

On J -folded alcove paths and combinatorial representations of affine Hecke algebras

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Abstract

We introduce the combinatorial model of J -folded alcove paths in an affine Weyl group and construct representations of affine Hecke algebras using this model. We study boundedness of these representations, and we state conjectures linking our combinatorial formulae to Kazhdan-Lusztig theory and Opdam’s Plancherel Theorem.

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Introduction

In this paper we introduce the combinatorial model of *J-folded alcove paths* in an affine Weyl group \widetilde{W} , and use this model to construct and analyse representations of affine Hecke algebras with arbitrary parameters. A main motivation for this work is towards extending to arbitrary dimension the methods of [16, 15] where a combinatorial proof of Lusztig’s conjectures **P1–P15** (see [22]) for 2-dimensional affine Hecke algebras with arbitrary parameters is given. While we do not discuss Lusztig’s conjectures directly in the present paper, we expect that the theory developed here, and in particular the combinatorial modules constructed, will play a role in future work on these conjectures and related combinatorial questions. In the final sections of the paper we give some indications of these future connections via a series of conjectures linking our combinatorial modules to Kazhdan-Lusztig theory, and we verify these conjectures in various cases.

More precisely, for each subset $J \subseteq I$ (with I an indexing set for the generators of the finite Weyl group) and each *J-parameter system* \mathbf{v} (see Definition 3.5) we construct a finite dimensional module $M_{J,\mathbf{v}}$ for the affine Hecke algebra $\widetilde{\mathcal{H}}$ (Theorem 4.2), and develop a formula for the matrix entries of $\pi_{J,\mathbf{v}}(T_w)$, $w \in \widetilde{W}$, with respect to a distinguished basis in terms of *J-folded alcove paths*, where $\{T_w \mid w \in \widetilde{W}\}$ is the standard basis of the affine Hecke algebra (Theorem 4.13). This construction is inspired by the work of Deodhar [5, 6, 7, 8] (see in particular [5, §2]) and Lusztig (see in particular [21, Lemma 4.7]).

Parabolic induction plays a central role in the representation theory of Hecke algebras (see [30] and the references within). In the case of affine Hecke algebras this parabolic induction takes the form of induction of finite dimensional representations of Levi subalgebras. We show that our combinatorial modules $M_{J,\mathbf{v}}$ are isomorphic to representations of the affine Hecke algebra induced from “generic” 1-dimensional representations of Levi subalgebras, and that all such induced representations arise from our combinatorial construction (Theorem 4.21).

In Section 5 we recall the notion of bounded representations from [16, 15], modelled on the work of Geck [11] in the finite dimensional case. This theory requires the parameters \mathbf{q}_i of the affine Hecke algebra to be of the form $\mathbf{q}_i = \mathbf{q}^{L(s_i)}$, where L is a *positive weight function* on the affine Weyl group. In these “weighted Hecke algebras” one has the notion of degree (in the parameter \mathbf{q}), and this is the setting of Kazhdan-Lusztig theory. Then a *bounded representation* is a finite dimensional representation π for which the degree of the matrix entries of $\pi(T_w)$ with respect to some basis is uniformly bounded for all $w \in \widetilde{W}$, and in this case the *bound* of the representation is the least upper bound \mathbf{a}_π of this degree (this statistic depends on the particular basis chosen).

In this paper we classify the subsets $J \subseteq I$ and the *J-parameter systems* \mathbf{v} such that $M_{J,\mathbf{v}}$ is bounded (Theorem 5.9), and we give a general conjectural formula for the bound $\mathbf{a}_{J,\mathbf{v}}$ (with respect to a certain distinguished basis) in terms of Macdonald’s *c-function* [23] and Opdam’s Plancherel Theorem [26] (see Conjecture 5.15 and Remark 5.21). The bound $\mathbf{a}_{J,\mathbf{v}}$ appears to be a very subtle and interesting statistic, and we make general conjectural links between $\mathbf{a}_{J,\mathbf{v}}$ and Kazhdan-Lusztig theory (see Conjecture 5.16). We prove our conjectures for various examples.

We now give an overview of the structure of the paper. In Section 1 we give background material on root systems, Weyl groups, positively folded alcove paths, and affine Hecke algebras.

In Section 2 we study the geometry of the fundamental J -alcove in preparation for Section 3 where we introduce and develop the combinatorial model of J -folded alcove paths. We also introduce the notion of a J -parameter system, which will play an important role in this work.

In Section 4 we construct our $\tilde{\mathcal{H}}$ -modules $M_{J,\mathbf{v}}$ (Theorem 4.2), and develop a combinatorial formula for the matrix entries of this representation in terms of J -folded alcove paths (Theorems 4.13 and 4.15). Moreover we show that $M_{J,\mathbf{v}}$ can be realised as a module induced from a “generic” 1-dimensional representation of the Levi subalgebra \mathcal{L}_J (Theorem 4.21). It is not completely straightforward to show that $M_{J,\mathbf{v}}$ is indeed a module for the affine Hecke algebra, and we note that another approach would be to define $M_{J,\mathbf{v}}$ from the onset as an induced module, and then prove the path formula for the matrix entries (similar to the approach taken in [16, 15]). However we note that in such an approach much of the difficulty in proving $M_{J,\mathbf{v}}$ is a module from the original approach is transferred to deducing the path formula in the second approach, and so we have decided to present this work taking the combinatorial approach, following Deodhar [5, §2] and Lusztig [21, Lemma 4.7] (see Remark 4.1 for more details).

In Section 5 we recall the notion of bounded representations from [16, 15] for weighted affine Hecke algebras, and we classify the subsets $J \subseteq I$ and J -parameter systems \mathbf{v} for which $M_{J,\mathbf{v}}$ is bounded (Theorem 5.9). We apply the theory of J -folded alcove paths to study the bound $\mathbf{a}_{J,\mathbf{v}}$ of $M_{J,\mathbf{v}}$, and state conjectural formulae for $\mathbf{a}_{J,\mathbf{v}}$ and conjectural links with Lusztig’s \mathbf{a} -function and Kazhdan-Lusztig cells (see Conjectures 5.15 and 5.16). We verify these conjectures in various cases (including 2-dimensional affine Hecke algebras with general parameters). In Section 6 we consider the $\tilde{\mathbf{A}}_n$ case with $J = \{1, 2, \dots, n-1\}$, giving both an example of the theory developed in the paper, and verifying our conjectures for this case.

Finally, we note that throughout the paper we work in the general setting of multiparameter affine Hecke algebras, with the non-reduced root systems of type \mathbf{BC}_n being employed to deal with the 3-parameter affine Hecke algebras with affine Weyl group of type $\tilde{\mathbf{C}}_n$ (and the 2-parameter $\tilde{\mathbf{A}}_1$ Hecke algebras, see Convention 1.7). At times this level of generality leads to more complicated formulae. However since this work is primarily directed towards understanding Kazhdan-Lusztig theory for Hecke algebras with unequal parameters (where the deep geometric interpretations of Kazhdan and Lusztig [18] and the associated positivity of Elias and Williamson [9] for the equal parameter case typically do not hold), and since the 3-parameter affine Hecke algebras are the most extreme cases of such algebras, we believe that this level of generality is warranted and worthwhile. As a general guide to the reader unacquainted with the non-reduced setting one may wish to assume the reduced case on first reading, in which case all symbols like $\mathbf{v}_{2\alpha}$ or $\mathbf{q}_{2\alpha}$ may be read as being 1.

1 Background

This section contains background on root systems, Weyl groups, positively folded alcove paths, affine Hecke algebras and the Bernstein-Lusztig presentation. Our main references are [3] (for root systems and Weyl groups, [27, 29] (for positively folded alcove paths), and [20, 25, 29] (for the affine Hecke algebra and Bernstein-Lusztig presentation).

1.1 Root systems

Let $I = \{1, 2, \dots, n\}$. Let Φ be an irreducible, not necessarily reduced, crystallographic root system of rank n in a real vector space V with bilinear form $\langle \cdot, \cdot \rangle$. For $\alpha \in V \setminus \{0\}$ let $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$, and let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ be the dual root system. Let $\{\alpha_i \mid i \in I\}$ be a system of simple roots, and let Φ^+ denote the corresponding set of positive roots. We will adopt Bourbaki conventions [3] when labelling the simple roots. If $\alpha = \sum_{i \in I} a_i \alpha_i \in \Phi$ let $\text{ht}(\alpha) = \sum_{i \in I} a_i$ denote the *height* of α . Let $\varphi \in \Phi$ denote the *highest root* of Φ , and define integers $m_i \geq 1$ by $\varphi = m_1 \alpha_1 + \dots + m_n \alpha_n$.

The *coroot lattice* of Φ is the \mathbb{Z} -lattice Q spanned by Φ^\vee . The *fundamental coweights* of Φ are the elements $\omega_i \in V$, $i \in I$, with $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$. The *coweight lattice* of Φ is the \mathbb{Z} -lattice P spanned by the fundamental coweights, that is $P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. The set of positive coweights is $P^+ = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n$. Note that $Q \subseteq P$.

Remark 1.1. A root system Φ is called *reduced* if $\alpha \in \Phi$ and $k\alpha \in \Phi$ implies that $k \in \{-1, 1\}$. In any irreducible reduced root system there are at most two root lengths (the *long roots* and the *short roots*, with all roots considered long if there is only one root length). For each $n \geq 1$ there is a unique non-reduced irreducible crystallographic root system Φ of rank n up to isomorphism, denoted BC_n . Explicitly we can take $V = \mathbb{R}^n$ with standard basis e_1, \dots, e_n , simple roots $\alpha_j = e_j - e_{j+1}$ for $1 \leq j \leq n-1$ and $\alpha_n = e_n$. Then

$$\Phi^+ = \{e_i - e_j, e_i + e_j, e_k, 2e_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n\}.$$

Note that there are three root lengths in Φ (if $n > 1$). The roots $2e_k$ are the *long roots*. Note that $P = Q$ for the BC_n root system, and that the highest root is $\varphi = 2e_1 = 2(\alpha_1 + \cdots + \alpha_n)$. See Figure 1.

To each root system Φ we associate reduced root systems Φ_0 and Φ_1 by

$$\Phi_0 = \{\alpha \in \Phi \mid \alpha/2 \notin \Phi\} \quad \text{and} \quad \Phi_1 = \{\alpha \in \Phi \mid 2\alpha \notin \Phi\}.$$

In particular if Φ is reduced then $\Phi_0 = \Phi_1 = \Phi$, and if Φ is of type BC_n then Φ_0 (respectively Φ_1) is a reduced root system of type B_n (respectively C_n).

1.2 Weyl groups, affine Weyl groups, and extended affine Weyl groups

Let Φ be as above. For each $\alpha \in \Phi$ let s_α be the orthogonal reflection in the hyperplane $H_\alpha = \{x \in V \mid \langle x, \alpha \rangle = 0\}$ orthogonal to α , thus $s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee$ for $x \in V$. Write $s_i = s_{\alpha_i}$ for $i \in I$. The *Weyl group* of Φ is the subgroup W_0 of $GL(V)$ generated by the reflections s_1, \dots, s_n . The *inversion set* of $w \in W_0$ is $\Phi(w) = \{\alpha \in \Phi_0^+ \mid w^{-1}\alpha \in -\Phi_0^+\}$ (note the convention that $\Phi(w) \subseteq \Phi_0$).

For each $\alpha \in \Phi$ and $k \in \mathbb{Z}$ let $H_{\alpha,k} = \{x \in V \mid \langle x, \alpha \rangle = k\}$, and let $s_{\alpha,k}$ be the orthogonal reflection in the affine hyperplane $H_{\alpha,k}$. Explicitly, $s_{\alpha,k}(x) = x - (\langle x, \alpha \rangle - k)\alpha^\vee$. The *affine Weyl group* is $W = \langle \{s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\} \rangle$ (a subgroup of $\text{Aff}(V)$). We have $W = Q \rtimes W_0$, where we identify $\lambda \in V$ with the translation $t_\lambda(x) = x + \lambda$. The *extended affine Weyl group* is $\widetilde{W} = P \rtimes W_0$. Since $Q \subseteq P$ we have $W \leq \widetilde{W}$. If $w \in \widetilde{W}$ we define the *linear part* $\theta(w) \in W_0$ and the *translation coweight* $\text{wt}(w) \in P$ by the equation

$$w = t_{\text{wt}(w)}\theta(w). \tag{1.1}$$

The Weyl group W_0 is a Coxeter group with Coxeter generators $\{s_i \mid i \in I\}$. The affine Weyl group W is a Coxeter group with Coxeter generators $S = \{s_i \mid i \in \{0\} \cup I\}$, where $s_0 = s_{\varphi,1} = t_{\varphi^\vee}s_\varphi$. If $\Phi = \text{BC}_n$ then $W = \widetilde{W}$ is a Coxeter group of type $\widetilde{\text{C}}_n$.

The extended affine Weyl group is typically not a Coxeter group. Writing $\Sigma = P/Q$ we have $\widetilde{W} \cong W \rtimes \Sigma$. Each $\sigma \in \Sigma$ induces a permutation (also denoted by σ) of $\{0\} \cup I$ by $\sigma s_i \sigma^{-1} = s_{\sigma(i)}$. In this way we can identify Σ with a subgroup of the group of automorphisms of the extended Dynkin diagram. For example, in type A_n the group Σ is generated by the permutation $i \mapsto i+1 \pmod{n+1}$ of $\{0\} \cup I$.

Let $\ell : W \rightarrow \mathbb{N}$ denote the length function on the Coxeter group W . We extend this length function to the extended affine Weyl group \widetilde{W} by setting $\ell(w\sigma) = \ell(w)$ for all $w \in W$ and $\sigma \in \Sigma$. Thus $\Sigma = \{w \in \widetilde{W} \mid \ell(w) = 0\}$. By a *reduced expression* for $w \in \widetilde{W}$ we shall mean a decomposition $w = s_{i_1} \cdots s_{i_\ell} \sigma$ with $\ell = \ell(w)$ and $\sigma \in \Sigma$.

The closures of the open connected components of $V \setminus \left(\bigcup_{\alpha, k} H_{\alpha, k} \right)$ are called *alcoves*. Let \mathbb{A} denote the set of all alcoves. The *fundamental alcove* is given by

$$A_0 = \{x \in V \mid 0 \leq \langle x, \alpha \rangle \leq 1 \text{ for all } \alpha \in \Phi^+\}.$$

The hyperplanes bounding A_0 are called the *walls* of A_0 . Explicitly these walls are $H_{\alpha_i, 0}$ with $i \in I$ and $H_{\varphi, 1}$. We say that a *panel* of A_0 (that is, a codimension 1 facet) has *type* i for $i \in I$ if it lies on the wall $H_{\alpha_i, 0}$, and type 0 if it lies on the wall $H_{\varphi, 1}$.

Note that if Φ is not reduced, and if $\alpha \in \Phi^+$ with $2\alpha \in \Phi$, then $H_{\alpha, 2k} = H_{2\alpha, k}$. Thus there are two distinct positive roots associated to this hyperplane. However note that every hyperplane H can be expressed uniquely as $H = H_{\alpha, k}$ for some $\alpha \in \Phi_1^+$ and $k \in \mathbb{Z}$.

The (non-extended) affine Weyl group W acts simply transitively on \mathbb{A} . We use the action of W to transfer the notions of walls, panels, and types of panels to arbitrary alcoves. Alcoves A and A' are called *i-adjacent* (written $A \sim_i A'$) if $A \neq A'$ and A and A' share a common type i panel (with $i \in \{0\} \cup I$). Thus the alcoves wA_0 and ws_iA_0 are *i-adjacent* for all $w \in W$ and $i \in \{0\} \cup I$.

The extended affine Weyl group \widetilde{W} acts transitively on \mathbb{A} , and the stabiliser of A_0 is Σ . The vertex set of A_0 is $\{x_i \mid i \in \{0\} \cup I\}$ where $x_0 = 0$ and $x_i = \omega_i/m_i$ for $i \in I$ (with m_i as in Section 1.1), and the action of $\sigma \in \Sigma$ on this set of vertices is given by $\sigma(x_i) = x_{\sigma(i)}$.

Each affine hyperplane $H_{\alpha, k}$ with $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$ divides V into two half-spaces, denoted

$$H_{\alpha, k}^+ = \{x \in V \mid \langle x, \alpha \rangle \geq k\} \quad \text{and} \quad H_{\alpha, k}^- = \{x \in V \mid \langle x, \alpha \rangle \leq k\}.$$

This ‘‘orientation’’ of the hyperplanes is called the *periodic orientation* (see Figure 1 for an illustration in the non-reduced BC_2 case).

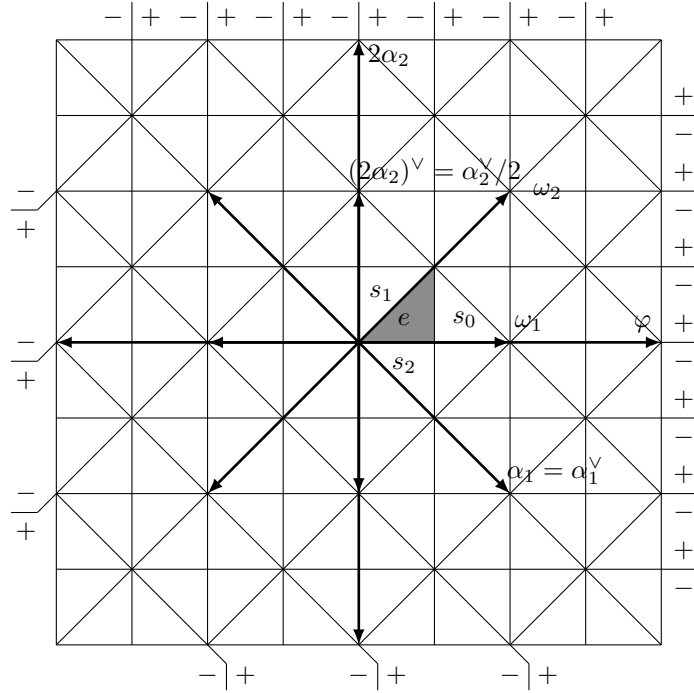


Figure 1: Root system of type BC_2

1.3 Affine root system

It is convenient to have the notion of the *affine root system*

$$\widetilde{\Phi} = \{\alpha + k\delta \mid \alpha \in \Phi, k \in \mathbb{Z}\}$$

in the space $V \oplus \mathbb{R}\delta$. Identifying V with its dual, one may regard δ as the (nonlinear) constant function $\delta : V \rightarrow \mathbb{R}$ with $\delta(v) = 1$ for all $v \in V$. Writing $\langle \lambda, \alpha + k\delta \rangle = \langle \lambda, \alpha \rangle + k$ (however note this is no longer bilinear) we have $H_{\alpha,k} = \{x \in V \mid \langle x, \alpha - k\delta \rangle = 0\}$, and so the hyperplane $H_{\alpha,k}$ corresponds to the affine roots $\pm(\alpha - k\delta)$.

The simple roots of the affine root system are $\alpha_0 = -\varphi + \delta$ and $\alpha_i + 0\delta$ with $i \in I$. These choices give the set of positive affine roots as

$$\tilde{\Phi}^+ = (\Phi^+ + \mathbb{Z}_{\geq 0}\delta) \cup (-\Phi^+ + \mathbb{Z}_{> 0}\delta).$$

Let $\tilde{\Phi}_0 = \Phi_0 + \mathbb{Z}\delta$ (and so $\tilde{\Phi}_0 = \tilde{\Phi}$ if Φ is reduced), and let $\tilde{\Phi}_0^+ = \tilde{\Phi}^+ \cap \tilde{\Phi}_0$.

The action of \tilde{W} on the affine root system (given by the action on half spaces) is given by

$$w(\alpha + k\delta) = w\alpha + k\delta \quad \text{and} \quad t_\lambda(\alpha + k\delta) = \alpha + (k - \langle \lambda, \alpha \rangle)\delta \quad \text{for } w \in W_0 \text{ and } \lambda \in P.$$

The (affine) inversion set of $w \in \tilde{W}$ is

$$\tilde{\Phi}(w) = \{\alpha + k\delta \in \tilde{\Phi}_0^+ \mid w^{-1}(\alpha + k\delta) \in -\tilde{\Phi}_0^+\}.$$

Moreover, for $w \in \tilde{W}$ and $i \in \{0\} \cup I$ we have

$$wA_0^{-|+} ws_i A_0 \quad \text{if and only if} \quad w\alpha_i \in -\Phi^+ + \mathbb{Z}\delta. \quad (1.2)$$

Here the notation $wA_0^{-|+} ws_i A_0$ indicates that $ws_i A_0$ is on the positive side of the hyperplane separating wA_0 and $ws_i A_0$. We will also say that $w \rightarrow ws_i$ is a *positive crossing* (or simply is *positive*), and similarly if $wA_0^{+|-} ws_i A_0$ we say that $w \rightarrow ws_i$ is negative (see also Section 1.5).

1.4 Parabolic subgroups

Let $J \subseteq I$. The *J-parabolic subgroup* of W_0 is the subgroup $W_J = \langle \{s_j \mid j \in J\} \rangle$. Since W_J is finite there exists a unique longest element of W_J , denoted w_J , and we have $\ell(w_J w) = \ell(w w_J) = \ell(w_J) - \ell(w)$ for all $w \in W_J$. We write $w_0 = w_J$. It is well known (see, for example [1, Proposition 2.20]) that each coset $W_J w$ with $w \in W_0$ contains a unique representative of minimal length. Let W^J be the transversal of these minimal length coset representatives. Then each $w \in W_0$ has a unique decomposition

$$w = yu \quad \text{with } y \in W_J, u \in W^J, \quad (1.3)$$

and moreover whenever $y \in W_J$ and $u \in W^J$ we have $\ell(yu) = \ell(y) + \ell(u)$.

The *support* of a root $\alpha \in \Phi$ is $\text{supp}(\alpha) = \{i \in I \mid c_i \neq 0\}$, where $\alpha = \sum_{i \in I} c_i \alpha_i$. For $J \subseteq I$ let $\Phi_J = \{\alpha \in \Phi \mid \text{supp}(\alpha) \subseteq J\}$, and for $w \in W_0$ write $\Phi_J(w) = \Phi(w) \cap \Phi_J$. The following lemma is well known (see, for example [17, Corollary 2.13]), however we provide a proof for completeness.

Lemma 1.2. *Let $J \subseteq I$. If $w = yu$ with $y \in W_J$ and $u \in W^J$ then $\Phi(y) = \Phi(w) \cap \Phi_J$. In particular, we have $W^J = \{u \in W \mid \Phi_J(u) = \emptyset\}$.*

Proof. Suppose there exists $\beta \in \Phi_J(w) \setminus \Phi(y)$. Since $\beta \notin \Phi(y)$ we have $\ell(s_\beta y) > \ell(y)$, and since $\beta \in \Phi_J(w)$ we have $\ell(s_\beta w) < \ell(w)$. Since $\beta \in \Phi_J$ we have $s_\beta \in W_J$, but then the element $y' = s_\beta y \in W_J$ satisfies $\ell(y'u) = \ell(s_\beta w) < \ell(w) = \ell(y) + \ell(u) < \ell(s_\beta y) + \ell(u) = \ell(y') + \ell(u)$, contradicting the fact that $u \in W^J$. \square

Definition 1.3. Let $w \in \tilde{W}$ and recall the definition of $\theta(w)$ from (1.1). Define $\theta_J(w) \in W_J$ and $\theta^J(w) \in W^J$ by the equation $\theta(w) = \theta_J(w)\theta^J(w)$.

For $J \subseteq I$ let

$$V_J = \sum_{j \in J} \mathbb{R}\alpha_j^\vee \quad \text{and} \quad V^J = \sum_{i \in I \setminus J} \mathbb{R}\omega_i.$$

Then $V = V_J \oplus V^J$ (orthogonal direct sum).

Let $\lambda \mapsto \lambda^J$ denote the orthogonal projection $V \rightarrow V^J$, and let

$$P^J = \{\lambda^J \mid \lambda \in P\} \subseteq V^J.$$

Note that in general the \mathbb{Z} -lattice P^J is not a subset of P . For example if Φ is of type A_2 and $J = \{1\}$ then $P^J \ni \omega_1^J = \frac{1}{2}\omega_2 \notin P$ (see Figure 2, where V^J is denoted as a solid line, and P^J is indicated by heavy dots). Let $\{\bar{\omega}_i \mid i \in I \setminus J\}$ be a choice of \mathbb{Z} -basis of P^J (in the example of Figure 2 we may take $\bar{\omega}_2 = \omega_2/2$).

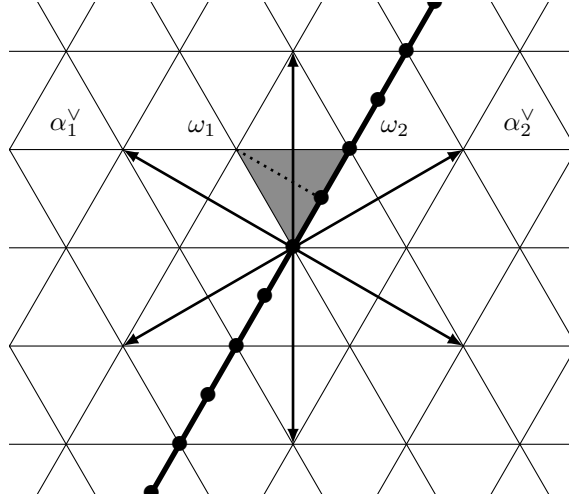


Figure 2: The set P^J for type A_2 with $J = \{1\}$

For $J \subseteq I$ let $\Phi_{J,l}$ and $\Phi_{J,s}$ be the long and short roots of $\Phi_J \cap \Phi_0$, respectively (with $\Phi_{J,s} = \emptyset$ if Φ is simply laced). Define

$$\rho_J = \frac{1}{2} \sum_{\alpha \in \Phi_{J,l}^+} \alpha \quad \text{and} \quad \rho'_J = \frac{1}{2} \sum_{\alpha \in \Phi_{J,s}^+} \alpha,$$

and let $\{\tilde{\omega}_j \mid j \in J\}$ be the basis of V_J dual to the basis $\{\alpha_j^\vee \mid j \in J\}$ (that is, $\langle \alpha_j^\vee, \tilde{\omega}_i \rangle = \delta_{i,j}$ for all $i, j \in J$).

Lemma 1.4. *We have*

$$\rho_J = \sum_{\{j \in J \mid \alpha_j \in \Phi_{J,l}\}} \tilde{\omega}_j \quad \text{and} \quad \rho'_J = \sum_{\{j \in J \mid \alpha_j \in \Phi_{J,s}\}} \tilde{\omega}_j.$$

In particular ρ_J (respectively ρ'_J) is orthogonal to all short (respectively long) simple roots of Φ_J .

Proof. In the simply laced case see [3, VI, §1, Proposition 29]. In the non-simply laced case the claim is readily checked from the classification (it is sufficient to check for irreducible J). \square

1.5 Positively folded alcove paths

Since the extended affine Weyl group \widetilde{W} does not act freely on the set of alcoves, it is convenient for our purposes to consider “alcove paths” as sequences of elements of \widetilde{W} rather than sequences of alcoves. Thus we make the following definition (see [29]).

Definition 1.5. Let $\vec{w} = s_{i_1} s_{i_2} \cdots s_{i_\ell} \sigma$ be an expression for $w \in \widetilde{W}$ (not necessarily reduced) with $\sigma \in \Sigma$, and let $v \in \widetilde{W}$. A *positively folded alcove path of type \vec{w} starting at v* is a sequence $p = (v_0, v_1, \dots, v_\ell, v_\ell \sigma)$ with $v_0, \dots, v_\ell \in \widetilde{W}$ such that, for $1 \leq k \leq \ell$,

- (1) $v_0 = v$ and $v_k \in \{v_{k-1}, v_{k-1} s_{i_k}\}$, and
- (2) if $v_{k-1} = v_k$ then the alcove $v_{k-1} A_0$ is on the positive side of the hyperplane separating the alcoves $v_{k-1} A_0$ and $v_{k-1} s_{i_k} A_0$.

The *end* of p is $\text{end}(p) = v_\ell \sigma$.

For $v \in \widetilde{W}$ let

$$\mathcal{P}(\vec{w}, v) = \{\text{all positively folded alcove paths of type } \vec{w} \text{ starting at } v\}.$$

Let $\vec{w} = s_{i_1} s_{i_2} \cdots s_{i_\ell} \sigma$ and let $p = (v_0, \dots, v_\ell, v_\ell \sigma) \in \mathcal{P}(\vec{w}, v)$. The index $k \in \{1, 2, \dots, \ell\}$ is called:

- (1) a *positive (respectively, negative) i_k -crossing* if $v_k = v_{k-1} s_{i_k}$ and $v_k A_0$ is on the positive (respectively, negative) side of the hyperplane separating the alcoves $v_{k-1} A_0$ and $v_k A_0$;
- (2) an *i_k -fold* if $v_k = v_{k-1}$ (in which case $v_{k-1} A_0$ is necessarily on the positive side of the hyperplane separating $v_{k-1} A_0$ and $v_{k-1} s_{i_k} A_0$).

If p has no folds we say that p is *straight*. Less formally, the above crossings and folds can be visualised as follows (where $x = v_{k-1}$):

$$\begin{array}{ccc} \begin{array}{c} - \\ | \\ x \longrightarrow x s_{i_k} \\ | \\ + \end{array} & \begin{array}{c} - \\ | \\ x s_{i_k} \longrightarrow x \\ | \\ + \end{array} & \begin{array}{c} - \\ | \\ x s_{i_k} \longleftarrow x \\ | \\ + \end{array} \\ \text{(positive } i_k\text{-crossing)} & \text{(} i_k\text{-fold)} & \text{(negative } i_k\text{-crossing)} \end{array}$$

If p is a positively folded alcove path, then for each $i \in \{0\} \cup I$ we define

$$f_i(p) = \#(i\text{-folds in } p) \quad \text{and} \quad f(p) = \#(\text{folds in } p) = \sum_{i=0}^n f_i(p).$$

The *coweight* and *final direction* of a positively folded alcove path p are (c.f. (1.1))

$$\text{wt}(p) = \text{wt}(\text{end}(p)) \quad \text{and} \quad \theta(p) = \theta(\text{end}(p)), \quad (1.4)$$

and we write $\theta_J(p) = \theta_J(\text{end}(p))$ and $\theta^J(p) = \theta^J(\text{end}(p))$.

