}$$

Lemma 4.6. *Suppose $x = x_0 \in \mathbb{R}$ is a conjugate point. For all x near x_0, the curve of matrices $x \mapsto R(x) \in \mathbb{R}^{4\times 4}$,*

$$R(x) = \mathbf{U}(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top - \mathbf{S}_+\left(\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}\mathbf{S}_+^\top, \tag{4.21}$$

is analytic and satisfies $\mathbb{E}^u_+(x,0) = \operatorname{graph}(R(x)) = \{q + R(x)q : q \in \mathbb{S}_+(0)\}$.

Proof. First, note that by continuity, $\mathbb{E}^u_+(x,0)$ and $\mathbb{S}_+(0)^\perp$ are transverse for all x near x_0. It follows that $\mathbf{S}_+^\top\mathbf{U}(x) = X(x) + S_+Y(x)$ is invertible for all x near x_0. Indeed, transversality of $\mathbb{E}^u_+(x,0)$ and $\mathbb{S}_+(0)^\perp$ implies that the 4×4 matrix whose columns consist of bases for these spaces is invertible. A frame for $\mathbb{S}_+(0)^\perp$ is given by $J(I,S_+) = (-S_+, I)$. Using Schur's formula, we therefore have

$$0 \neq \det\begin{pmatrix} X(x) & -S_+ \\ Y(x) & I \end{pmatrix} = \det\left(X(x) + S_+Y(x)\right).$$

Analyticity of $x \mapsto R(x)$ now follows from the analyticity of $x \mapsto \mathbf{U}(x)$, the entries of which are solutions to (4.3).

Now for any $q \in \mathbb{S}_+(0)$ we have $q = \mathbf{S}_+ k_0$ for some $k_0 \in \mathbb{R}^2$. Then

$$\begin{aligned} q + R(x)q &= \mathbf{S}_+ k + \mathbf{U}(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ k - \mathbf{S}_+\left(\mathbf{S}_+^\top \mathbf{S}_+\right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ k, \\ &= \mathbf{U}(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ k \in \mathbb{E}_+^u(x,0). \end{aligned}$$

For the opposite inclusion, for any $v \in \mathbb{E}_+^u(x,0)$ we may write $v = \mathbf{U}(x)h$ for some $h \in \mathbb{R}^2$. Now set $k = \left(\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+\right)^{-1} h$. Then

$$v = \mathbf{U}(x)h = \mathbf{S}_+ k + \mathbf{U}(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ k - \mathbf{S}_+\left(\mathbf{S}_+^\top \mathbf{S}_+\right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ k,$$

and setting $q = \mathbf{S}_+ k$ we have $v = q + R(x)q \in \operatorname{graph}(R(x))$. □

Proof of Lemma 4.4. We will prove that crossings of the path $x \mapsto (\mathbb{E}_+^u(x,0), \mathbb{S}_+(0))$ are negative in two cases: (1) $\dim \mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0) = 1$ and Hypothesis 4.3 holds, and (2) $\dim \mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0) = 2$.

For the first case, we need to show that the first order form $\mathfrak{m}_{x_0}$ is negative definite. From (3.24) we have

$$\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q) = \frac{d}{dx}\omega\left(R(x)q, q\right)\big|_{x=x_0}, \tag{4.22}$$

where $q \in \mathbb{E}_+^u(\cdot,0) \cap \mathbb{S}_+(0)$ is fixed, and $R(x)$ is given in Lemma 4.6. Note that for any $q \in \mathbb{E}_+^u(\cdot,0) \cap \mathbb{S}_+(0)$ we may write $q = \mathbf{U}(x_0)h = \mathbf{S}_+ k$ for some $h = (c,d) \in \mathbb{R}^2$, $k = (a,b) \in \mathbb{R}^2$.

We will require the first derivatives of the matrices $X(x)$, $Y(x)$ and $R(x)$. Since the columns of the frame $\mathbf{U}(x) = (X(x), Y(x))$ satisfy (4.3), we have

$$X'(x) = B_+ Y(x), \qquad Y'(x) = C_+(x)X(x). \tag{4.23}$$

(Recall $C_+(x) = C_+(x,0)$.) We also have

$$R'(x) = \mathbf{U}'(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1} \mathbf{S}_+^\top - \mathbf{U}(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x)\left(\mathbf{S}_+^\top \mathbf{U}(x)\right)^{-1} \mathbf{S}_+^\top. \tag{4.24}$$

Denoting

$$R_0 = \left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1} \mathbf{S}_+^\top \mathbf{U}'(x_0)\left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1} \mathbf{S}_+^\top, \tag{4.25}$$

we now compute:

$$\begin{aligned} \mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q) &= \omega(R'(x_0)q, q) = \left\langle JR'(x_0)\mathbf{S}_+ k, \mathbf{S}_+ k\right\rangle_{\mathbb{R}^4}, \\ &= \left\langle J\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1} \mathbf{S}_+^\top \mathbf{S}_+ k, \mathbf{S}_+ k\right\rangle_{\mathbb{R}^4} \\ &\qquad + \left\langle J\mathbf{U}(x_0)R_0\, \mathbf{S}_+ k, \mathbf{S}_+ k\right\rangle_{\mathbb{R}^4}, \\ &= \left\langle J\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top \mathbf{U}(x_0)\right)^{-1} \mathbf{S}_+^\top \mathbf{U}(x_0)h, \mathbf{U}(x_0)h\right\rangle_{\mathbb{R}^4} \\ &\qquad + \left\langle \mathbf{U}(x_0)^\top J\mathbf{U}(x_0)R_0\, \mathbf{U}(x_0)h, h\right\rangle_{\mathbb{R}^4}, \\ &= \left\langle J\mathbf{U}'(x_0)h, \mathbf{U}(x_0)h\right\rangle_{\mathbb{R}^4}, \\ &= -\left\langle C_+(x_0)X(x_0)h, X(x_0)h\right\rangle_{\mathbb{R}^2} + \left\langle B_+ Y(x_0)h, Y(x_0)h\right\rangle_{\mathbb{R}^2}, \\ &= \left\langle\, (-C_+(x_0) + S_+ B_+ S_+)\, k, k\right\rangle_{\mathbb{R}^2}, \end{aligned}$$

where $\mathbf{U}(x_0)^\top J\mathbf{U}(x_0) = -X(x_0)^\top Y(x_0) + Y(x_0)^\top X(x_0) = 0$ because $\mathbf{U}(x_0)$ is the frame for a Lagrangian plane, and we used (4.13) and the symmetry of S_+. (Recall that $q = \mathbf{U}(x_0)h = \mathbf{S}_+k$.) Recalling (4.13) and (4.15), we have

$$C_+(x) - S_+B_+S_+ = \widehat{C}_+(x_0) + \widetilde{C}_+ - S_+B_+S_+ = \widehat{C}_+(x_0), \tag{4.26}$$

and therefore, under Hypotheses 4.2 and 4.3,

$$\mathfrak{m}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(q) = -\big\langle \widehat{C}_+(x_0)k,k\big\rangle_{\mathbb{R}^2} = -3\phi(x_0)^2b^2 < 0. \tag{4.27}$$

Hence $n_-(\mathfrak{m}_{x_0}) = 1$, and crossings are negative in this case. By (3.30) their contribution to the Maslov index is $-\dim(\mathbb{E}^u_+(\cdot,0)\cap\mathbb{S}_+(0)) = -1$.

Next, we treat the case $\dim \mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0) = 2$. We have already seen that $\mathfrak{m}_{x_0}$ is degenerate (but not identically zero), and thus we cannot possibly have $n_-(\mathfrak{m}_{x_0}) = 2$. Therefore, recalling that a crossing is negative if (3.42) satisfied, our goal will be to show that $n_-(\mathfrak{m}_{x_0}) = 1$, $\mathfrak{m}^{(2)}_{x_0}$ is identically zero, and $n_-(\mathfrak{m}^{(3)}_{x_0}) = 1$.

By definition, we have

$$\mathfrak{m}^{(k)}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(q_0) = \frac{d^k}{dx^k}\,\omega\big(R(x)q(x),q_0\big)\Big|_{x=x_0}, \qquad q_0\in W_k, \tag{4.28}$$

where

$$\begin{aligned} W_k = \{q_0\in\mathbb{S}_+(0) :\ &\exists \text{ a generalised Jordan chain of length } k,\\ &\text{starting at } q_0, \text{ for the curve of matrices } JR(x) \text{ at } x = x_0\}. \end{aligned}\tag{4.29}$$

To compute the forms for $k = 1,2,3$, we will work instead with the smooth curve of symmetric matrices

$$[x_0-\varepsilon,x_0+\varepsilon]\ni x\mapsto L(x) := \mathbf{S}_+^\top JR(x)\mathbf{S}_+ \in\mathbb{R}^{2\times2}. \tag{4.30}$$

If there exists a generalised Jordan chain $\{k_i\}_i$ for the curve $L(x)$ at $x =_0$, then $\{q_i\}_i = \{\mathbf{S}_+k_i\}_i$ is a generalised Jordan chain for the family $x\mapsto JR(x) : \mathbb{S}_+(0)\to\mathbb{S}_+(0)$ at $x = x_0$. We can thus write the crossing forms as

$$\mathfrak{m}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(q_0) = \langle L'(x_0)k_0,k_0\rangle, \tag{4.31a}$$

$$\mathfrak{m}^{(2)}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(q_0) = \langle L''(x_0)k_1,k_0\rangle + \langle L'(x_0)k_0,k_0\rangle, \tag{4.31b}$$

$$\mathfrak{m}^{(3)}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(q_0) = \langle L'''(x_0)k_2,k_0\rangle + \langle L''(x_0)k_1,k_0\rangle + \langle L'(x_0)k_0,k_0\rangle. \tag{4.31c}$$

Let us first compute the derivatives $L'(x_0), L''(x_0), L'''(x_0)$. Differentiating (4.23),

$$X''(x) = B_+C_+(x)X(x), \qquad Y''(x) = C_+'(x)X(x) + C_+(x)B_+Y(x),$$

and

$$\begin{aligned} X'''(x) &= B_+C_+'(x)X(x) + B_+C_+(x)B_+Y(x),\\ Y'''(x) &= C_+''(x)X(x) + 2C_+'(x)B_+Y(x) + C_+(x)B_+C_+(x)X(x). \end{aligned}$$

Since $\dim\mathbb{E}^u_+(x_0,0)\cap\mathbb{S}_+(0) = 2$, we have $\mathbb{E}^u_+(x_0,0) = \mathbb{S}_+(0)$, and from (3.18),

$$\ker\omega(R(x)\cdot,\cdot) = \mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0) = \mathbb{S}_+(0). \tag{4.32}$$