1.6 Affine Hecke algebras and the Bernstein-Lusztig presentation

Let $(\mathbf{q}_i)_{i \in \{0\} \cup I}$ be a family of commuting invertible indeterminates with the property that $\mathbf{q}_i = \mathbf{q}_j$ whenever s_i and s_j are conjugate in \widetilde{W} . Let $\mathbf{R} = \mathbb{Z}[(\mathbf{q}_i^{\pm 1})_{i \in \{0\} \cup I}]$. The *extended affine Hecke algebra* is the \mathbf{R} -algebra $\widetilde{\mathcal{H}}$ with basis $\{T_w \mid w \in \widetilde{W}\}$ and multiplication given by (for $w, v \in \widetilde{W}$ and $i \in \{0\} \cup I$)

$$\begin{aligned} T_w T_v &= T_{wv} && \text{if } \ell(wv) = \ell(w) + \ell(v) \\ T_w T_{s_i} &= T_{ws_i} + (\mathbf{q}_i - \mathbf{q}_i^{-1}) T_w && \text{if } \ell(ws_i) = \ell(w) - 1. \end{aligned} \quad (1.5)$$

Note that each T_{s_i} is invertible with $T_{s_i}^{-1} = T_{s_i} - (\mathbf{q}_i - \mathbf{q}_i^{-1})$, and that each T_σ (with $\sigma \in \Sigma$) is invertible with $T_\sigma^{-1} = T_{\sigma^{-1}}$. It follows that each T_w with $w \in \widetilde{W}$ is invertible. The (non-extended) affine Hecke algebra is the subalgebra \mathcal{H} spanned by $\{T_w \mid w \in W\}$.

We often write T_i in place of T_{s_i} . For $w \in \widetilde{W}$ we write $\mathbf{q}_w = \mathbf{q}_{i_1} \cdots \mathbf{q}_{i_k}$ whenever $w = s_{i_1} \cdots s_{i_k} \sigma$ is a reduced expression of w (this can easily be seen to be independent of the choice

of reduced expression using Tits' solution to the Word Problem). In particular, note that $\mathbf{q}_\sigma = 1$ for all $\sigma \in \Sigma$.

Let $w \in \widetilde{W}$ and choose any expression $w = s_{i_1} \cdots s_{i_\ell} \sigma$ (not necessarily reduced). Let $v_0 = e$ and $v_k = s_{i_1} \cdots s_{i_k}$ for $1 \leq k \leq \ell$ (thus $(v_0, v_1, \dots, v_\ell)$ is the straight alcove path of type $s_{i_1} \cdots s_{i_\ell}$ starting at e). Let $A_k = v_k A_0$, and let $\epsilon_1, \dots, \epsilon_\ell$ be the sequence of signs of the crossings, defined by (see (1.2)):

$$\epsilon_k = \begin{cases} +1 & \text{if } A_{k-1}^- |^+ A_k \text{ (that is, } v_{k-1} \alpha_{i_k} \in -\Phi^+ + \mathbb{Z}\delta) \\ -1 & \text{if } A_{k-1}^+ |^- A_k \text{ (that is, } v_{k-1} \alpha_{i_k} \in \Phi^+ + \mathbb{Z}\delta). \end{cases}$$

Then the element

$$X_w = T_{i_1}^{\epsilon_1} \cdots T_{i_k}^{\epsilon_k} T_\sigma \in \widetilde{\mathcal{H}}$$

does not depend on the particular expression $w = s_{i_1} \cdots s_{i_k} \sigma$ chosen (see [13]).

From the defining relations (1.5) it follows that $X_w - T_w$ is a linear combination of terms T_v with $v < w$ (in extended Bruhat order), and hence $\{X_w \mid w \in \widetilde{W}\}$ is a basis of $\widetilde{\mathcal{H}}$.

If $\lambda \in P$ we write

$$X^\lambda = X_{t_\lambda}.$$

Then $X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda$ for all $\lambda, \mu \in P$, and for $w \in \widetilde{W}$ we have

$$X_w = X_{t_\lambda u} = X^\lambda X_u = X^\lambda T_{u^{-1}}^{-1} \quad (1.6)$$

where $\lambda = \text{wt}(w)$ and $u = \theta(w)$ (the second equality follows since t_λ is on the positive side of every hyperplane through λ , and the third equality follows since $X_u = T_{u^{-1}}^{-1}$ for all $u \in W_0$).

Thus the set $\{X^\lambda T_{u^{-1}}^{-1} \mid \lambda \in P, u \in W_0\}$ is a basis of $\widetilde{\mathcal{H}}$ (called the *Bernstein-Lusztig basis*).

Since $s_0 = t_{\varphi^\vee} s_\varphi$ equation (1.6) gives

$$T_0 = X^{\varphi^\vee} T_{s_\varphi}^{-1}. \quad (1.7)$$

The combinatorics of positively folded alcove paths encode the change of basis from the standard basis $(T_w)_{w \in \widetilde{W}}$ of $\widetilde{\mathcal{H}}$ to the Bernstein-Lusztig basis $(X_w)_{w \in \widetilde{W}}$. This is seen by taking $u = e$ in the following proposition.

Proposition 1.6. [29, Theorem 3.3] *Let $u, w \in \widetilde{W}$ and let \vec{w} be any reduced expression for w . Then*

$$X_u T_w = \sum_{p \in \mathcal{P}(\vec{w}, u)} \mathcal{Q}(p) X_{\text{end}(p)} \quad \text{where} \quad \mathcal{Q}(p) = \prod_{i \in \{0\} \cup I} (\mathbf{q}_i - \mathbf{q}_i^{-1})^{f_i(p)}.$$

Convention 1.7. It is convenient to make the following convention, and we will do so henceforth: If $\Phi = A_1$ then $\mathbf{q}_0 = \mathbf{q}_1$, and if $\Phi = C_n$ ($n \geq 2$) then $\mathbf{q}_0 = \mathbf{q}_n$ (note that there is no loss of generality as the case $\mathbf{q}_0 \neq \mathbf{q}_n$ is covered by the non-reduced system BC_n for $n \geq 1$). Thus if $\sigma \in \Sigma$ we have $\mathbf{q}_{\sigma(i)} = \mathbf{q}_i$ for all $i \in \{0\} \cup I$ (recall that Σ is trivial if $\Phi = \text{BC}_n$).

With Convention 1.7 in force, a complete set of relations of $\widetilde{\mathcal{H}}$ in the Bernstein-Lusztig basis is given by (for $i, j \in I$ with $i \neq j$ and $\lambda, \mu \in P$):

$$\begin{aligned} T_i^2 &= I + (\mathbf{q}_i - \mathbf{q}_i^{-1}) T_i \\ T_i T_j T_i \cdots &= T_j T_i T_j \cdots \quad (m_{ij} \text{ terms on each side}) \\ X^\lambda X^\mu &= X^{\lambda+\mu} \\ T_i X^\lambda &= \begin{cases} X^{s_i \lambda} T_i + (\mathbf{q}_i - \mathbf{q}_i^{-1}) \frac{X^\lambda - X^{s_i \lambda}}{1 - X^{-\alpha_i^\vee}} & \text{if } (\Phi, i) \neq (\text{BC}_n, n) \\ X^{s_n \lambda} T_n + [\mathbf{q}_n - \mathbf{q}_n^{-1} + (\mathbf{q}_0 - \mathbf{q}_0^{-1}) X^{-\alpha_n^\vee/2}] \frac{X^\lambda - X^{s_n \lambda}}{1 - X^{-\alpha_n^\vee}} & \text{if } (\Phi, i) = (\text{BC}_n, n). \end{cases} \end{aligned}$$

The final relation is known as the Bernstein-Lusztig relation (see [20, Proposition 3.6]). Note that since $s_i \lambda = \lambda - \langle \lambda, \alpha_i \rangle \alpha_i^\vee$ and $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$ the right hand side of the Bernstein-Lusztig relation is in $\widetilde{\mathcal{H}}$.

2 The fundamental J -alcove \mathcal{A}_J

In this section we extend the definition of “strips” in [16, 15] to arbitrary affine type, and study the geometry of these subsets of V (following Lusztig [21, §2] we call these generalisations J -alcoves for reasons that will be explained below). It turns out that the geometry of the fundamental J -alcove \mathcal{A}_J will be crucial to our combinatorial formulae, and so we develop some of the basic properties of \mathcal{A}_J and its symmetries here.

2.1 J -alcoves and J -affine Weyl groups

Definition 2.1. Let $J \subseteq I$. The *fundamental J -alcove* is the set

$$\mathcal{A}_J = \{x \in V \mid 0 \leq \langle x, \alpha \rangle \leq 1 \text{ for all } \alpha \in \Phi_J^+\}.$$

Let us explain this terminology. To begin with, in the case $J = I$ we have $\mathcal{A}_I = A_0$, the fundamental alcove. For general $J \subseteq I$ let \mathbb{H}_J be the set of all hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi_J^+$ and $k \in \mathbb{Z}$. Let \mathbb{A}_J denote the set of “alcoves” of this hyperplane arrangement: by definition these are the closures of the open connected components of $V \setminus \left(\bigcup_{H \in \mathbb{H}_J} H\right)$. We shall call elements of \mathbb{A}_J J -alcoves, and it is clear that \mathcal{A}_J is indeed a J -alcove.

The J -affine Weyl group is

$$W_J^{\text{aff}} = \langle s_{\alpha,k} \mid \alpha \in \Phi_J^+, k \in \mathbb{Z} \rangle.$$

By the general theory of Section 1.2 (c.f. [3, §V.3]) the group W_J^{aff} acts simply transitively on the set \mathbb{A}_J , and \mathcal{A}_J is a fundamental domain for the action of W_J^{aff} on V , see [3, §V.3].

Let $\mathcal{K}(J)$ denote the set of connected components of J (in the Coxeter graph of W_0). For example, in type \tilde{A}_6 if $J = \{1, 3, 4, 6\}$ then $\mathcal{K}(J) = \{\{1\}, \{3, 4\}, \{6\}\}$. For connected subsets $K \subseteq I$ let φ_K be the highest root of Φ_K . Note that

$$\Phi_J = \bigsqcup_{K \in \mathcal{K}(J)} \Phi_K. \quad (2.1)$$

Note that if Φ_J is not reduced, then Φ is of type BC_n , and J contains a connected component $K = \{k+1, k+2, \dots, n\}$ (of type BC_{n-k}).

Lemma 2.2. Let $J \subseteq I$. The walls of the fundamental J -alcove \mathcal{A}_J are $H_{\alpha_j,0}$ and $H_{\varphi_K,1}$ with $j \in J$ and $K \in \mathcal{K}(J)$. That is,

$$\mathcal{A}_J = \{x \in V \mid \langle x, \alpha_j \rangle \geq 0 \text{ for all } j \in J \text{ and } \langle x, \varphi_K \rangle \leq 1 \text{ for all } K \in \mathcal{K}(J)\}.$$

Proof. Write \mathcal{A}'_J for the right hand side of the displayed equation in the statement of the lemma. Clearly $\mathcal{A}_J \subseteq \mathcal{A}'_J$, and so it suffices to show that $\mathcal{A}'_J \subseteq \mathcal{A}_J$. Thus suppose that $x \in \mathcal{A}'_J$. Let $\alpha \in \Phi_J$. Writing $\alpha = \sum_{j \in J} a_j \alpha_j$ we have $\langle x, \alpha \rangle = \sum_{j \in J} a_j \langle x, \alpha_j \rangle \geq 0$. Since $\alpha \in \Phi_J$ we have $\alpha \in \Phi_K$ for some $K \in \mathcal{K}(J)$, by (2.1). Since $\langle x, \alpha_j \rangle \geq 0$ for all $j \in J$ and $\varphi_K - \alpha$ is a nonnegative linear combination of roots α_k with $k \in K \subseteq J$ we have $\langle x, \varphi_K - \alpha \rangle \geq 0$, and so $\langle x, \alpha \rangle \leq \langle x, \varphi_K \rangle \leq 1$, hence $x \in \mathcal{A}_J$. \square

The reflections in the walls of \mathcal{A}_J generate the J -affine Coxeter group, and we have

$$W_J^{\text{aff}} = \prod_{K \in \mathcal{K}(J)} W_K^{\text{aff}}. \quad (2.2)$$

Let $s'_j = s_j$ for $j \in J$, and let $s'_{0_K} = s_{\varphi_K,1}$ for $K \in \mathcal{K}(J)$. Then $\{s'_{0_K}\} \cup \{s'_k \mid k \in K\}$ are the Coxeter generators of W_K^{aff} . Write $J^{\text{aff}} = \{0_K \mid K \in \mathcal{K}(J)\} \cup J$, and so $\{s'_j \mid j \in J^{\text{aff}}\}$ is the set of Coxeter generators of W_J^{aff} .

2.2 The set $P^{(J)} = \mathcal{A}_J \cap P$

In this section we determine the set $P^{(J)} = \mathcal{A}_J \cap P$ of coweights contained in the fundamental J -alcove. Let

$$Q_J = \sum_{\alpha \in \Phi_J} \mathbb{Z}\alpha^\vee.$$

Note that if Φ_J is not reduced (hence Φ is of type BC_n and $n \in J$) then $\alpha_n^\vee/2 = (2\alpha_n)^\vee \in Q_J$.

Lemma 2.3. *Let $J \subseteq I$ and let $\lambda \in P$. There exists a unique $\lambda^* \in (\lambda + Q_J) \cap \mathcal{A}_J$.*

Proof. Let $\lambda \in P$. Then $\lambda \in \mathcal{A}$ for some J -alcove $\mathcal{A} \in \mathbb{A}_J$. Since W_J^{aff} acts transitively on \mathbb{A}_J , and since $\mathcal{A}_J \in \mathbb{A}_J$, there is $w \in W_J^{\text{aff}}$ such that $w\mathcal{A} = \mathcal{A}_J$ (and in particular, $w\lambda \in \mathcal{A}_J$). Since w is a product of reflections in hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi_J^+$ and $k \in \mathbb{Z}$, and since $s_{\alpha,k}(\lambda) = \lambda - (\langle \lambda, \alpha \rangle - k)\alpha^\vee$ it follows that $w\lambda \in \lambda + Q_J$, proving the existence of λ^* . Uniqueness follows from the fact that \mathcal{A}_J is a fundamental domain for the action of W_J^{aff} on V . \square

Definition 2.4. Let $\lambda^{(J)}$ denote the unique element λ^* of $(\lambda + Q_J) \cap \mathcal{A}_J$ (c.f. Lemma 2.3). Note that $\lambda^{(J)}$ and λ^J are, in general, distinct. See Example 3.3.

The following lemma gives a useful characterisation of $P^{(J)}$. For $\lambda \in P$ let

$$J_\lambda = \{j \in J \mid \langle \lambda, \alpha_j \rangle \neq 0\}. \quad (2.3)$$

Lemma 2.5. *Let $J \subseteq I$ and $\lambda \in P$. Then $\lambda \in \mathcal{A}_J$ if and only if the set J_λ has the following properties:*

- (1) if $j \in J_\lambda$ then $\langle \lambda, \alpha_j \rangle = 1$;
- (2) for each $K \in \mathcal{K}(J)$ we have $|J_\lambda \cap K| \leq 1$;
- (3) if $j \in J_\lambda$ then $\langle \omega_j, \varphi_K \rangle = 1$ where $K \in \mathcal{K}(J)$ is the connected component of J containing j .

Proof. Write $\lambda = \sum_{i \in I} c_i \omega_i$. If $\lambda \in P \cap \mathcal{A}_J$ then $c_j \in \{0, 1\}$ for all $j \in J$. Moreover, for each $K \in \mathcal{K}(J)$ there is at most one $j \in K$ with $c_j = 1$, for otherwise $\langle \lambda, \varphi_K \rangle \geq 2$. Moreover, if $j \in K$ with $c_j = 1$ then for $\alpha \in \Phi_K^+$ we have $\langle \lambda, \alpha \rangle = \langle \omega_j, \alpha \rangle$. In particular, taking $\alpha = \varphi_K \in \Phi_J$ we have $\langle \omega_j, \varphi_K \rangle = 1$. The converse is clear, using $\langle \omega_j, \alpha \rangle \leq \langle \omega_j, \varphi_K \rangle$ for all $\alpha \in \Phi_K^+$. \square

Lemma 2.5 shows that $P^{(J)}$ consists precisely of the elements

$$\lambda = \sum_{i \in I \setminus J} a_i \omega_i + \sum_{j \in J'} \omega_j \quad (2.4)$$

with $a_i \in \mathbb{Z}$ and where $J' \subseteq J$ is a set satisfying $|J' \cap K| \leq 1$ for all $K \in \mathcal{K}(J)$ and if $j \in J' \cap K$ then the coefficient of α_j in φ_K is 1 (then $J' = J_\lambda$).

2.3 The set \mathbb{W}^J and the elements y_λ and τ_λ

Define a subset $\mathbb{W}^J \subseteq \widetilde{W}$ by

$$\mathbb{W}^J = \{w \in \widetilde{W} \mid wA_0 \subseteq \mathcal{A}_J\}.$$

Thus $\mathbb{W}^J \cap W$ is in bijection with the (classical) alcoves contained in the fundamental J -alcove.

We note, in passing, that \mathbb{W}^J has the following characterisations (in particular, the third characterisation below shows that \mathbb{W}^J may be regarded as an affine analogue of W^J , see Lemma 1.2).

Lemma 2.6. *We have*

$$\begin{aligned} \mathbb{W}^J &= \{w \in \widetilde{W} \mid \ell(s_{\beta,k}w) > \ell(w) \text{ for all } \beta + k\delta \in \Phi_J + \mathbb{Z}\delta\} \\ &= \{w \in \widetilde{W} \mid \ell(s'_j w) > \ell(w) \text{ for all } j \in J^{\text{aff}}\} \\ &= \{w \in \widetilde{W} \mid \widetilde{\Phi}(w) \cap (\Phi_J + \mathbb{Z}\delta) = \emptyset\}. \end{aligned}$$

Proof. We have $w \in \mathbb{W}^J$ if and only if the alcove wA_0 lies on the same side of $H_{\beta,k}$ as A_0 for all $(\beta, k) \in \Phi_J \times \mathbb{Z}$, if and only if $\ell(s_{\beta,k}w) > \ell(w)$. Hence the first equality. For the second equality, we similarly have $w \in \mathbb{W}^J$ if and only if the alcove wA_0 lies on the same side of $H_{\alpha_j,0}$ and $H_{\varphi_K,1}$ as A_0 for all $j \in J$ and $K \in \mathcal{K}(J)$ (see Lemma 2.2). The third characterisation follows from the first characterisation and the fact that if $\beta - k\delta \in \tilde{\Phi}^+$ then $\ell(s_{\beta,k}w) < \ell(w)$ if and only if $\beta + k\delta \in \tilde{\Phi}(w)$. \square

Definition 2.7. For $\lambda \in P^{(J)}$ let

$$y_\lambda = w_{J \setminus J_\lambda} w_J \quad \text{and} \quad \tau_\lambda = t_\lambda y_\lambda,$$

where J_λ is as in (2.3).

Note that both y_λ and τ_λ depend on the subset J , however this dependence is suppressed in the notation.

Lemma 2.8. For $\lambda \in P^{(J)}$ we have $\Phi(y_\lambda) = \Phi_J^+ \setminus \Phi_{J \setminus J_\lambda}^+ = \{\alpha \in \Phi_J^+ \mid \langle \lambda, \alpha \rangle = 1\}$.

Proof. It is clear from the definition that $\Phi(y_\lambda) = \Phi_J^+ \setminus \Phi_{J \setminus J_\lambda}^+$. A root $\alpha \in \Phi_J^+$ does not lie in $\Phi_{J \setminus J_\lambda}^+$ if and only if there exists $j \in J_\lambda$ such that the coefficient of α_j in α is strictly positive. By Lemma 2.5 this occurs if and only if the coefficient of α_j in α is 1, and hence the result. \square

The following theorem gives an explicit decomposition of \mathbb{W}^J .

Theorem 2.9. We have $\mathbb{W}^J = \{\tau_\lambda u \mid \lambda \in P^{(J)} \text{ and } u \in W^J\}$.

Proof. We first show that $\tau_\lambda \in \mathbb{W}^J$ for all $\lambda \in P^{(J)}$. The hyperplane in the parallelism class of $\alpha \in \Phi_J$ passing through λ is $H_{\alpha, \langle \lambda, \alpha \rangle}$, and we have $\langle \lambda, \alpha \rangle \in \{0, 1\}$ (as $\lambda \in P^{(J)}$). Thus the hyperplanes in the parallelism classes of Φ_J separating the alcove $t_\lambda A_0$ from \mathcal{A}_J are precisely the hyperplanes $H_{\alpha,1}$ with $\alpha \in \Phi_J^+$ such that $\langle \lambda, \alpha \rangle = 1$. By Lemma 2.8 these are precisely the hyperplanes separating $t_\lambda A_0$ from $t_\lambda y_\lambda A_0$ (because the corresponding linear hyperplanes $H_{\alpha,0}$ are the hyperplanes separating e from y_λ , and translation by λ preserves orientation and shifts these hyperplanes). Hence $\tau_\lambda A_0 \subseteq \mathcal{A}_J$ and so $\tau_\lambda \in \mathbb{W}^J$.

Let $u \in W^J$ and suppose that $\tau_\lambda u \notin \mathbb{W}^J$. That is, $\tau_\lambda u A_0 \not\subseteq \mathcal{A}_J$. Since $\tau_\lambda A_0 \subseteq \mathcal{A}_J$ (from the previous paragraph) it follows that there is a hyperplane $H_{\alpha,k}$ separating $t_\lambda y_\lambda A_0$ from $t_\lambda y_\lambda u A_0$ with $\alpha \in \Phi_J^+$ and $k \in \{0, 1\}$. Translating by $t_{-\lambda}$ implies that the hyperplane $H_{\alpha,0}$ separates y_λ from $y_\lambda u$, contradicting Lemma 1.2. Thus $\{\tau_\lambda u \mid \lambda \in P^{(J)} \text{ and } u \in W^J\} \subseteq \mathbb{W}^J$.

For the reverse containment, suppose that $w \in \mathbb{W}^J$. Write $w = t_\lambda u_1 u_2$ with $\lambda = \text{wt}(w)$, $u_1 \in W_J$, and $u_2 \in W^J$. It is clear that $\lambda \in P^{(J)}$ (as $w(0) = \lambda$ and $0 \in A_0$). Since $\Phi_J(u_2) = \emptyset$ (by Lemma 1.2) there are no walls of \mathcal{A}_J separating $t_\lambda u_1$ from $t_\lambda u_1 u_2$. Thus $t_\lambda u_1 A_0 \subseteq \mathcal{A}_J$. The argument of the first paragraph, combined with the fact that $\Phi_J(u_2) = \emptyset$, shows that the hyperplanes separating t_λ from $t_\lambda u_1$ are precisely the hyperplanes $H_{\alpha,1}$ with $\alpha \in \Phi_J^+$ with $\langle \lambda, \alpha \rangle = 1$, and it follows that $u_1 = y_\lambda$. So $w = \tau_\lambda u_2$ with $u_2 \in W^J$. \square

Remark 2.10. The first paragraph of the proof of Theorem 2.9 shows that if $\lambda \in P^{(J)}$ then y_λ may be characterised as the unique element of W_J such that $t_\lambda y_\lambda \in \mathbb{W}^J$.

The following immediate consequence of Theorem 2.9 will play a role later. Recall the definition of $\theta_J(w)$ and $\theta^J(w)$ from Definition 1.3.

Corollary 2.11. If $w \in \mathbb{W}^J$ then $\theta_J(w) = y_{\text{wt}(w)}$, and so $w = \tau_{\text{wt}(w)} \theta^J(w)$.

Proof. By Theorem 2.9 $w = \tau_\lambda u = t_\lambda y_\lambda u$ with $\lambda = \text{wt}(w)$ and $u \in W^J$, hence the result. \square

2.4 The J -translation group \mathbb{T}_J

We have seen in Lemma 2.3 that $P^{(J)}$ is in bijection with P/Q_J and hence the former inherits a group structure of the later via $\lambda \mapsto \lambda + Q_J$. In this section we introduce the J -translation group

$$\mathbb{T}_J = \{\tau_\lambda \mid \lambda \in P^{(J)}\}$$

and show that this subset of \widetilde{W} is a group realising the above group structure (see Corollary 2.13). This group interpolates between P (when $J = \emptyset$) and Σ (when $J = I$). The following lemma gives some important basic properties of the elements y_λ and τ_λ .

Lemma 2.12. *Let $\lambda, \mu \in P^{(J)}$. Then*

- (1) $(\lambda + \mu)^{(J)} = \lambda + y_{\lambda\mu}$ and $(-\lambda)^{(J)} = -y_\lambda^{-1}\lambda$;
- (2) $y_\lambda y_\mu = y_{(\lambda+\mu)^{(J)}} = y_\mu y_\lambda$ and $y_\lambda^{-1} = y_{(-\lambda)^{(J)}}$;
- (3) $\tau_\lambda \tau_\mu = \tau_{(\lambda+\mu)^{(J)}} = \tau_\mu \tau_\lambda$ and $\tau_\lambda^{-1} = \tau_{(-\lambda)^{(J)}}$.

Proof. (1) Since $y_\lambda \in W_J$ we have $\lambda + y_{\lambda\mu} \in \lambda + \mu + Q_J$, and so to prove that $(\lambda + \mu)^{(J)} = \lambda + y_{\lambda\mu}$ it suffices, by the uniqueness in Lemma 2.3, to show that $\lambda + y_{\lambda\mu} \in P^{(J)}$. To do this, let $\alpha \in \Phi_J^+$, and write $\beta = y_\lambda^{-1}\alpha$. Then $\langle \lambda + y_{\lambda\mu}, \alpha \rangle = \langle \lambda, \alpha \rangle + \langle \mu, \beta \rangle$. Since $\lambda \in P^{(J)}$ we have $\langle \lambda, \alpha \rangle \in \{0, 1\}$. If $\langle \lambda, \alpha \rangle = 0$ then $\alpha \notin \Phi(y_\lambda)$ (by Lemma 2.8) and so $\beta = y_\lambda^{-1}\alpha \in \Phi_J^+$, and so $\langle \lambda + y_{\lambda\mu}, \alpha \rangle = \langle \mu, \beta \rangle$, giving $0 \leq \langle \lambda + y_{\lambda\mu}, \alpha \rangle \leq 1$ (as $\beta \in \Phi_J^+$ and $\mu \in \mathcal{A}_J$). If $\langle \lambda, \alpha \rangle = 1$ then $\alpha \in \Phi(y_\lambda)$ and so $\beta \in -\Phi_J^+$. Thus $\langle \lambda + y_{\lambda\mu}, \alpha \rangle = 1 + \langle \mu, \beta \rangle$ and so $0 \leq \langle \lambda + y_{\lambda\mu}, \alpha \rangle \leq 1$ (as $-1 \leq \langle \mu, \beta \rangle \leq 0$ as $\beta \in -\Phi_J^+$ and $\mu \in \mathcal{A}_J$). Hence $\lambda + y_{\lambda\mu} \in P^{(J)}$.

To show that $(-\lambda)^{(J)} = -y_\lambda^{-1}\lambda$ one shows, in a similar way, that $-y_\lambda^{-1}\lambda \in P^{(J)}$.

(2) By (1) it suffices to show that $y_\lambda y_\mu = y_{\lambda+y_{\lambda\mu}}$ and $y_\lambda^{-1} = y_{-y_\lambda^{-1}\lambda}$. To prove the first statement we shall show that $\Phi(y_{\lambda+y_{\lambda\mu}}) = \Phi(y_\lambda y_\mu)$. It follows from Lemma 2.8 that

$$\Phi(y_{\lambda+y_{\lambda\mu}}) = \{\alpha \in \Phi_J^+ \mid \langle \lambda, \alpha \rangle = 0 \text{ and } \langle \mu, y_\lambda^{-1}\alpha \rangle = 1, \text{ or } \langle \lambda, \alpha \rangle = 1 \text{ and } \langle \mu, y_\lambda^{-1}\alpha \rangle = 0\}.$$

Suppose that $\alpha \in \Phi(y_{\lambda+y_{\lambda\mu}})$. If $\langle \lambda, \alpha \rangle = 1$ and $\langle \mu, y_\lambda^{-1}\alpha \rangle = 0$ then $\alpha \in \Phi(y_\lambda)$ and $-y_\lambda^{-1}\alpha \notin \Phi(y_\mu)$, and so $\alpha \in \Phi(y_\lambda y_\mu)$. If $\langle \lambda, \alpha \rangle = 0$ and $\langle \mu, y_\lambda^{-1}\alpha \rangle = 1$ then $\alpha \in \Phi_J^+ \setminus \Phi(y_\lambda)$ and $y_\lambda^{-1}\alpha \in \Phi(y_\mu)$, giving $\alpha \in \Phi(y_\lambda y_\mu)$. Conversely, suppose that $\alpha \in \Phi(y_\lambda y_\mu)$. Then $\alpha \in \Phi_J^+$ with $y_\mu^{-1}y_\lambda^{-1}\alpha < 0$, and there are two cases. If $y_\lambda^{-1}\alpha > 0$ and $y_\mu^{-1}(y_\lambda^{-1}\alpha) < 0$ then $\langle \lambda, \alpha \rangle = 0$ (as $\alpha \notin \Phi(y_\lambda)$) and $\langle \mu, y_\lambda^{-1}\alpha \rangle = 1$, so $\alpha \in \Phi(y_{\lambda+y_{\lambda\mu}})$. If $y_\lambda^{-1}\alpha < 0$ and $y_\mu^{-1}(-y_\lambda^{-1}\alpha) > 0$ then $\langle \lambda, \alpha \rangle = 1$ and $\langle \mu, -y_\lambda^{-1}\alpha \rangle = 0$, and so again $\alpha \in \Phi(y_{\lambda+y_{\lambda\mu}})$. Hence $y_\lambda y_\mu = y_{\lambda+y_{\lambda\mu}} = y_{(\lambda+\mu)^{(J)}} = y_\mu y_\lambda$.

The statement $y_\lambda^{-1} = y_{-y_\lambda^{-1}\lambda}$ follows from the general formula $\Phi(w^{-1}) = \{-w\alpha \mid \alpha \in \Phi(w)\}$.

(3) By (1) and (2) we have

$$\tau_\lambda \tau_\mu = t_\lambda y_\lambda t_\mu y_\mu = t_{\lambda+y_{\lambda\mu}} y_\lambda y_\mu = t_{(\lambda+\mu)^{(J)}} y_{(\lambda+\mu)^{(J)}} = \tau_{(\lambda+\mu)^{(J)}}.$$

Similarly $\tau_\lambda^{-1} = \tau_{(-\lambda)^{(J)}}$. □

Corollary 2.13. *We have $\mathbb{T}_J \cong P/Q_J$. In particular, \mathbb{T}_J is an abelian group of rank $|I \setminus J|$. Moreover \mathbb{T}_J acts freely on \mathbb{W}^J with fundamental domain W^J .*

Proof. Lemma 2.12(3) shows that \mathbb{T}_J is a group, and that the map $f : P \rightarrow \mathbb{T}_J$ given by $f(\lambda) = \tau_{\lambda^{(J)}}$ is a group homomorphism. This homomorphism is surjective (as $\lambda^{(J)} = \lambda$ for $\lambda \in P^{(J)}$) and $\ker(f) = Q_J$.