Moreover, $\mathbf{S}_+ = (I,S_+)$ and $\mathbf{U}(x_0) = (X(x_0),Y(x_0))$ are frames for the same Lagrangian plane, meaning there exists an invertible 2×2 matrix F so that $\mathbf{U}(x_0) = \mathbf{S}_+F$. Looking at the upper 2×2 block of this equation, this means that $X(x_0) = F$ is invertible, and therefore $X(x)$ is invertible for nearby x. Right multiplying by $X(x)^{-1}$, we can thus take

$$\mathbf{U}(x) = \begin{pmatrix} I\\ U(x)\end{pmatrix}, \qquad U(x) := Y(x)X(x)^{-1}, \tag{4.33}$$

to be a frame for $\mathbb{E}^u_+(x,0)$, where now $\mathbf{U}(x_0) = \mathbf{S}_+$ and $U(x_0) = Y(x_0)X(x_0)^{-1} = S_+$. The first derivative of $U(x)$ is given by

$$U'(x) = Y'(x)X(x)^{-1} - Y(x)X(x)^{-1}X'(x)X(x)^{-1} = C_+(x) - U(x)B_+U(x),$$

hence

$$U'(x_0) = C_+(x) - S_+B_+S_+ = \widehat{C}_+(x_0), \tag{4.34}$$

recalling (4.26). Using (4.34) and (4.15), the second and third derivatives are shown to be

$$U''(x_0) = C'_+(x_0) - \widehat{C}_+(x_0)B_+S_+ - S_+B_+\widehat{C}_+(x_0), \tag{4.35}$$

$$\begin{aligned} U'''(x_0) = C''_+(x_0) &- 2\widehat{C}_+(x_0)B_+\widehat{C}_+(x_0) + 2S_+B_+\widehat{C}_+(x_0)B_+S_+ \\ &- C'_+(x_0)B_+S_+ - S_+B_+C'_+(x_0) + \widehat{C}_+(x_0)B_+\widetilde{C}_+ + \widetilde{C}_+B_+\widehat{C}_+(x_0). \end{aligned} \tag{4.36}$$

We are ready to compute derivatives of $L(x)$. Using (4.24) and (4.25), and that $\mathbf{U}(x_0) = \mathbf{S}_+$, $\mathbf{S}_+^\top J\mathbf{S}_+ = 0$ and $\mathbf{U}'(x) = (0, U'(x))$, we have

$$L'(x_0) = \mathbf{S}_+^\top JR'(x_0)\mathbf{S}_+ = \mathbf{S}_+^\top J\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top\mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+ - \mathbf{S}_+^\top J\mathbf{U}(x_0)R_0\mathbf{S}_+ = -\widehat{C}_+(x_0).$$

Differentiating (4.24),

$$\begin{aligned} R''(x) = \mathbf{U}''(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top &- 2\mathbf{U}'(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{U}'(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top \\ &+ \mathbf{U}(x)\frac{d^2}{dx^2}\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top, \end{aligned}$$

thus

$$\begin{aligned} L''(x_0) = \mathbf{S}_+^\top JR''(x_0)\mathbf{S}_+ &= \mathbf{S}_+^\top J\mathbf{U}''(x_0)\left(\mathbf{S}_+^\top\mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+ \\ &\quad - 2\mathbf{S}_+^\top J\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top\mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top\mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+ \\ &\quad - \mathbf{S}_+^\top J\mathbf{U}(x_0)\frac{d^2}{dx^2}\left(\mathbf{S}_+^\top\mathbf{U}(x_0)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}, \\ &= \mathbf{S}_+^\top J\mathbf{U}''(x_0) - 2\mathbf{S}_+^\top J\mathbf{U}'(x_0)\left(\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}\mathbf{S}_+^\top\mathbf{U}'(x_0), \\ &= -U''(x_0) + 2U'(x_0)(I + S_+^2)^{-1}S_+U'(x_0). \end{aligned} \tag{4.37}$$

Differentiating again,

$$\begin{aligned} R'''(x) = \mathbf{U}'''(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top &- 3\mathbf{U}''(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{U}'(x)\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top, \\ &+ 3\mathbf{U}'(x)\frac{d^2}{dx^2}\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top + \mathbf{U}(x)\frac{d^3}{dx^3}\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top, \end{aligned}$$

hence

$$\begin{aligned} L'''(x_0) = \mathbf{S}_+^\top JR'''(x_0)\mathbf{S}_+ &= \mathbf{S}_+^\top J\mathbf{U}'''(x_0) - 3\mathbf{S}_+^\top J\mathbf{U}''(x_0)\left(\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}\mathbf{S}_+^\top\mathbf{U}'(x_0), \\ &\quad + 3\mathbf{S}^\top J\mathbf{U}'(x_0)\frac{d^2}{dx^2}\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+\Big|_{x=x_0} \\ &= -U'''(x_0) + 3U''(x_0)\left(\mathbf{S}_+^\top\mathbf{S}_+\right)^{-1}\mathbf{S}_+^\top\mathbf{U}'(x_0) \\ &\quad - 3U'(x_0)\frac{d^2}{dx^2}\left(\mathbf{S}_+^\top\mathbf{U}(x)\right)^{-1}\mathbf{S}_+^\top\mathbf{S}_+\Big|_{x=x_0}. \end{aligned}$$

Some algebra shows that

$$L'''(x_0) = -U'''(x_0) + 3U''(x_0)\left(I+S_+^2\right)^{-1}S_+U'(x_0) + 3U'(x_0)(I+S_+U(x_0))^{-1}S_+U''(x_0) \\ - 6U'(x_0)(I+S_+U(x_0))^{-1}S_+U'(x_0)(I+S_+U(x_0))^{-1}S_+U'(x_0). \tag{4.38}$$

Let us examine the above expressions more closely. For $L''(x_0)$ we have

$$U''(x_0) = C_+'(x_0) - \widehat{C}_+(x_0)B_+S_+ - S_+B_+\widehat{C}_+(x_0) = \begin{pmatrix} 0 & \frac{3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \\ \frac{3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} & * \end{pmatrix}. \tag{4.39}$$

Noting that $U'(x_0) = \widehat{C}(x_0)$ and

$$\widehat{C}_+(x_0)M\widehat{C}_+(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \tag{4.40}$$

for any 2×2 matrix M, the second term of (4.37) is of the form of (4.40). For $L'''(x_0)$, the first two terms of $U'''(x_0)$ in (4.36) and the last term of $L'''(x_0)$ in (4.38) all have the form of (4.40). The third term of $U'''(x_0)$ has the form

$$2S_+B_+\widehat{C}_+(x_0)B_+S_+ = \begin{pmatrix} \frac{6\phi(x)^2}{2\sqrt{\beta}-\sigma_2} & * \\ * & * \end{pmatrix}. \tag{4.41}$$

The remaining terms in $U'''(x_0)$, i.e.

$$-C_+'(x_0)B_+S_+ - S_+B_+C_+'(x_0) + \widehat{C}_+(x_0)B_+\widetilde{C}_+ + \widetilde{C}_+B_+\widehat{C}_+(x_0), \tag{4.42}$$

as well as the second and third terms of $L''(x_0)$ in (4.38), can all be shown to have the form

$$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}. \tag{4.43}$$

In summary, we have

$$L'(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & -3\phi(x_0)^2 \end{pmatrix},\quad L''(x_0) = \begin{pmatrix} 0 & \frac{-3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \\ \frac{-3\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} & * \end{pmatrix},\quad L'''(x_0) = \begin{pmatrix} \frac{-6\phi(x)^2}{2\sqrt{\beta}-\sigma_2} & * \\ * & * \end{pmatrix}. \tag{4.44}$$

The expressions (4.44) are sufficient to determine the partial signatures of (4.31). To do so, we need to compute any generalised Jordan chains for the curve $L(x)$. Define $k_i = (a_i, b_i)^\top \in \mathbb{R}^2$ for $i = 0,1,2,3$. That $\dim(\mathbb{E}_+^u(x,0)\cap\mathbb{S}_+(0)) = 2$ means that

$$\ker L(x_0) = \mathbb{R}^2, \tag{4.45}$$

and therefore $\{k_0\}$ is a chain of length one for any $k_0 \in \ker L(x_0)$. Next, there exists solutions $k_1 = (a_1, b_1)^\top$ to

$$L(x_0)k_1 + L'(x_0)k_0 = \begin{pmatrix} 0 \\ 3\,b_0\,\phi(x_0)^2 \end{pmatrix} = 0 \tag{4.46}$$

if and only if $b_0 = 0$. Hence, $\{k_0, k_1\}$ is a chain of length two if and only if $b_0 = 0$. Now taking $k_0 = (a_0, 0)^\top$, there exists solutions k_2 to

$$L''(x_0)k_0 + L'(x_0)k_1 + L(x_0)k_2 = \begin{pmatrix} 0 \\ 3\,b_1\,\phi(x_0)^2 + \frac{3\,a_0\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \end{pmatrix} = 0 \tag{4.47}$$

if and only if $b_1 = -\frac{a_0}{\sqrt{2\sqrt{\beta}-\sigma_2}}$. Thus $\{k_0,k_1,k_2\}$ is a chain of length three if and only if $b_0 = 0$ and $b_1 = -\frac{a_0}{\sqrt{2\sqrt{\beta}-\sigma_2}}$. Finally, note that for nontrivial $k_0 = (a_0, 0)^\top$ there are no solutions k_3 to

$$L'''(x_0)k_0 + L''(x_0)k_1 + L'(x_0)k_2 + L(x_0)k_3 = \begin{pmatrix} \frac{3a_0\phi(x_0)^2}{2\sqrt{\beta}-\sigma_2} \\ * \end{pmatrix} = 0. \tag{4.48}$$

In other words, the chain $\{k_0, k_1, k_2\}$ is maximal. We are ready to compute the partial signatures. For $k_0 = (a_0, b_0)^\top \in \ker L(x_0)$, we have

$$\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = \langle L'(x_0)k_0, k_0 \rangle = -3b_0^2\phi(x_0)^2 < 0, \tag{4.49}$$

while for $k_0 = (a_0, 0)^\top$ and $k_1 = (a_1, -\frac{a_0}{\sqrt{2\sqrt{\beta}-\sigma_2}})^\top$, we have

$$\mathfrak{m}_{x_0}^{(2)}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = \langle L''(x_0)k_0 + L'(x_0)k_1, k_0 \rangle = 0, \tag{4.50}$$

$$\mathfrak{m}_{x_0}^{(3)}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = \langle L'''(x_0)k_0 + L''(x_0)k_1 + L'(x_0)k_2, k_0 \rangle = -\frac{3a_0^2\phi(x_0)^2}{2\sqrt{\beta}-\sigma_2} < 0. \tag{4.51}$$

The right hand sides of (4.49) and (4.51) are negative due to our assumptions (1.8) (which implies that $2\sqrt{\beta} - \sigma_2 > 0$) and Hypothesis 4.2. We have just shown that $n_-(\mathfrak{m}_{x_0}) = n_-(\mathfrak{m}_{x_0}^{(3)}) = 1$ and $n_+(\mathfrak{m}_{x_0}) = n_+(\mathfrak{m}_{x_0}^{(3)}) = n_\pm(\mathfrak{m}_{x_0}^{(2)}) = 0$. Therefore each crossing $x_0 \in \mathbb{R}$ where $\dim(\mathbb{E}_+^u(\cdot,0) \cap \mathbb{S}_+(0)) = 2$ is negative, because in such cases

$$\dim(\mathbb{E}_+^u(\cdot,0) \cap \mathbb{S}_+(0)) = n_-(\mathfrak{m}_{x_0}) + n_-(\mathfrak{m}_{x_0}^{(3)}). \tag{4.52}$$

By (3.30) the contribution of each crossing $x_0 \in \mathbb{R}$ is therefore $-\dim(\mathbb{E}_+^u(\cdot,0) \cap \mathbb{S}_+(0))$. This completes the proof for the L_+ problem.

The proof for the L_- problem is similar. The case of one-dimensional crossings under Hypothesis 4.3 is almost identical, while for the case of two-dimensional crossings we'll have

$$L'(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & \phi(x_0)^2 \end{pmatrix}, \quad L''(x_0) = \begin{pmatrix} 0 & \frac{\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} \\ \frac{\phi(x)^2}{\sqrt{2\sqrt{\beta}-\sigma_2}} & * \end{pmatrix}, \quad L'''(x_0) = \begin{pmatrix} \frac{2\phi(x)^2}{2\sqrt{\beta}-\sigma_2} & * \\ * & * \end{pmatrix}.$$

Computing the generalised Jordan chains as above leads to

$$\mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = b_0^2\phi(x_0)^2 > 0,$$
$$\mathfrak{m}_{x_0}^{(2)}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = 0,$$
$$\mathfrak{m}_{x_0}^{(3)}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(q_0) = \frac{a_0^2\phi(x_0)^2}{2\sqrt{\beta}-\sigma_2} > 0,$$

for some $a_0, b_0 \in \mathbb{R}$, with positivity under Hypothesis 4.2. Each crossing $x \in \mathbb{R}$ of the path $x \mapsto \mathbb{E}_-^u(x,0) \cap \mathbb{S}_-(0)$ thus contributes $\dim \mathbb{E}_-^u(x,0) \cap \mathbb{S}_-(0)$ to its Maslov index. □

Remark 4.7. That the matrix $L'(x_0)$ is degenerate, i.e. that crossings $x_0 \in \mathbb{R}$ are non-regular, is the reason for using the partial signatures approach of [GPP04b] to compute the Maslov index.

Remark 4.8. If Hypothesis 4.2 fails, i.e. for any crossing $x_0 \in \mathbb{R}$ such that $\phi(x_0) = 0$, the forms $\mathfrak{m}_{x_0}$ and $\mathfrak{m}_{x_0}^{(3)}$ are degenerate, and higher order crossing forms are needed.

Remark 4.9. Proposition 4.1 will also hold for any power-law fourth-order NLS equation, i.e. (1.19) for any $p \in \mathbb{N}$. In these cases the crossing forms $\mathfrak{m}_{x_0}^{(k)}(q_0)$ will be the same as those above, but scaled by a positive constant, and with $\phi(x_0)^2$ replaced by $\phi(x_0)^{2p}$. The signs are therefore preserved.

The following lemma shows that crossings along Γ_1 are isolated.

Lemma 4.10. *There are finitely-many isolated intersections of the path* $[-1,1] \ni \tau \mapsto \widehat{\mathbb{E}}^u_\pm(\tau,0)$ *with the trains* $\mathcal{T}(\mathbb{S}_\pm(0))$ *and* $\mathcal{T}(\mathbb{E}^s_\pm(\ell,0))$.

Proof. First, note that because $\mathbb{U}(0)_\pm \cap \mathbb{S}_\pm(0) = \{0\}$ and $\lim_{\tau\to -1^+} \widehat{\mathbb{E}}^u_\pm(\tau,0) = \mathbb{U}_\pm(0)$, by continuity there exists a $\hat\tau$ close to -1 such that $\widehat{\mathbb{E}}^u_\pm(\tau,0)\cap\mathbb{S}_\pm(0) = \{0\}$ for all $\tau\in[-1,\hat\tau]$. Now consider the compactly-defined path $\tau\mapsto\widehat{\mathbb{E}}^u_\pm(\tau,0)$, $\tau\in[-\hat\tau,\tau_\ell]\subset(-1,1)$. Since the elements of $\widehat{\mathbb{E}}^u_\pm(\cdot,0)$ are solutions to a differential equation and therefore analytic on $(-1,1)$, we can form an analytic path of frames $\tau\mapsto\widehat{\mathbf{U}}_\pm(\tau)$ on $[-\hat\tau,\tau_\ell]$. Now collecting the columns of $\mathbf{U}_\pm(\tau)$ and the columns of a frame for $\mathbb{E}^s_\pm(\ell,0)$ into a 4×4 matrix $F(\tau)$, the function $\tau\mapsto\det D(\tau)$ is real-valued and analytic on $[-\hat\tau,\tau_\ell]$. It therefore has finitely-many isolated zeroes, which correspond to intersections of $\tau\mapsto\widehat{\mathbb{E}}^u_\pm(\tau,0)$ with $\mathcal{T}(\mathbb{E}^s_\pm(\ell,0))$. It will follow from the perturbative arguments in the proof of Lemma 4.11 that the crossings of $\tau\mapsto\widehat{\mathbb{E}}^u_\pm(\tau,0)$ with $\mathcal{T}(\mathbb{E}^s_\pm(\ell,0))$ over $\tau\in[-1,\tau_\ell]$ and the crossings of $\tau\mapsto\widehat{\mathbb{E}}^u_\pm(\tau,0)$ with $\mathcal{T}(\mathbb{S}_\pm(0))$ over $\tau\in[-1,1]$ are in one-to-one correspondance. This completes the proof. ☐

Lemma 4.11. *For the Lagrangian path* $x\mapsto\big(\mathbb{E}^u_+(x,0),\mathbb{E}^s_+(\ell,0)\big)$ *we have*

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[-\infty,\ell-\varepsilon]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0);[-\infty,\infty)) \tag{4.53}$$

for $\varepsilon>0$ *small enough. A similar statement holds for the path* $x\mapsto(\mathbb{E}^u_-(x,0),\mathbb{E}^s_-(\ell,0))$.

Proof. First, we show that

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[-\infty,\ell]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0);[-\infty,\infty]). \tag{4.54}$$

In order to do so, it will be convenient to compactify $\mathbb{R}$ via the change of variables in Remark 3.5. Thus, defining

$$\widehat{\mathbb{E}}^{s,u}_\pm(\tau,0) := \mathbb{E}^{s,u}_\pm\left(\ln\left(\frac{1+\tau}{1-\tau}\right),0\right), \tag{4.55}$$

(4.54) is equivalent to

$$\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0),\widehat{\mathbb{E}}^s_+(\tau_\ell,0);[-1,\tau_\ell]) = \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0),\widehat{\mathbb{E}}^s_+(1,0);[-1,1]), \tag{4.56}$$

where $\ell=\ln((1+\tau_\ell)/(1-\tau_\ell))$, i.e. $\tau_\ell=(e^\ell-1)/(e^\ell+1)$, and we have used that $\widehat{\mathbb{E}}^s_+(1,0) = \mathbb{E}^s_+(+\infty,0) := \mathbb{S}_+(0)$. Rescaling further, we can map $[-1,1]$ to $[-1,\tau_\ell]$ via the function

$$g(\tau) = \left(\frac{1+\tau_\ell}{2}\right)\tau + \left(\frac{\tau_\ell-1}{2}\right),$$

where $g(-1)=-1$ and $g(1)=\tau_\ell$. This allows us to write both Lagrangian paths in (4.56) over $[-1,1]$, i.e.

$$\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0),\widehat{\mathbb{E}}^s_+(\tau_\ell,0);[-1,1]) = \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0),\widehat{\mathbb{E}}^s_+(1,0);[-1,1]). \tag{4.57}$$

To prove (4.57), we set

$$\Lambda_1(s,\tau) := \widehat{\mathbb{E}}^u_+(\tau+(g(\tau)-\tau)s,0), \qquad \Lambda_2(s,\tau) := \widehat{\mathbb{E}}^s_+(1+(\tau_\ell-1)s,0). \tag{4.58}$$

(Λ_2 is independent of τ.) Both maps $(s,\tau)\to\Lambda_{1,2}(s,\tau)$ are continuous on $[0,1]\times[-1,1]$. In addition,

$$\Lambda_1(s,-1) = \widehat{\mathbb{E}}^u_+(-1,0) = \mathbb{U}_+(0), \qquad \Lambda_2(s,-1) = \widehat{\mathbb{E}}^s_+(1+(\tau_\ell-1)s,0),$$

where we used that $g(-1)=-1$. Since $\mathbb{U}_+(0)\cap\mathbb{E}^s_+(x,0)=\{0\}$ for all $x\ge\ell$ (see (3.48)) and $\mathbb{U}_+(0)\cap\mathbb{S}_+(0)=\{0\}$, we have $\mathbb{U}_+(0)\cap\widehat{\mathbb{E}}^s_+(\tau,0)=\{0\}$ for all $\tau\in[\tau_\ell,1]$, and hence

$$\Lambda_1(s,-1)\cap\Lambda_2(s,-1)=\{0\}$$

for all $s \in [0,1]$. Furthermore,

$$\Lambda_1(s,1) = \widehat{\mathbb{E}}^u_+(1+(\tau_\ell - 1)s, 0), \qquad \Lambda_2(s,1) = \widehat{\mathbb{E}}^s_+(1+(\tau_\ell - 1)s, 0),$$

and therefore

$$\dim \Lambda_1(s,1) \cap \Lambda_2(s,1) = 1$$

for all $s \in [0,1]$ by Hypothesis 1.1. Equation (4.57) (and thus (4.54)) now follows from Lemma 3.4.

By additivity under concatenation (see Proposition 3.3), we can write (4.57) as

$$\begin{aligned}\mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1, 1-\varepsilon]) + \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon, 1]) \\ = \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1, 1-\varepsilon_0]) + \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [1-\varepsilon_0, 1])\end{aligned} \quad (4.59)$$

for $\varepsilon, \varepsilon_0 > 0$ small. Because crossings of the path $x \mapsto \mathbb{E}^u_+(x,0)$ with $\mathcal{T}(\mathbb{S}_+(0))$ and $\mathcal{T}(\mathbb{E}^s_+(\ell,0))$ are isolated (see Lemma 4.10), we can choose $\varepsilon, \varepsilon_0 > 0$ small enough so that $\tau = 1$ is the only crossing in the intervals $[1-\varepsilon, 1]$ and $[1-\varepsilon_0, 1]$ for the paths in (4.59). To prove Lemma 4.11, it thus suffices to show that

$$\mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon, 1]) = \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [1-\varepsilon_0, 1]), \quad (4.60)$$

i.e. that the conjugate points occuring at the final points of each of the paths

$$\tau \mapsto \left(\widehat{\mathbb{E}}^u_+(g(\tau),0), \widehat{\mathbb{E}}^s(\tau_\ell,0)\right), \quad \tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \mathbb{S}_+(0)\right), \quad \tau \in [-1,1], \quad (4.61)$$

have the same contribution to their respective Maslov indices. To this end, notice that the arguments of the unstable bundles appearing in (4.61) are arbitrarily close: by choosing ℓ large enough, so that $\tau_\ell = 1-\delta$ for $\delta > 0$ small enough, we have

$$|g(\tau) - \tau| = \left(\frac{1-\tau_\ell}{2}\right)(\tau+1) \le \delta$$

uniformly for $\tau \in [-1,1]$. Thus, the paths in (4.61) are arbitrarily small perturbations of one another. In addition, since $\mathbb{E}^s(\tau,0)$ can be taken as close to $\mathbb{S}_+(0)$ (as points in $\mathcal{L}(2)$) as we like, the trains $\mathcal{T}(\mathbb{E}^s(\tau,0))$ and $\mathcal{T}(\mathbb{S}_+(0))$ are also arbitrarily small perturbations of one another. From these two facts, it follows that the paths in (4.61) approach the trains $\mathcal{T}(\mathbb{E}^s(\tau,0))$ and $\mathcal{T}(\mathbb{S}_+(0))$ from the same direction as $\tau \to 1^-$. The contributions of the associated conjugate points to their respective Maslov indices are therefore the same, i.e. (4.60) holds, and by (4.59) we have

$$\mathrm{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1, 1-\varepsilon]) = \mathrm{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1, 1-\varepsilon_0]).$$

Recalling Remark 4.5, this is exactly (4.53) (for a different but still arbitrarily small ε). The proof for the L_- problem is similar. □

We remark here that Lemma 4.4 does not apply to the conjugate point at $\tau = 1$ ($x = +\infty$). This is because the functions in the unstable bundle used in the crossing form calculations either blow up to infinity or decay to zero there. Nonetheless, recalling the definition given by Arnol'd (see Section 3.2), we can still compute the Maslov indices in (4.60). Undoing the scaling by g, the paths in (4.61) are given by

$$\tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0)\right), \ \tau \in [-1,\tau_\ell], \qquad \tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \mathbb{S}_+(0)\right), \ \tau \in [-1,1]. \quad (4.62)$$

We know from Hypothesis 1.1 that the final crossing of each path in (4.62) is one-dimensional. In particular, we have $\widehat{\mathbb{E}}^u_+(\tau_\ell,0) \in \mathcal{T}_1(\widehat{\mathbb{E}}^s_+(\tau_\ell,0))$. From the arguments in the proof of Lemma 4.11, $\widehat{\mathbb{E}}^u_+(\tau_\ell,0)$ is therefore arbitrarily close to $\mathcal{T}_1(\mathbb{S}_+(0))$. Lemma 4.4 implies that at all interior one-dimensional crossings $\tau \in (-1,1)$, the path $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ passes through $\mathcal{T}_1(\mathbb{S}_+(0))$ in the negative direction (i.e. from the positive to the negative side of $\mathcal{T}_1(\mathbb{S}_+(0))$).