If $\lambda \in P^{(J)}$ and $w \in \mathbb{W}^J$ then by Theorem 2.9 we have $w = \tau_\mu u$ with $\mu = \text{wt}(w) \in P^{(J)}$ and $u \in W^J$. By Lemma 2.12(3) we have $\tau_\lambda \cdot w = \tau_\lambda \tau_\mu u = \tau_{(\lambda+\mu)^{(J)}} u$, and hence $\tau_\lambda \cdot w \in \mathbb{W}^J$ by Theorem 2.9. Thus \mathbb{T}_J acts on \mathbb{W}^J . It is clear that this action is free, and that W^J is a fundamental domain. □

Remark 2.14. By Corollary 2.13 the map $\mathbb{T}_J \rightarrow P/Q_J$ with $\tau_\lambda \rightarrow \lambda^{(J)} + Q_J$ is an isomorphism. In contrast, we note that the map $\mathbb{T}_J \rightarrow P^J$ with $\tau_\lambda \mapsto \lambda^J$ is a surjective, not necessarily injective, group homomorphism (the fact that the map is a surjective homomorphism follows from Lemma 2.12(3) and the obvious fact that $(\lambda^{(J)})^J = \lambda^J$, and to see that the map is not necessarily injective consider the extreme case $J = I$ where $P^J = \{0\}$ while $\mathbb{T}_J = P/Q = \Sigma$ may be nontrivial).

Recall the classical formula $\ell(t_\lambda) = \sum_{\alpha \in \Phi_1^+} |\langle \lambda, \alpha \rangle|$ (obtained by counting hyperplanes crossed). The following proposition gives an analogous formula for the pseudo-translations.

Proposition 2.15. *For $\lambda \in P^{(J)}$ we have*

$$\ell(\tau_\lambda) = \sum_{\alpha \in \Phi_1^+ \setminus \Phi_J} |\langle \lambda, \alpha \rangle|.$$

Proof. By [24, (2.4.1)] we have (for $\lambda \in P$ and $w \in W_0$)

$$\ell(t_\lambda w) = \sum_{\alpha \in \Phi_1^+} |\langle \lambda, \alpha \rangle - \chi^-(w^{-1}\alpha)|,$$

where $\chi^-(\cdot)$ is the characteristic function of $-\Phi^+$. Now suppose that $\lambda \in P^{(J)}$ and consider the contribution $|\langle \lambda, \alpha \rangle - \chi^-(y_\lambda^{-1}\alpha)|$ to $\ell(\tau_\lambda)$ in the above sum from $\alpha \in \Phi_1^+$. If $\alpha \in \Phi_J^+$ then either $\langle \lambda, \alpha \rangle = 0$ in which case $\chi^-(y_\lambda^{-1}\alpha) = 0$, or $\langle \lambda, \alpha \rangle = 1$ in which case $\chi^-(y_\lambda^{-1}\alpha) = 1$ (in both cases using Lemma 2.8). Thus if $\alpha \in \Phi_J^+$ then the contribution to the sum is 0. If $\alpha \in \Phi_1^+ \setminus \Phi_J$ then $\chi^-(y_\lambda^{-1}\alpha) = 0$, hence the result. \square

Example 2.16. Figure 3 illustrates the decomposition of \widetilde{W} (which equals W in this case) into J -alcoves for type G_2 with $J = \{1\}$.

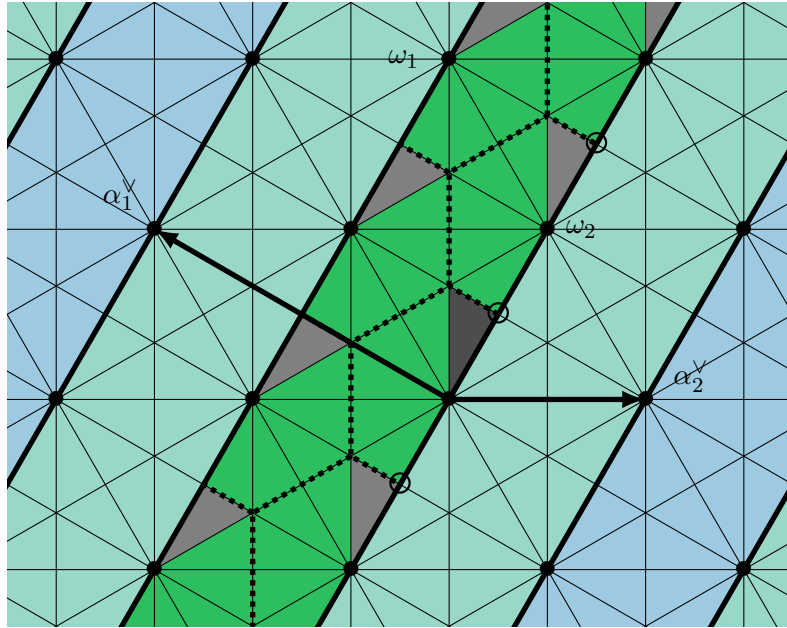


Figure 3: J -alcoves and the group \mathbb{T}_J for $\Phi = G_2$ and $J = \{1\}$

The J alcoves are shaded blue and green, with the dark green region denoting the fundamental J -alcove \mathcal{A}_J . Moreover the elements τ_λ with $\lambda \in P^{(J)}$ are shaded grey, and the decomposition of \mathcal{A}_J given by Theorem 2.9 and Corollary 2.13 is illustrated by dotted lines. We have $\omega_1 - \omega_2 = \frac{1}{2}\omega_2$ and so $P^J = \mathbb{Z}\bar{\omega}_2$ where $\bar{\omega}_2 = \frac{1}{2}\omega_2$ (this lattice is denoted as the union of the solid and open circles on the hyperplane $H_{\alpha_1,0}$). We have $P_J/Q_J = \mathbb{Z}/2\mathbb{Z}$.

2.5 Symmetries of the fundamental J -alcove

In this section we determine the subgroup of \widetilde{W} stabilising the fundamental J -alcove \mathcal{A}_J . The results of this section are not directly used elsewhere in this paper, however since they play an important role in future work we record them here.

Definition 2.17. Let $G_J = \{g \in W_0 \mid g\mathcal{A}_J = \mathcal{A}_J\}$ be the subgroup of W_0 stabilising \mathcal{A}_J .

Theorem 2.18. *We have $G_J = \{g \in W_0 \mid g\Phi_J^+ = \Phi_J^+\}$. Moreover, for $g \in G_J$ and $\alpha \in \Phi_J^+$ we have $\text{ht}(g\alpha) = \text{ht}(\alpha)$. In particular g maps the simple roots of Φ_J to the simple roots of Φ_J , and hence induces a permutation of J . This permutation maps connected components of J to connected components of J .*

Proof. Let $g \in G_J$. Then for all $\alpha \in \Phi_J^+$ and all $\lambda \in \mathcal{A}_J$ we have $0 \leq \langle g\lambda, \alpha \rangle \leq 1$. We claim that this forces $g^{-1}\Phi_J^+ = \Phi_J^+$. For if $g^{-1}\alpha \notin \Phi_J^+$ then either $g^{-1}\alpha \in -\Phi_J^+$, or there is $i \in \text{supp}(g^{-1}\alpha)$ with $i \notin J$. In the former case, choose any $j \in \text{supp}(g^{-1}\alpha)$ (and so $j \in J$) and let m_j^K be the coefficient of α_j in φ_K , where K is the connected component of J containing j . Then $\lambda = \omega_j/m_j^K \in \mathcal{A}_J$ (for if $\beta \in \Phi_J^+ \setminus \Phi_K^+$ then $\langle \lambda, \beta \rangle = 0$ and if $\beta \in \Phi_K^+$ then $0 \leq \langle \lambda, \beta \rangle \leq \langle \lambda, \varphi_K \rangle = 1$), and $\langle \lambda, g^{-1}\alpha \rangle < 0$ gives a contradiction. In the latter case, taking $\lambda = 2\omega_i \in \mathcal{A}_J$ gives $|\langle g\lambda, \alpha \rangle| = |\langle \lambda, g^{-1}\alpha \rangle| \geq 2$, again a contradiction. Thus $g\Phi_J^+ = \Phi_J^+$. On the other hand, if $g\Phi_J^+ = \Phi_J^+$ and $\lambda \in \mathcal{A}_J$ then for $\alpha \in \Phi_J^+$ we have $\langle g\lambda, \alpha \rangle = \langle \lambda, g^{-1}\alpha \rangle$, which lies between 0 and 1 (since $g^{-1}\alpha \in \Phi_J^+$ and $\lambda \in \mathcal{A}_J$), and so $g\mathcal{A}_J = \mathcal{A}_J$.

Let $g \in G_J$. Since $g\Phi_J^+ = \Phi_J^+$ we have $\text{ht}(g\alpha) \geq \text{ht}(\alpha)$ (because each $g\alpha_j$ has height at least 1 and g is linear). Since $g^{-1} \in G_J$ we also have $\text{ht}(g^{-1}\beta) \geq \text{ht}(\beta)$ for all $\beta \in \Phi_J^+$, and by hypothesis we have $\beta = g\alpha$ for some $\alpha \in \Phi_J^+$, and so $\text{ht}(\alpha) \geq \text{ht}(g\alpha)$. Thus $\text{ht}(g\alpha) = \text{ht}(\alpha)$ for all $\alpha \in \Phi_J^+$. In particular each $g \in G_J$ maps simple roots of Φ_J to simple roots of Φ_J , and hence induces a permutation of J by $\alpha_{g(j)} = g\alpha_j$ for $j \in J$.

We now show that for each $g \in G_J$ the permutation $g : J \rightarrow J$ maps connected components to connected components. Let K be a connected component of J , and let φ_K be the highest root of Φ_K . We claim that $g\varphi_K = \varphi_{K'}$ is the highest root of some connected component $K' \subseteq J$. For if not, then there is $i \in J$ such that $\alpha = g\varphi_K + \alpha_i$ is a root of Φ_J . But then $g^{-1}\alpha = \varphi_K + \alpha_{g^{-1}(i)}$ is a root, necessarily of Φ_K , of height exceeding $\text{ht}(\varphi_K)$, a contradiction. Thus g maps highest roots to highest roots, and so g maps K to K' . \square

Lemma 2.19. *If $\lambda \in P^{(J)}$ and $g \in G_J$ then*

$$gy\lambda g^{-1} = y_g\lambda, \quad g\tau\lambda g^{-1} = \tau_g\lambda, \quad \text{and} \quad \ell(\tau_g\lambda) = \ell(\tau\lambda).$$

Proof. To prove that $gy\lambda g^{-1} = y_g\lambda$ it is sufficient to show that $\Phi(gy\lambda g^{-1}) = \Phi(y_g\lambda)$. If $\alpha \in \Phi(y_g\lambda)$ then $\alpha \in \Phi_J^+$ with $\langle g\lambda, \alpha \rangle = 1$. Thus $\langle \lambda, g^{-1}\alpha \rangle = 1$ and so $g^{-1}\alpha \in \Phi(y\lambda)$. Hence $gy\lambda g^{-1}\alpha \in -\Phi_J^+$, showing that $\Phi(y_g\lambda) \subseteq \Phi(gy\lambda g^{-1})$. On the other hand, suppose that $\alpha \in \Phi(gy\lambda g^{-1})$. Since g maps simple roots of Φ_J to simple roots of Φ_J we have $gs_j g^{-1} = s_{g(j)} \in W_J$ for all $j \in J$, and so $gy\lambda g^{-1} \in W_J$. Thus $\alpha \in \Phi_J$, and so $g^{-1}\alpha \in \Phi_J^+$. If $g^{-1}\alpha \notin \Phi(y\lambda)$ then $gy\lambda g^{-1}\alpha \in \Phi_J^+$, a contradiction, and so $g^{-1}\alpha \in \Phi(y\lambda)$. Thus $\langle \lambda, g^{-1}\alpha \rangle = 1$, and so $\langle g\lambda, \alpha \rangle = 1$, giving $\alpha \in \Phi(y_g\lambda)$ as required.

It then follows that $g\tau\lambda g^{-1} = t_{g\lambda}(gy\lambda g^{-1}) = t_{g\lambda}y_g\lambda = \tau_g\lambda$ for all $\lambda \in P^{(J)}$. By Proposition 2.15 we have $\ell(\tau_g\lambda) = \sum_{\alpha \in \Phi_1^+ \setminus \Phi_J} |\langle \lambda, g^{-1}\alpha \rangle| = \sum_{\alpha \in \Phi_1^+ \setminus \Phi_J} |\langle \lambda, \alpha \rangle| = \ell(\tau\lambda)$. \square

Corollary 2.20. *The subgroup of \widetilde{W} stabilising \mathcal{A}_J is $\mathbb{T}_J \rtimes G_J$.*

Proof. Let $w \in \widetilde{W}$ and suppose that $w\mathcal{A}_J = \mathcal{A}_J$. Let $\lambda = \text{wt}(w)$, and so $w(0) = \lambda \in P^{(J)}$. Then $\tau_\lambda^{-1}w(0) = 0$, and so $g = \tau_\lambda^{-1}w \in W_0$ with $g\mathcal{A}_J = \mathcal{A}_J$, and so $g \in G_J$. Thus $w = \tau_\lambda g \in \mathbb{T}_J G_J$. \square

Note that the group $\mathbb{T}_J \rtimes G_J$ plays the role of the ‘‘extended affine Weyl group’’ of \mathcal{A}_J in the sense that if $J = \emptyset$ we have $\mathbb{T}_J \rtimes G_J = \widetilde{W}$.

3 J -folded alcove paths and J -parameter systems

In this section we introduce J -folded alcove paths, generalising the positively folded alcove paths of Ram [29] and the 2-dimensional theory from [16, 15]. We also introduce the notion of a J -parameter system, a combinatorial object that will be useful in indexing later objects in this paper.

3.1 J -folded alcove paths

We introduce the following definition, giving a J -relative version of positively folded alcove paths.

Definition 3.1. Let $\vec{w} = s_{i_1} s_{i_2} \cdots s_{i_\ell} \sigma$ be an expression for $w \in \widetilde{W}$ (not necessarily reduced) with $\sigma \in \Sigma$, and let $v \in \mathbb{W}^J$. A J -folded alcove path of type \vec{w} starting at v is a sequence $p = (v_0, v_1, \dots, v_\ell, v_\ell \sigma)$ with $v_0, \dots, v_\ell \in \mathbb{W}^J$ such that, for $1 \leq k \leq \ell$,

- (1) $v_0 = v$ and $v_k \in \{v_{k-1}, v_{k-1} s_{i_k}\}$, and
- (2) if $v_{k-1} = v_k$ then either:
 - (a) $v_{k-1} s_{i_k} A_0 \not\subseteq \mathcal{A}_J$, or
 - (b) $v_{k-1} s_{i_k} A_0 \subseteq \mathcal{A}_J$ and the alcove $v_{k-1} A_0$ is on the positive side of the hyperplane separating the alcoves $v_{k-1} A_0$ and $v_{k-1} s_{i_k} A_0$.

The end of the J -folded alcove path $p = (v_0, \dots, v_\ell, v_\ell \sigma)$ is $\text{end}(p) = v_\ell \sigma$.

For $v \in \mathbb{W}^J$ let

$$\mathcal{P}_J(\vec{w}, v) = \{\text{all } J\text{-folded alcove paths of type } \vec{w} \text{ starting at } v\}.$$

Note that if $J = \emptyset$ then $\mathbb{W}^J = \widetilde{W}$ and so Definition 3.1(2)(a) is vacuous, and hence \emptyset -folded alcove paths are the same as positively folded alcove paths.

Let $\vec{w} = s_{i_1} s_{i_2} \cdots s_{i_\ell} \sigma$ and let $p = (v_0, \dots, v_\ell, v_\ell \sigma) \in \mathcal{P}_J(\vec{w}, v)$. The index $k \in \{1, 2, \dots, \ell\}$ is:

- (1) a *positive (respectively, negative) i_k -crossing* if $v_k = v_{k-1} s_{i_k}$ and $v_k A_0$ is on the positive (respectively, negative) side of the hyperplane separating the alcoves $v_{k-1} A_0$ and $v_k A_0$;
- (2) a *(positive) i_k -fold* if $v_k = v_{k-1}$ and $v_{k-1} s_{i_k} A_0 \subseteq \mathcal{A}_J$ (in which case $v_{k-1} A_0$ is necessarily on the positive side of the hyperplane separating $v_{k-1} A_0$ and $v_{k-1} s_{i_k} A_0$);
- (3) a *positive (respectively, negative) bounce* if $v_k = v_{k-1}$ and $v_{k-1} s_{i_k} A_0 \not\subseteq \mathcal{A}_J$ and $v_{k-1} A_0$ is on the positive (respectively, negative) side of the hyperplane H separating $v_{k-1} A_0$ and $v_{k-1} s_{i_k} A_0$. Necessarily $H = H_{\alpha,0}$ (respectively, $H = H_{\alpha,1}$) for some $\alpha \in \Phi_J^+$ and we say that k *occurs* on the hyperplane $H_{\alpha,0}$ (respectively, $H_{\alpha,1}$).

Less formally, these steps are denoted as follows (where $x = v_{k-1}$ and $s = s_{i_k}$):

$\begin{array}{c} - \quad + \\ xA_0 \quad \quad xsA_0 \\ \quad \quad \quad \rightarrow \end{array}$	$\begin{array}{c} - \quad + \\ xsA_0 \quad \quad xA_0 \\ \quad \quad \quad \leftarrow \end{array}$	$\begin{array}{c} - \quad + \\ xsA_0 \quad \quad xA_0 \\ \quad \quad \quad \leftarrow \end{array}$	$\begin{array}{c} H_{\alpha,0} \\ - \quad + \\ xsA_0 \quad \quad xA_0 \\ \quad \quad \quad \leftarrow \end{array}$	$\begin{array}{c} H_{\alpha,1} \\ - \quad + \\ xA_0 \quad \quad xsA_0 \\ \quad \quad \quad \leftarrow \end{array}$
positive s -crossing	s -fold	negative s -crossing	positive bounce	negative bounce
(a) The case $xsA_0 \subseteq \mathcal{A}_J$			(b) The case $xsA_0 \not\subseteq \mathcal{A}_J$	

It will turn out that bounces play a very different role in the theory to folds, and so we emphasise the distinction between these two concepts. Put briefly, all of the interactions a path makes with the walls of \mathcal{A}_J are bounces, and the folds can only occur in the ‘‘interior’’ of \mathcal{A}_J .

Let p be a J -folded alcove path. For each $i \in \{0\} \cup I$ and $\alpha \in \Phi_J^+$ we write

$$\begin{aligned} f_i(p) &= \#(i\text{-folds in } p) \\ b_\alpha^+(p) &= \#(\text{positive bounces in } p \text{ occurring on } H_{\alpha,0}) \\ b_\alpha^-(p) &= \#(\text{negative bounces in } p \text{ occurring on } H_{\alpha,1}) \\ b_\alpha(p) &= b_\alpha^+(p) + b_\alpha^-(p). \end{aligned}$$

Remark 3.2. By Lemma 2.2 we have $b_{\alpha}^{-}(p) = 0$ unless $\alpha = \varphi_K$ for some $K \in \mathcal{K}(J)$, and $b_{\alpha}^{+}(p) = 0$ unless $\alpha = \alpha_j$ or $\alpha = 2\alpha_j$ for some $j \in J$. Of course the case $\alpha = 2\alpha_j$ only occurs if Φ_J is not reduced and $j = n$, and in this case $b_{\alpha_n}^{+}(p) = b_{2\alpha_n}^{+}(p)$.

3.2 Coweights and final directions of J -folded alcove paths

By Corollary 2.13 the set W^J is a fundamental domain for the action of \mathbb{T}_J on \mathbb{W}^J . While in many respects this choice of fundamental domain is “natural”, in examples and applications it turns out to be important to have additional flexibility in order to better incorporate symmetries present (see, for example, Section 6).

Let F be a fundamental domain for the action of \mathbb{T}_J on \mathbb{W}^J . Thus each $w \in \mathbb{W}^J$ has a unique expression as

$$w = \tau_{\lambda}u \quad \text{with } \lambda \in P^{(J)} \text{ and } u \in F,$$

and we define the *coweight of w relative to F* and the *final direction of w relative to F* by

$$\text{wt}(w, F) = \lambda \quad \text{and} \quad \theta(w, F) = u.$$

If p is a J -folded alcove path then the *coweight of p relative to F* and the *final direction of p relative to F* are

$$\text{wt}(p, F) = \text{wt}(\text{end}(p), F) \quad \text{and} \quad \theta(p, F) = \theta(\text{end}(p), F). \quad (3.1)$$

In particular, note that $\text{wt}(p, W^J) = \text{wt}(p)$ and $\theta(p, W^J) = \theta^J(p)$, where $\text{wt}(p) = \text{wt}(\text{end}(p))$ and $\theta^J(p) = \theta^J(\text{end}(p))$ (see Definition 1.3).

Example 3.3. Figure 5 illustrates a J -folded alcove path in type G_2 with $J = \{1\}$ with two choices of fundamental domain F shaded blue (see Figure 3 for the root system conventions). In both cases the path has 3 bounces (two positive, and one negative), and 2 folds (a 2-fold and a 0-fold). In Figure 5(a) we have $\theta^J(p) = 21$ and $\text{wt}(p, W^J) = \text{wt}(p) = \omega_1$. In Figure 5(b) we have $\theta(p, F) = 020$ and $\text{wt}(p, F) = \omega_2$.

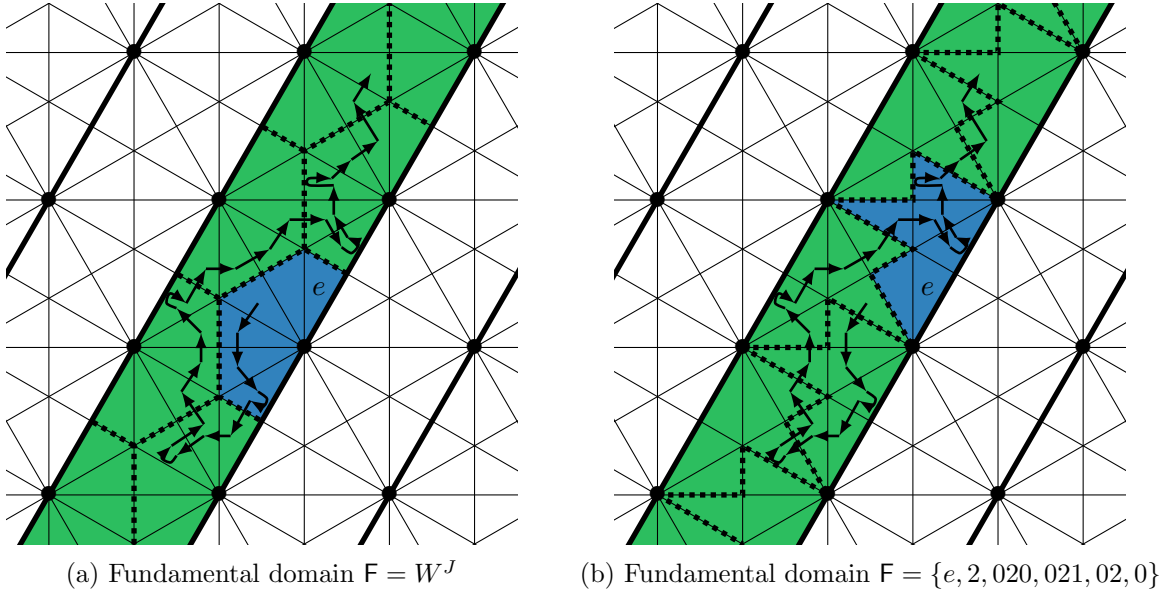


Figure 5: J -folded alcove path for $\Phi = G_2$ with $J = \{1\}$

There is a natural action of \mathbb{T}_J on J -folded alcove paths, as follows (recall the definition of $\lambda^{(J)}$ from Definition 2.4).

Lemma 3.4. *If $p = (v_0, v_1, \dots, v_\ell, v_\ell\sigma)$ is a J -folded alcove path of type $\vec{w} = s_{i_1} \cdots s_{i_\ell}\sigma$ and $\lambda \in P^{(J)}$ then*

$$\tau_\lambda \cdot p = (\tau_\lambda v_0, \tau_\lambda v_1, \dots, \tau_\lambda v_\ell, \tau_\lambda v_\ell\sigma)$$

is a J -folded alcove path of type \vec{w} , and folds are mapped to folds and bounces to bounces. For any fundamental domain F we have

$$\text{wt}(\tau_\lambda \cdot p, F) = (\lambda + \text{wt}(p, F))^{(J)} \quad \text{and} \quad \theta(\tau_\lambda \cdot p, F) = \theta(p, F).$$

Proof. Since \mathbb{T}_J acts on \mathbb{W}^J we have $\tau_\lambda v_0, \dots, \tau_\lambda v_\ell, \tau_\lambda v_\ell\sigma \in \mathbb{W}^J$, and moreover bounces are mapped to bounces, and folds to folds (because if $v_k = v_{k-1}$ then $v_{k-1}s_{i_k} \in \mathbb{W}^J$ if and only if $\tau_\lambda v_{k-1}s_{i_k} \in \mathbb{W}^J$). It remains to show that the image of a fold remains positively oriented. Thus, suppose that $v_{k-1} = v_k$ and $v_{k-1}s_{i_k} \in \mathbb{W}^J$. Then $v_{k-1}A_0$ is on the positive side of the hyperplane separating the alcoves $v_{k-1}A_0$ and $v_{k-1}s_{i_k}A_0$, and this hyperplane has a linear root $\alpha \in \Phi^+$ with $\alpha \notin \Phi_J$. We have $\tau_\lambda = t_\lambda y_\lambda$, and since $y_\lambda \in W_J$ we have $y_\lambda \alpha \in \Phi^+$, and hence the fold remains positively oriented. The statements about the coweight and final direction are then clear using Lemma 2.12. \square

3.3 J -parameter systems and the \mathbf{v} -mass of a path

Recall the definition of the parameters \mathbf{q}_i , $i \in \{0\} \cup I$ from Section 1.6 and Convention 1.7.

Definition 3.5. A J -parameter system is a family $\mathbf{v} = (v_\alpha)_{\alpha \in \Phi_J}$ with:

- (1) $v_\alpha = v_\beta$ if $\beta \in W_J \alpha$;
- (2) $v_{\alpha_j} \in \{\mathbf{q}_j, -\mathbf{q}_j^{-1}\}$ for $j \in J$ with $2\alpha_j \notin \Phi_J$;
- (3) $v_{2\alpha_n} \in \{\mathbf{q}_0, -\mathbf{q}_0^{-1}\}$ and $v_{\alpha_n} v_{2\alpha_n} \in \{\mathbf{q}_n, -\mathbf{q}_n^{-1}\}$ if Φ_J is not reduced.

By convention we set $v_\alpha = 1$ if $\alpha \notin \Phi_J$. Thus we have $v_{\alpha_j} v_{2\alpha_j} \in \{\mathbf{q}_j, -\mathbf{q}_j^{-1}\}$ for all $j \in J$.

Example 3.6. If $\Phi = C_{13}$ and $J = \{1, 2, 3, 6, 7, 9, 11, 12, 13\}$ as depicted below



then there are 32 distinct J -parameter systems. If, instead, $\Phi = BC_{13}$ then there are 64 distinct J -parameter systems.

If $\mathbf{v} = (v_\alpha)_{\alpha \in \Phi_J}$ is a J -parameter system, for $\lambda \in P$ and $y \in W_J$ we write

$$v^\lambda = \prod_{\alpha \in \Phi_J^+} v_\alpha^{\langle \lambda, \alpha \rangle} \quad \text{and} \quad v(y) = \prod_{\alpha \in \Phi(y)} v_\alpha v_{2\alpha}. \quad (3.2)$$

Note that if $\lambda \in P \cap V^J$ then $v^\lambda = 1$ (as $\langle \lambda, \alpha \rangle = 0$ for all $\alpha \in \Phi_J^+$).

In the following definition we introduce the \mathbf{v} -mass of a J -folded alcove path. This quantity will play an important role in the combinatorial formula of Theorem 4.13.

Definition 3.7. Let $\mathbf{v} = (v_\alpha)_{\alpha \in \Phi_J}$ be a J -parameter system. The \mathbf{v} -mass of a J -folded alcove path p is

$$\mathcal{Q}_{J, \mathbf{v}}(p) = \left[\prod_{\alpha \in \Phi_J^+} v_\alpha^{b_\alpha(p)} \right] \left[\prod_{i \in \{0\} \cup I} (\mathbf{q}_i - \mathbf{q}_i^{-1})^{f_i(p)} \right].$$

Lemma 3.11. *Let \mathbf{v} be a J -parameter system.*

- (1) *If $j \in J$ with $2\alpha_j \notin \Phi_J$ then $\mathbf{v}^{\alpha_j^\vee} = \mathbf{v}_{\alpha_j}^2$.*
- (2) *If Φ_J is not reduced then $\mathbf{v}^{\alpha_n^\vee/2} = \mathbf{v}_{\alpha_n} \mathbf{v}_{2\alpha_n}^2$.*

Proof. Let $K \in \mathcal{K}(J)$ be the connected component with $j \in K$. If Φ_K is reduced then there are at most two root lengths in Φ_K (with all roots long in the simply laced case), and since all roots of the same length are conjugate in W_K (as K is connected), for $j \in K$ we have

$$\mathbf{v}^{\alpha_j^\vee} = \prod_{\alpha \in \Phi_J^+} \mathbf{v}_\alpha^{\langle \alpha_j^\vee, \alpha \rangle} = \prod_{\alpha \in \Phi_K^+} \mathbf{v}_\alpha^{\langle \alpha_j^\vee, \alpha \rangle} = \mathbf{v}_{\text{sh}}^{\langle \alpha_j^\vee, 2\rho'_K \rangle} \mathbf{v}_{\text{lo}}^{\langle \alpha_j^\vee, 2\rho_K \rangle},$$

where \mathbf{v}_{sh} (respectively \mathbf{v}_{lo}) is the constant value of \mathbf{v}_β on short (respectively long) roots β in Φ_K , and where ρ'_K and ρ_K are as Section 1.4. It follows by Lemma 1.4 that $\mathbf{v}^{\alpha_j^\vee} = \mathbf{v}_{\alpha_j}^2$.