It follows that at $\tau_\ell \in (-1,1)$, the path $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ must arrive at $\mathcal{T}(\widehat{\mathbb{E}}^s_+(\tau_\ell,0))$ in the negative direction as $\tau \to \tau_\ell^-$. The final crossings of the paths in (4.61) are thus both negative. By our convention the final crossings may only contribute positively, and therefore

$$\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon,1]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [\ell-\varepsilon,\ell]) = 0. \tag{4.63}$$

Lemma 4.12. *Each crossing $\lambda = \lambda_0$ of the path of Lagrangian pairs $\lambda \mapsto (\mathbb{E}^u_+(\ell,\lambda), \mathbb{E}^s_+(\ell,\lambda))$ is positive. Thus,*

$$\operatorname{Mas}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot); [\varepsilon,\lambda_\infty]) = P. \tag{4.64}$$

for $\varepsilon > 0$ small enough. Similarly, each crossing $\lambda = \lambda_0$ of the path $\lambda \mapsto (\mathbb{E}^u_-(\ell,\lambda), \mathbb{E}^s_-(\ell,\lambda))$ is negative, and we have

$$\operatorname{Mas}(\mathbb{E}^u_-(\ell,\cdot), \mathbb{E}^s_-(\ell,\cdot); [\varepsilon,\lambda_\infty]) = -Q. \tag{4.65}$$

Proof. We begin with the first two statements. We proceed by computing the relative crossing form of Robbin and Salamon [RS93] at each crossing $\lambda = \lambda_0$, given by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot))(q) = \mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\lambda_0))(q) - \mathfrak{m}_{\lambda_0}(\mathbb{E}^s_+(\ell,\cdot), \mathbb{E}^u_+(\ell,\lambda_0))(q), \tag{4.66}$$

where $q \in \mathbb{E}^u_+(\ell,\lambda_0) \cap \mathbb{E}^s_+(\ell,\lambda_0)$ is fixed. We compute each of the crossing forms on the right hand side separately.

For the first, we consider the path $\lambda \mapsto \mathbb{E}^u_+(\ell,\lambda)$ over $\lambda \in [\lambda_0-\varepsilon, \lambda_0+\varepsilon]$ for $\varepsilon > 0$ small with reference plane $\mathbb{E}^s_+(\ell,\lambda_0)$. We have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\lambda_0))(q) = \frac{d}{d\lambda}\omega(R^u_+(\lambda)q,q)\big|_{\lambda=\lambda_0}, \quad q \in \mathbb{E}^u_+(\ell,\lambda_0) \cap \mathbb{E}^s_+(\ell,\lambda_0), \tag{4.67}$$

where $R^u_+(\lambda) : \mathbb{E}^s_+(\ell,\lambda_0) \to \mathbb{E}^s_+(\ell,\lambda_0)^\perp$ is the unique family of matrices such that $\mathbb{E}^u_+(\ell,\lambda) = \operatorname{graph}(R^u_+(\lambda)) = \{q + R^u_+(\lambda)q : q \in \mathbb{E}^s_+(\ell,\lambda_0)\}$ for all $\lambda \in [\lambda_0-\varepsilon, \lambda_0+\varepsilon]$. Fixing $q \in \mathbb{E}^s_+(\ell,\lambda_0) \cap \mathbb{E}^u_+(\ell,\lambda_0)$, let $h(\lambda) = q + R^u_+(\lambda)q \in \mathbb{E}^u_+(\ell,\lambda)$. From the definition of $\mathbb{E}^u_+(\ell,\lambda)$, there exists a one-parameter family of solutions $\lambda \mapsto \mathbf{u}(\cdot;\lambda)$ to (4.3) satisfying $\mathbf{u}(x;\lambda) \to 0$ as $x \to -\infty$, such that $h(\lambda) = \mathbf{u}(\ell;\lambda)$. Moreover, $h(\lambda_0) = q = \mathbf{u}(\ell;\lambda_0)$ because $q \in \mathbb{E}^u_+(\ell,\lambda_0) \cap \mathbb{E}^s_+(\ell,\lambda_0) = \big(\ker R^u_+(\lambda_0)\big) \cap \mathbb{E}^s_+(\ell,\lambda_0)$. This allows us to write

$$\begin{aligned}\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\lambda_0))(q) &= \frac{d}{d\lambda}\omega(R^u_+(\lambda)q,q)\big|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\omega(q + R^u_+(\lambda)q,q)\big|_{\lambda=\lambda_0},\\ &= \omega\Big(\frac{d}{d\lambda}\mathbf{u}(\ell,\lambda), \mathbf{u}(\ell,\lambda_0)\Big)\big|_{\lambda=\lambda_0}.\end{aligned}$$

Now

$$
\begin{aligned}
\omega\Big(\frac{d}{d\lambda}\mathbf{u}(\ell;\lambda),\mathbf{u}(\ell;\lambda)\Big) &= \int_{-\infty}^{\ell} \partial_x\,\omega(\partial_\lambda \mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\,dx,\\
&= \int_{-\infty}^{\ell} \omega\left(\partial_\lambda\big[A_+(x;\lambda)\mathbf{u}(x;\lambda)\big],\mathbf{u}(x;\lambda)\right)\\
&\qquad\qquad + \omega(\partial_\lambda \mathbf{u}(x;\lambda), A_+(x;\lambda)\mathbf{u}(x;\lambda))\,dx,\\
&= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\\
&\qquad\qquad + \omega(A_+(x;\lambda)\partial_\lambda\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\\
&\qquad\qquad\qquad\qquad + \omega(\partial_\lambda \mathbf{u}(x;\lambda), A_+(x;\lambda)\mathbf{u}(x;\lambda))\,dx,\\
&= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\\
&\qquad + \big\langle [A_+(x;\lambda)^\top J + JA_+(x;\lambda)]\partial_\lambda\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda)\big\rangle\,dx,\\
&= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\,dx,
\end{aligned}
\tag{4.68}
$$

where we used that $\lim_{x\to-\infty}\mathbf{u}(x;\lambda)=0$ in the first line and (2.14) in the last line. Since

$$
\partial_\lambda A_+(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix},
\tag{4.69}
$$

and $\mathbf{u}=(u_1,u_2,u_3,u_4)^\top$, evaluating the last line of (4.68) at $\lambda=\lambda_0$ we have

$$
\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\lambda_0))(q) = \int_{-\infty}^{\ell} u_2(x;\lambda_0)^2\,dx.
\tag{4.70}
$$

For the second term of the relative crossing form we use a similar argument. We have

$$
\mathfrak{m}_{\lambda_0}(\mathbb{E}^s_+(\ell,\cdot),\mathbb{E}^u_+(\ell,\lambda_0))(q) = \frac{d}{d\lambda}\omega(R^s_+(\lambda)q,q)\big|_{\lambda=\lambda_0},\quad q\in\mathbb{E}^u_+(\ell,\lambda_0)\cap\mathbb{E}^s_+(\ell,\lambda_0),
\tag{4.71}
$$

where $R^s_+(\lambda):\mathbb{E}^u_+(\ell,\lambda_0)\to\mathbb{E}^u_+(\ell,\lambda_0)^\perp$ is the unique family of matrices such that $\mathbb{E}^s_+(\ell,\lambda)=\operatorname{graph}(R^s_+(\lambda))=\{q+R^s_+(\lambda)q : q\in\mathbb{E}^s_+(\ell,\lambda_0)\}$. For the *same* fixed $q\in\mathbb{E}^u_+(\ell,\lambda_0)\cap\mathbb{E}^s_+(\ell,\lambda_0)$ as in the paragraph following (4.67), we can construct a curve $g(\lambda)=q+R^s_+(\lambda)q\in\mathbb{E}^s_+(\ell,\lambda)$ for which there exists a one-parameter family of solutions $\lambda\mapsto\widetilde{\mathbf{u}}(\cdot;\lambda)$ to (4.3) such that $g(\lambda)=\widetilde{\mathbf{u}}(\ell;\lambda)$ and $g(\lambda_0)=q=\widetilde{\mathbf{u}}(\ell;\lambda_0)$. Arguing as previously, but noting that now $\widetilde{\mathbf{u}}(x;\lambda)\to 0$ as $x\to+\infty$, we have

$$
\mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,\lambda_0))(q) = \omega\Big(\frac{d}{d\lambda}\widetilde{\mathbf{w}}(\ell;\lambda),\widetilde{\mathbf{w}}(\ell;\lambda)\Big)\big|_{\lambda=\lambda_0} = -\int_{\ell}^{\infty}\widetilde{u}_2(x;\lambda_0)^2\,dx
\tag{4.72}
$$

(where $\widetilde{\mathbf{u}}=(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{u}_4)^\top$). Importantly, by uniqueness of solutions we have $\widetilde{\mathbf{u}}(\cdot;\lambda_0)=\mathbf{u}(\cdot;\lambda_0)$, so that the integrands in (4.72) and (4.70) are the same. Therefore, (4.66) becomes

$$
\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot))(q) = \int_{-\infty}^{\infty} u_2(x;\lambda_0)^2\,dx > 0.
\tag{4.73}
$$

As the form is positive definite, each crossing contributes $\dim\mathbb{E}^u_+(\ell,\lambda_0)\cap\mathbb{E}^s_+(\ell,\lambda_0)$. It follows that the Maslov index counts the number of crossings (up to dimension) of the path of Lagrangian pairs $\lambda\mapsto(\mathbb{E}^u_+(\ell,\lambda),\mathbb{E}^s_+(\ell,\lambda))$, $\lambda\in[\varepsilon,\lambda_\infty]$, for $\varepsilon>0$ small enough. But this is precisely a count (with negative sign) of the number of positive eigenvalues of L_+ up to multiplicity, i.e. equation (4.64) holds.

For the path $\lambda \mapsto (\mathbb{E}^u_-(\ell,\lambda), \mathbb{E}^s_-(\ell,\lambda))$, $\lambda \in [0,\lambda_\infty]$ the argument is similar, where now the Maslov index counts, with *negative* sign, the number of crossings along Γ_2. The sign change results from the fact that λ now appears with positive sign in the first order system (4.6), so that

$$\partial_\lambda A_-(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.74}$$

The associated crossing form will then be negative, and by the same reasoning as before equation (4.65) follows. □

The following lemma shows that there are no crossings along Γ_3 and Γ_4.

Lemma 4.13. *We have $\mathbb{E}^u_+(x,\lambda_\infty) \cap \mathbb{E}^s_+(\ell,\lambda_\infty) = \{0\}$ for all $x \in \mathbb{R}$, provided both $\lambda_\infty > 0$ and $\ell > 0$ are large enough. In addition, $\mathbb{U}_+(\lambda) \cap \mathbb{E}^s_+(\ell,\lambda) = \{0\}$ for all $\lambda \geq 0$ provided $\ell > 0$ is large enough. Therefore*

$$\mathrm{Mas}(\mathbb{E}^u_+(\cdot,\lambda_\infty), \mathbb{E}^s_+(\ell,\lambda_\infty); [-\infty,\ell]) = \mathrm{Mas}(\mathbb{U}_+(\cdot), \mathbb{E}^s_+(\ell,\cdot); [0,\lambda_\infty]) = 0. \tag{4.75}$$

Similar statements hold for the paths $x \mapsto \big(\mathbb{E}^u_-(x,\lambda_\infty), \mathbb{E}^s_-(\ell,\lambda_\infty)\big)$ and $\lambda \mapsto \big(\mathbb{U}_+(\lambda), \mathbb{E}^s_+(\ell,\lambda)\big)$.

Proof. The strategy of the following proof mirrors the one given in [Cor19, §4] (see also [AGJ90, §3 and §5.B]).

For the first statement, we begin by noting that $\mathrm{Spec}(L_+)$ is bounded from above. To see this, note that we can write

$$L_+ = D + V, \qquad D = -\partial_{xxxx} - \sigma_2 \partial_{xx}, \qquad V = -\beta + 3\phi(x)^2, \tag{4.76}$$

where $\mathrm{dom}(D) = \mathrm{dom}(V) = \mathrm{dom}(L_+)) = H^4(\mathbb{R})$, so that $D = D^*$ is selfadjoint and V is bounded and symmetric on $L^2(\mathbb{R})$. It can be shown that D has no point spectrum, and moreover, $\mathrm{Spec}(D) = \mathrm{Spec}_{\mathrm{ess}}(D) = (-\infty, 1/4]$ if $\sigma_2 = 1$, and $\mathrm{Spec}(D) = (-\infty, 0]$ if $\sigma = -1$. It then follows from [Kat80, Theorem V.4.10, p.291] that

$$\mathrm{dist}\left(\mathrm{Spec}(L_+), \mathrm{Spec}(D)\right) \leq \|V\|, \tag{4.77}$$

so that $\mathrm{Spec}(L_+) \subseteq (-\infty, \|V\|]$. Consequently, we have $\mathbb{E}^u_+(\ell,\lambda) \cap \mathbb{E}^s_+(\ell,\lambda) = \{0\}$ for all $\lambda > \|V\|$.

Next, we claim that there exists a $\lambda_\infty > \|V\|$ such that

$$\mathbb{E}^u_+(x,\lambda) \cap \mathbb{S}_+(\lambda) = \{0\} \tag{4.78}$$

for all $x \in \mathbb{R}$ and all $\lambda \geq \lambda_\infty$. Once this is shown, it follows that there exists an $\ell_\infty \gg 1$ such that

$$\mathbb{E}^u_+(x,\lambda_\infty) \cap \mathbb{E}^s_+(\ell,\lambda_\infty) = \{0\} \tag{4.79}$$

for all $x \in \mathbb{R}$ and all $\ell \geq \ell_\infty$, because $\lim_{x\to\infty} \mathbb{E}^s_+(x,\lambda) = \mathbb{S}_+(\lambda)$. It remains to prove the claim. We mimic the proof of [Cor19, Lemma 4.1]. Consider then the change of variables:

$$y = \lambda^{1/4} x, \qquad \widetilde{u}_1 = u_1, \quad \widetilde{u}_2 = \lambda^{1/2} u_2, \quad \widetilde{u}_3 = \lambda^{1/4} u_3, \quad \widetilde{u}_4 = \lambda^{-1/4} u_4, \tag{4.80}$$

under which the system (4.3) becomes

$$\frac{d}{dy}\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \\ \widetilde{u}_3 \\ \widetilde{u}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\sigma_2}{\sqrt{\lambda}} & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & 0 \\ -\frac{\sigma_2}{\sqrt{\lambda}} & \frac{\alpha\left(\frac{y}{\sqrt[4]{\lambda}}\right)}{\lambda} - 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \\ \widetilde{u}_3 \\ \widetilde{u}_4 \end{pmatrix} \tag{4.81}$$

(recall that $\alpha\left(\frac{y}{\sqrt[4]{\lambda}}\right) = 3\phi(\frac{y}{\sqrt[4]{\lambda}})^2 - \beta + 1$). Taking $y \to \pm\infty$, the asymptotic system for (4.81) is given by

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\sigma_2}{\sqrt{\lambda}} & 1\\ 0 & 0 & 1 & 0\\ 1 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & 0\\ -\frac{\sigma_2}{\sqrt{\lambda}} & \frac{-\beta+1}{\lambda} - 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix}. \tag{4.82}$$

Denote the stable and unstable subspaces for (4.82) by $\widetilde{\mathbb{S}}_+(\lambda)$ and $\widetilde{\mathbb{U}}_+(\lambda)$ respectively, and denote the unstable bundle of (4.81) by $\widetilde{\mathbb{E}}^u_+(y, \lambda)$. Then, we have

$$\mathbb{E}^u_+(x, \lambda) \cap \mathbb{S}_+(\lambda) = \{0\} \iff \widetilde{\mathbb{E}}^u_+(\lambda^{1/4}x, \lambda) \cap \widetilde{\mathbb{S}}_+(\lambda) = \{0\}, \tag{4.83}$$

since $\widetilde{\mathbb{E}}^u_+(\lambda^{1/4}x, \lambda) = M\cdot\mathbb{E}^u(x, \lambda)$ and $\widetilde{\mathbb{S}}_+(\lambda) = M\cdot\mathbb{S}_+(\lambda)$, where $M = \text{diag}\{1, \lambda^{1/2}, \lambda^{1/4}, \lambda^{-1/4}\}$ is the (nonsingular) linear transformation of the dependent variables in (4.80), and "$\cdot$" represents the induced action of M on a subspace of $\mathbb{R}^4$.

Both the nonautonomous system (4.81) and the autonomous system (4.82) induce flows on $\text{Gr}_2(\mathbb{R}^4)$, the Grassmannian of two dimensional subspaces of $\mathbb{R}^4$. For the flow associated with (4.82), it is known [AGJ90] that $\widetilde{\mathbb{U}}_+(\lambda)$, the invariant subspace associated with eigenvalues of positive real part, is an attracting fixed point. Thus, since $\mathcal{L}(2) \subset \text{Gr}_2(\mathbb{R}^4)$, there exists a trapping region $\mathcal{R} \subset \Lambda(2)$ containing $\widetilde{\mathbb{U}}_+(\lambda)$. By taking λ large enough, we can ensure that the flow induced by (4.81) is as close as we like to that induced by (4.82), because $\phi\left(\frac{y}{\sqrt[4]{\lambda}}\right)^2/\lambda$ – the nonautonomous part of (4.81) – is close to zero. It follows that $\mathcal{R} \subset \mathcal{L}(2)$ is also a trapping region for (4.81). Furthermore, we can choose $\mathcal{R}$ small enough such that $\mathbb{V} \cap \widetilde{\mathbb{S}}_+(\lambda) = \{0\}$ for all $\mathbb{V} \in \mathcal{R}$, uniformly for λ large enough. To see this, note that clearly $\widetilde{\mathbb{S}}_+(\lambda) \cap \widetilde{\mathbb{U}}_+(\lambda) = \{0\}$, while taking $\lambda \to +\infty$ in (4.82) yields

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix}, \tag{4.84}$$

which has stable and unstable subspaces $\widetilde{\mathbb{S}}_{+\infty}$ and $\widetilde{\mathbb{U}}_{+\infty}$ with respective frames $(I, -W)$ and (I, W), where

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Thus, in the limit we also have $\widetilde{\mathbb{S}}_{+\infty} \cap \widetilde{\mathbb{U}}_{+\infty} = \{0\}$, so we can choose $\mathcal{R}$ as stated. Finally, we note that if $\lambda > \|V\|$ so that $\lambda \notin \text{Spec}(L_+)$, then by [AGJ90, Lemma 3.7] we have $\lim_{y\to\infty} \widetilde{\mathbb{E}}^u_+(y, \lambda) = \widetilde{\mathbb{U}}_+(\lambda)$. All in all, we conclude that for any $\lambda = \lambda_\infty > \|V\|$ large enough, the trajectory $\widetilde{\mathbb{E}}^u_+(\cdot, \lambda_\infty) : [-\infty, \infty] \to \mathcal{L}(2)$, which starts and finishes at $\widetilde{\mathbb{U}}_+(\lambda_\infty)$, will remain inside $\mathcal{R}$ and thus always be disjoint from $\widetilde{\mathbb{S}}_+(\lambda_\infty)$. This proves the claim.

For the second statement of the lemma, the facts that $\mathbb{U}_+(\lambda) \cap \mathbb{S}_+(\lambda) = \{0\}$ and $\lim_{x\to\infty} \mathbb{E}^s_+(x, \lambda) = \mathbb{S}_+(\lambda)$ imply that there exists an $\ell_0 \gg 1$ such that $\mathbb{U}_+(\lambda) \cap \mathbb{E}^s_+(x, \lambda) = \{0\}$ for all $x \geq \ell_0$. Taking $\ell > \ell_0$ gives the result. □

We are now ready to prove Proposition 4.1. In what follows, we choose $\ell > 0$ and $\lambda_\infty > 0$ large enough so that the statements of Lemma 4.13 hold.

Proof of Proposition 4.1. By homotopy invariance and additivity under concatenation, we have

$$\begin{aligned}\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[-\infty,\ell])+\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\lambda_\infty])\\ -\mathrm{Mas}(\mathbb{E}^u_+(\cdot,\lambda_\infty),\mathbb{E}^s_+(\ell,\lambda_\infty);[-\infty,\ell])-\mathrm{Mas}(\mathbb{E}^u_+(-\infty,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\lambda_\infty])=0. \quad (4.85)\end{aligned}$$

From Lemma 4.13 the third and fourth terms on the left hand side vanish. Again using the concatenation property, we find that

$$\begin{aligned}\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[-\infty,\ell-\varepsilon])+\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[\ell-\varepsilon,\ell])\\ +\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\varepsilon])+\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[\varepsilon,\lambda_\infty])=0 \quad (4.86)\end{aligned}$$

where $\varepsilon>0$ is small. The second and third terms of (4.86) represent the contributions to the Maslov index from the conjugate point $(x,\lambda)=(\ell,0)$ at the top left corner of the Maslov box in the x and λ directions respectively. From (4.63), Lemma 4.12 and Definition 3.2 we have

$$\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[\ell-\varepsilon,\ell])=\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\varepsilon])=0. \quad (4.87)$$

Lemmas 4.4 and 4.11 imply that

$$\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[-\infty,\ell-\varepsilon])=-\sum_{x\in\mathbb{R}}\dim\left(\mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0)\right). \quad (4.88)$$

The previous three equations along with Lemma 4.12 now yield (4.1).