If Φ_K is not reduced then direct analysis of the BC_n root system gives

$$\mathbf{v}^{\alpha_j^\vee} = \prod_{\alpha \in \Phi_K^+} \mathbf{v}_\alpha^{\langle \alpha_j^\vee, \alpha \rangle} = \mathbf{v}_{\alpha_1}^{\langle \alpha_j^\vee, 2\rho'_K \rangle} (\mathbf{v}_{\alpha_n} \mathbf{v}_{2\alpha_n}^2)^{\langle \alpha_j^\vee, 2\rho_K \rangle}.$$

If $j \in K \setminus \{n\}$ we have $\langle \alpha_j^\vee, 2\rho_K \rangle = 0$ and $\langle \alpha_j^\vee, 2\rho'_K \rangle = 2$ and so $\mathbf{v}^{\alpha_j^\vee} = \mathbf{v}_{\alpha_1}^2 = \mathbf{v}_{\alpha_j}^2$, and if $j = n$ then $\langle \alpha_n^\vee, 2\rho'_K \rangle = 0$ and $\langle \alpha_n^\vee, 2\rho_K \rangle = 2$ giving $\mathbf{v}^{\alpha_n^\vee/2} = \mathbf{v}_{\alpha_n} \mathbf{v}_{2\alpha_n}^2$ as required. \square

Lemma 3.12. *Let $K \in \mathcal{K}(J)$ and let $\alpha \in \Phi_K^+$ be a long root of Φ_K (with all roots long if Φ_K is simply laced). Then $\mathbf{v}^{\alpha^\vee} \mathbf{v}(s_\alpha)^{-1} = \mathbf{v}_\alpha$.*

Proof. Let $w = s_{j_1} \cdots s_{j_\ell} \in W_K$ be of minimal length subject to $w^{-1}\alpha = \alpha_k$ for some $k \in K$ (if Φ_K is reduced) or $w^{-1}\alpha = 2\alpha_n$ (if Φ_K is not reduced). Let $\beta_0 = \alpha$ and define $\beta_1, \dots, \beta_\ell \in \Phi_K^+$ by $\beta_r = s_{j_r} \beta_{r-1}$ for $1 \leq r \leq \ell$, and so $\beta_\ell = \alpha_k$ (or $\beta_\ell = 2\alpha_n$ if Φ_K is not reduced). Since each β_r is a long root of Φ_K we have $\langle \beta_{r-1}^\vee, \alpha_{j_r} \rangle \in \{-1, 0, 1\}$ (see [3, IV, §1, Proposition 8]) and so

$$\beta_r^\vee = s_{j_r} \beta_{r-1}^\vee = \beta_{r-1}^\vee - \langle \beta_{r-1}^\vee, \alpha_{j_r} \rangle \alpha_{j_r}^\vee = \beta_{r-1}^\vee - \alpha_{j_r}^\vee$$

(because $\langle \beta_{r-1}^\vee, \alpha_{j_r} \rangle \in \{0, -1\}$ contradicts minimality of w , see for example [4, Lemma 1.7]).

We claim that

$$\mathbf{v}^{\beta_r^\vee} \mathbf{v}(s_{\beta_r})^{-1} = \mathbf{v}_\alpha \quad \text{for } 0 \leq r \leq \ell.$$

We argue by downward induction on r . If $r = \ell$ then $\beta_\ell = \alpha_k$ (if Φ_K is reduced) or $\beta_\ell = 2\alpha_n$ (if Φ_K is not reduced), and the result follows from Lemma 3.11, starting the induction.

Since $s_{\beta_{r-1}} = s_{j_r} s_{\beta_r} s_{j_r}$ (with length adding) we have

$$\mathbf{v}^{\beta_{r-1}^\vee} \mathbf{v}(s_{\beta_{r-1}})^{-1} = \mathbf{v}^{\alpha_{j_r}^\vee} \mathbf{v}(s_{j_r})^{-2} \mathbf{v}^{\beta_r^\vee} \mathbf{v}(s_{\beta_r})^{-1}.$$

Note that if Φ_K is not reduced then $j_r \neq n$ for $1 \leq r \leq \ell$ (because $s_{n-1} \cdots s_k(2e_k) = 2e_n$), and so by Lemma 3.11 we have $\mathbf{v}^{\alpha_{j_r}^\vee} \mathbf{v}(s_{j_r})^{-2} = 1$ for all $1 \leq r \leq \ell$, and the result follows by induction. \square

Lemma 3.13. *Let \mathbf{v} be a J -parameter system. If $K \in \mathcal{K}(J)$ and $y \in W_J$ then*

$$\frac{\mathbf{v}(y)}{\mathbf{v}(y s_{\varphi_K})} \mathbf{v}^{y\varphi_K^\vee} = \begin{cases} \mathbf{v}_{\varphi_K} & \text{if } y\varphi_K \in \Phi_J^+ \\ \mathbf{v}_{\varphi_K}^{-1} & \text{if } y\varphi_K \in -\Phi_J^+. \end{cases}$$

Proof. We may assume without loss of generality that $y \in W_K$ (making use of Lemma 3.12). Let $\alpha \in \Phi_K^+$. If $y^{-1}\alpha \neq \pm\varphi_K$ we claim that

$$\langle \varphi_K^\vee, y^{-1}\alpha \rangle = \begin{cases} 1 & \text{if } y^{-1}\alpha \in \Phi(s_{\varphi_K}) \\ -1 & \text{if } y^{-1}\alpha \in -\Phi(s_{\varphi_K}) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.15. *Let $w \in \widetilde{W}$ and let \vec{w} be any expression for w (not necessarily reduced). Let $v \in \mathbb{W}^J$ and let $p \in \mathcal{P}_J(\vec{w}, v)$. The J -straightening map $p \mapsto p_J$ is a bijection from $\mathcal{P}_J(\vec{w}, v)$ to the set $\{p' \in \mathcal{P}(\vec{w}, v) \mid p' \text{ has no folds on hyperplanes } H_{\alpha, k} \text{ with } \alpha \in \Phi_J^+ \text{ and } k \in \mathbb{Z}\}$.*

Proof. Consider the effect of straightening a bounce. Let $\alpha \in \Phi_J^+$ be the linear root associated to the wall on which the bounce occurs. A later positive fold that occurs on a β -wall (necessarily with $\beta \in \Phi^+ \setminus \Phi_J$) now occurs on a $s_\alpha \beta$ wall. Since $\beta \notin \Phi_J$ and $s_\alpha \in W_J$ we have $s_\alpha \beta \in \Phi^+ \setminus \Phi_J$ (as s_α permutes the set Φ_J) and hence this reflected fold remains a positive fold (see (1.2)), and it does not occur on a hyperplane $H_{\gamma, k}$ with $\gamma \in \Phi_J$ (a “ Φ_J -wall”). This shows that p_J is positively folded, with no folds occurring on Φ_J -walls.

A similar argument shows that one may apply the reverse procedure, starting with a positively folded alcove path $p' \in \mathcal{P}(\vec{w}, v)$ with no folds on Φ_J -walls and forcing the bounces on the Φ_J -walls. These operations are mutually inverse procedures, hence the result. \square

We now give some more precise information on the J -straightening relating to the v -mass. If p is a positively folded alcove path, for each $\alpha - k\delta \in \widetilde{\Phi}^+$ we define

$$c_{\alpha, k}(p) = \#(\text{crossings in } p \text{ that occur on the hyperplane } H_{\alpha, k}).$$

Note that in the non-reduced case, if $2\alpha \in \Phi$ then $c_{2\alpha, 2k}(p) = c_{\alpha, k}$.

Given a J -parameter system $\mathbf{v} = (v_\alpha)_{\alpha \in \Phi_J}$ we define

$$v_{\alpha+k\delta} = v_\alpha \quad \text{if } \alpha \in \Phi_J \text{ and } k \in \mathbb{Z}, \quad (3.3)$$

and we set $v_{\alpha+k\delta} = 1$ if $\alpha + k\delta \notin \Phi_J + \mathbb{Z}\delta$.

Proposition 3.16. *Let p be a J -folded alcove path (not necessarily of reduced type) and let $\mathbf{v} = (v_\alpha)_{\alpha \in \Phi_J}$ be a J -parameter system. Then*

$$\mathcal{Q}_{J, \mathbf{v}}(p) = \mathcal{Q}(p_J) \prod_{\alpha - k\delta \in \widetilde{\Phi}^+} v_{\alpha - k\delta}^{c_{\alpha, k}(p_J)}$$

where $\mathcal{Q}(\cdot)$ is as in Proposition 1.6 (note that the product has finitely many terms $\neq 1$).

Proof. First, note that under the bijection $p \mapsto p_J$ the types of the folds are preserved (that is, i -folds in p are mapped to i -folds in p_J) because the J -affine Weyl group W_J^{aff} is type preserving. This shows that

$$\prod_{i \in \{0\} \cup I} (q_i - q_i^{-1})^{f_i(p)} = \prod_{i \in \{0\} \cup I} (q_i - q_i^{-1})^{f_i(p_J)} = \mathcal{Q}(p_J).$$

Thus it remains to show that if $p \in \mathcal{P}_J(\vec{w}, v)$ then

$$\prod_{\alpha \in \Phi_J^+} v_\alpha^{b_\alpha(p)} = \prod_{\alpha - k\delta \in \widetilde{\Phi}^+} v_{\alpha - k\delta}^{c_{\alpha, k}(p_J)},$$

which in turn follows from the fact that each crossing of a $\Phi_J + \mathbb{Z}\delta$ wall in the J -straightened path p_J corresponds to a bounce on a wall of \mathcal{A}_J in the path p , and from the definition of J -straightening if p_J has a crossing on $H_{\beta, k}$ with $\beta \in \Phi_J$ then there is $v \in W_J^{\text{aff}}$ with $vH_{\beta, k}$ equal to the wall $H_{\gamma, r}$ of \mathcal{A}_J on which the corresponding bounce occurred. \square

4 The $\tilde{\mathcal{H}}$ -modules $M_{J,\nu}$

In this section we construct finite dimensional $\tilde{\mathcal{H}}$ -modules $(\pi_{J,\nu}, M_{J,\nu})$ (Theorem 4.2), and develop a combinatorial formula for the matrix entries of $\pi_{J,\nu}(T_w)$ in terms of J -folded alcove paths (Theorems 4.13 and 4.15). We prove that these modules are irreducible (Corollary 4.19) and realise them as induced representations from 1-dimensional representations of Levi subalgebras (Theorem 4.21). In fact we will show that all representations of $\tilde{\mathcal{H}}$ that are induced from 1-dimensional representations of Levi subalgebras arise from our construction (after specialising certain “variables” appropriately, see Remark 4.23).

Our $\tilde{\mathcal{H}}$ -modules are inspired by the modules constructed by Lusztig [21, Lemma 4.7] and Deodhar [5, Corollary 2.3], however note that those modules are infinite dimensional when applied to our setting. To motivate the general philosophy of Theorem 4.2, recall Lusztig’s *periodic Hecke module* $M = \bigoplus_{A \in \mathbb{A}} \mathbb{R}A$ with basis indexed by the (classical) alcoves and $\tilde{\mathcal{H}}$ -action given by (see [21, §3.2])

$$(wA_0) \cdot T_i = \begin{cases} ws_i A_0 & \text{if } wA_0^{-|+} ws_i A_0 \\ ws_i A_0 + (\mathfrak{q}_i - \mathfrak{q}_i^{-1})wA_0 & \text{if } wA_0^{+|-} ws_i A_0 \end{cases}$$

for $i \in \{0\} \cup I$. The idea behind Theorem 4.2 is to adapt this action to create a J -analogue that also incorporates the action of the translation group \mathbb{T}_J on \mathcal{A}_J to create a finite dimensional $\tilde{\mathcal{H}}$ -module. We shall be able to achieve this goal for each subset $J \subseteq I$ and each J -parameter system ν , thus constructing many “combinatorial” finite dimensional $\tilde{\mathcal{H}}$ -modules.

Remark 4.1. Starting from the definition of $M_{J,\nu}$ in Theorem 4.2, it requires quite a lot of work to show that we indeed obtain a $\tilde{\mathcal{H}}$ -module. It could seem more natural to define $M_{J,\nu}$ as an induced module from a 1-dimensional representation of Levi subalgebras (as shown in Theorem 4.21). Let us briefly explain why we have not followed this approach in this paper. First of all, we wanted to follow Deodhar’s original idea because of its combinatorial nature. Next, it is in fact rather complicated to derive the path formula when $M_{J,\nu}$ is defined as an induced module, and most of the work after Theorem 4.2 still remains to be done with such an approach. Finally, since the authors used induction to define $M_{J,\nu}$ in [16, 15] in the rank 2 case, we believe that it is more interesting to provide a different proof of the path formula here.

4.1 Construction of the module $M_{J,\nu}$

Let $\{\zeta_i \mid i \in I\}$ be a family of commuting invertible indeterminates, and for $\lambda \in P$ let $\zeta^\lambda = \prod_{i \in I} \zeta_i^{a_i}$ if $\lambda = \sum_{i \in I} a_i \omega_i$. For $J \subseteq I$, let \mathcal{I}_J denote the ideal of the Laurent polynomial ring $\mathbb{R}[\{\zeta_i^{\pm 1} \mid i \in I\}]$ generated by the elements $\zeta_j^{\alpha_j^\vee} - 1$ for $j \in J$. Let

$$\zeta_J^\lambda = \zeta^\lambda + \mathcal{I}_J \quad \text{and write} \quad \mathbb{R}[\zeta_J] = \mathbb{R}[\{\zeta_i^{\pm 1} \mid i \in I\}] / \mathcal{I}_J.$$

In particular, note that $\zeta_J^\gamma = 1$ for all $\gamma \in Q_J$, and thus $\zeta_J^\lambda = \zeta_J^{\lambda^{(J)}}$ for all $\lambda \in P$.

The following theorem introduces the main objects of study in this paper. Recall that by (3.3) we extend the definition of ν_α to $\nu_{\alpha+k\delta}$ for all $\alpha + k\delta \in \tilde{\Phi}$, and that $\alpha_0 = -\varphi + \delta$.

Theorem 4.2. Let $J \subseteq I$ and let \mathbf{v} be a J -parameter system. Let $M_{J,\mathbf{v}}$ be a module over the ring $\mathbb{R}[\zeta_J]$ with basis $\{\mathbf{m}_u \mid u \in W^J\}$, and for $i \in \{0\} \cup I$ and $\sigma \in \Sigma$ define

$$\mathbf{m}_u \cdot T_i = \begin{cases} \zeta_J^{\text{wt}(us_i)} \mathbf{m}_{\theta^J(us_i)} & \text{if } u \rightarrow us_i \text{ positive and } us_i \in \mathbb{W}^J \\ \zeta_J^{\text{wt}(us_i)} \mathbf{m}_{\theta^J(us_i)} + (\mathbf{q}_i - \mathbf{q}_i^{-1}) \mathbf{m}_u & \text{if } u \rightarrow us_i \text{ is negative and } us_i \in \mathbb{W}^J \\ \mathbf{v}_{u\alpha_i} \mathbf{v}_{2u\alpha_i} \mathbf{m}_u & \text{if } us_i \notin \mathbb{W}^J \text{ (hence } u\alpha_i \in \Phi_J + \mathbb{Z}\delta) \end{cases}$$

$$\mathbf{m}_u \cdot T_\sigma = \zeta_J^{\text{wt}(u\sigma)} \mathbf{m}_{\theta^J(u\sigma)}.$$

This extends to a right action of $\widetilde{\mathcal{H}}$ on the module $M_{J,\mathbf{v}}$.

Before proving Theorem 4.2 we give some lemmas and auxiliary results.

Lemma 4.3. Let $w, v \in \widetilde{W}$. Then

- (1) $\theta^J(wv) = \theta^J(\theta^J(w)v)$.
- (2) $\theta_J(wv) = \theta_J(w)\theta_J(\theta^J(w)v)$
- (3) $\text{wt}(wv) = \text{wt}(w) + \theta_J(w)\text{wt}(\theta^J(w)v)$.

Proof. These all follow immediately from the definitions. □

Lemma 4.4. Let $w \in \widetilde{W}$ and $i \in \{0\} \cup I$. Let $u = \theta^J(w)$. Then

$$\frac{\mathbf{v}(\theta_J(w))}{\mathbf{v}(\theta_J(ws_i))} \mathbf{v}^{\text{wt}(ws_i) - \text{wt}(w)} = \begin{cases} 1 & \text{if } us_i \in \mathbb{W}^J \\ \mathbf{v}_{u\alpha_i}^\epsilon \mathbf{v}_{2u\alpha_i}^\epsilon & \text{if } us_i \notin \mathbb{W}^J. \end{cases}$$

where $\epsilon \in \{-1, 1\}$ is the sign of the crossing $w \rightarrow ws_i$.

Proof. The expression on the left hand side of the equation is invariant under replacing w with $t_\lambda w$ for any $\lambda \in P$, and the sign of the crossing $w \rightarrow ws_i$ is the same as the sign of the crossing $t_\lambda w \rightarrow t_\lambda ws_i$. Therefore we can assume without loss of generality that $w \in W_0$.

So $w = yu \in W_0$ with $y = \theta_J(w)$ and $u = \theta^J(w)$. Suppose first that $i \in I$. If $us_i \in \mathbb{W}^J$ then $us_i \in W^J$ and so $\theta^J(ws_i) = us_i$, and hence $\theta_J(ws_i) = y$, and the result follows. If $us_i \notin \mathbb{W}^J$ then $us_i = s_j u$ for a unique $j \in J$, and so $\theta_J(ws_i) = ys_j$. If $w \rightarrow ws_i$ is positive then $\ell(ws_i) = \ell(w) - 1$ (as $w \in W_0$) and so $\ell(ys_j) = \ell(y) - 1$, and if $w \rightarrow ws_i$ is negative then $\ell(ys_j) = \ell(y) + 1$. Thus

$$\frac{\mathbf{v}(\theta_J(w))}{\mathbf{v}(\theta_J(ws_i))} \mathbf{v}^{\text{wt}(ws_i) - \text{wt}(w)} = \frac{\mathbf{v}(y)}{\mathbf{v}(y)\mathbf{v}_{\alpha_j}^{-\epsilon} \mathbf{v}_{2\alpha_j}^{-\epsilon}} = \mathbf{v}_{\alpha_j}^\epsilon \mathbf{v}_{2\alpha_j}^\epsilon,$$

as required.

Suppose now that $i = 0$. If $us_0 \in \mathbb{W}^J$ then $u\varphi^\vee \in P^{(J)}$, and so by Corollary 2.11 we have $\theta_J(us_0) = y_{u\varphi^\vee}$. Thus $\theta_J(ws_0) = yy_{u\varphi^\vee}$ and so

$$\frac{\mathbf{v}(\theta_J(w))}{\mathbf{v}(\theta_J(ws_i))} \mathbf{v}^{\text{wt}(ws_i) - \text{wt}(w)} = \frac{\mathbf{v}(y)}{\mathbf{v}(yy_{u\varphi^\vee})} \mathbf{v}^{yu\varphi^\vee},$$

and since $u\varphi^\vee \in P^{(J)}$ the result follows by Lemma 3.10.

If $us_0 \notin \mathbb{W}^J$ then $u\varphi^\vee \in \Phi_J$. We have $\text{wt}(ws_0) = w\varphi^\vee$, and $ws_0 = t_{w\varphi^\vee} y u s_\varphi = t_{y u \varphi^\vee} (y s_{u\varphi}) u$ and so $\theta_J(ws_0) = y s_{u\varphi}$ (as $s_{u\varphi} \in W_J$). Thus

$$\frac{\mathbf{v}(\theta_J(w))}{\mathbf{v}(\theta_J(ws_i))} \mathbf{v}^{\text{wt}(ws_i) - \text{wt}(w)} = \frac{\mathbf{v}(y)}{\mathbf{v}(y s_{u\varphi})} \mathbf{v}^{y u \varphi^\vee}.$$

Since $us_0 \notin \mathbb{W}^J$ we have $u\alpha_0 \in \Phi_J + \mathbb{Z}\delta$. Thus $u\varphi \in \Phi_J$ (since $\alpha_0 = -\varphi + \delta$). Hence by Lemma 2.2 we have $u\varphi = \varphi_K$ for some $K \in \mathcal{K}(J)$. Moreover $w \rightarrow ws_0$ is positive if and only if $y u \varphi \in \Phi_J^+$, if and only if $y \varphi_K \in \Phi_J^+$. The result now follows from Lemma 3.13. □

Lemma 4.5. *Let $w \in \widetilde{W}$ and $i \in \{0\} \cup I$. Write $u = \theta^J(w)$, and suppose that $us_i \in \mathbb{W}^J$. Then $w \rightarrow ws_i$ is a positive crossing if and only if $u \rightarrow us_i$ is a positive crossing.*

Proof. Since $us_i \in \mathbb{W}^J$ we have $u\alpha_i \notin \Phi_J + \mathbb{Z}\delta$. Suppose that $w \rightarrow ws_i$ is positive, and so $w\alpha_i = -\alpha + k\delta \in -\Phi^+ + \mathbb{Z}\delta$ (recall (1.2)). Write $w = t_\lambda y u$ with $\lambda = \text{wt}(w)$, $y = \theta_J(w)$, and $u = \theta^J(w)$. Then $u\alpha_i = y^{-1}t_{-\lambda}(w\alpha_i) = y^{-1}t_{-\lambda}(-\alpha + k\delta) \in -y^{-1}\alpha + \mathbb{Z}\delta$. Since $y^{-1}\alpha \notin \Phi_J$ and $y \in W_J$ we have $\alpha \notin \Phi_J$. Then $y^{-1}\alpha$ is necessarily a positive (linear) root, and so $u \rightarrow us_i$ is positive.

Conversely, suppose that $u \rightarrow us_i$ is positive. So $u\alpha_i = -\beta + k\delta$ with $\beta \in \Phi^+$, and since $us_i \in \mathbb{W}^J$ we have $\beta \notin \Phi_J$. Then $w\alpha_i = t_\lambda y u \alpha_i = t_\lambda(-y\beta + k\delta) \in -y\beta + \mathbb{Z}\delta$, and since $\beta \notin \Phi_J$ we have $y\beta > 0$, hence the result. \square

The following proposition is the key to proving Theorem 4.2. Define a linear map $\varpi_{J,\nu} : \widetilde{\mathcal{H}} \rightarrow M_{J,\nu}$ by linearly extending the definition

$$\varpi_{J,\nu}(X_w) = (\nu\zeta_J)^{\text{wt}(w)} \nu(\theta_J(w))^{-1} \mathbf{m}_{\theta^J(w)}$$

for $w \in \widetilde{W}$. (To motivate this definition, note that if we assume for the moment that the proposed action given in Theorem 4.2 is indeed an action, then $\varpi_{J,\nu}(X_w) = \mathbf{m}_e \cdot X_w$.)

Proposition 4.6. *For $i \in \{0\} \cup I$ and $\sigma \in \Sigma$ we have*

$$\varpi_{J,\nu}(hT_i) = \varpi_{J,\nu}(h) \cdot T_i \quad \text{and} \quad \varpi_{J,\nu}(hT_\sigma) = \varpi_{J,\nu}(h) \cdot T_\sigma$$

for all $h \in \widetilde{\mathcal{H}}$ (with $\mathbf{m} \cdot T_i$ and $\mathbf{m} \cdot T_\sigma$ as in the statement of Theorem 4.2 for $\mathbf{m} \in M_{J,\nu}$).

Proof. By linearity it is sufficient to prove that $\varpi_{J,\nu}(X_w T_i) = \varpi_{J,\nu}(X_w) \cdot T_i$ and $\varpi_{J,\nu}(X_w T_\sigma) = \varpi_{J,\nu}(X_w) \cdot T_\sigma$ for all $w \in \widetilde{W}$, $i \in \{0\} \cup I$, and $\sigma \in \Sigma$. Consider the second formula. Let $\mu \in P$ be such that $t_\mu w \in \mathbb{W}^J$. Then $\theta(t_\mu w) = \theta(w)$ and so by Corollary 2.11 and Lemma 3.10 we have

$$\nu(\theta_J(w)) = \nu(\theta_J(t_\mu w)) = \nu(y_{\text{wt}(t_\mu w)}) = \nu^{\text{wt}(t_\mu w)} = \nu^{\mu + \text{wt}(w)}.$$

Since $t_\mu w \sigma \in \mathbb{W}^J$ (as $\sigma A_0 = A_0$) we similarly have $\nu(\theta_J(w\sigma)) = \nu^{\mu + \text{wt}(w\sigma)}$, and it follows that

$$\nu^{\text{wt}(w\sigma)} \nu(\theta_J(w\sigma))^{-1} = \nu^{-\mu} = \nu^{\text{wt}(w)} \nu(\theta_J(w))^{-1}.$$

Using this formula, along with the definition of $\varpi_{J,\nu}$, we have

$$\begin{aligned} \varpi_{J,\nu}(X_w T_\sigma) &= \varpi_{J,\nu}(X_{w\sigma}) = (\nu\zeta_J)^{\text{wt}(w\sigma)} \nu(\theta_J(w\sigma))^{-1} \mathbf{m}_{\theta^J(w\sigma)} \\ &= \zeta_J^{\text{wt}(w\sigma) - \text{wt}(w)} (\nu\zeta_J)^{\text{wt}(w)} \nu(\theta_J(w))^{-1} \mathbf{m}_{\theta^J(w\sigma)} \end{aligned}$$

and so $\varpi_{J,\nu}(X_w T_\sigma) = \varpi_{J,\nu}(X_w) \cdot T_\sigma$ as required.

We now prove the first formula, $\varpi_{J,\nu}(X_w T_i) = \varpi_{J,\nu}(X_w) \cdot T_i$. Write $u = \theta^J(w)$. Since $T_i = T_i^{-1} + (\mathbf{q}_i - \mathbf{q}_i^{-1})$ we have $X_w T_i = X_{ws_i}$ if $w \rightarrow ws_i$ is positive, and $X_w T_i = X_{ws_i} + (\mathbf{q}_i - \mathbf{q}_i^{-1})X_w$ if $w \rightarrow ws_i$ is negative. Thus, since $\theta^J(ws_i) = \theta^J(us_i)$ (by Lemma 4.3) we have

$$\varpi_{J,\nu}(X_w T_i) = \begin{cases} (\nu\zeta_J)^{\text{wt}(ws_i)} \nu(\theta_J(ws_i))^{-1} \mathbf{m}_{\theta^J(us_i)} & \text{if } \epsilon = 1 \\ (\nu\zeta_J)^{\text{wt}(ws_i)} \nu(\theta_J(ws_i))^{-1} \mathbf{m}_{\theta^J(us_i)} + (\mathbf{q}_i - \mathbf{q}_i^{-1}) (\nu\zeta_J)^{\text{wt}(w)} \nu(\theta_J(w))^{-1} \mathbf{m}_u & \text{if } \epsilon = -1 \end{cases}$$

where $\epsilon \in \{-1, 1\}$ is the sign of the crossing $w \rightarrow ws_i$.

We now compute $\varpi_{J,\nu}(X_w) \cdot T_i$. There are various cases to consider.

Case 1: Suppose that $u \rightarrow us_i$ is positive and $us_i \in \mathbb{W}^J$. Then

$$\varpi_{J,\nu}(X_w) \cdot T_i = (\nu\zeta_J)^{\text{wt}(w)} \nu(\theta_J(w))^{-1} \mathbf{m}_u \cdot T_i = (\nu\zeta_J)^{\text{wt}(w)} \zeta_J^{\text{wt}(us_i)} \nu(\theta_J(w))^{-1} \mathbf{m}_{\theta^J(us_i)}$$

By Lemma 4.5 we have that $w \rightarrow ws_i$ is necessarily positive. By Lemma 4.3 (with $v = s_i$) we have $\text{wt}(ws_i) = \text{wt}(w) + \theta_J(w)\text{wt}(us_i)$, and since $\theta_J(w) \in W_J$ we have $\text{wt}(ws_i) = \text{wt}(w) + \text{wt}(us_i) + \gamma$ for some $\gamma \in Q_J$. Since $\zeta_J^\gamma = 1$ it follows that $\varpi_{J,\nu}(X_w) \cdot T_i = \varpi_{J,\nu}(X_w T_i)$ if and only if

$$\mathbf{v}^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} = \mathbf{v}^{\text{wt}(ws_i)} \mathbf{v}(\theta_J(ws_i))^{-1},$$

and the result follows from Lemma 4.4.

Case 2: Suppose that $u \rightarrow us_i$ is negative and $us_i \in \mathbb{W}^J$. Then Lemma 4.5 gives that $w \rightarrow ws_i$ is also negative. We compute

$$\begin{aligned} \varpi_{J,\nu}(X_w) \cdot T_i &= (\mathbf{v}\zeta_J)^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \mathbf{m}_u \cdot T_i \\ &= (\mathbf{v}\zeta_J)^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \left[\zeta_J^{\text{wt}(us_i)} \mathbf{m}_{\theta^J(us_i)} + (\mathbf{q}_i - \mathbf{q}_i^{-1}) \mathbf{m}_u \right]. \end{aligned}$$

The same analysis as in Case 1 deals with the ζ_J factors, and the result again follows from Lemma 4.4.

Case 3: Suppose that $us_i \notin \mathbb{W}^J$, and so $\theta^J(us_i) = u$. Then we have

$$\varpi_{J,\nu}(X_w) \cdot T_i = (\mathbf{v}\zeta_J)^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \mathbf{v}_{u\alpha_i} \mathbf{v}_{2u\alpha_i} \mathbf{m}_u.$$

Since $\theta^J(us_i) = u$, the calculation of $\varpi_{J,\nu}(X_w T_i)$ from above gives

$$\varpi_{J,\nu}(X_w T_i) = \begin{cases} (\mathbf{v}\zeta_J)^{\text{wt}(ws_i)} \mathbf{v}(\theta_J(ws_i))^{-1} \mathbf{m}_u & \text{if } \epsilon = 1 \\ \left[(\mathbf{v}\zeta_J)^{\text{wt}(ws_i)} \mathbf{v}(\theta_J(ws_i))^{-1} + (\mathbf{q}_i - \mathbf{q}_i^{-1}) (\mathbf{v}\zeta_J)^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \right] \mathbf{m}_u & \text{if } \epsilon = -1, \end{cases}$$

where, as before, ϵ is the sign of the crossing $w \rightarrow ws_i$. Thus if $\epsilon = 1$ then $\varpi_{J,\nu}(X_w) \cdot T_i = \varpi_{J,\nu}(X_w T_i)$ if and only if

$$(\mathbf{v}\zeta_J)^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \mathbf{v}_{u\alpha_i} \mathbf{v}_{2u\alpha_i} = (\mathbf{v}\zeta_J)^{\text{wt}(ws_i)} \mathbf{v}(\theta_J(ws_i))^{-1}$$

Recall that $\text{wt}(ws_i) = \text{wt}(w) + \theta_J(w)\text{wt}(us_i)$. Moreover, since $us_i \notin \mathbb{W}^J$ we have $u\alpha_i \in \Phi_J + \mathbb{Z}\delta$, and hence $\text{wt}(us_i) \in Q_J$. Thus $\zeta_J^{\text{wt}(ws_i)} = \zeta_J^{\text{wt}(w)}$, and the result follows from Lemma 4.4.