The proof for the Morse index of the L_- operator is similar. This time, crossings along Γ_1 are positive, while crossings along Γ_2 are negative. Arguing as we did for (4.63), we can show that

$$\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[\ell-\varepsilon,\ell])=\dim\left(\mathbb{E}^u_+(\ell,0)\cap\mathbb{E}^s_+(\ell,0)\right)=1, \quad (4.89)$$

and from Lemma 4.12 and Definition 3.2 we have

$$\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\varepsilon])=\dim\left(\mathbb{E}^u_+(\ell,0)\cap\mathbb{E}^s_+(\ell,0)\right)=-1. \quad (4.90)$$

The contributions (4.89) and (4.90) to the Maslov index coming from $(x,\lambda)=(\ell,0)$ thus cancel each other out. Applying the same homotopy argument as above yields the formula for Q in the proposition. □

5. Proofs of the main results

We now return to the computation of the Maslov indices appearing on the left hand side of (3.55). After computing each, we provide the proofs of Theorems 1.2 and 1.5. We begin with Γ_1 (excluding its endpoint at $x=\ell$).

Lemma 5.1. $\mathrm{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell-\varepsilon])=Q-P$, *where* $\varepsilon>0$ *is small.*

Proof. Recall that when $\lambda=0$ the eigenvalue equations (1.11) decouple. Consequently, the equations for the u and v components in the first order system (2.2) also decouple. Hence, for each $x\in\mathbb{R}$,

$$\mathbb{E}^u(x,0)=\mathbb{E}^u_+(x,0)\oplus\mathbb{E}^u_-(x,0), \quad (5.1)$$

in the sense that for any $\mathbf{w} \in \mathbb{E}^u(x,0)$ we have

$$\mathbf{w} = \begin{pmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ 0 \\ u_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_1 \\ 0 \\ v_2 \\ 0 \\ v_3 \\ 0 \\ v_4 \end{pmatrix}, \tag{5.2}$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)^\top \in \mathbb{E}^u_+(x,0)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)^\top \in \mathbb{E}^u_-(x,0)$. By the same reasoning, for the reference plane we have

$$\mathbb{E}^s(\ell,0) = \mathbb{E}^s_+(\ell,0) \oplus \mathbb{E}^s_-(\ell,0). \tag{5.3}$$

Now using property (3) of Proposition 3.3, we have

$$\begin{aligned} \operatorname{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [-\infty, \ell-\varepsilon]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [-\infty, \ell-\varepsilon]) \\ + \operatorname{Mas}(\mathbb{E}^u_-(\cdot,0), \mathbb{E}^s_-(\ell,0); [-\infty, \ell-\varepsilon]), \end{aligned} \tag{5.4}$$

and the result follows combining equations (4.88) and (4.1) (and the accompanying statements for L_-). ☐

Next, we show that there are no crossings along Γ_3 and Γ_4.

Lemma 5.2. *There exists $\ell_1 \gg 1$ such that $\mathbb{E}^u(x,\lambda_\infty) \cap \mathbb{E}^s(\ell,\lambda_\infty) = \{0\}$ for all $x \in \mathbb{R}$ and all $\ell \geq \ell_1$, provided $\lambda_\infty > 0$ is large enough. Therefore, for all $\ell \geq \ell_1$,*

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,\lambda_\infty), \mathbb{E}^s(\ell,\lambda_\infty); [-\infty, \ell]) = 0.$$

In addition, $\mathbb{U}(\lambda) \cap \mathbb{E}^s(\ell,\lambda) = \{0\}$ for all $\lambda \geq 0$ provided $\ell > 0$ is large enough. Consequently,

$$\operatorname{Mas}(\mathbb{U}(\cdot), \mathbb{E}^s(\ell,\cdot); [0, \lambda_\infty]) = 0.$$

Proof. For the first assertion, note that N is a bounded perturbation of a skew-selfadjoint operator, so that its spectrum lies in a vertical strip around the imaginary axis in the complex plane. More precisely, we have that

$$iN = \widetilde{D} + \widetilde{V}, \quad \widetilde{D} = i\begin{pmatrix} 0 & \partial_{xxxx} + \sigma_2 \partial_{xx} \\ -\partial_{xxxx} - \sigma_2 \partial_{xx} & 0 \end{pmatrix}, \quad \widetilde{V} = i\begin{pmatrix} 0 & \beta - \phi^2 \\ -\beta + 3\phi^2 & 0, \end{pmatrix} \tag{5.5}$$

where, with $\operatorname{dom}(\widetilde{D}) = \operatorname{dom}(\widetilde{V}) = \operatorname{dom}(N)$, $\widetilde{D}^* = \widetilde{D}$ is selfadjoint and $\widetilde{V}$ is bounded. Now using [Kat80, Remark 3.2, p.208] and [Kat80, eq. (3.16), p.272], we may conclude that

$$\zeta \in \operatorname{Spec}(\widetilde{D} + \widetilde{V}) \implies |\operatorname{Im}(\zeta)| \leq \|\widetilde{V}\|. \tag{5.6}$$

By the spectral mapping theorem, $\operatorname{Spec}(iN) = i\operatorname{Spec}(N)$. It follows that

$$\lambda \in \operatorname{Spec}(N) \implies |\operatorname{Re}(\lambda)| \leq \|\widetilde{V}\|. \tag{5.7}$$

Thus, for all $\lambda > \|\widetilde{V}\|$ we have $\mathbb{E}^u(\ell,\lambda) \cap \mathbb{E}^s(\ell,\lambda) = \{0\}$.

The proof now follows from the same arguments used to prove the first assertion in Lemma 4.13. Namely, via the change of variables (4.80) along with

$$\widetilde{v}_1 = v_1, \quad \widetilde{v}_2 = \lambda^{1/2} v_2, \quad \widetilde{v}_3 = \lambda^{1/4} v_3, \quad \widetilde{v}_4 = \lambda^{-1/4} v_4 \tag{5.8}$$

we can rewrite (2.2) as

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{v}_1\\ \widetilde{u}_2\\ \widetilde{v}_2\\ \widetilde{u}_3\\ \widetilde{v}_3\\ \widetilde{u}_4\\ \widetilde{v}_4\end{pmatrix}=\left(\begin{array}{cccc|cccc} & & & & \frac{\sigma_2}{\sqrt{\lambda}} & 0 & 1 & 0\\ & 0 & & & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & 1\\ & & & & 1 & 0 & 0 & 0\\ & & & & 0 & 1 & 0 & 0\\ \hline 1 & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & & & & \\ 0 & -1 & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & & 0 & & \\ -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & \frac{\alpha(x)}{\lambda} & 1 & & & & \\ 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 1 & \frac{\eta(x)}{\lambda} & & & & \end{array}\right)\begin{pmatrix}\widetilde{u}_1\\ \widetilde{v}_1\\ \widetilde{u}_2\\ \widetilde{v}_2\\ \widetilde{u}_3\\ \widetilde{v}_3\\ \widetilde{u}_4\\ \widetilde{v}_4\end{pmatrix}. \tag{5.9}$$

Again, the flow of the associated asymptotic system is close to that of (5.9) for large λ. From the transversality of the four dimensional stable and unstable subspaces of the limiting system of (5.9) as $\lambda \to \infty$, i.e.

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{v}_1\\ \widetilde{u}_2\\ \widetilde{v}_2\\ \widetilde{u}_3\\ \widetilde{v}_3\\ \widetilde{u}_4\\ \widetilde{v}_4\end{pmatrix}=\left(\begin{array}{cccc|cccc} & & & & 0 & 0 & 1 & 0\\ & 0 & & & 0 & 0 & 0 & 1\\ & & & & 1 & 0 & 0 & 0\\ & & & & 0 & 1 & 0 & 0\\ \hline 1 & 0 & 0 & 0 & & & & \\ 0 & -1 & 0 & 0 & & 0 & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \end{array}\right)\begin{pmatrix}\widetilde{u}_1\\ \widetilde{v}_1\\ \widetilde{u}_2\\ \widetilde{v}_2\\ \widetilde{u}_3\\ \widetilde{v}_3\\ \widetilde{u}_4\\ \widetilde{v}_4\end{pmatrix}, \tag{5.10}$$

one can show that there exists a $\lambda_\infty > \|\widetilde{V}\|$ such that $\mathbb{E}^u(x,\lambda)$ and $\mathbb{S}(\lambda)$ are transverse for all $x \in \mathbb{R}$ and all $\lambda \geq \lambda_\infty$. Hence $\mathbb{E}^u(x,\lambda)$ and $\mathbb{E}^s(\ell,\lambda_\infty)$ are transverse for all $x \in \mathbb{R}, \ell \geq \ell_\infty$ and $\lambda \geq \lambda_\infty$. The second assertion follows from the same arguments used to prove the second assertion in Lemma 4.13. □

For the proof of Theorem 1.2, it remains to compute

$$\mathfrak{c} := \operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[\ell-\varepsilon,\ell]) + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\varepsilon]), \tag{5.11}$$

the contribution to the Maslov index from the conjugate point $(x,\lambda) = (\ell,0)$. For the contribution in the x direction, i.e. the arrival along Γ_1, again using property (3) of Proposition 3.3 and equations (4.87) and (4.89), we have

$$\begin{aligned}\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[\ell-\varepsilon,\ell]) &= \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[\ell-\varepsilon,\ell])\\ &\quad+\operatorname{Mas}(\mathbb{E}^u_-(\cdot,0),\mathbb{E}^s_-(\ell,0);[\ell-\varepsilon,\ell]),\\ &= 1.\end{aligned} \tag{5.12}$$

To determine the contribution in the λ direction given by the departure along Γ_2 (the second term on the right hand side of (5.11)), we will compute crossing forms. To that end, suppose $\lambda = \lambda_0$ is a crossing of the Lagrangian pair $\lambda \mapsto (\mathbb{E}^u(\ell,\lambda),\mathbb{E}^s(\ell,\lambda))$, $\lambda \in [0,\lambda_\infty]$. The first-order relative crossing form ((3.37) with $k=1$) is given by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = \mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q) - \mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,\lambda_0))(q), \tag{5.13}$$

where $q \in \mathbb{E}^u(\ell,\lambda_0) \cap \mathbb{E}^s(\ell,\lambda_0)$ is fixed. We compute each of these terms separately.

The first term concerns the path $\lambda \mapsto \mathbb{E}^u(\ell,\lambda)$ with reference plane $\mathbb{E}^s(\ell,\lambda_0)$. The first-order form (3.24) is given here by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q) = \frac{d}{d\lambda}\omega(R^u(\lambda)q,q)\big|_{\lambda=\lambda_0}, \quad q \in \mathbb{E}^u(\ell,\lambda_0)\cap\mathbb{E}^s(\ell,\lambda_0), \tag{5.14}$$

where $R^u(\lambda) : \mathbb{E}^s(\ell,\lambda_0) \to \mathbb{E}^s(\ell,\lambda_0)^\perp$ is the unique family of matrices such that $\mathbb{E}^u(\ell,\lambda) = \text{graph}(R^u(\lambda)) = \{q + R^u(\lambda)q : q \in \mathbb{E}^s(\ell,\lambda_0)\}$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. Fixing some $q \in \mathbb{E}^s(\ell,\lambda_0) \cap \mathbb{E}^u(\ell,\lambda_0)$, let $r(\lambda) = q + R^u(\lambda)q \in \mathbb{E}^u(\ell,\lambda)$. From the definition of $\mathbb{E}^u(\ell,\lambda)$, there exists a one-parameter family of solutions $\lambda \mapsto \mathbf{w}(\cdot;\lambda)$ to (2.2) satisfying $\mathbf{w}(x;\lambda) \to 0$ as $x \to -\infty$ such that $r(\lambda) = \mathbf{w}(\ell;\lambda)$. Furthermore, $r(\lambda_0) = q = \mathbf{w}(\ell;\lambda_0)$ because $(\ker R(x_0)) \cap \mathbb{E}^s(\ell,\lambda_0) = \mathbb{E}^u(\ell,\lambda_0) \cap \mathbb{E}^s(\ell,\lambda_0)$ (recall (3.18)). With this family we can write

$$\begin{aligned}\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q) &= \frac{d}{d\lambda}\omega(R^u(\lambda)q,q)\big|_{\lambda=\lambda_0} = \frac{d}{d\lambda}\omega(q + R^u(\lambda)q,q)\big|_{\lambda=\lambda_0},\\ &= \omega\Big(\frac{d}{d\lambda}\mathbf{w}(\ell,\lambda),\mathbf{w}(\ell,\lambda_0)\Big)\big|_{\lambda=\lambda_0}.\end{aligned}$$

A calculation similar to (4.68) with

$$\partial_\lambda A(x;\lambda) = \begin{pmatrix} 0_4 & 0_4 \\ M & 0_4 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.15}$$

and $\mathbf{w} = (u_1, v_2, u_2, v_2, u_3, v_3, u_4, v_4)^\top$ yields

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q_0) = -2\int_{-\infty}^{\ell} u_2(x;\lambda_0)v_2(x;\lambda_0)\,dx.$$

The second term in (5.13) concerns the path $\lambda \mapsto \mathbb{E}^s(\ell,\lambda)$ with reference plane $\mathbb{E}^u(\ell,0)$. We have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q) = \frac{d}{d\lambda}\omega(R^s(\lambda)q,q)\big|_{\lambda=\lambda_0}, \quad q \in \mathbb{E}^s(\ell,\lambda_0) \cap \mathbb{E}^u(\ell,\lambda_0), \tag{5.16}$$

where $R^s(\lambda) : \mathbb{E}^u(\ell,\lambda_0) \to \mathbb{E}^u(\ell,\lambda_0)^\perp$ uniquely satisfies $\mathbb{E}^s(\ell,\lambda) = \text{graph}\,(R^s(\lambda))$. For the same fixed $q \in \mathbb{E}^u(\ell,\lambda_0) \cap \mathbb{E}^s(\ell,0)$ as before, associated to the curve $t(\lambda) = q + R^s(\lambda)q \in \mathbb{E}^s(\ell,\lambda)$ is a family of solutions $\lambda \mapsto \widetilde{\mathbf{w}}(\cdot;\lambda)$ to (2.2), such that $t(\lambda) = \widetilde{\mathbf{w}}(\ell;\lambda)$ and $t(\lambda_0) = q = \widetilde{\mathbf{w}}(\ell;\lambda_0)$. Arguing as for the first term of (5.13), but noting that now $\widetilde{\mathbf{w}}(x;\lambda) \to 0$ as $x \to +\infty$, we have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,\lambda_0))(q) = \omega\Big(\frac{d}{d\lambda}\widetilde{\mathbf{w}}(\ell;\lambda),\widetilde{\mathbf{w}}(\ell;\lambda)\Big)\Big|_{\lambda=\lambda_0} = 2\int_{\ell}^{\infty} \widetilde{u}_2(x;\lambda_0)\widetilde{v}_2(x;\lambda_0)\,dx.$$

Using uniqueness of solutions as in the proof of Lemma 4.12, we conclude

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = -2\int_{-\infty}^{\infty} u_2(x;\lambda_0)v_2(x;\lambda_0)\,dx. \tag{5.17}$$

Remark 5.3. The form (5.17) is *not* sign definite, and therefore the Maslov index does not afford an exact count of the crossings of the path $\lambda \mapsto (\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))$ for $\lambda \in [0,\lambda_\infty]$. This will be the reason for the inequality (and not an equality) in (1.17) in Theorem 1.2.

Let us now evaluate the form (5.17) at $\lambda = 0$. Note that because $\dim(\mathbb{E}^u(x,0) \cap \mathbb{E}^s(x,0)) = 2$ (c.f. Hypothesis 1.1) where

$$\mathbb{E}^u(x,0) \cap \mathbb{E}^s(x,0) = \text{span}\{\boldsymbol{\phi}(x),\boldsymbol{\varphi}(x)\},$$

it suffices to evaluate (5.17) on the vectors $\boldsymbol{\phi}(x)$ and $\boldsymbol{\varphi}(x)$ from (3.50). Writing $\mathbf{w}(x;0) = \boldsymbol{\phi}(x)k_1 + \boldsymbol{\varphi}(x)k_2$ for some $k_1, k_2 \in \mathbb{R}$, so that $u_2(x;0) = \phi'(x)k_1$ and $v_2(x;0) = -\phi(x)k_2$, we have

$$\mathfrak{m}_0(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = 2\int_{-\infty}^{\infty} \phi'\phi\,dx\,k_1k_2 = \int_{-\infty}^{\infty} \frac{d}{dx}\phi^2\,dx\,k_1k_2 = 0, \tag{5.18}$$

since $\phi \in H^4(\mathbb{R})$. That is, the two dimensional crossing form (5.13) is identically zero at $\lambda_0 = 0$, and the conjugate point $(\ell,0)$ is non-regular in the λ direction. We therefore need to compute higher order crossing forms.

As discussed in Section 3.2, in the case that the first-order form is identically zero, the second-order relative crossing form is given by

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q)=\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,0))(q)-\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,0))(q), \tag{5.19}$$

where $q\in W_2=\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0)$. Each of the crossing forms on the right hand side are computed separately with (3.28). For the first, using the same one-parameter family $\lambda\to\mathbf{w}(\cdot;\lambda)$ as we did for the corresponding first-order form (5.14) (i.e. such that $\mathbf{w}(\ell;\lambda)=r(\lambda)$; see the paragraph following (5.14)), we have

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\lambda_0))(q)=\frac{d^2}{d\lambda^2}\omega(R^u(\lambda)q,q)\big|_{\lambda=\lambda_0}=\omega\Big(\frac{d^2}{d\lambda^2}\mathbf{w}(\ell,\lambda),\mathbf{w}(\ell,\lambda_0)\Big)\big|_{\lambda=\lambda_0}.$$

Now

$$\begin{aligned}\omega\Big(\frac{d^2}{d\lambda^2}\mathbf{w}(\ell;\lambda),\mathbf{w}(\ell;\lambda)\Big)&=\int_{-\infty}^{\ell}\partial_x\,\omega(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))dx,\\ &=\int_{-\infty}^{\ell}\omega\left(\partial_{\lambda\lambda}\left[A(x;\lambda)\mathbf{w}(x;\lambda)\right],\mathbf{w}(x;\lambda)\right)\\ &\qquad+\omega\left(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),A(x;\lambda)\mathbf{w}(x;\lambda)\right)dx,\\ &=\int_{-\infty}^{\ell}\omega(A_{\lambda\lambda}(x;\lambda)\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\\ &\qquad+2\,\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\\ &\qquad\quad+\omega(A(x;\lambda)\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\\ &\qquad\qquad+\omega\left(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),A(x;\lambda)\mathbf{w}(x;\lambda)\right)\,dx,\\ &=\int_{-\infty}^{\ell}\langle[A(x;\lambda)^\top J+JA(x;\lambda)]\partial_{\lambda\lambda}\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda)\rangle\\ &\qquad+2\,\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))\,dx,\\ &=2\int_{-\infty}^{\ell}\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda),\mathbf{w}(x;\lambda))dx,\end{aligned}\tag{5.20}$$

where we used (2.14) and $A_{\lambda\lambda}(x;\lambda)=0$. Using (5.15) and evaluating at $\lambda=0$, we see that

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,0))(q)=-2\int_{-\infty}^{\ell}u_2(x;0)\partial_\lambda v_2(x;0)+v_2(x;0)\partial_\lambda u_2(x;0)\,dx. \tag{5.21}$$

For the second form in the right hand side of (5.19), we use the same one-parameter family $\lambda\to\widetilde{\mathbf{w}}(\cdot;\lambda)$ defined in the paragraph following (5.16) (i.e. such that $\widetilde{\mathbf{w}}(\ell;\lambda)=t(\lambda)$) and the same argument used to arrive at (5.21) to obtain

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,0))(q)=2\int_{\ell}^{\infty}\widetilde{u}_2(x;0)\partial_\lambda\widetilde{v}_2(x;0)+\widetilde{v}_2(x;0)\partial_\lambda\widetilde{u}_2(x;0)\,dx. \tag{5.22}$$

By uniqueness of solutions we have $\mathbf{w}(\cdot;0)=\widetilde{\mathbf{w}}(\cdot;0)$. On the other hand, it is not immediately obvious whether the same is true for the functions $\widehat{u}_2(x)=\partial_\lambda u_2(x;0)$ and $\widehat{v}_2(x)=\partial_\lambda v_2(x;0)$. However, observe that with (5.14) and (5.16), we can write the relative crossing form (5.13) as

$$\mathfrak{m}_{\lambda_0}\left(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot)\right)(q)=\omega(q,\big(\dot{R}^u(0)-\dot{R}^s(0)\big)q),\qquad q\in\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0),$$

where dot denotes $d/d\lambda$. This form is identically zero if and only if $J(\dot{R}^u(0)-\dot{R}^s(0))$ is the zero operator on $\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0)$. From the invertibility of J, it follows that $\dot{R}^u(0)q=\dot{R}^s(0)q$ for all $q\in\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0)$. Recalling that $\mathbf{w}(\ell;\lambda)=r(\lambda)=q+R^u(\lambda)q$ and $\widetilde{\mathbf{w}}(\ell;\lambda)=t(\lambda)=q+R^s(\lambda)q$, taking λ derivatives and evaluating at $\lambda=0$ yields

$$\partial_\lambda\mathbf{w}(\ell;0)=\dot{r}(0)=\dot{R}^u(0)q=\dot{R}^s(0)q=\dot{t}(0)=\partial_\lambda\widetilde{\mathbf{w}}(\ell;0). \tag{5.23}$$

Now, both $\partial_\lambda \mathbf{w}(\cdot;0)$ and $\partial_\lambda \widetilde{\mathbf{w}}(\cdot;0)$ solve the inhomogeneous differential equation

$$\frac{d}{dx}\left(\partial_\lambda \mathbf{w}\right) = A\left(\partial_\lambda \mathbf{w}\right) + A_\lambda\left(\boldsymbol{\phi}\, k_1 + \boldsymbol{\varphi}\, k_2\right), \tag{5.24}$$

obtained by differentiating (2.3) with respect to λ and evaluating at $\lambda = 0$, and using that $\mathbf{w}(\cdot;0) = \boldsymbol{\phi}\, k_1 + \boldsymbol{\varphi}\, k_2$. (Note that $k_1, k_2 \in \mathbb{R}$ are determined by the fixed vector q, where $q = \mathbf{w}(\ell;0) = \boldsymbol{\phi}(\ell)k_1 + \boldsymbol{\varphi}(\ell)k_2$.) It follows from (5.23) and uniqueness of solutions of (5.24) that indeed $\partial_\lambda \mathbf{w}(x;0) = \partial_\lambda \widetilde{\mathbf{w}}(x;0)$ for *all* $x \in \mathbb{R}$. Collecting (5.21) and (5.22) together, (5.19) becomes