Finally, if $\epsilon = -1$ then $\varpi_{J,\nu}(X_w) \cdot T_i = \varpi_{J,\nu}(X_w T_i)$ if and only if

$$\mathbf{v}^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \mathbf{v}_{u\alpha_i} \mathbf{v}_{2u\alpha_i} = \left[\mathbf{v}^{\text{wt}(ws_i)} \mathbf{v}(\theta_J(ws_i))^{-1} + (\mathbf{q}_i - \mathbf{q}_i^{-1}) \mathbf{v}^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \right]$$

Rearranging, and noting that $\mathbf{v}_{u\alpha_i} \mathbf{v}_{2u\alpha_i} - \mathbf{q}_i + \mathbf{q}_i^{-1} = \mathbf{v}_{u\alpha_i}^{-1} \mathbf{v}_{2u\alpha_i}^{-1}$, we have $\varpi_{J,\nu}(X_w) \cdot T_i = \varpi_{J,\nu}(X_w T_i)$ if and only if

$$\mathbf{v}^{\text{wt}(w)} \mathbf{v}(\theta_J(w))^{-1} \mathbf{v}_{u\alpha_i}^{-1} \mathbf{v}_{2u\alpha_i}^{-1} = \mathbf{v}^{\text{wt}(ws_i)} \mathbf{v}(\theta_J(ws_i))^{-1}$$

and the result again follows from Lemma 4.4. \square

We now give the proof of Theorem 4.2.

Proof of Theorem 4.2. It is sufficient to check that the relations (for $i, j \in \{0\} \cup I$, $\sigma, \sigma' \in \Sigma$)

$$\cdots T_i T_j T_i = \cdots T_j T_i T_j, \quad T_i^2 = 1 + (\mathbf{q}_i - \mathbf{q}_i^{-1}) T_i, \quad T_\sigma T_i = T_{\sigma(i)} T_\sigma, \quad T_\sigma T_{\sigma'} = T_{\sigma\sigma'}$$

are respected by the proposed action (with m_{ij} terms on either side in the first relation). For example, to verify that the braid relation is respected one use Proposition 4.6 repeatedly to get

$$\varpi_{J,\nu}(X_u T_i T_j T_i \cdots) = (\cdots (((\mathbf{m}_u \cdot T_i) \cdot T_j) \cdot T_i) \cdots)$$

for all $u \in W^J$, and note that $\varpi_{J,\nu}(X_u T_i T_j T_i \cdots) = \varpi_{J,\nu}(X_u T_j T_i T_j \cdots)$. The remaining relations follow in the same way. \square

Corollary 4.7. *The map $\varpi_{J,\nu} : \tilde{\mathcal{H}} \rightarrow M_{J,\nu}$ satisfies*

$$\varpi_{J,\nu}(hh') = \varpi_{J,\nu}(h) \cdot h' \quad \text{for all } h, h' \in \tilde{\mathcal{H}}.$$

Proof. This is immediate from Theorem 4.2. \square

4.2 The multiplicative character $\psi_{J,\mathfrak{v}}$

The J -Levi subalgebra of $\tilde{\mathcal{H}}$ is the subalgebra \mathcal{L}_J generated by the elements T_j , $j \in J$, and X^λ , $\lambda \in P$. By construction of the module $M_{J,\mathfrak{v}}$ it is clear that $\mathbb{R}[\zeta_J]\mathfrak{m}_e$ is stable under the action of \mathcal{L}_J , and thus we may define a map $\psi_{J,\mathfrak{v}} : \mathcal{L}_J \rightarrow \mathbb{R}[\zeta_J]$ by

$$\mathfrak{m}_e \cdot h = \psi_{J,\mathfrak{v}}(h)\mathfrak{m}_e \quad \text{for } h \in \mathcal{L}_J.$$

Recall the definition of τ_λ and y_λ from Definition 2.7.

Proposition 4.8. *We have the following.*

- (1) *The map $\psi_{J,\mathfrak{v}}$ is a multiplicative character (a 1-dimensional representation) of \mathcal{L}_J .*
- (2) *$\psi_{J,\mathfrak{v}}(T_j) = \mathfrak{v}_{\alpha_j}\mathfrak{v}_{2\alpha_j} \in \{\mathfrak{q}_j, -\mathfrak{q}_j^{-1}\}$ for all $j \in J$.*
- (3) *$\psi_{J,\mathfrak{v}}(T_y) = \mathfrak{v}(y)$ for all $y \in W_J$.*
- (4) *$\psi_{J,\mathfrak{v}}(X^\lambda) = \mathfrak{v}^\lambda \zeta_J^\lambda$ for all $\lambda \in P$.*
- (5) *If $2\alpha_j \notin \Phi_J$ then $\psi(X^{\alpha_j^\vee}) = \psi(T_j)^2$.*
- (6) *If Φ_J is not reduced then $\psi(X^{\alpha_n^\vee/2}T_n^{-1}) \in \{\mathfrak{q}_0, -\mathfrak{q}_0^{-1}\}$.*
- (7) *$\psi_{J,\mathfrak{v}}(T_{y_\lambda}) = \mathfrak{v}^\lambda$ for all $\lambda \in P^{(J)}$.*
- (8) *$\psi_{J,\mathfrak{v}}(X_{\tau_\lambda}) = \zeta_J^\lambda$ for all $\lambda \in P^{(J)}$.*

Proof. (1) follows from Corollary 4.7, and (2), (3) and (4) follow directly from the definition of the action. For (5) and (6), if $2\alpha_j \notin \Phi_J$ then the Bernstein-Lusztig relation gives $T_j X^{\omega_j} = X^{\omega_j - \alpha_j^\vee} T_j + (\mathfrak{q}_j - \mathfrak{q}_j^{-1}) X^{\omega_j}$, and so $T_j^{-1} X^{\omega_j} = X^{\omega_j - \alpha_j^\vee} T_j$, from which it follows that $\psi(X^{\alpha_j^\vee}) = \psi(T_j)^2$. If Φ_J is not reduced then the Bernstein-Lusztig relation gives $T_n^{-1} X^{\omega_n} = X^{\omega_n - \alpha_n^\vee} T_n + (\mathfrak{q}_0 - \mathfrak{q}_0^{-1}) X^{\omega_n - \alpha_n^\vee/2}$, and it follows that $\psi(X^{\alpha_n^\vee/2} T_n^{-1})$ satisfies a quadratic relation, giving $\psi(X^{\alpha_n^\vee/2} T_n^{-1}) \in \{\mathfrak{q}_0, -\mathfrak{q}_0^{-1}\}$ as required.

To prove (7) and (8), since $\psi_{J,\mathfrak{v}}(X^\lambda) = \mathfrak{v}^\lambda \zeta_J^\lambda$ it is sufficient to prove that $\psi_{J,\mathfrak{v}}(T_{y_\lambda}) = \mathfrak{v}^\lambda$, and since $\psi_{J,\mathfrak{v}}(T_{y_\lambda}) = \mathfrak{v}(y_\lambda)$ this follows from Lemma 3.10 (with $y = e$). \square

We also record the following fact.

Lemma 4.9. *Let $K \in \mathcal{K}(J)$ and let $\alpha \in \Phi_K^+$ be a long root of Φ_K (with all roots long if Φ_K is simply laced). Then $\psi_{J,\mathfrak{v}}(X^{\alpha^\vee} T_{s_\alpha}^{-1}) = \mathfrak{v}_\alpha$.*

Proof. Since $\alpha \in \Phi_J$ we have $\zeta_J^{\alpha^\vee} = 1$, and so by Proposition 4.8(3) and (4) we have

$$\psi_{J,\mathfrak{v}}(X^{\alpha^\vee} T_{s_\alpha}^{-1}) = \mathfrak{v}^{\alpha^\vee} \mathfrak{v}(s_\alpha)^{-1},$$

and the result follows from Lemma 3.12. \square

4.3 The J -affine Hecke algebra

Recall the definition of W_J^{aff} and its Coxeter generators $\{s'_j \mid j \in J^{\text{aff}}\}$ from Section 2.1. For $K \in \mathcal{K}(J)$ and $j \in J$ let

$$T'_j = T_j \quad \text{and} \quad T'_{0_K} = X^{\varphi_K^\vee} T_{s_{\varphi_K}}^{-1}.$$

Let $\mathcal{H}_J^{\text{aff}}$ be the subalgebra of $\tilde{\mathcal{H}}$ generated by $\{T'_j \mid j \in J^{\text{aff}}\}$. By the general theory of Section 1.6 (in particular (1.7)) the algebra $\mathcal{H}_J^{\text{aff}}$ is an affine Hecke algebra of type W_J^{aff} , and

$$\mathcal{H}_J^{\text{aff}} = \prod_{K \in \mathcal{K}(J)} \mathcal{H}_K^{\text{aff}}$$

with each $\mathcal{H}_K^{\text{aff}}$ an affine Hecke algebra of irreducible type W_K^{aff} .

Since $s_{\varphi_K} \in W_J$ for $K \in \mathcal{K}(J)$ the algebra $\mathcal{H}_J^{\text{aff}}$ is a subalgebra of the Levi subalgebra \mathcal{L}_J . Thus if $\mathfrak{v} = (\mathfrak{v}_\alpha)_{\alpha \in \Phi_J}$ is a J -parameter system then $\psi_{J,\mathfrak{v}}$, restricted to $\mathcal{H}_J^{\text{aff}}$, gives a 1-dimensional representation of $\mathcal{H}_J^{\text{aff}}$. By definition we have $\psi_{J,\mathfrak{v}}(T'_j) = \mathfrak{v}_{\alpha_j} \mathfrak{v}_{2\alpha_j}$ for all $j \in J$. The following corollary allows us to compute $\psi_{J,\mathfrak{v}}(T'_{0_K})$ for $K \in \mathcal{K}(J)$.

Corollary 4.10. For $K \in \mathcal{K}(J)$ we have $\psi_{J,\nu}(T'_{0_K}) = \nu_{\varphi_K} = \nu_{\varphi_K} \nu_{2\varphi_K}$.

Proof. This follows from Lemma 4.9 and the fact that $\nu_{2\varphi_K} = 1$. \square

4.4 The path formula for $\pi_{J,\nu}(T_w)$

We write $(\pi_{J,\nu}, M_{J,\nu})$ for the representation of $\widetilde{\mathcal{H}}$ in Theorem 4.2. In this section we prove the path formula (Theorem 4.13) for the representations $(\pi_{J,\nu}, M_{J,\nu})$. We will need the following simple lemma.

Lemma 4.11. If a wall H contains panels of type i and j then $\mathbf{q}_i = \mathbf{q}_j$.

Proof. Let $v, w \in W$ be such that $vA_0 \cap vs_iA_0$ and $wA_0 \cap ws_jA_0$ are panels of H (of types i, j respectively). Applying v^{-1} we may assume that $v = e$ (as W acts in a type preserving way), and replacing w with ws_j if necessary we may assume that eA_0 and wA_0 lie in the same halfspace determined by H (equivalently, $\ell(ws_j) = \ell(w) + 1$). Then $ws_j = s_iw$ (because these elements lie on the same side of all walls), and so s_j and s_i are conjugate in W . Hence $\mathbf{q}_i = \mathbf{q}_j$. \square

The following theorem is a preliminary step to proving the path formula, and gives a J -analogue of Proposition 1.6.

Theorem 4.12. For $u \in W^J$ and $w \in \widetilde{W}$ we have

$$\varpi_{J,\nu}(X_u T_w) = \sum_{p \in \mathcal{P}_J(\vec{w}, u)} \mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X_{\text{end}(p)}),$$

where \vec{w} is any reduced expression for w .

Proof. Induction on $\ell(w)$, with the case $\ell(w) = 0$ true by definition of the action $\mathbf{m}_u \cdot T_\sigma$. Suppose that $\ell(ws_k) = \ell(w) + 1$. Then by the induction hypothesis

$$\varpi_{J,\nu}(X_u T_{ws_k}) = \varpi_{J,\nu}(X_u T_w) \cdot T_k = \sum_{p \in \mathcal{P}_J(\vec{w}, u)} \mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X_{\text{end}(p)} T_k).$$

Let $p \in \mathcal{P}_J(\vec{w}, u)$ and write $v = \text{end}(p)$. There are 4 cases to consider.

Case 1: If $vA_0^- |^+ vs_k A_0$ with $vs_k \in \mathbb{W}^J$ then $\varpi_{J,\nu}(X_v T_k) = \varpi_{J,\nu}(X_{vs_k})$. Writing $p\epsilon_k^+$ for the path obtained from p by appending a positive s_k -crossing we have $\mathcal{Q}_{J,\nu}(p) = \mathcal{Q}_{J,\nu}(p\epsilon_k^+)$ and $vs_k = \text{end}(p\epsilon_k^+)$, and so

$$\mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X_v T_k) = \mathcal{Q}_{J,\nu}(p\epsilon_k^+) \varpi_{J,\nu}(X_{\text{end}(p\epsilon_k^+)}).$$

Case 2: If $vA_0^+ |^- vs_k A_0$ with $vs_k \in \mathbb{W}^J$ then using $T_k = T_k^{-1} + (\mathbf{q}_k - \mathbf{q}_k^{-1})$ gives

$$\mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X_v T_k) = \mathcal{Q}_{J,\nu}(p\epsilon_k^-) \varpi_{J,\nu}(X_{\text{end}(p\epsilon_k^-)}) + \mathcal{Q}_{J,\nu}(pf_k) \varpi_{J,\nu}(X_{\text{end}(pf_k)}),$$

where $p\epsilon_k^-$ denotes the path obtained from p by appending a negative s_k -crossing and pf_k denotes the path obtained from p by appending an s_k -fold.

Case 3: If $vA_0^- |^+ vs_k A_0$ with $vs_k \notin \mathbb{W}^J$ then by Lemma 2.2 the panel $vA_0 \cap vs_k A_0$ is contained in $H_{\varphi_K, 1}$ for some $K \in \mathcal{K}(J)$. Then $vs_k = s_{\varphi_K, 1} v$, and since $s_{\varphi_K, 1} = t_{\varphi_K} \check{s}_{\varphi_K}$ we have

$$X_v T_k = X_{vs_k} = X_{t_{\varphi_K} \check{s}_{\varphi_K} v} = X^{\varphi_K} X_{s_{\varphi_K} v}.$$

We claim that $X_{s_{\varphi_K} v} = T_{s_{\varphi_K}}^{-1} X_v$. To see this, let \vec{v} be a path of reduced type from e to v , and consider the reflected path $s_{\varphi_K}(\vec{v})$ (a path joining s_{φ_K} to $s_{\varphi_K} v$). Since $v \in \mathbb{W}^J$ no reduced path from e to v crosses any hyperplanes parallel to any wall H_α with $\alpha \in \Phi_J^+$, and since

$\Phi(s_{\varphi_K}) \subseteq \Phi_K^+ \subseteq \Phi_J^+$ it follows that each positive (respectively negative) crossing in \vec{v} is mapped to a positive (respectively negative) crossing in $s_{\varphi_K}(\vec{v})$. Thus $X_{s_{\varphi_K}v} = X_{s_{\varphi_K}}X_v$, and since $s_{\varphi_K} \in W_0$ we have $X_{s_{\varphi_K}} = T_{s_{\varphi_K}}^{-1}$, and hence the claim.

Thus $X_v T_k = X \varphi_K^{\vee} T_{s_{\varphi_K}}^{-1} X_v$, and since φ_K is long in Φ_K Lemma 4.9, and Proposition 4.8, gives

$$\begin{aligned} \mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X_v T_k) &= \mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X \varphi_K^{\vee} T_{s_{\varphi_K}}^{-1} X_v) \\ &= \mathcal{Q}_{J,\nu}(p) \psi_{J,\nu}(X \varphi_K^{\vee} T_{s_{\varphi_K}}^{-1}) \varpi_{J,\nu}(X_v) \\ &= \mathcal{Q}_{J,\nu}(p) \nu_{\varphi_K} \varpi_{J,\nu}(X_v) \\ &= \mathcal{Q}_{J,\nu}(pb_{\varphi_K}^-) \varpi_{J,\nu}(X_{\text{end}(pb_{\varphi_K}^-)}), \end{aligned}$$

where $pb_{\varphi_K}^-$ denotes the path obtained from p by appending a negative bounce on the wall $H_{\varphi_K;1}$ (note also that $\nu_{\varphi_K} = \nu_{\varphi_K} \nu_{2\varphi_K}$ to calculate $\mathcal{Q}_{J,\nu}(pb_{\varphi_K}^-)$).

Case 4: If $vA_0^+ |^- vs_k A_0$ with $vs_k \notin \mathbb{W}^J$ then by Lemma 2.2 the panel $vA_0 \cap vs_k A_0$ is contained in $H_{\alpha_j,0}$ for some $j \in J$. Using the formula $T_k = T_k^{-1} + (\mathbf{q}_k - \mathbf{q}_k^{-1})$ we have

$$\varpi_{J,\nu}(X_v T_k) = \varpi_{J,\nu}(X_{vs_k}) + (\mathbf{q}_k - \mathbf{q}_k^{-1}) \varpi_{J,\nu}(X_v).$$

Since $vs_k = s_j v$ we have $\varpi_{J,\nu}(X_v T_k) = \varpi_{J,\nu}(X_{s_j v}) + (\mathbf{q}_k - \mathbf{q}_k^{-1}) \varpi_{J,\nu}(X_v)$. Similar arguments to Case 3 shows that $X_{s_j v} = T_{s_j}^{-1} X_v$, and hence

$$\varpi_{J,\nu}(X_v T_k) = (\psi_{J,\nu}(T_j)^{-1} + \mathbf{q}_k - \mathbf{q}_k^{-1}) \varpi_{J,\nu}(X_v).$$

By Lemma 4.11 we have $\mathbf{q}_k = \mathbf{q}_j$, and so $\psi_{J,\nu}(T_j^{-1}) + \mathbf{q}_k - \mathbf{q}_k^{-1} = \psi_{J,\nu}(T_j^{-1}) + \mathbf{q}_j - \mathbf{q}_j^{-1} = \psi_{J,\nu}(T_j)$. If either Φ_J is reduced, or Φ_J is not reduced and $j \neq n$, then $\psi_{J,\nu}(T_j) = \nu_{\alpha_j}$. If Φ_J is not reduced and $j = n$ then $\psi_{J,\nu}(T_n) = \nu_{\alpha_n} \nu_{2\alpha_n}$. In all cases we have $\mathcal{Q}_{J,\nu}(p) \psi_{J,\nu}(T_j) = \mathcal{Q}_{J,\nu}(pb_{\alpha_j}^+)$ where $pb_{\alpha_j}^+$ denotes the path obtained from p by appending a positive bounce on the wall $H_{\alpha_j,0}$. Thus we have

$$\mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X_v T_k) = \mathcal{Q}_{J,\nu}(p) \psi_{J,\nu}(T_j) \varpi_{J,\nu}(X_v) = \mathcal{Q}_{J,\nu}(pb_{\alpha_j}^+) \varpi_{J,\nu}(X_{\text{end}(pb_{\alpha_j}^+)}).$$

Hence the result. \square

We now prove the path formula for the matrix entries of $\pi_{J,\nu}(T_w)$ with respect to the basis

$$\mathbf{B}_{J,\nu} = \{\mathbf{m}_u \mid u \in W^J\}$$

of $M_{J,\nu}$. A version of this formula for more general bases will be given in Theorem 4.15.

Theorem 4.13. *Let $w \in \widetilde{W}$. The matrix entries of $\pi_{J,\nu}(T_w)$ with respect to the basis $\mathbf{B}_{J,\nu}$ are*

$$[\pi_{J,\nu}(T_w)]_{u,v} = \sum_{\{p \in \mathcal{P}_J(\vec{w}, u) \mid \theta^J(p) = v\}} \mathcal{Q}_{J,\nu}(p) \zeta_J^{\text{wt}(p)} \quad \text{for } u, v \in \mathbf{F},$$

where \vec{w} is any choice of reduced expression for w .

Proof. Writing $\lambda = \text{wt}(p)$ we have, by Corollary 2.11 and (1.6),

$$X_{\text{end}(p)} = X \lambda T_{y_\lambda}^{-1} T_{\theta^J(p)}^{-1}.$$

Then by Theorem 4.12 and Proposition 4.8

$$\begin{aligned}
\varpi_{J,\nu}(X_u) \cdot T_w &= \sum_{p \in \mathcal{P}_J(\vec{w}, u)} \mathcal{Q}_{J,\nu}(p) \varpi_{J,\nu}(X^{\text{wt}(p)} T_{y_{\text{wt}(p)}^{-1}}^{-1} T_{\theta^J(p)^{-1}}^{-1}) \\
&= \sum_{p \in \mathcal{P}_J(\vec{w}, u)} \mathcal{Q}_{J,\nu}(p) \psi_{J,\nu}(X^{\text{wt}(p)} T_{y_{\text{wt}(p)}^{-1}}^{-1}) \varpi_{J,\nu}(T_{\theta^J(p)^{-1}}^{-1}) \\
&= \sum_{p \in \mathcal{P}_J(\vec{w}, u)} \mathcal{Q}_{J,\nu}(p) \zeta_J^{\text{wt}(p)} \varpi_{J,\nu}(T_{\theta^J(p)^{-1}}^{-1}),
\end{aligned}$$

completing the proof since $\varpi_{J,\nu}(T_{\theta^J(p)^{-1}}^{-1}) = \mathbf{m}_{\theta^J(p)}$. \square

4.5 Changing fundamental domains

It will be useful to have a more general version of Theorem 4.13 adapted to other choices of basis (for example, see Section 6).

Proposition 4.14. *If F is a fundamental domain for the action of \mathbb{T}_J on \mathbb{W}^J then*

$$\mathbf{B}_F = \{\varpi_{J,\nu}(X_u) \mid u \in F\}$$

is a basis of $M_{J,\nu}$.

Proof. By construction of the module $M_{J,\nu}$ the result holds for the fundamental domain $F = W^J$. Now let F be an arbitrary fundamental domain. For $u \in F$ define $u' \in W^J$ and $\lambda \in P^{(J)}$ by $u = \tau_\lambda u'$ (so $\lambda = \text{wt}(u)$ and $u' = \theta^J(u)$). Thus by Proposition 4.8 we have

$$\varpi_{J,\nu}(X_u) = \varpi_{J,\nu}(X_{\tau_\lambda} X_{u'}) = \mathbf{m}_e \cdot X_{\tau_\lambda} X_{u'} = \zeta_J^\lambda \mathbf{m}_e \cdot X_{u'} = \zeta_J^\lambda \varpi_{J,\nu}(X_{u'}),$$

and hence the result. \square

Note that $\mathbf{B}_{W^J} = \mathbf{B}_{J,\nu}$. A version of Theorem 4.13 for arbitrary fundamental domains is given below. Recall the definition of $\text{wt}(p, F)$ and $\theta(p, F)$ from (3.1).

Theorem 4.15. *Let F be a fundamental domain for the action of \mathbb{T}_J on \mathbb{W}^J . With respect to the basis \mathbf{B}_F of $M_{J,\nu}$ from Proposition 4.14, the matrix entries of $\pi_{J,\nu}(T_w)$, with $w \in \widetilde{W}$, are*

$$[\pi_{J,\nu}(T_w)]_{u,v} = \sum_{\{p \in \mathcal{P}_J(\vec{w}, u) \mid \theta(p, F) = v\}} \mathcal{Q}_{J,\nu}(p) \zeta_J^{\text{wt}(p, F)} \quad \text{for } u, v \in F,$$

where \vec{w} is any choice of reduced expression for w .

Proof. Using the fact that F is a fundamental domain, define functions $g : F \rightarrow W^J$ and $h : F \rightarrow P^{(J)}$ by the equation

$$u = \tau_{h(u)} g(u) \quad \text{for } u \in F.$$

Then, as in the proof of Proposition 4.14, we have $\varpi_{J,\nu}(X_u) = \zeta_J^{h(u)} \varpi_{J,\nu}(X_{g(u)})$, and by changing basis from the fundamental domain W^J case (proved in Theorem 4.13) we have

$$[\pi_{J,\nu}(T_w)]_{u,v} = \sum_{\{p \in \mathcal{P}_J(\vec{w}, g(u)) \mid \theta(p, W^J) = g(v)\}} \mathcal{Q}_{J,\nu}(p) \zeta_J^{\text{wt}(p) + h(u) - h(v)}.$$

Using Lemma 3.4 and Lemma 3.9 it follows that

$$\begin{aligned}
[\pi_{J,\nu}(T_w)]_{u,v} &= \sum_{\{p \in \mathcal{P}_J(\vec{w}, g(u)) \mid \theta(p, W^J) = g(v)\}} \mathcal{Q}_{J,\nu}(\tau_{h(u)} \cdot p) \zeta_J^{\text{wt}(\tau_{h(u)} \cdot p) - h(v)} \\
&= \sum_{\{p \in \mathcal{P}_J(\vec{w}, u) \mid \theta(p, F) = v\}} \mathcal{Q}_{J,\nu}(p) \zeta_J^{\text{wt}(p) - h(v)},
\end{aligned}$$

and the result follows since $\text{wt}(p, F) = \text{wt}(p) - h(v)$ if $\theta(p, F) = v$. \square

4.6 Intertwiners and irreducibility

For $i \in I$ define *intertwiners* $U_i \in \tilde{\mathcal{H}}$ by

$$U_i = \begin{cases} (1 - X^{-\alpha_i^\vee})T_i - (\mathfrak{q}_i - \mathfrak{q}_i^{-1}) & \text{if } 2\alpha_i \notin \Phi \\ (1 - X^{-\alpha_n^\vee})T_n - (\mathfrak{q}_n - \mathfrak{q}_n^{-1} + (\mathfrak{q}_0 - \mathfrak{q}_0^{-1})X^{-\alpha_n^\vee/2}) & \text{if } \Phi = \text{BC}_n \text{ and } i = n \text{ and } \mathfrak{q}_0 \neq \mathfrak{q}_n \\ (1 - X^{-\alpha_n^\vee/2})T_n - (\mathfrak{q}_n - \mathfrak{q}_n^{-1}) & \text{if } \Phi = \text{BC}_n \text{ and } i = n \text{ and } \mathfrak{q}_0 = \mathfrak{q}_n \end{cases}$$

The terminology comes from the fact that these elements “intertwine” the weight spaces of $\tilde{\mathcal{H}}$ -modules (see Proposition 4.18(1)). A direct calculation, using the Bernstein-Lusztig relation, gives

$$U_i^2 = \mathfrak{q}_i^2(1 - \mathfrak{q}_i^{-2}X^{-\alpha_i^\vee})(1 - \mathfrak{q}_i^{-2}X^{\alpha_i^\vee}) \quad \text{if } 2\alpha_i \notin \Phi, \quad (4.1)$$

while if $\Phi = \text{BC}_n$ and $i = n$ then if $\mathfrak{q}_0 \neq \mathfrak{q}_n$ we have

$$U_n^2 = \mathfrak{q}_n^2(1 - \mathfrak{q}_0^{-1}\mathfrak{q}_n^{-1}X^{-\alpha_n^\vee/2})(1 + \mathfrak{q}_0\mathfrak{q}_n^{-1}X^{-\alpha_n^\vee/2})(1 - \mathfrak{q}_0^{-1}\mathfrak{q}_n^{-1}X^{\alpha_n^\vee/2})(1 + \mathfrak{q}_0\mathfrak{q}_n^{-1}X^{\alpha_n^\vee/2})$$

and if $\mathfrak{q}_0 = \mathfrak{q}_n$ then $U_n^2 = \mathfrak{q}_n^2(1 - \mathfrak{q}_n^{-2}X^{-\alpha_n^\vee/2})(1 - \mathfrak{q}_n^{-2}X^{\alpha_n^\vee/2})$.

The elements U_i satisfy the braid relations (see [28, Proposition 2.14], however note that we have normalised these elements so that they are elements of the Hecke algebra), and hence for $w \in W_0$ we may define

$$U_w = U_{i_1} \cdots U_{i_\ell}$$

whenever $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression. From the Bernstein-Lusztig relation we have $U_i X^\lambda = X^{s_i \lambda} U_i$, and hence we have the very useful relation

$$U_w X^\lambda = X^{w\lambda} U_w \quad \text{for all } w \in W_0 \text{ and } \lambda \in P.$$

Lemma 4.16. *If $\alpha \in \Phi \setminus \Phi_J$ then $\psi_{J,\nu}(X^{\alpha^\vee}) \notin \mathbb{R}$.*

Proof. Let $\alpha \in \Phi$. If $\mathbb{R} \ni \psi_{J,\nu}(X^{\alpha^\vee}) = \nu^\lambda \zeta_J^{\alpha^\vee}$ then $\zeta_J^{\alpha^\vee} = 1$. Thus $(\alpha^\vee)^J = 0$, and so $\alpha^\vee \in V_J$, and so $\alpha \in \Phi_J$. \square

Proposition 4.17. *The module $M_{J,\nu}$ has basis $\{\varpi_{J,\nu}(U_u) \mid u \in W^J\}$.*

Proof. Let $u = s_{i_1} \cdots s_{i_\ell} \in W_0$ be a reduced expression. From the Bernstein-Lusztig relation we have

$$\begin{aligned} U_u &= (1 - X^{-\alpha_{i_1}^\vee})T_{i_1}(1 - X^{-\alpha_{i_2}^\vee})T_{i_2} \cdots (1 - X^{-\alpha_{i_\ell}^\vee})T_{i_\ell} + \text{lower terms} \\ &= (1 - X^{-\alpha_{i_1}^\vee})(1 - X^{-s_{i_1}\alpha_{i_2}^\vee}) \cdots (1 - X^{-s_{i_1} \cdots s_{i_{\ell-1}}\alpha_{i_\ell}^\vee})T_u + \text{lower terms} \\ &= \left[\prod_{\alpha \in \Phi(u)} (1 - X^{-\alpha^\vee}) \right] T_u + \text{lower terms,} \end{aligned}$$

where on each line “lower terms” denotes a linear combination of terms $p_v(X)T_v$ with $v < u$ (in Bruhat order). Since $T_u = X_u + (\text{terms } X_v \text{ with } v < u)$ and $\varpi_{J,\nu}(X_u) = \mathbf{m}_u$ for $u \in W^J$ we have

$$\varpi_{J,\nu}(U_u) = \left[\prod_{\alpha \in \Phi(u)} (1 - \psi_{J,\nu}(X^{\alpha^\vee})^{-1}) \right] \mathbf{m}_u + \text{lower terms.}$$

For $u \in W^J$ we have $\Phi_J(u) = \emptyset$ (by Lemma 1.2), and so by Lemma 4.16 the coefficient

$$\prod_{\alpha \in \Phi(u)} (1 - \psi_{J,\nu}(X^{-\alpha^\vee}))$$

does not vanish, and the result follows. \square

The following proposition gives the decomposition of $M_{J,\mathfrak{v}}$ into weight spaces.

Proposition 4.18. *Let $J \subseteq I$ and let \mathfrak{v} be a J -parameter system. For $u \in W_0$ let*

$$M_u = \{\mathfrak{m} \in M_{J,\mathfrak{v}} \mid \mathfrak{m} \cdot X^\lambda = \psi_{J,\mathfrak{v}}(X^{u\lambda})\mathfrak{m} \text{ for all } \lambda \in P\}.$$

- (1) *If $u, us_i \in W^J$ then the map $\tilde{U}_i : M_u \rightarrow M_{us_i}$ with $\tilde{U}_i(\mathfrak{m}) = \mathfrak{m} \cdot U_i$ is bijective.*
- (2) *For $u \in W^J$ we have $M_u = \{r\varpi_{J,\mathfrak{v}}(U_u) \mid r \in \mathbb{R}[\zeta_J]\}$, and*

$$M_{J,\mathfrak{v}} = \bigoplus_{u \in W^J} M_u.$$

Proof. (1) For $u \in W_0$ let $z_u : P \rightarrow \mathbb{R}[\zeta_J]$ be the map $z_u^\lambda = \psi_{J,\mathfrak{v}}(X^{u\lambda})$. Then $z_u^\lambda = z^{u\lambda}$, where we write $z = z_e$, and $M_u = \{\mathfrak{m} \in M_{J,\mathfrak{v}} \mid \mathfrak{m} \cdot X^\lambda = z_u^\lambda \mathfrak{m} \text{ for all } \lambda \in P\}$. For any $u \in W_0$ and $i \in I$, if $\mathfrak{m} \in M_u$ then

$$(\mathfrak{m} \cdot U_i)X^\lambda = (\mathfrak{m} \cdot X^{s_i\lambda})U_i = z_u^{s_i\lambda}(\mathfrak{m} \cdot U_i) = z_{us_i}^\lambda(\mathfrak{m} \cdot U_i),$$

and so $\mathfrak{m} \cdot U_i \in M_{us_i}$. It follows that there are operators $\tilde{U}_i : M_u \rightarrow M_{us_i}$ and $\tilde{U}_i : M_{us_i} \rightarrow M_u$ given by $\tilde{U}_i(\mathfrak{m}) = \mathfrak{m} \cdot U_i$. Thus $\tilde{U}_i^2 : M_u \rightarrow M_u$. If $2\alpha_i \notin \Phi$ then by (4.1) we have

$$\tilde{U}_i^2(\mathfrak{m}) = \mathfrak{m} \cdot U_i^2 = \mathfrak{q}_i^2(1 - \mathfrak{q}_i^{-2}z^{-u\alpha_i^\vee})(1 - \mathfrak{q}_i^{-2}z^{u\alpha_i^\vee})\mathfrak{m}$$

and so if $z^{u\alpha_i^\vee} \neq \mathfrak{q}_i^{\pm 2}$ then the operators $\tilde{U}_i : M_u \rightarrow M_{us_i}$ and $\tilde{U}_i : M_{us_i} \rightarrow M_u$ are bijective. If Φ is not reduced and $i = n$ then the same result holds for \tilde{U}_n provided $z^{u\alpha_n^\vee/2} \neq (\mathfrak{q}_0\mathfrak{q}_n)^{\pm 1}, -(\mathfrak{q}_0^{-1}\mathfrak{q}_n)^{\pm 1}$ (with the $-(\mathfrak{q}_0^{-1}\mathfrak{q}_n)^{\pm 1}$ case omitted if $\mathfrak{q}_0 = \mathfrak{q}_n$).

Suppose now that $u \in W^J$ and $i \in I$ with $us_i \in W^J$. Then $u\alpha_i \notin \Phi_J$, and thus Lemma 4.16 gives $z^{u\alpha_i^\vee} \notin \mathbb{R}$ (and also $z^{u\alpha_n^\vee/2} \notin \mathbb{R}$ in the non reduced case) and so by the previous paragraph $\tilde{U}_i : M_u \rightarrow M_{us_i}$ is bijective, proving (1).

(2) It is clear that $\{r\varpi_{J,\mathfrak{v}}(U_u) \mid r \in \mathbb{R}[\zeta_J]\} \subseteq M_u$ for each $u \in W^J$, and thus by Proposition 4.17 the spaces M_u , $u \in W^J$, span $M_{J,\mathfrak{v}}$. Thus to prove (2) it is sufficient to show that if $u_1, u_2 \in W^J$ with $M_{u_1} = M_{u_2}$ then $u_1 = u_2$. To see this, $M_{u_1} = M_{u_2}$ implies that $\psi_{J,\mathfrak{v}}(X^{u_1\lambda}) = \psi_{J,\mathfrak{v}}(X^{u_2\lambda})$ for all $\lambda \in P$. Replacing λ by $u_2^{-1}\lambda$ we have $\psi_{J,\mathfrak{v}}(X^{u_1u_2^{-1}\lambda}) = 1$ for all $\lambda \in P$. Thus $(u_1u_2^{-1}\lambda - \lambda)^J = 0$ for all $\lambda \in P$, and so $u_1u_2^{-1}\lambda \in \lambda + V_J$ for all $\lambda \in P$. Applying this to $\lambda = \alpha^\vee \in \Phi_J^\vee$ it follows that $u_1u_2^{-1}\alpha^\vee \in \Phi_J^\vee$ for all $\alpha^\vee \in \Phi_J^\vee$, and hence $u_1u_2^{-1} \in W_J$. So $u_1 \in W_Ju_2$, forcing $u_1 = u_2$ (as $u_1, u_2 \in W^J$). \square

Corollary 4.19. *Let $J \subseteq I$ and let \mathfrak{v} be a J -parameter system. The representation $(\pi_{J,\mathfrak{v}}, M_{J,\mathfrak{v}})$ is irreducible.*

Proof. Let N be a nonzero $\tilde{\mathcal{H}}$ -invariant submodule of $M_{J,\mathfrak{v}}$. Since N is invariant under the action of the elements X^λ , $\lambda \in P$, it follows from Proposition 4.18(2) that $N \cap M_u \neq \emptyset$ for some $u \in W^J$. Since M_u is 1-dimensional we have $M_u \subseteq N$. But then Proposition 4.18(1), along with $\tilde{\mathcal{H}}$ -invariance, forces $N = M_{J,\mathfrak{v}}$. \square

4.7 Generic induced representations

In this subsection we realise the combinatorial modules $M_{J,\mathfrak{v}}$ introduced in Theorem 4.2 as induced representations from 1-dimensional representations of a Levi subalgebra. Let $\psi_{J,\mathfrak{v}}$ be as in Section 4.2, and let $\xi_{J,\mathfrak{v}}$ be a generator of the 1-dimensional \mathcal{L}_J module $\mathbb{R}[\zeta_J]\xi_{J,\mathfrak{v}}$ affording the character $\psi_{J,\mathfrak{v}}$. That is,

$$\xi_{J,\mathfrak{v}} \cdot h = \psi_{J,\mathfrak{v}}(h)\xi_{J,\mathfrak{v}} \quad \text{for all } h \in \mathcal{L}_J.$$

Let $M'_{J,\mathfrak{v}} = \text{Ind}_{\mathcal{L}_J}^{\tilde{\mathcal{H}}}(\psi_{J,\mathfrak{v}}) = (\mathbb{R}[\zeta_J]\xi_{J,\mathfrak{v}}) \otimes_{\mathcal{L}_J} \tilde{\mathcal{H}}$.

Proposition 4.20. *The module $M'_{J,\mathfrak{v}}$ has basis $\{\xi_{J,\mathfrak{v}} \otimes X_u \mid u \in W^J\}$.*

Proof. Since $\{X_w \mid w \in \widetilde{W}\}$ is a basis of $\widetilde{\mathcal{H}}$ the set $\{\xi_{J,\mathfrak{v}} \otimes X_w \mid w \in \widetilde{W}\}$ spans $M'_{J,\mathfrak{v}}$. If $w \in \widetilde{W}$ then by (1.6) we have $X_w = X^\lambda T_{u_1}^{-1}$ where $\lambda = \text{wt}(w)$ and $u = \theta(w)$. Write $u = u_1 u_2 \in W_J W^J$. Since $X^\lambda T_{u_1}^{-1} \in \mathcal{L}_J$ we have

$$\xi_{J,\mathfrak{v}} \otimes X_w = \xi_{J,\mathfrak{v}} \otimes X^\lambda T_{u_1}^{-1} T_{u_2}^{-1} = \psi_{J,\mathfrak{v}}(X^\lambda T_{u_1}^{-1})(\xi_{J,\mathfrak{v}} \otimes X_{u_2}).$$

Thus $M'_{J,\mathfrak{v}}$ is spanned by $\{\xi_{J,\mathfrak{v}} \otimes X_u \mid u \in W^J\}$, and these elements are linearly independent because $\{X^\lambda T_{v_1}^{-1} T_{v_2}^{-1} \mid \lambda \in P, v \in W_J, u \in W^J\}$ is a basis of $\widetilde{\mathcal{H}}$. \square

The following theorem shows that $(\pi_{J,\mathfrak{v}}, M_{J,\mathfrak{v}})$ is isomorphic to the representation $(\pi'_{J,\mathfrak{v}}, M'_{J,\mathfrak{v}})$, and moreover identifies bases of each module giving an isomorphism of matrix representations.

Theorem 4.21. *We have $M_{J,\mathfrak{v}} \cong M'_{J,\mathfrak{v}}$. Moreover*

$$[\pi_{J,\mathfrak{v}}(h)]_{u,v} = [\pi'_{J,\mathfrak{v}}(h)]_{u,v} \quad \text{for all } h \in \widetilde{\mathcal{H}} \text{ and } u, v \in W^J,$$

where $M_{J,\mathfrak{v}}$ and $M'_{J,\mathfrak{v}}$ are endowed with the bases $\{\mathbf{m}_u \mid u \in W^J\}$ and $\{\xi_{J,\mathfrak{v}} \otimes X_u \mid u \in W^J\}$ respectively.

Proof. We will show that the action of T_i ($i \in \{0\} \cup I$) and T_σ ($\sigma \in \Sigma$) on $M'_{J,\mathfrak{v}}$ with respect to the basis $\{\xi_{J,\mathfrak{v}} \otimes X_u \mid u \in W^J\}$ agrees with the action from Theorem 4.2. The analysis is easy for the cases T_i with $i \in I$ and T_σ with $\sigma \in \Sigma$, and so we focus on the action of T_0 .

Case 1: Suppose that $uA_0^- |^+ us_0A_0$ with $us_0 \in \mathbb{W}^J$. Then $X_u T_0 = X_{us_0}$, and since $us_0 \in \mathbb{W}^J$ Corollary 2.11 gives $us_0 = \tau_\lambda \theta^J(uw_0)$ where $\lambda = \text{wt}(us_0) = u\varphi^\vee$. Then

$$(\xi_{J,\mathfrak{v}} \otimes X_u) \cdot T_0 = \xi_{J,\mathfrak{v}} \otimes X_{us_0} = \xi_{J,\mathfrak{v}} \otimes X_{\tau_{u\varphi^\vee} X_{\theta^J(us_0)}} = \psi_{J,\mathfrak{v}}(X_{\tau_{u\varphi^\vee}})(\xi_{J,\mathfrak{v}} \otimes X_{\theta^J(us_0)}),$$

and Proposition 4.8(8) gives $(\xi_{J,\mathfrak{v}} \otimes X_u) \cdot T_0 = \zeta_J^{u\varphi^\vee} (\xi_{J,\mathfrak{v}} \otimes X_{\theta^J(us_0)})$, and hence the result in this case.

Case 2: Suppose that $uA_0^+ |^- us_0A_0$ with $us_0 \in \mathbb{W}^J$. Since $T_0 = T_0^{-1} + (\mathfrak{q}_0 - \mathfrak{q}_0^{-1})$ we have

$$\begin{aligned} (\xi_{J,\mathfrak{v}} \otimes X_u) \cdot T_0 &= (\xi_{J,\mathfrak{v}} \otimes X_{us_0}) + (\mathfrak{q}_0 - \mathfrak{q}_0^{-1})(\xi_{J,\mathfrak{v}} \otimes X_u) \\ &= \zeta_J^{u\varphi^\vee} (\xi_{J,\mathfrak{v}} \otimes X_{\theta^J(us_0)}) + (\mathfrak{q}_0 - \mathfrak{q}_0^{-1})(\xi_{J,\mathfrak{v}} \otimes X_u), \end{aligned}$$

and hence the result in this case.

Case 3: If $uA_0^- |^+ us_0A_0$ with $us_0 \notin \mathbb{W}^J$ then, exactly as in Case 3 of the proof of Theorem 4.12, we have

$$X_u T_0 = X^{\varphi_K^\vee} T_{s_{\varphi_K}}^{-1} X_u$$

for some $K \in \mathcal{K}(J)$. Since φ_K is long in Φ_K Lemma 4.9 gives

$$(\xi_{J,\mathfrak{v}} \otimes X_u) \cdot T_0 = \xi_{J,\mathfrak{v}} \otimes X^{\varphi_K^\vee} T_{s_{\varphi_K}}^{-1} X_u = \psi_{J,\mathfrak{v}}(X^{\varphi_K^\vee} T_{s_{\varphi_K}}^{-1})(\xi_{J,\mathfrak{v}} \otimes X_u) = \mathfrak{v}_\alpha \mathfrak{v}_{2\alpha} (\xi_{J,\mathfrak{v}} \otimes X_u),$$

as required.

Case 4: If $uA_0^+ |^- us_0A_0$ with $us_0 \notin \mathbb{W}^J$ then by Lemma 2.2 the panel $uA_0 \cap us_0A_0$ is contained in $H_{\alpha_j,0}$ for some $j \in J$. Thus this case is impossible, as $u\alpha_0 = -u\varphi + \delta \neq \pm\alpha_i$ for any $i \in I$. \square

Remark 4.22. If $J = \emptyset$ then \mathfrak{v} is vacuous, and $(\pi'_{\emptyset,\mathfrak{v}}, M'_{\emptyset,\mathfrak{v}})$ is the *principal series representation* of $\widetilde{\mathcal{H}}$ with central character $\zeta = \zeta_I$.

Remark 4.23. Up to specialising the “variables” ζ_i and extending scalars, all representations of $\widetilde{\mathcal{H}}$ induced from 1-dimensional representations of a Levi subalgebra can be realised by the construction in Theorem 4.2. Let us briefly explain this. Let $\psi : \mathcal{L}_J \rightarrow \mathbf{R}'$ be a one dimensional representation of \mathcal{L}_J over an integral domain \mathbf{R}' containing \mathbf{R} . For $\alpha \in \Phi_J$ let

$$\mathbf{v}_\alpha = \begin{cases} \psi(T_j) & \text{if } \alpha \in W_J\alpha_j \text{ with } j \in J \text{ and } \alpha \in \Phi_0 \cap \Phi_1 \\ \psi(X^{\alpha_n^\vee/2})\psi(T_n)^{-1} & \text{if } \Phi_J \text{ not reduced and } \alpha \in W_J(2\alpha_n) \\ \psi(X^{\alpha_n^\vee/2})^{-1}\psi(T_n)^2 & \text{if } \Phi_J \text{ is not reduced and } \alpha \in W_J\alpha_n, \end{cases}$$

and define $z : P \rightarrow \mathbf{R}'$ by $z^\lambda = \mathbf{v}^{-\lambda}\psi(X^\lambda)$, with \mathbf{v}^λ as in (3.2). It is not difficult to see that \mathbf{v} is a J -parameter system and that $z^\gamma = 1$ for all $\gamma \in Q_J$.

After extending the ring \mathbf{R}' to a ring \mathbf{R}'' if necessary one can choose a specialisation $\zeta_i \mapsto z_i \in \mathbf{R}''$ such that $\zeta_j^\lambda \mapsto z^\lambda$ for all $\lambda \in P$. Then, by Theorem 4.21, the combinatorial module $M_{J,\mathbf{v}}$ constructed in Theorem 4.2 specialises to $\text{Ind}_{\mathcal{L}_J}^{\widetilde{\mathcal{H}}}(z)$ (after extending scalars if necessary).

5 Bounded representations

Let $L : W \rightarrow \mathbb{N}$ be a *positive weight function* on W . That is, $L(w) > 0$ for $w \in W \setminus \{e\}$ and $L(uv) = L(u) + L(v)$ whenever $\ell(uv) = \ell(u) + \ell(v)$. Let \mathbf{q} be an invertible indeterminate, and specialise $\mathbf{q}_i = \mathbf{q}^{L(s_i)}$ for $i \in \{0\} \cup I$. The ring \mathbf{R} specialises to $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$, and one can consider the associated *weighted Hecke algebras* $\mathcal{H}(L)$ and $\widetilde{\mathcal{H}}(L)$ over the ring $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$. In this context a J -parameter system becomes a *weighted J -parameter system*, where each occurrence of \mathbf{q}_j in Definition 3.5 (with $j \in \{0\} \cup J$) is replaced by $\mathbf{q}^{L(s_j)}$.

It is convenient to make the following convention throughout this section (by the natural symmetry present in BC_n this convention does not result in a loss of generality).

Convention 5.1. If Φ is of type BC_n we assume that $L(s_n) \geq L(s_0)$.

5.1 Boundedness

In [16, 15] the authors introduced the notion of a *balanced system of cell representations*, inspired by the work of Geck [10, 11] in the finite case, and we used this notion to prove Lusztig’s Conjectures **P1–P15** for rank 3 affine Coxeter systems with arbitrary positive weight function L . A key part of this concept was the property of *boundedness* of a matrix representation. We recall this theory here.

Every nonzero rational function $f(\mathbf{q}) = a(\mathbf{q})/b(\mathbf{q})$ in \mathbf{q} can be written under the form $f(\mathbf{q}) = \mathbf{q}^N a'(\mathbf{q}^{-1})/b'(\mathbf{q}^{-1})$ with $N \in \mathbb{Z}$ and with $a'(\mathbf{q}^{-1})$ and $b'(\mathbf{q}^{-1})$ polynomials in \mathbf{q}^{-1} with nonzero constant term. The integer N in this expression is unique, and we shall call it the *degree* of the rational function $f(\mathbf{q})$, written $\deg f(\mathbf{q}) = N$. For example, $\deg((\mathbf{q}^2+1)(\mathbf{q}^3+1)/(\mathbf{q}^7-\mathbf{q}+1)) = -2$, and if $f(\mathbf{q})$ is a polynomial then $\deg f(\mathbf{q})$ agrees with the usual degree. We set $\deg(0) = -\infty$.

By a *matrix representation* of $\widetilde{\mathcal{H}}(L)$ we shall mean a triple (π, M, \mathbf{B}) where M is a right $\widetilde{\mathcal{H}}(L)$ -module over an $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ -polynomial ring \mathbf{S} , and \mathbf{B} is a basis of M . We write (for $h \in \widetilde{\mathcal{H}}(L)$ and $u, v \in \mathbf{B}$)

$$\pi(h, \mathbf{B}) \quad \text{and} \quad [\pi(h, \mathbf{B})]_{u,v}$$

for the matrix of $\pi(h)$ with respect to the basis \mathbf{B} , and the $(u, v)^{\text{th}}$ entry of $\pi(h, \mathbf{B})$. When the basis \mathbf{B} is clear from context we omit it from the notation.

Definition 5.2. A matrix representation (π, M, \mathbf{B}) is called *bounded* if $\deg([\pi(T_w, \mathbf{B})]_{u,v})$ is bounded from above for all $u, v \in \mathbf{B}$ and all $w \in \widetilde{W}$. In this case we call the integer

$$\mathbf{a}_{\pi, M, \mathbf{B}} = \max\{\deg([\pi(T_w, \mathbf{B})]_{u,v}) \mid u, v \in \mathbf{B}, w \in W\} \quad (5.1)$$

the *bound* of the matrix representation.

Definition 5.3. If (π, M, \mathbf{B}) is a bounded matrix representation with bound $\mathbf{a}_{\pi, M, \mathbf{B}}$ then the cell recognised by (π, M, \mathbf{B}) is the set

$$\Gamma_{\pi, M, \mathbf{B}} = \{w \in \widetilde{W} \mid \deg([\pi(T_w, \mathbf{B})]_{u,v}) = \mathbf{a}_{\pi, M, \mathbf{B}} \text{ for some } u, v \in \mathbf{B}\}.$$

If $w \in \Gamma_{\pi, M, \mathbf{B}}$ the *leading matrix* of w is defined by

$$\mathbf{c}_{\pi, M, \mathbf{B}}(w) = \text{sp}_{\mathbf{q}^{-1}=0}(\mathbf{q}^{-\mathbf{a}_{\pi, M, \mathbf{B}}} \pi(T_w, \mathbf{B})),$$

where $\text{sp}_{\mathbf{q}^{-1}=0}$ is specialisation at $\mathbf{q}^{-1} = 0$.

The notion of leading matrices comes from the work of Geck [10, 11], and played a crucial role in [16, 15]. We will not discuss leading matrices further in this work, until the final example in Section 6.

Remark 5.4. If a finite dimensional representation (π, M, \mathbf{B}) is bounded then (π, M, \mathbf{B}') is bounded for all bases \mathbf{B}' of M , because the (finite) change of basis matrix has a bound on the degrees of its entries. Thus we may talk of a “bounded (finite dimensional) representation” without specifying the basis. However we note that both the value of the bound, and the cell recognised, are highly dependent on the particular basis. In this paper we will study the matrix representations $(\pi_{J,v}, M_{J,v}, \mathbf{B}_F)$ where \mathbf{B}_F is the basis from Proposition 4.14 associated to a fundamental domain F . These bases appear to have some remarkable and beautiful properties (see Conjectures 5.15 and 5.16). We note that the basis from Proposition 4.17, while easier to work with in many respects, does not appear to have the same remarkable properties.

Let F and F' be fundamental domains for the action \mathbb{T}_J on \mathbb{W}^J . By Remark 5.4 $(\pi_{J,v}, M_{J,v}, \mathbf{B}_F)$ is bounded if and only if $(\pi_{J,v}, M_{J,v}, \mathbf{B}_{F'})$ is bounded. However the connection is much stronger, as the following proposition shows.

Proposition 5.5. *Let F and F' be fundamental domains for the action of \mathbb{T}_J on \mathbb{W}^J such that the associated matrix representations are bounded. Then the matrix representations have the same bound, and recognise the same cell.*

Proof. Define $g : F \rightarrow W^J$ and $h : F \rightarrow P^{(J)}$ by the equation $u = \tau_{h(u)}g(u)$ for $u \in F$ (as in the proof of Theorem 4.15). Then $\varpi_{J,v}(X_u) = \zeta_J^{h(u)} \mathbf{m}_{g(u)}$, showing that the change of basis matrix from \mathbf{B}_F to $\mathbf{B}_{J,v} = \mathbf{B}_{W^J}$ is a monomial matrix with entries independent of \mathbf{q} . It follows that the matrix representations with respect to the bases \mathbf{B}_F and $\mathbf{B}_{J,v}$ have the same bound and recognise the same elements, hence the result. \square

Thus if $(\pi_{J,v}, M_{J,v})$ is bounded we write $\mathbf{a}_{J,v} = \mathbf{a}_{\pi_{J,v}, M_{J,v}, \mathbf{B}_F}$ and $\Gamma_{J,v} = \Gamma_{\pi_{J,v}, M_{J,v}, \mathbf{B}_F}$ (for any choice of fundamental domain F for the action of \mathbb{T}_J on \mathbb{W}^J).

Example 5.6. The concept of boundedness is already interesting and delicate for 1-dimensional representations. Note that the particular choice of basis is irrelevant in the 1-dimensional case. Consider the $\tilde{\mathbf{G}}_2$ case with weight function $L(s_0) = L(s_2) = b$ and $L(s_1) = a$.

$$\begin{array}{ccc} b & b & a \\ \bullet & \bullet & \bullet \\ 0 & 2 & 1 \end{array}$$

Let $\psi_{I,v}$ be the 1-dimensional representation of $\tilde{\mathcal{H}}(L)$ with $\mathbf{v}_{\alpha_1} = \psi_{I,v}(T_1) = \mathbf{q}^a$ and $\mathbf{v}_{\alpha_2} = \psi_{I,v}(T_2) = \psi_{I,v}(T_0) = -\mathbf{q}^{-b}$. It is a (not entirely trivial) exercise to show that:

- (1) If $a/b < 1$ then $\psi_{I,v}$ is bounded with $\mathbf{a}_{I,v} = a$ and $\Gamma_{I,v} = \{1\}$ (recall $1 = s_1$).
- (2) If $a/b = 1$ then $\psi_{I,v}$ is bounded with $\mathbf{a}_{I,v} = a = 3a - 2b$ and $\Gamma_{I,v} = \{1, 121, 12121\}$.
- (3) If $1 < a/b < 3/2$ then $\psi_{I,v}$ is bounded with $\mathbf{a}_{I,v} = 3a - 2b$ and $\Gamma_{I,v} = \{12121\}$

- (4) If $a/b = 3/2$ then $\psi_{I,v}$ is bounded with $\mathbf{a}_{I,v} = 3a - 2b$ and $\Gamma_{I,v} = \{(12121) \cdot (02121)^k \mid k \geq 0\}$.
(5) If $a/b > 3/2$ then $\psi_{I,v}$ is not bounded.

(One approach is to consider reduced expressions in $\tilde{\mathbf{G}}_2$).

Example 5.7. If $J = \emptyset$ then $(\pi_{\emptyset,v}, M_{\emptyset,v}, \mathbf{B}_{\emptyset,v}) = (\pi, M, \mathbf{B})$ is the principal series representation (see Theorem 4.21). By [16, Lemma 6.2] we have $\deg \mathcal{Q}(p) \leq L(\mathbf{w}_0)$ for all positively folded paths p of reduced type, and hence by Theorem 4.15 (π, M, \mathbf{B}) is bounded with bound $L(\mathbf{w}_0)$.

For each $\lambda \in P$ let W_λ be the stabiliser of λ in W and let w_λ be the longest element of W_λ . By an analysis similar to that in [16, Theorem 6.6] it can be shown that the set of elements of W that are recognised by (π, M, \mathbf{B}) is

$$\Gamma = \{w \in W \mid w = w_1 \cdot w_\lambda \cdot w_2, w_1, w_2 \in W, \lambda \in P\},$$

where the notation $u \cdot v$ means $\ell(uv) = \ell(u) + \ell(v)$ (recall Conventions 1.7 and 5.1 are in force, and we further assume that $L(s_n) > L(s_0)$ for the above statement, and so in particular for the BC_n case $\{w_\lambda \mid \lambda \in P\} = \{w_0\}$).

Remark 5.8. A main motivation for our path formula in Theorem 4.15 is to give a combinatorial approach to studying bounded representations. Indeed J -alcove paths are very useful in studying boundedness (see, for example, Theorems 5.9 and 5.13, and the work in [16, 15]), however before proceeding we briefly discuss some complications and subtleties in the theory.

Let $u, v \in W^J$ and let \vec{w} be a reduced expression for $w \in \widetilde{W}$. If there is $N > 0$ such that $\deg \mathcal{Q}_{J,v}(p) \leq N$ for all $p \in \mathcal{P}_J(\vec{w}, u)$ with $\theta^J(p) = v$ then by Theorem 4.15 we have $\deg[\pi_{J,v}(T_w)]_{u,v} \leq N$, showing that boundedness of paths leads directly to boundedness of matrix entries. It is important to note that the reverse direction is more subtle, as this fact significantly complicates the theory.

For example, let Φ be of type A_3 with $J = \{1\}$ and let \mathbf{v} be the J -parameter system with $\mathbf{v}_{\alpha_1} = -\mathbf{q}^{-1}$. Let $\vec{w}_0 = 323123$ (a reduced expression for the longest element of W_0). The paths $p \in \mathcal{P}_J(\vec{w}_0, e)$ with $\theta^J(p) = e$, along with their respective \mathbf{v} -masses, are as follows (where \hat{i} denotes a fold, and \check{i} denotes a bounce):

$$\begin{array}{ll} p_1 = \hat{3}\hat{2}\hat{3}\check{1}\hat{2}\hat{3} & \mathcal{Q}_{J,v}(p_1) = -\mathbf{q}^{-1}(\mathbf{q} - \mathbf{q}^{-1})^5 \\ p_2 = \hat{3}\hat{2}\hat{3}\hat{1}\hat{2}\hat{3} & \mathcal{Q}_{J,v}(p_2) = (\mathbf{q} - \mathbf{q}^{-1})^4 \\ p_3 = \hat{3}\hat{2}\hat{3}\check{1}\hat{2}\hat{3} & \mathcal{Q}_{J,v}(p_3) = -\mathbf{q}^{-1}(\mathbf{q} - \mathbf{q}^{-1})^3 \\ p_4 = \hat{3}\hat{2}\hat{3}\check{1}\hat{2}\hat{3} & \mathcal{Q}_{J,v}(p_4) = -\mathbf{q}^{-1}(\mathbf{q} - \mathbf{q}^{-1})^3 \\ p_5 = 3\hat{2}\hat{3}\hat{1}\hat{2}\hat{3} & \mathcal{Q}_{J,v}(p_5) = (\mathbf{q} - \mathbf{q}^{-1})^2. \end{array}$$

Note that $\deg \mathcal{Q}_{J,v}(p_1) = \deg \mathcal{Q}_{J,v}(p_2) = 4$. By Theorem 4.13 we have

$$[\pi_{J,v}(T_{w_0})]_{e,e} = \sum_{i=1}^5 \mathcal{Q}_{J,v}(p_i) = \mathbf{q}^{-4}(\mathbf{q} - \mathbf{q}^{-1})^2,$$

which has degree -2 .