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = -2\int_{-\infty}^{\infty} u_2(x;0)\partial_\lambda v_2(x;0) + v_2(x;0)\partial_\lambda u_2(x;0)\, dx. \tag{5.25}$$

We need to understand the function $\partial_\lambda \mathbf{w}(\cdot;0)$. Notice that it solves the inhomogeneous equation (5.24) if and only if its third and fourth entries $\partial_\lambda u_2(\cdot;0)$ and $\partial_\lambda v_2(\cdot;0)$ solve

$$N\begin{pmatrix}\partial_\lambda u_2(\cdot;0)\\ -\partial_\lambda v_2(\cdot;0)\end{pmatrix} = \begin{pmatrix}\phi_x\, k_1\\ -\phi\, k_2\end{pmatrix}. \tag{5.26}$$

This follows from differentiating the eigenvalue equation (1.12) with respect to λ, evaluating at $\lambda = 0$ and making the substitutions

$$\partial_\lambda u(\cdot;0) = \partial_\lambda u_2(\cdot;0), \quad \partial_\lambda v(\cdot;0) = -\partial_\lambda v_2(\cdot;0), \quad u(\cdot;0) = \phi_x\, k_1, \quad v(\cdot;0) = -\phi\, k_2.$$

Now, both equations

$$\begin{aligned} -L_-\partial_\lambda v_2(\cdot;0) &= -\phi_x\, k_1,\\ L_+\partial_\lambda u_2(\cdot;0) &= -\phi\, k_2, \end{aligned} \tag{5.27}$$

are solvable by virtue of the Fredholm alternative, since $\langle \phi', \phi\rangle_{L^2(\mathbb{R})} = 0$ and hence $\phi_x \in \ker(L_-)^\perp$ and $\phi \in \ker(L_+)^\perp$. Denoting by $\widehat{v}$ and $\widehat{u}$ any solutions to

$$-L_-v = \phi_x \qquad \text{and} \qquad L_+u = \phi \tag{5.28}$$

in $H^4(\mathbb{R})$ respectively (note the sign change in both equations from (5.27)), (5.25) becomes

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = 2\left(\int_{-\infty}^{\infty} \phi_x\, \widehat{v}\, dx\right)k_1^2 - 2\left(\int_{-\infty}^{\infty} \phi\, \widehat{u}\, dx\right)k_2^2, \tag{5.29}$$

recalling that $u_2 = \phi_x k_1$ and $v_2 = -\phi\, k_2$. Having computed the form, we count the number of negative squares. Using (3.40), and defining $\mathcal{I}_1$ and $\mathcal{I}_2$ to be the integrals appearing in the first and second terms of (5.29) respectively (as in (1.16)), we find that

$$\mathrm{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\varepsilon]) = -n_-(\mathfrak{m}^{(2)}_{\lambda_0}) = \begin{cases} 0 & \mathcal{I}_1 > 0,\, \mathcal{I}_2 < 0,\\ -1 & \mathcal{I}_1\mathcal{I}_2 > 0,\\ -2 & \mathcal{I}_1 < 0,\, \mathcal{I}_2 > 0.\end{cases} \tag{5.30}$$

Recalling the definition of $\mathfrak{c}$ in (5.11) and using (5.12) yields the following.

Lemma 5.4. *The value of* $\mathfrak{c}$ *is given by*

$$\mathfrak{c} = \begin{cases} 1 & \mathcal{I}_1 > 0,\, \mathcal{I}_2 < 0,\\ 0 & \mathcal{I}_1\mathcal{I}_2 > 0,\\ -1 & \mathcal{I}_1 < 0,\, \mathcal{I}_2 > 0.\end{cases} \tag{5.31}$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By homotopy invariance and additivity under concatenation, we have

$$\mathrm{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell]) + \mathrm{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\lambda_\infty])$$

$$- \operatorname{Mas}(\mathbb{E}^u(\cdot, \lambda_\infty), \mathbb{E}^s(\ell, \lambda_\infty); [-\infty, \ell]) - \operatorname{Mas}(\mathbb{E}^u(-\infty, \cdot), \mathbb{E}^s(\ell, \cdot); [0, \lambda_\infty]) = 0.$$

By Lemma 5.2 the last two terms on the left hand side vanish. Recalling the definition of $\mathfrak{c}$ from (3.56) and using the concatenation property once more,

$$\operatorname{Mas}(\mathbb{E}^u(\cdot, 0), \mathbb{E}^s(\ell, 0); [-\infty, \ell - \varepsilon]) + \mathfrak{c} + \operatorname{Mas}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot); [\varepsilon, \lambda_\infty]) = 0. \tag{5.32}$$

Since the Maslov index counts *signed* crossings, the number of crossings along Γ_2 for $\lambda > 0$ is bounded from below by the absolute value of the Maslov index of this piece, i.e.

$$n_+(N) \geq |\operatorname{Mas}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot); [\varepsilon, \lambda_\infty])|. \tag{5.33}$$

Combining (5.32) and (5.33) with Lemma 5.1, the inequality (1.17) follows. The statement of the theorem then follows from the computation of $\mathfrak{c}$ in Lemma 5.4. □

Remark 5.5. It may be more tractable to compute P and Q via Proposition 4.1. Thus, an alternate form of (1.17), which may be more useful in practice, is given by

$$n_+(N) \geq \left| \sum_{x \in \mathbb{R}} \dim \left(\mathbb{E}^u_+(x, 0) \cap \mathbb{S}_+(0) \right) - \sum_{x \in \mathbb{R}} \dim \left(\mathbb{E}^u_-(x, 0) \cap \mathbb{S}_-(0) \right) - \mathfrak{c} \right|. \tag{5.34}$$

We conclude with the proof of Theorem 1.5, for which we will need the following lemma. The first assertion gives a sufficient condition for monotonicity of the Maslov index along Γ_2, and is adapted from [CCLM23, Lemma 5.1]. The second assertion is given in [CCLM23, Lemma 5.2].

Lemma 5.6. *If L_- is a nonpositive operator, then each crossing $\lambda = \lambda_0 > 0$ of the path $\lambda \mapsto (\mathbb{E}^u(\ell, \lambda), \mathbb{E}^s(\ell, \lambda))$ is positive. Moreover, in this case* $\operatorname{Spec}(N) \subset \mathbb{R} \cup i\mathbb{R}$.

Proof. If $\lambda = \lambda_0$ is a crossing then the eigenvalue equations

$$- L_- v = \lambda_0 u, \qquad L_+ u = \lambda_0 v \tag{5.35}$$

are satisfied for some $\tilde{u}, \tilde{v} \in H^4(\mathbb{R})$. Notice that $\lambda_0 > 0$ necessitates that *both* $\tilde{u}$ and $\tilde{v}$ are nontrivial.

Solving the first equation in (5.35) yields $\tilde{v} = \alpha \phi + \tilde{v}_\perp$ for some $\alpha \in \mathbb{R}$, where $\ker(L_-) = \operatorname{span}\{\phi\}$ and $\tilde{v}_\perp \in \ker(L_-)^\perp$. Therefore

$$\langle L_- \tilde{v}, \tilde{v} \rangle_{L^2(\mathbb{R})} = \langle L_-(\alpha\phi + \tilde{v}_\perp), \alpha\phi + \tilde{v}_\perp \rangle_{L^2(\mathbb{R})} = \langle L_- \tilde{v}_\perp, \tilde{v}_\perp \rangle_{L^2(\mathbb{R})} < 0 \tag{5.36}$$

because L_- is nonpositive and $\hat{v}_\perp \in \ker(L_-)^\perp$. Now analysing the crossing form (5.17) for the path $\lambda \mapsto (\mathbb{E}^u(\ell, \lambda), \mathbb{E}^s(\ell, \lambda))$, where $v_2 = -\tilde{v}$ and $u_2 = \tilde{u}$, we have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell, \cdot), \mathbb{E}^s(\ell, \cdot))(q) = -\frac{2}{\lambda_0} \int_{-\infty}^{\infty} (\lambda_0 \, u_2) \, v_2 \, dx = -\frac{2}{\lambda_0} \langle L_- \tilde{v}, \tilde{v} \rangle_{L^2(\mathbb{R})} > 0,$$

which was to be proven. The second statement may be proven using similar arguments as in the proof of [CCLM23, Lemma 5.1]. Namely, we can rewrite (1.12) as the selfadjoint eigenvalue problem

$$(-L_-|_{X_c})^{1/2} \, \Pi L_+ \Pi \, (-L_-|_{X_c})^{1/2} \, w = \lambda^2 w, \tag{5.37}$$

where $X_c = \ker(L_-)^\perp$, Π is the orthogonal projection in $L^2(\mathbb{R})$ onto X_c, $(-L_-|_{X_c})^{1/2}$ is well-defined because $-L_-$ is nonnegative, and $w = (-L_-|_{X_c})^{1/2} \, \Pi v$. It follows that $\lambda^2 \in \mathbb{R}$. For more details on the equivalence of (1.12) with (5.37), see [CCLM23, Lemma 3.21]. We omit the details here. □

Proof of Theorem 1.5. If $Q = 0$ then it follows from Lemma 5.6 that

$$\mathrm{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[\varepsilon,\lambda_\infty]) = n_+(N) \tag{5.38}$$

for ε small enough. Using this and Lemma 5.1 in (5.32), we obtain

$$n_+(N) = P - Q - \mathfrak{c} = 1 - \mathfrak{c}. \tag{5.39}$$

For the evaluation of $\mathfrak{c}$, using (5.28) we can write

$$\mathcal{I}_1 = \int_{-\infty}^{\infty} \phi_x \,\widehat{v}\, dx = -\int_{-\infty}^{\infty} (L_-\widehat{v})\,\widehat{v}\, dx, \tag{5.40}$$

so that if $Q = 0$ then $\mathcal{I}_1 \geq 0$. An argument similar to (5.36) shows that in fact $\mathcal{I}_1 > 0$. Lemma 5.4 now yields the value of $\mathfrak{c}$. In particular, if $\mathcal{I}_2 > 0$ then $\mathfrak{c} = 0$ and $n_+(N) = 1$, and the standing wave $\widehat{\psi}$ is unstable. If, on the other hand, $\mathcal{I}_2 < 0$, then $\mathfrak{c} = 1$ and $n_+(N) = 0$. By the second assertion of Lemma 5.6, this means $\mathrm{Spec}(N) \subset i\mathbb{R}$, so that $\widehat{\psi}$ is spectrally stable. □

Remark 5.7. If either $\mathcal{I}_1 = 0$ or $\mathcal{I}_2 = 0$, the second order form (5.29) is degenerate. In this case one would need to determine the signature of crossing forms $\mathfrak{m}^{(k)}_{\lambda_0}(q)$ with $k \geq 3$ in order to compute $\mathfrak{c}$. If both $\mathcal{I}_1 = \mathcal{I}_2 = 0$ then (5.29) is identically zero. In this case the third order form will in fact also be identically zero. One would then need to determine the number of negative squares of the fourth-order form, provided it is nondegenerate.

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