Note that the leading terms have cancelled. The combinatorics of this cancellation of leading terms in matrix coefficients appears to be rather delicate. In [16, 15] the authors were able to deal with this phenomena in affine rank 3 Hecke algebras, as the cancellations are rather rare and tame in that low dimension case. Understanding these cancellations in arbitrary rank would lead to a significant advance in understanding boundedness, with implications to Kazhdan-Lusztig theory (as illustrated by the discussion in the following sections, and in the work [16, 15]). For example, in Section 6 we are able to sufficiently control the cancellations that occur in the A_n case with $J = \{1, 2, \dots, n-1\}$.

5.2 Classification of bounded modules $M_{J,\mathbf{v}}$

In this section we classify the weighted J -parameter systems for which $(\pi_{J,\mathbf{v}}, M_{J,\mathbf{v}})$ is bounded (c.f. Remark 5.4). Recall from Section 4.3 that the J -affine Hecke algebra is a subalgebra of \mathcal{L}_J , and hence $\psi_{J,\mathbf{v}}$ restricts to a 1-dimensional representation of $\mathcal{H}_J^{\text{aff}}$ (see Corollary 4.10).

Theorem 5.9. *Let \mathbf{v} be a weighted J -parameter system. The following are equivalent.*

- (1) *The representation $(\pi_{J,\mathbf{v}}, M_{J,\mathbf{v}})$ is bounded.*
- (2) *We have $\deg \mathbf{v}^\lambda \leq 0$ for all $\lambda \in P^+$.*
- (3) *We have $\deg \mathbf{v}^{\omega_j} \leq 0$ for all $j \in J$.*
- (4) *We have $\deg \mathbf{v}^{u\lambda} \leq 0$ for all $\lambda \in P^+$ and all $u \in W^J$.*
- (5) *The associated 1-dimensional representation $\psi_{J,\mathbf{v}}$ of $\mathcal{H}_J^{\text{aff}}(L)$ is bounded.*
- (6) *There is a uniform bound $\deg \mathcal{Q}_{J,\mathbf{v}}(p) \leq N$ for all J -folded alcove paths of reduced type.*

Proof. (1) \implies (2). Suppose that $(\pi_{J,\mathbf{v}}, M_{J,\mathbf{v}})$ is bounded. Since

$$\mathbf{m}_e \cdot X^\lambda = \psi_{J,\mathbf{v}}(X^\lambda)\mathbf{m}_e = \mathbf{v}^\lambda \zeta_J^\lambda \mathbf{m}_e$$

we have $[\pi_{J,\mathbf{v}}(X^\lambda)]_{e,e} = \mathbf{v}^\lambda \zeta_J^\lambda$ and so $\deg \mathbf{v}^\lambda \leq 0$ for all $\lambda \in P^+$ (for if not the degree of $[\pi_{J,\mathbf{v}}(T_{t_{N\lambda}})]_{e,e} = [\pi_{J,\mathbf{v}}(X^{N\lambda})]_{e,e}$ is unbounded for $N \in \mathbb{N}$).

(2) \implies (3) is trivial.

(3) \implies (4). Since $\mathbf{v}^{\omega_i} = 1$ for $i \in I \setminus J$ we have

$$\mathbf{v}^{u\lambda} = \prod_{j \in J} \mathbf{v}^{\langle u\lambda, \alpha_j \rangle \omega_j}.$$

If $u \in W^J$ then $u^{-1}\alpha \in \Phi^+$ for all $\alpha \in \Phi_J^+$, and hence $\langle u\lambda, \alpha \rangle = \langle \lambda, u^{-1}\alpha \rangle \geq 0$ for all $\lambda \in P^+$, $u \in W^J$, and $\alpha \in \Phi_J^+$. Thus $\deg \mathbf{v}^{u\lambda} \leq 0$.

(4) \implies (1). For $\lambda \in P^+$ let m_λ denote the unique minimal length element of $W_0 t_\lambda W_0$. Write $t_\lambda = m_\lambda w_\lambda$ where $w_\lambda \in W_0$ and $\ell(t_\lambda) = \ell(m_\lambda) + \ell(w_\lambda)$. Each $w \in \widetilde{W}$ can be written as $w = um_\lambda v$ with $u, v \in W_0$ and $\lambda \in P^+$, and moreover $\ell(w) = \ell(u) + \ell(m_\lambda) + \ell(v)$. Thus

$$T_w = T_u T_{m_\lambda} T_v = T_u T_{t_\lambda} T_{w_\lambda}^{-1} T_v = T_u X^\lambda T_{w_\lambda}^{-1} T_v.$$

Let $\mathbf{B}'_{J,\mathbf{v}} = \{\xi_{J,\mathbf{v}} \otimes U_u \mid u \in W^J\}$ denote the basis of $M_{J,\mathbf{v}}$ from Proposition 4.17, and write $\pi'_{J,\mathbf{v}}(h) = \pi_{J,\mathbf{v}}(h, \mathbf{B}'_{J,\mathbf{v}})$. We have $\pi'_{J,\mathbf{v}}(T_w) = \pi'_{J,\mathbf{v}}(X^\lambda) \pi'_{J,\mathbf{v}}(T_u) \pi'_{J,\mathbf{v}}(T_v) \pi'_{J,\mathbf{v}}(T_{w_\lambda}^{-1})$. Since $u, v, w_\lambda \in W_0$, and W_0 is finite, there is a global bound on the degree of the entries of the last three matrices in this product. Moreover, since $U_u X^\lambda = X^{u\lambda} U_u$ the matrix $\pi'_{J,\mathbf{v}}(X^\lambda)$ is diagonal with entries $\psi_{J,\mathbf{v}}(X^{u\lambda}) = \mathbf{v}^{u\lambda} \zeta_J^{u\lambda}$ for $u \in W^J$, and hence by assumption the degree of the entries of this matrix is bounded for $\lambda \in P^+$. Thus the degree of the entries of $\pi'_{J,\mathbf{v}}(T_w)$ is bounded. Hence the result (c.f. Remark 5.4).

(3) \iff (5). We have established the equivalence of (1) and (3). Applying this to the case $J = I$ we see that a 1-dimensional representation $\psi_{I,\mathbf{v}}$ of $\mathcal{H}(L)$ is bounded if and only if $\deg \mathbf{v}^{\omega_i} \leq 0$ for all $i \in I$. Applying this to the case of the weighted J -affine Hecke algebra we see that $\psi_{J,\mathbf{v}}$ is a bounded 1-dimensional representation of this algebra if and only if $\deg \mathbf{v}^{\omega_j} \leq 0$ for all $j \in J$, hence the result.

(1) \iff (6). Theorem 4.13 shows that (6) implies (1). The reverse implication is complicated by the cancellations discussed in Remark 5.8, and we argue as follows. Suppose that (1) holds. Let p be a J -folded path of reduced type, and let p_J be the associated J -straightening. Since p_J is positively folded a well known result (see [19, Lemma 7.7] or [16, Lemma 6.2]) gives $f(p_J) \leq \ell(\mathbf{w}_0)$, where $f(p_J)$ denotes the number of folds in p_J . Decompose p_J as $p_0 \cdot f_1 \cdot p_1 \cdot f_2 \cdots f_k \cdot p_k$, where p_0, \dots, p_k are straight (ie, no folds) and f_1, \dots, f_k are the folds (so $k \leq \ell(\mathbf{w}_0)$). Consider the path p_j . Let $x, y \in W_J^{\text{aff}}$ such that p_j starts in $x\mathcal{A}_J$ and ends in $y\mathcal{A}_J$. It is not difficult

to see, using Proposition 3.16, that the contribution of p_j to $\mathcal{Q}_{J,\mathbf{v}}(p)$ is $\psi_{J,\mathbf{v}}(T'_{x^{-1}y})$, where T'_w denotes the standard basis of $\mathcal{H}_J^{\text{aff}}$ (see Section 4.3). We have already seen that (1) implies (5), and hence there is a bound $\deg \psi_{J,\mathbf{v}}(T'_w) \leq N'$ for all $w \in W_J^{\text{aff}}$. Thus

$$\deg \mathcal{Q}_{J,\mathbf{v}}(p) \leq \sum_{i \in \{0\} \cup I} L(s_i) f_i(p) + (\ell(\mathbf{w}_0) + 1) N'$$

(with the first sum coming from the folds) hence (6). \square

Remark 5.10. On extending scalars to \mathbb{C} and specialising $\mathfrak{q} \rightarrow q$ and $\zeta_i \rightarrow z_i$ for $i \in I \setminus J$ with $q > 1$ and $z_i \in \mathbb{C}$ with $|z_i| = 1$, Casselman's criteria for temperedness (see [26, Lemma 2.20]) along with the equivalence of (1) and (2) in Theorem 5.9 shows that $(\pi_{J,\mathbf{v}}, M_{J,\mathbf{v}})$ is bounded if and only if the specialised representation is tempered.

By Theorem 5.9 determining boundedness of $(\pi_{J,\mathbf{v}}, M_{J,\mathbf{v}})$ is equivalent to determining boundedness of the associated 1-dimensional representation $\psi_{J,\mathbf{v}}$ of $\mathcal{H}_J^{\text{aff}}$. The latter is a much simpler task. Since it suffices to consider each irreducible component of $\mathcal{H}_J^{\text{aff}}$, it is sufficient to determine the bounded 1-dimensional representations of an affine Hecke algebra $\tilde{\mathcal{H}}(L)$ with irreducible root system Φ . Since Φ is irreducible the degrees of freedom in choosing a weighted I -parameter system is equal to the number of root lengths in Φ . If Φ is simply laced then $\mathbf{v}_\alpha = \mathbf{v}$ for all $\alpha \in \Phi$ and we write $b = L(s_i)$ for any $i \in I$. If Φ is reduced with two distinct root lengths we write $\mathbf{v}_{\text{sh}} = \mathbf{v}_\alpha$ for any short root α , and $\mathbf{v}_{\text{lo}} = \mathbf{v}_\alpha$ for any long root α , and we write $a = L(s_j)$ if α_j is short, and $b = L(s_j)$ if α_j is long. Then $\mathbf{v}_{\text{sh}} \in \{\mathfrak{q}^a, -\mathfrak{q}^{-a}\}$ and $\mathbf{v}_{\text{lo}} \in \{\mathfrak{q}^b, -\mathfrak{q}^{-b}\}$.

Proposition 5.11. *The bounded 1-dimensional representations of $\tilde{\mathcal{H}}(L)$ are the maps $\psi_{I,\mathbf{v}}$ where $\mathbf{v} = (\mathbf{v}_\alpha)_{\alpha \in \Phi}$ is a weighted I -parameter system appearing in the list below.*

- (1) *If Φ is simply laced then $\mathbf{v}_\alpha = -\mathfrak{q}^{-b}$ for all $\alpha \in \Phi$.*
- (2) *If Φ is reduced and not simply laced then the possible values of $(\mathbf{v}_{\text{sh}}, \mathbf{v}_{\text{lo}})$ are as follows, with the stated constraints on a, b :*

$(\mathbf{v}_{\text{sh}}, \mathbf{v}_{\text{lo}})$	\mathbf{B}_n	\mathbf{C}_n	\mathbf{F}_4	\mathbf{G}_2
$(-\mathfrak{q}^{-a}, -\mathfrak{q}^{-b})$	$a, b \geq 1$	$a, b \geq 1$	$a, b \geq 1$	$a, b \geq 1$
$(\mathfrak{q}^a, -\mathfrak{q}^{-b})$	$a/b \leq n - 1$	$a/b \leq 1/(n - 1)$	$a/b \leq 6/5$	$a/b \leq 3/2$
$(-\mathfrak{q}^{-a}, \mathfrak{q}^b)$	$a/b \geq 2(n - 1)$	$a/b \geq 2/(n - 1)$	$a/b \geq 5/3$	$a/b \geq 2$

- (3) *If Φ is of type \mathbf{BC}_n then the possible values of $(\mathbf{v}_{\alpha_1}, \mathbf{v}_{\alpha_n}, \mathbf{v}_{2\alpha_n}, \mathbf{v}_{2\alpha_n})$ are as follows, with the stated constraints on $a = L(s_n)$, $b = L(s_1)$, and $c = L(s_0)$ (and Convention 5.1 in force):*

$(\mathbf{v}_{\alpha_1}, \mathbf{v}_{\alpha_n}, \mathbf{v}_{2\alpha_n}, \mathbf{v}_{2\alpha_n})$	\mathbf{BC}_n
$(-\mathfrak{q}^{-b}, -\mathfrak{q}^{-a}, -\mathfrak{q}^{-c})$	$a, b, c \geq 1$
$(-\mathfrak{q}^{-b}, -\mathfrak{q}^{-a}, \mathfrak{q}^c)$	$a, b, c \geq 1$
$(\mathfrak{q}^b, -\mathfrak{q}^{-a}, -\mathfrak{q}^{-c})$	$a/b + c/b \geq 2(n - 1)$
$(\mathfrak{q}^b, -\mathfrak{q}^{-a}, \mathfrak{q}^c)$	$a/b - c/b \geq 2(n - 1)$
$(-\mathfrak{q}^{-b}, \mathfrak{q}^a, -\mathfrak{q}^{-c})$	$a/b - c/b \leq n - 1$
$(-\mathfrak{q}^{-b}, \mathfrak{q}^a, \mathfrak{q}^c)$	$a/b + c/b \leq n - 1$

Proof. By Theorem 5.9 it is sufficient to determine the weighted I -parameter systems \mathbf{v} with $\deg \mathbf{v}^\lambda \leq 0$ for all $\lambda \in P^+$, and this in turn is equivalent to $\deg \mathbf{v}^{\omega_i} \leq 0$ for all $i \in I$. Write $\rho = \rho_I$ and $\rho' = \rho'_I$ (see Section 1.4). If Φ is simply laced then $\mathbf{v}_\alpha = \mathbf{v}$ is constant for all $\alpha \in \Phi$, and hence $\mathbf{v}^{\omega_i} = \mathbf{v}^{\langle \omega_i, 2\rho \rangle}$ for all $i \in I$. Since $\mathbf{v} \in \{\mathfrak{q}^b, -\mathfrak{q}^{-b}\}$ and $\langle \omega_i, 2\rho \rangle > 0$ the result follows in this case.

Suppose now that Φ is reduced with two distinct root lengths. Then $\mathbf{v}^{\omega_i} = \mathbf{v}_{\text{sh}}^{\langle \omega_i, 2\rho' \rangle} \mathbf{v}_{\text{lo}}^{\langle \omega_i, 2\rho \rangle}$. Thus if $(\mathbf{v}_{\text{sh}}, \mathbf{v}_{\text{lo}}) = (\mathfrak{q}^a, -\mathfrak{q}^{-b})$ we require $a/b \leq \langle \omega_i, 2\rho \rangle / \langle \omega_i, 2\rho' \rangle$ for all $1 \leq i \leq n$, and if $(\mathbf{v}_{\text{sh}}, \mathbf{v}_{\text{lo}}) = (-\mathfrak{q}^{-a}, \mathfrak{q}^b)$ we require $a/b \geq \langle \omega_i, 2\rho \rangle / \langle \omega_i, 2\rho' \rangle$ for all $1 \leq i \leq n$.

The result now follows by considering each case. If Φ is of type B_n then in Bourbaki conventions [3] we have $2\rho = 2(n-1)e_1 + 2(n-2)e_2 + \cdots + 2e_{n-1}$ and $2\rho' = e_1 + e_2 + \cdots + e_n$. Since $\omega_i = e_1 + \cdots + e_i$ we have $\langle \omega_i, 2\rho' \rangle = i$ and $\langle \omega_i, 2\rho \rangle = i(2n-i-1)$ for $1 \leq i \leq n$. Thus if $(\mathbf{v}_{\text{sh}}, \mathbf{v}_{\text{lo}}) = (\mathbf{q}^a, -\mathbf{q}^{-b})$ we have $a/b \leq 2n-i-1$ for all $1 \leq i \leq n$, and so $a/b \leq n-1$. If $(\mathbf{v}_{\text{sh}}, \mathbf{v}_{\text{lo}}) = (-\mathbf{q}^{-a}, \mathbf{q}^b)$ then $a/b \geq 2n-i-1$ for all $1 \leq i \leq n$, and so $a/b \geq 2(n-1)$.

The remaining reduced cases are similar. In type C_n we have $2\rho = 2(e_1 + \cdots + e_n)$, $2\rho' = 2(n-1)e_1 + 2(n-2)e_2 + \cdots + 2e_{n-1}$, $\omega_i = e_1 + \cdots + e_i$ for $1 \leq i < n$, and $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. In type F_4 we have $2\rho' = 5e_1 + e_2 + e_3 + e_4$, $2\rho = 6e_1 + 4e_2 + 2e_3$, $\omega_1 = e_1 + e_2$, $\omega_2 = 2e_1 + e_2 + e_3$, $\omega_3 = 3e_1 + e_2 + e_3 + e_4$, and $\omega_4 = 2e_1$. In type G_2 we have $2\rho' = -e_2 + e_3$, $2\rho = -e_1 - e_2 + 2e_3$, $\omega_1 = -e_2 + e_3$, and $\omega_2 = \frac{1}{3}(-e_1 - e_2 + 2e_3)$.

Finally, consider the BC_n case. We have

$$\mathbf{v}^\lambda = \mathbf{v}_{\alpha_1}^{\langle \lambda, 2(n-1)e_1 + 2(n-2)e_2 + \cdots + 2e_{n-1} \rangle} \mathbf{v}_{\alpha_n}^{\langle \lambda, e_1 + \cdots + e_n \rangle} \mathbf{v}_{2\alpha_n}^{2\langle \lambda, e_1 + \cdots + e_n \rangle}.$$

Since $\omega_i = e_1 + \cdots + e_i$ we require $\deg(\mathbf{v}_{\alpha_1}^{i(2n-i-1)} \mathbf{v}_{\alpha_n}^i \mathbf{v}_{2\alpha_n}^{2i}) \leq 0$ for all $1 \leq i \leq n$, and the result follows by considering the possibilities $\mathbf{v}_{\alpha_1} \in \{\mathbf{q}^b, -\mathbf{q}^{-b}\}$, $\mathbf{v}_{\alpha_n} \mathbf{v}_{2\alpha_n} \in \{\mathbf{q}^a, -\mathbf{q}^{-a}\}$, and $\mathbf{v}_{2\alpha_n} \in \{\mathbf{q}^c, -\mathbf{q}^{-c}\}$. \square

Example 5.12. Combining Theorem 5.9 and Proposition 5.11 gives a very explicit classification of the bounded modules $M_{J,\mathbf{v}}$. For example, in type F_4 the bounded representations are listed below. We encode the representations by the Dynkin diagram of type F_4 in which the nodes $j \in J$ are encircled. An encircled node j is white if $\psi_{J,\mathbf{v}}(T_j) = -\mathbf{q}^{-L(s_j)}$ and black if $\psi_{J,\mathbf{v}}(T_j) = \mathbf{q}^{L(s_j)}$. Let $L(s_i) = a$ if α_i is short and $L(s_i) = b$ if α_i is long (thus $L(s_1) = L(s_2) = b$ and $L(s_3) = L(s_4) = a$). In some cases there are constraints on a/b for the representation to be bounded, and these constraints are indicated under the diagram.

$a/b \geq 1$	$a/b \leq 1/2$	$a/b \geq 2$	$a/b \leq 1$	$a/b \geq 5/3$	$a/b \leq 6/5$

5.3 The bound $\mathbf{a}_{J,\mathbf{v}}$

In this section we use the path formula (Theorem 4.13) to give an upper bound for the bound $\mathbf{a}_{J,\mathbf{v}}$ for the representations associated to the simplest weighted J -parameter system $\hat{\mathbf{v}} = (\mathbf{v}_\alpha)_{\alpha \in \Phi_J}$ given by

$$\begin{aligned} \mathbf{v}_{\alpha_j} &= -\mathbf{q}^{-L(s_j)} && \text{whenever } j \in J \text{ with } 2\alpha_j \notin \Phi_J \\ \mathbf{v}_{\alpha_n} \mathbf{v}_{2\alpha_n} &= -\mathbf{q}^{-L(s_n)} && \text{if } \Phi_J \text{ is not reduced} \\ \mathbf{v}_{2\alpha_n} &= -\mathbf{q}^{-L(s_0)} && \text{if } \Phi_J \text{ is not reduced.} \end{aligned}$$

With Convention 5.1 in force we have the following result.

Theorem 5.13. *The bound of $(\pi_{J,\hat{\mathbf{v}}}, M_{J,\hat{\mathbf{v}}}, \mathbf{B}_{J,\hat{\mathbf{v}}})$ satisfies $\mathbf{a}_{J,\hat{\mathbf{v}}} \leq L(\mathbf{w}_0)$.*

Proof. By Theorem 4.13 we have, for $w \in \widetilde{W}$ and $u, v \in \mathbf{B}_{J,\hat{\mathbf{v}}}$,

$$\deg[\pi_{J,\hat{\mathbf{v}}}(T_w)]_{u,v} \leq \max\{\deg \mathcal{Q}_{J,\hat{\mathbf{v}}}(p) \mid p \in \mathcal{P}_J(\vec{w}, u) \text{ with } \theta^J(p) = v\}, \quad (5.2)$$

and by Proposition 3.16 we have

$$\mathcal{Q}_{J,\check{\nu}}(p) = \mathcal{Q}(p_J) \prod_{\alpha - k\delta \in \check{\Phi}^+} v_{\alpha - k\delta}^{c_{\alpha,k}(p_J)} \quad (5.3)$$

for all J -folded paths p . It is well known that $\deg \mathcal{Q}(p') \leq L(\mathbf{w}_0)$ for all positively folded alcove paths p' (see [19, Lemma 7.7] or [16, Lemma 6.2]). Since the second term in (5.3) has degree bounded above by 0 we have $\deg \mathcal{Q}_{J,\check{\nu}}(p) \leq L(\mathbf{w}_0)$. \square

5.4 Conjectures

The precise value of the bound $\mathbf{a}_{J,\check{\nu}}$ appears to be a very subtle statistic, and we conjecture (based on the analysis in [16, 15] and the examples below) that it is intimately connected to Lusztig's \mathbf{a} -function [22], Macdonald's c -function [23], and Opdam's Plancherel Theorem [26]. Again Convention 5.1 is assumed to be in force. Firstly, we believe that the upper bound $L(\mathbf{w}_0)$ for $\mathbf{a}_{J,\check{\nu}}$ given in Theorem 5.13 applies more generally.

Conjecture 5.14. *If $(\pi_{J,\check{\nu}}, M_{J,\check{\nu}}, \mathbf{B}_{J,\check{\nu}})$ is bounded then the bound $\mathbf{a}_{J,\check{\nu}}$ satisfies $\mathbf{a}_{J,\check{\nu}} \leq L(\mathbf{w}_0)$ with equality if and only if $J = \emptyset$.*

In fact, we shall state a considerably stronger conjecture giving a formula for $\mathbf{a}_{J,\check{\nu}}$. For $\alpha \in \Phi$ we define $\mathbf{q}_\alpha \in \mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ as follows. If Φ is reduced, let

$$\mathbf{q}_\alpha = \mathbf{q}^{L(s_i)} \quad \text{if } \alpha \in W_0\alpha_i,$$

and if Φ is not reduced let

$$\mathbf{q}_\alpha = \begin{cases} \mathbf{q}^{L(s_i)} & \text{if } \alpha \in W_0\alpha_i \text{ with } i \neq n \\ \mathbf{q}^{L(s_n) - L(s_0)} & \text{if } \alpha \in W_0\alpha_n \\ \mathbf{q}^{L(s_0)} & \text{if } \alpha \text{ is long.} \end{cases}$$

Conjecture 5.15. *If $(\pi_{J,\check{\nu}}, M_{J,\check{\nu}}, \mathbf{B}_{J,\check{\nu}})$ is bounded then the bound $\mathbf{a}_{J,\check{\nu}}$ is given by*

$$\mathbf{a}_{J,\check{\nu}} = L(\mathbf{w}_0) - \frac{1}{2} \deg \prod'_{\alpha \in \Phi} \frac{1 - \mathbf{q}_{\alpha/2}^{-1} v^{\alpha^\vee}}{1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_\alpha^{-2} v^{\alpha^\vee}}$$

where \prod' indicates that any factors in the numerator or denominator that are 0 are omitted.

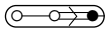


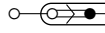


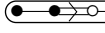
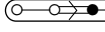
We make the following conjecture, linking bounded representations to Lusztig's \mathbf{a} function (see [22] for the definition of the \mathbf{a} -function). In particular, note that Conjecture 5.16 combined with Conjecture 5.15 give a conjectural formula for the value of Lusztig's \mathbf{a} -function at elements $w \in W$ that are recognised by some bounded representation $(\pi_{J,\check{\nu}}, M_{J,\check{\nu}}, \mathbf{B}_{J,\check{\nu}})$.

Conjecture 5.16. *Let $\Gamma_{J,\check{\nu}}$ be the cell recognised by the bounded representation $(\pi_{J,\check{\nu}}, M_{J,\check{\nu}}, \mathbf{B}_{J,\check{\nu}})$. Then*

- (1) *Lusztig's \mathbf{a} -function satisfies $\mathbf{a}(w) = \mathbf{a}_{J,\check{\nu}}$ for all $w \in \Gamma_{J,\check{\nu}} \cap W$.*
- (2) *The set $\Gamma_{J,\check{\nu}} \cap W$ is contained in a two sided Kazhdan-Lusztig cell of the weighted Coxeter group (W, L) .*

Remark 5.17. It is not necessarily true that $\Gamma_{J,\check{\nu}}$ equals a two sided Kazhdan-Lusztig cell. For example, in Example 5.6(2) and (4) the set $\Gamma_{J,\check{\nu}}$ is strictly contained in a two sided cell (see [16, Figure 2] for the cell decomposition of $\tilde{\mathbf{G}}_2$).

Example 5.18. Consider the case $\Phi = F_4$, with notation as in Example 5.12. Writing $r = a/b$, the conjectural bounds \mathbf{a}_π (from Conjecture 5.15) for a selection of the bounded representations of \tilde{F}_4 are as follows:

- (1)  $\mathbf{a}_\pi = 2a + 2b, 5a, 6a - b, 11a - 7b$ for $r \in (0, 2/3], [2/3, 1], [1, 6/5], [6/5, 2]$.
- (2)  $\mathbf{a}_\pi = 4a + 12b$ for $r \in [4, \infty)$.
- (3)  $\mathbf{a}_\pi = -2a + 11b, a + 6b, 2a + 3b$ for $r \in [1, 5/3], [5/3, 3], [3, \infty)$.
- (4)  $\mathbf{a}_\pi = 6a + 4b, 9a + 3b$ for $r \in (0, 1/3], [1/3, 1/2]$.
- (5)  $\mathbf{a}_\pi = 4a + 12b, 6a + 4b$ for $r \in [2, 4], [4, \infty)$.
- (6)  $\mathbf{a}_\pi = 3a + 6b, 11a + 2b$ for $r \in (0, 1/2], [1/2, 1]$.
- (7)  $\mathbf{a}_\pi = -2a + 11b, -a + 9b, 6b$ for $r \in [5/3, 2], [2, 3], [3, \infty)$.
- (8)  $\mathbf{a}_\pi = 3a, 5a - b, 11a - 7b$ for $r \in (0, 1/2], [1/2, 1], [1, 6/5]$.

Thus Conjecture 5.16 predicts the existence of elements of the affine Weyl group of type \tilde{F}_4 with the above \mathbf{a} -function values in the respective parameter ranges.

Proposition 5.19. *Conjecture 5.15 implies Conjecture 5.14.*

Proof. We have

$$\prod'_{\alpha \in \Phi} \frac{1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{\alpha^\vee}}{1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{\alpha^\vee}} = \prod'_{\alpha \in \Phi^+} \frac{(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{\alpha^\vee})(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{-\alpha^\vee})}{(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{\alpha^\vee})(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{-\alpha^\vee})}.$$

Suppose that $\alpha \in \Phi_0 \cap \Phi_1$ (that is, $\alpha/2, 2\alpha \notin \Phi$). Then $\mathbf{q}_\alpha = \mathbf{q}^a$ for some $a > 0$, and writing $\mathbf{v}^{\alpha^\vee} = \mathbf{q}^k$ for some $k \in \mathbb{Z}$ gives

$$\frac{(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{\alpha^\vee})(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{-\alpha^\vee})}{(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{\alpha^\vee})(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{-\alpha^\vee})} = \frac{(1 - \mathbf{q}^k)(1 - \mathbf{q}^{-k})}{(1 - \mathbf{q}^{k-2a})(1 - \mathbf{q}^{k-2a})}$$

Recalling the convention that any factors that are identically 0 are removed, it follows that

$$\deg \frac{(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{\alpha^\vee})(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{-\alpha^\vee})}{(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{\alpha^\vee})(1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{-\alpha^\vee})} = \begin{cases} |k| & \text{if } 0 \leq |k| \leq 2a \\ 2a & \text{if } 2a \leq |k|. \end{cases}$$

If $\alpha \notin \Phi_0 \cap \Phi_1$ (this only occurs in the non-reduced case) then we pair the four terms in the product related to α and $\alpha/2$ (if $\alpha \in \Phi_1$) or 2α (if $\alpha \in \Phi_0$). We may thus assume that $\alpha \in \Phi_0$ (and so $\mathbf{q}_{\alpha/2} = 1$, $\mathbf{q}_\alpha = \mathbf{q}^{L(s_n) - L(s_0)}$, and $\mathbf{q}_{2\alpha} = \mathbf{q}^{L(s_0)}$), and the four terms combine to give

$$\begin{aligned} C_\alpha &= \frac{(1 - \mathbf{v}^{\alpha^\vee})(1 - \mathbf{v}^{-\alpha^\vee})(1 - \mathbf{q}_{\alpha}^{-1} \mathbf{v}^{\alpha^\vee/2})(1 - \mathbf{q}_{\alpha}^{-1} \mathbf{v}^{-\alpha^\vee/2})}{(1 - \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{\alpha^\vee})(1 - \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{-\alpha^\vee})(1 - \mathbf{q}_{2\alpha}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{\alpha^\vee/2})(1 - \mathbf{q}_{2\alpha}^{-1} \mathbf{q}_{\alpha}^{-2} \mathbf{v}^{-\alpha^\vee/2})} \\ &= \frac{(1 - \mathbf{v}^{\alpha^\vee})(1 - \mathbf{v}^{-\alpha^\vee})}{(1 + \mathbf{q}^{c-a} \mathbf{v}^{\alpha^\vee/2})(1 + \mathbf{q}^{c-a} \mathbf{v}^{-\alpha^\vee/2})(1 - \mathbf{q}^{-a-c} \mathbf{v}^{\alpha^\vee/2})(1 - \mathbf{q}^{-a-c} \mathbf{v}^{-\alpha^\vee/2})}, \end{aligned}$$

where $a = L(s_n)$ and $c = L(s_0)$. Recall that by Convention 5.1 we have $a - c > 0$. Writing $\mathbf{v}^{\alpha^\vee} = \mathbf{q}^{2k}$ for some $k \in \mathbb{Z}$ then

$$\deg C_\alpha = \begin{cases} 2|k| & \text{if } 0 \leq |k| \leq a - c \\ |k| + a - c & \text{if } a - c \leq |k| \leq a + c \\ 2a & \text{if } 2a \leq |k|. \end{cases}$$

In summary, once the terms in the product are suitably grouped together we may write the product as $\prod_{\alpha \in \Phi_0^+} C_\alpha$ where $\deg C_\alpha \geq 0$ with $\deg C_\alpha = 0$ if and only if $v^{\alpha^\vee} = 1$. Thus

$$L(w_0) - \frac{1}{2} \deg \prod_{\alpha \in \Phi} \frac{1 - q_{\alpha/2}^{-1} v^{\alpha^\vee}}{1 - q_{\alpha/2}^{-1} q_{\alpha}^{-2} v^{\alpha^\vee}} \leq L(w_0)$$

with equality if and only if $v^{\alpha^\vee} = 1$ for all $\alpha \in \Phi_0^+$, which in turn forces $J = \emptyset$. \square

Theorem 5.20. *Conjectures 5.15 and 5.16 hold in the following cases.*

- (1) *All weighted affine Hecke algebras in the case $J = \emptyset$.*
- (2) *All weighted affine Hecke algebras of dimension 1 or 2 (rank 2 or 3).*

Proof. (1) follows from Example 5.7 and [14, Theorem 4.6], which shows that the set Γ from Example 5.7 is the lowest two sided Kazhdan-Lusztig cell of (W, L) , and it is well known that $\mathbf{a}(w) = L(w_0)$ for all elements of this cell.

(2) The cases $\Phi \in \{A_1, BC_1, A_2\}$ are easy exercises and are omitted (see [19, Figure 1] and [22, §7] for the decomposition of A_2, A_1 , and BC_1 into cells). Conjecture 5.16 for $\Phi \in \{C_2, BC_2, G_2\}$ follows from the results of [16, 15]. Thus it only remains to verify Conjecture 5.15 in the cases $\Phi \in \{C_2, BC_2, G_2\}$.

Consider the case $\Phi = G_2$. The case $J = \emptyset$ is covered by (1). Suppose that $J = \{1\}$. The only bounded representation with $J = \{1\}$ has $v_{\alpha_1} = -q^{-a}$, where we use the conventions of Example 5.6 ($L(s_1) = a$ and $L(s_2) = L(s_0) = b$). Thus we have $v_{\alpha_1}^\vee = q^{-2a}$ and $v_{\alpha_2}^\vee = -q^a$. Since $\Phi^\vee = \pm\{\alpha_1^\vee, \alpha_1^\vee + 3\alpha_2^\vee, 2\alpha_1^\vee + 3\alpha_2^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee\}$ we have

$$\prod_{\alpha \in \Phi} \frac{1 - q_{\alpha/2}^{-1} v^{\alpha^\vee}}{1 - q_{\alpha/2}^{-1} q_{\alpha}^{-2} v^{\alpha^\vee}} = \frac{(1 - q^{-2a})(1 - q^{2a})(1 + q^a)^4(1 + q^{-a})^2}{(1 - q^{-4a})(1 + q^{-3a})^2(1 - q^{-2b})^2(1 + q^{a-2b})^2(1 + q^{-a-2b})^2}$$

Thus, since $L(w_0) = 3a + 3b$, we have

$$L(w_0) - \frac{1}{2} \deg \prod_{\alpha \in \Phi} \frac{1 - q_{\alpha/2}^{-1} v^{\alpha^\vee}}{1 - q_{\alpha/2}^{-1} q_{\alpha}^{-2} v^{\alpha^\vee}} = \begin{cases} 3b & \text{if } a - 2b < 0 \\ a + b & \text{if } a - 2b \geq 0. \end{cases}$$

By [16, Theorem 7.10] these are equal to the bound of the induced representation $\pi_{\{1\}, v}$, confirming the Conjecture 5.15 in this case.

The case $J = \{2\}$ is similar (using again the values of the bounds from [16, Theorem 7.10]) and we omit the details.

Consider now $\Phi = G_2$ and $J = I = \{1, 2\}$. By Proposition 5.11 there are three bounded I -parameter systems. The case $(v_{\alpha_1}, v_{\alpha_2}) = (-q^{-a}, -q^{-b})$ is clearly bounded by 0 and recognises the trivial Kazhdan-Lusztig cell $\{e\}$. Consider the case $(v_{\alpha_1}, v_{\alpha_2}) = (q^a, -q^{-b})$ (which is bounded if and only if $a/b \leq 3/2$). We have $v_{\alpha_1}^\vee = q^{2a}$ and $v_{\alpha_2}^\vee = q^{-2b}$, and hence

$$\begin{aligned} & \prod_{\alpha \in \Phi} \frac{1 - q_{\alpha/2}^{-1} v^{\alpha^\vee}}{1 - q_{\alpha/2}^{-1} q_{\alpha}^{-2} v^{\alpha^\vee}} \\ &= \frac{(1 - q^{2a})(1 - q^{-2a+6b})(1 - q^{4a-6b})(1 - q^{-2b})(1 - q^{2b})(1 - q^{2a-2b})(1 - q^{-2a+4b})}{(1 - q^{-4a})(1 - q^{-6b})(1 - q^{-6a+6b})(1 - q^{-4b})(1 - q^{2a-6b})} \end{aligned}$$

(with the convention on removing zero products if required). It follows that

$$L(w_0) - \frac{1}{2} \deg \prod_{\alpha \in \Phi} \frac{1 - q_{\alpha/2}^{-1} v^{\alpha^\vee}}{1 - q_{\alpha/2}^{-1} q_{\alpha}^{-2} v^{\alpha^\vee}} = \begin{cases} 3a - 2b & \text{if } 1 \leq a/b \leq 3/2 \\ a & \text{if } a/b \leq 1, \end{cases}$$

agreeing with the bounds given in Example 5.6. Moreover, the cell recognised by this representation is given in Example 5.6, and by [16, Figure 2] these cells are contained in Kazhdan-Lusztig cells for the relevant parameter values.

The case $(\mathbf{v}_{\alpha_1}, \mathbf{v}_{\alpha_2}) = (-\mathbf{q}^{-a}, \mathbf{q}^b)$ is similar, and we omit the details. Moreover, the analysis for the cases $\Phi = \mathbf{C}_2$ and \mathbf{BC}_2 is similar, using [15, Theorems 6.15, 6.21, 6.22], and we again omit the calculations. \square

Remark 5.21. We provide some comments on the origin of Conjecture 5.15. Recall that for a finite dimensional weighted Hecke algebra \mathcal{H}_0 the “canonical trace” $\text{Tr}(\sum a_w T_w) = a_e$ on \mathcal{H}_0 decomposes as a sum of irreducible characters as $\text{Tr} = \sum m_\lambda \chi_\lambda$ where the elements m_λ are rational functions in \mathbf{q} known as the “generic degrees” of \mathcal{H}_0 (see [12, Chapter 11]). There is a connection between the generic degrees and Lusztig’s \mathbf{a} -function, stated roughly that if w is in the cell “corresponding” to λ then $\deg m_\lambda = 2\mathbf{a}(w)$ (see [11]).

In the affine case, the decomposition of the canonical trace takes the form of an integral over tempered representations of the affine Hecke algebra, and the generic degrees are replaced by the “Plancherel measure”. The *Macdonald c -function* is (see Macdonald [23])

$$c(X) = \prod_{\alpha \in \Phi^+} \frac{1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_\alpha^{-2} X^{-\alpha^\vee}}{1 - \mathbf{q}_{\alpha/2}^{-1} X^{-\alpha^\vee}}.$$

By Opdam’s work [26, Theorem 3.25] the reciprocal of the term $\psi_{J,\mathbf{v}}(\mathbf{q}^{2L(\mathbf{w}_0)} c(X) c(X^{-1}))'$ (where the prime indicates that any factors that are 0 on evaluation by $\psi_{J,\mathbf{v}}$ are to be omitted) appears as the mass of the character $\chi_{J,\mathbf{v}}$ of $\pi_{J,\mathbf{v}}$ in the Plancherel formula for $\tilde{\mathcal{H}}(L)$ (once the parameters \mathbf{q} and ζ_i are specialised appropriately and scalars are extended, see [16, §9.4] for further discussion). Thus since

$$L(\mathbf{w}_0) - \frac{1}{2} \deg \prod_{\alpha \in \Phi} \frac{1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{v}^{\alpha^\vee}}{1 - \mathbf{q}_{\alpha/2}^{-1} \mathbf{q}_\alpha^{-2} \mathbf{v}^{\alpha^\vee}} = \frac{1}{2} \deg \psi_{J,\mathbf{v}} \left(\mathbf{q}^{2L(\mathbf{w}_0)} c(X) c(X^{-1}) \right)'$$

our conjecture can be seen as an affine analogue of the finite dimensional situation, giving conjectural connections between Kazhdan-Lusztig Theory and Opdam’s Plancherel Theorem.

6 The case $\Phi = \mathbf{A}_n$ with $J = \{1, 2, \dots, n-1\}$

In this section we illustrate the theory in the case $\Phi = \mathbf{A}_n$ with $J = \{1, 2, \dots, n-1\}$. We will apply Theorem 4.15 to prove Conjectures 5.15 and 5.16 in this case.

The only J -parameter system for which $\pi_{J,\mathbf{v}}$ is bounded is $\mathbf{v} = (-\mathbf{q}^{-1})_{\alpha \in \Phi_J}$ (see Theorem 5.9 and Proposition 5.11), and so the symbol \mathbf{v} will be suppressed in the notation. Thus, for example, we shall write $\pi_J = \pi_{J,\mathbf{v}}$ and $\varpi_J = \varpi_{J,\mathbf{v}}$.

We have $\mathbf{R}[\zeta_J] = \mathbf{R}[\zeta]$ where $\zeta = \zeta_J^{\omega_1}$, since $\zeta_J^{\omega_i} = \zeta^i$ for $1 \leq i \leq n$. Since $y_{\omega_1} = \mathbf{W}_{\{2,3,\dots,n-1\}} \mathbf{W}_{\{1,2,\dots,n-1\}} = s_1 \cdots s_{n-1}$ we have

$$\tau_{\omega_1} = t_{\omega_1} y_{\omega_1} = (s_0 s_n s_{n-1} \cdots s_2 \sigma)(s_1 \cdots s_{n-1}) = s_0 \sigma,$$

where $\sigma \in \Sigma$ is given by $\sigma s_i \sigma^{-1} = s_{i+1}$ (with indices read cyclically modulo $n+1$, and so in particular $s_{n+1} = s_0$).

The following choice of fundamental domain leads to a matrix representation with very symmetric matrices.

Lemma 6.1. *The set Σ is a fundamental domain for the action of \mathbb{T}_J on \mathbb{W}^J .*

Proof. We have $W^J = \{e, s_n, s_n s_{n-1}, \dots, s_n s_{n-1} \cdots s_2 s_1\}$, and for $1 \leq i \leq n$ we compute

$$\tau_{\omega_1}^i s_n s_{n-1} \cdots s_{n-i+1} = \sigma^i,$$

hence the result □

Fix the following order on the basis:

$$(\varpi_J(X_{\sigma^{-1}}), \varpi_J(X_{\sigma^{-2}}), \dots, \varpi_J(X_{\sigma^{-n}}), \varpi_J(X_{\sigma^{-n-1}})). \quad (6.1)$$

For $1 \leq i \leq n+1$ we have $\varpi_J(X_{\sigma^{-i}}) \cdot T_w = \varpi_J(X_e) \cdot T_{\sigma^{-i}(w)} X_{\sigma^{-i}}$, and it follows that

$$[\pi_J(T_w)]_{i,j} = [\pi_J(T_{\sigma^{-i}(w)})]_{n+1, \sigma^{-i}(j)} \quad \text{for } 1 \leq i, j \leq n+1. \quad (6.2)$$

Corollary 6.2. *The matrices for $\pi_J(T_i)$, $0 \leq i \leq n$, with respect to the ordered basis (6.1) are*

$$\pi_J(T_i) = \begin{bmatrix} -\mathbf{q}^{-1} \mathbf{l}_{i-1} & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 \\ 0 & \zeta^{-1} & \mathbf{q} - \mathbf{q}^{-1} & 0 \\ 0 & 0 & 0 & -\mathbf{q}^{-1} \mathbf{l}_{n-i} \end{bmatrix} \quad \pi_J(T_0) = \begin{bmatrix} \mathbf{q} - \mathbf{q}^{-1} & 0 & \zeta^{-1} \\ 0 & -\mathbf{q}^{-1} \mathbf{l}_{n-1} & 0 \\ \zeta & 0 & 0 \end{bmatrix}$$

where \mathbf{l}_k is the $k \times k$ identity matrix, and $\zeta = \zeta_J^{\omega_1}$.

Proof. This follows directly from Theorem 4.15. □

Definition 6.3. Recall that W is the non-extended affine Weyl group. Let

$$\Gamma_{\uparrow} = \{w \in W \setminus \{e\} \mid w \text{ has a unique reduced expression}\}.$$

By [2, Chapter 12] for any Coxeter group (W, S) with weight function $L = \ell$ (equal parameters) the set Γ_{\uparrow} forms a two-sided Kazhdan-Lusztig cell, and the right cells in Γ_{\uparrow} are the sets $\{w \in \Gamma_{\uparrow} \mid D_L(w) = s\}$ for $s \in S$ (with D_L the left descent set). In type \tilde{A}_n the elements of Γ_{\uparrow} are precisely the nontrivial elements with a reduced expression with no subwords $s_i s_j$ with $m_{ij} = 2$ or $s_i s_j s_i$ with $m_{ij} = 3$ (by Tits' solution to the Word Problem).

Remark 6.4. For irreducible affine Coxeter groups W the set Γ_{\uparrow} is infinite if and only if W is of type \tilde{A}_n or \tilde{C}_n (see [2, Proposition 12.1.14]). We expect that the techniques of this section could be applied to the \tilde{C}_n case, with the relevant set in that case being $J = \{2, 3, \dots, n\}$.

If $x \in \Gamma_{\uparrow}$ then the unique reduced expression of x is of the form $s_i s_{i+1} s_{i+2} \cdots$ or $s_i s_{i-1} s_{i-2} \cdots$ for some $0 \leq i \leq n$ (recall the indices are read cyclically). The final generator that appears in the reduced expression will be important to keep track of, and so we denote

$$\begin{aligned} x_{i,j,\ell}^{\uparrow} &= s_i s_{i+1} s_{i+2} \cdots s_j \\ x_{i,j,\ell}^{\downarrow} &= s_i s_{i-1} s_{i-2} \cdots s_j \end{aligned}$$

where $\ell = \ell(x)$ (note that many cycles are allowed in these expressions, and that there is a compatibility condition between i, j, ℓ , however this condition will not play a role). Sometimes either the length or the final generator will not be important to keep track of, and in this case we will abbreviate $x_{i,j,\ell}^{\uparrow}$ to $x_{i,j}^{\uparrow}$ or x_i^{\uparrow} , and similarly for $x_{i,j,\ell}^{\downarrow}$. Thus, for example, x_5^{\downarrow} represents a work of the form $s_5 s_4 s_3 s_2 s_1 s_0 s_n s_{n-1} \cdots$ with no constraint on the length or final generator.

In the following lemmas we will be dealing with the combinatorics of alcove paths. We will often start with a ‘‘straight’’ path $p = s_{i_1} s_{i_2} s_{i_3} \cdots$, and produce the associated J -folded alcove paths. We will often abbreviate notation and write $p = i_1 i_2 i_3 \cdots$. Moreover, we will use the

following colour-coding to represent (partially) J -folded alcove paths: red for a bounce (hence degree contribution -1 to $\mathcal{Q}_J(p)$), green for a fold (hence degree contribution $+1$ to $\mathcal{Q}_J(p)$), and blue for a crossing (hence zero degree contribution). Parts of the path that remain black are understood to remain “straight” (and hence may not be J -folded). For example, given that p starts at e , the expression $p = n(n-1)n101234$ represents a path where the first 4 steps have been J -folded (a negative crossing, followed by a fold, a positive crossing, a bounce, and a positive crossing), and the remaining 4 steps have not yet been folded.

Lemma 6.5. *Let $x \in \Gamma_1$ and let p be a J -folded alcove path of type x . Then $\deg \mathcal{Q}_J(p) \leq 1$ with equality if and only if p has exactly one fold and no bounces. Moreover if $1 \leq i \leq n+1$ then the paths p of type $x \in \Gamma_1$ starting at σ^{-i} with $\deg \mathcal{Q}_J(p) = 1$ are the paths*

- (1) $p = (i-1)i(i+1)(i+2) \cdots n12 \cdots$, and
- (2) $p = (i-1)(i-2) \cdots 210n \cdots (j+1)j$ for some j .

Proof. We argue by induction on the length of the path, with the result clear for paths of length 1. It is sufficient to consider paths starting at e , because starting at σ^{-i} for some i is equivalent to starting e and performing a cyclic shift of the generators of the expression x . Thus we consider paths of x_i^\uparrow and x_i^\downarrow starting at e for each choice of $0 \leq i \leq n$.

Consider first paths p of type $x = x_i^\uparrow$ starting at e . If $i = 0$ then $p = 012 \cdots$ consists entirely of positive crossings staying in \mathcal{A}_J , and hence is J -folded with $\deg \mathcal{Q}_J(p) = 0$.

If $i = n$ then either $p = n012 \cdots$ or $p = n012 \cdots$. In the first case the remainder of the path consists entirely of positive crossings staying in \mathcal{A}_J , and so $\deg \mathcal{Q}_J(p) = 1$ and the path is as in case (1) of the statement of the lemma. We must show that in the second case the degree is bounded by 0. In the second case the next $n-1$ steps are forced bounces, giving $p = n012 \cdots (n-2)(n-1)n01 \cdots$. The remainder of the path is a path p' of type x_{n-1}^\uparrow starting at s_n . Note that $\tau_{\omega_1} s_n = s_0 \sigma s_n = \sigma$, and so by the action of \mathbb{T}_J on paths (see Lemma 3.9) the path $\tau_{\omega_1} \cdot p'$ starts at σ and $\deg \mathcal{Q}_J(p') = \deg \mathcal{Q}_J(\tau_{\omega_1} \cdot p') \leq 1$ (by the induction hypothesis). Since $\mathcal{Q}_J(p) = (-q^{-1})^{n-1} \mathcal{Q}_J(p')$ we have $\deg \mathcal{Q}_J(p) \leq 2 - n \leq 0$ as required.

If $i \in J$ then the first $n-i$ steps are forced bounces, and so $p = i \cdots (n-1)n012 \cdots$. The remainder of the path is a path p' of type $n012 \cdots$ starting at e , and by the $i = n$ case in the previous paragraph we have $\deg \mathcal{Q}_J(p) \leq i - n + 1 \leq 0$.

Now consider paths p of type $x = x_i^\downarrow$ starting at e . If $i = 0$ then $p = 0n(n-1) \cdots$. The next $n-1$ steps are necessarily bounces, giving $p = 0n(n-1) \cdots 210n(n-1) \cdots$. The remainder of the path is a path p' of type x_1^\downarrow starting at s_0 . Since $\tau_{\omega_1}^{-1} s_0 = \sigma^{-1}$ the path $\tau_{\omega_1}^{-1} \cdot p'$ starts at σ^{-1} and by induction $\deg \mathcal{Q}_J(p') = \deg \mathcal{Q}_J(\tau_{\omega_1}^{-1} \cdot p') \leq 1$. Then $\deg \mathcal{Q}_J(p) = -n + 1 + \deg \mathcal{Q}_J(p') \leq 2 - n \leq 0$.

If $i = n$ then $p = n(n-1)(n-2) \cdots$ has each step negative and stays in \mathcal{A}_J . Thus we either have no folds, and so $p = n(n-1)(n-2) \cdots$ has $\deg \mathcal{Q}_J(p) = 0$, or we fold at some step. Thus suppose that $p = n(n-1) \cdots (j+1)j(j-1) \cdots 10n(n-1) \cdots$ (possibly with multiple cycles before the fold). Either the path terminates immediately after the fold (and so $\deg \mathcal{Q}_J(p) = 1$ and p is as in case (2) of the statement of the lemma), or there is a sequence of forced bounces. In fact the following $n-1$ steps (or up to the end of the path, whichever is first) are forced to be bounces. To see this, one checks that $s_n s_{n-1} \cdots s_{j+1} \alpha \in \Phi_J$ for $\alpha \in \{\alpha_{j-1}, \dots, \alpha_1, \varphi, \alpha_n, \dots, \alpha_{j+1}\}$. Thus

$$p = n(n-1) \cdots (j+1)j(j-1) \cdots 10n(n-1) \cdots (j+1)j(j-1) \cdots$$

The remainder of the path is a path p' of type x_j^\downarrow starting at $s_n \cdots s_{j+1}$, and since $\tau_{\omega_1}^{n-j} s_n \cdots s_{j+1} = \sigma^{n-j}$ we use Lemma 3.9 and the induction hypothesis to give

$$\deg \mathcal{Q}_J(p) = 1 - n + \deg \mathcal{Q}_J(\tau_{\omega_1}^{n-j} \cdot p') \leq 2 - n \leq 0.$$

Finally, if $i \in J$ then $p = i(i-1) \cdots 10n(n-1) \cdots$ and we return to the $i = 0$ case, concluding the proof. \square

To avoid multiple subscripts (and superscripts in subscripts), we will abbreviate notation and write

$$\pi_J(w) = \pi_J(T_w) \quad \text{for } w \in \widetilde{W}.$$

Let $\text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}])$ denote the ring of $(n+1) \times (n+1)$ matrices with entries in $\mathbb{Z}[\mathfrak{q}^{-1}]$. For $1 \leq i, j \leq n+1$ let $E_{i,j}$ be the $(n+1) \times (n+1)$ matrix with 1 in the (i, j) -th place, and zeros elsewhere.

Corollary 6.6. *We have*

$$\begin{aligned} \pi_J(x_{i,j,\ell}^\uparrow) &\in \mathfrak{q}\zeta^{\ell-1}E_{i+1,j+1} + \text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}]) \\ \pi_J(x_{i,j,\ell}^\downarrow) &\in \mathfrak{q}\zeta^{-\ell+1}E_{i+1,j+1} + \text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}]) \end{aligned}$$

Proof. This follows from Theorem 4.15 and Lemma 6.5 and the fact that in (1) of Lemma 6.5 we have $\text{wt}(p, \Sigma)^J = (\ell(p) - 1)\bar{\omega}_n$, and in (2) we have $\text{wt}(p, \Sigma)^J = (-\ell(p) + 1)\bar{\omega}_n$. \square

Lemma 6.7. *Let $0 \leq i, j \leq n$ with $m_{ij} \in \{2, 3\}$ and let w_{ij} be the longest element of the parabolic subgroup $\langle s_i, s_j \rangle$. Let $y \in \Gamma_1 \cup \{e\}$ and suppose that $\ell(w_{ij}y) = \ell(w_{ij}) + \ell(y)$. Then*

- (1) $\pi_J(T_{w_{ij}y}) \in \text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}])$, and
- (2) if $\deg[\pi_J(T_{w_{ij}y})]_{k,l} = 0$ then $k \in \{i+1, j+1\}$.

Proof. By (6.2) it is sufficient to prove the result for $k = n+1$. That is, we must show that if $0 \leq i, j \leq n$ with $m_{ij} \in \{2, 3\}$, and if $y \in \Gamma_1 \cup \{e\}$ with $\ell(w_{ij}y) = \ell(w_{ij}) + \ell(y)$, then $\deg[\pi_J(T_{w_{ij}y})]_{n+1,l} \leq 0$ for all $1 \leq l \leq n+1$, and if $\deg[\pi_J(T_{w_{ij}y})]_{n+1,l} = 0$ then either $i = n$ or $j = n$. By Theorem 4.15 we have

$$[\pi_J(T_{w_{ij}y})]_{n+1,l} = \sum_p \mathcal{Q}_J(p) \zeta_J^{\text{wt}(p, \Sigma)}$$

where the sum is over paths $p \in \mathcal{P}_J(w_{ij}y, e)$ with $\theta(p, \Sigma) = \sigma^{-l}$. To fix notation, we will choose the reduced expression $w_{ij} = s_i s_j$ or $w_{ij} = s_i s_j s_i$ with $j < i$. It will turn out to be sufficient to bound the degree of $\mathcal{Q}_J(p)$ for each path $p \in \mathcal{P}_J(w_{ij}y, e)$ with the exception of case (4) below, where cancellations come into play (meaning that the path-by-path degree is higher than the degree of the associated matrix entry, and more care is required).

Let $p \in \mathcal{P}_J(w_{ij}y, e)$. Suppose first that $m_{ij} = 3$.

- (1) Suppose that $1 \leq j < i \leq n-1$. Then $p = \mathbf{ijiy}$. The remaining path is of type $y \in \Gamma_1$, and hence has degree bounded by 1 by Lemma 6.5, and hence $\deg \mathcal{Q}_J(p) \leq -2$.
- (2) Suppose that $(j, i) = (0, 1)$. Then $p = \mathbf{101y}$. Let s_r be the first generator in the unique reduced expression for y , and write $y = s_r y'$. Since $w_{ij}y$ is reduced we have $r \notin \{0, 1\}$. If $r \neq 2$ or $r \neq n$ then $p = \mathbf{101ry}'$ and $\deg \mathcal{Q}_J(p) \leq -1$ by Lemma 6.5. If $r = 2$ then $p = \mathbf{1012y}'$ and either $y' = 34\dots$ or $y' = 10\dots$. In the first case all remaining steps are positive crossings staying in \mathcal{A}_J , and so $p = \mathbf{10123}\dots$ and $\deg \mathcal{Q}_J(p) = -1$. In the second case we must have $p = \mathbf{10121}\dots$ and $\deg \mathcal{Q}_J(p) \leq -1$ by Lemma 6.5. The case $r = 0$ is similar.
- (3) Suppose that $(j, i) = (0, n)$. Then either $p = \mathbf{n0ny}$ or $p = \mathbf{n0ny}$. In the first case Lemma 6.5 gives $\deg \mathcal{Q}_J(p) \leq 0$, and so consider the second case. Let s_r be the first generator in the unique reduced expression for y , and write $y = s_r y'$. Since $w_{ij}y$ is reduced we have $r \notin \{0, n\}$. If $2 \leq r \leq n-1$ then $s_\varphi \alpha_r = \alpha_r \in \Phi_J$ and so $s_0 s_r \notin \mathcal{A}_J$, giving $p = \mathbf{n0nr}'$ and so again $\deg \mathcal{Q}_J(p) \leq 0$. If $r = 1$ then $p = \mathbf{n0n1y}'$. Either $y' = 234\dots$ or $y' = 0n(n-1)\dots$. In the first case all remaining steps are positive crossings staying in \mathcal{A}_J , and so $p = \mathbf{n0n1234}\dots$ with $\deg \mathcal{Q}_J(p) = 0$. In the second case since $s_0 s_1 s_0 \notin \mathcal{A}_J$ we have $p = \mathbf{n0n10n(n-1)}\dots$ and Lemma 6.5 gives $\deg \mathcal{Q}_J(p) \leq 0$ as required.

- (4) Suppose that $(j, i) = (n-1, n)$. The possibilities are $p = n(n-1)ny$, $p = n(n-1)ny$, $p = n(n-1)ny$, or $p = n(n-1)ny$. In the first case Lemma 6.5 gives $\deg \mathcal{Q}_J(p) \leq 0$. In the second case, with $y = s_r y'$ as above, since $r \notin \{n, n-1\}$ the next step is forced to be a bounce, giving $p = n(n-1)nr y'$ with $\deg \mathcal{Q}_J(p) \leq 0$. Thus consider the third and fourth cases. Note that the initial segments $n(n-1)n$ and $n(n-1)n$ both end at e . Thus for each $p' \in \mathcal{P}_J(y, e)$ we have a pair of paths $p_1, p_2 \in \mathcal{P}_J(w_{n-1,ny}, e)$ given by appending each initial segment to the beginning of p' . Then

$$\mathcal{Q}_J(p_1) + \mathcal{Q}_J(p_2) = (\mathfrak{q} - \mathfrak{q}^{-1})\mathcal{Q}_J(p') - \mathfrak{q}^{-1}(\mathfrak{q} - \mathfrak{q}^{-1})^2\mathcal{Q}_J(p') = (\mathfrak{q}^{-1} + \mathfrak{q}^{-3})\mathcal{Q}_J(p').$$

By Lemma 6.5 we have $\deg \mathcal{Q}_J(p') \leq 1$, and so the combined contribution to the matrix entry $[\pi_J(T_{w_{n-1,ny}})]_{n+1,l}$ (where $\sigma^{-l} = \theta(p', \Sigma)$) has degree at most 0, as required.

Now suppose that $m_{ij} = 2$.

- (5) Suppose that $1 \leq j < i \leq n-1$. Then $p = ijy$, and by Lemma 6.5 we have $\deg \mathcal{Q}_J(p) \leq -1$.
- (6) Suppose that $j = 0$ and $1 \leq i \leq n-1$. Then $p = i0y$. Let s_r be the first generator in the unique reduced expression for y . Then $r \neq 0, i$ (as $w_{ij}y$ is reduced). If $r \in \{1, 2, \dots, n\} \setminus \{1, i\}$ then writing $y = s_r y'$ we have $p = i0r y'$, and as in the previous case the degree is bounded by -1 . So suppose that $r = 1$. Then $p = i01y'$. Either $y' = 234 \dots$ or $y' = 0n(n-1) \dots$. In the first case all remaining steps are positive crossings staying in \mathcal{A}_J , and so we have $p = i0123 \dots$ with $\deg \mathcal{Q}_J(p) = -1$. In the second case, since $s_\varphi s_{\alpha_1} \varphi \in \Phi_J$, we have $p = i010n(n-1) \dots$. By Lemma 6.5 the degree contribution of the remainder of the path is bounded by 1, hence $\deg \mathcal{Q}_J(p) \leq -1$.
- (7) Suppose that $i = n$. Then $j \in J \setminus \{n-1\}$ (because the cases $j = 0, n-1$ are impossible as $m_{ij} = 2$). Thus $s_{\alpha_n} \alpha_j \in \Phi_J$ and so $p = njy$ or $p = njy$. In the second case Lemma 6.5 gives $\deg \mathcal{Q}_J(p) \leq 0$ as required. Thus consider the first case. As before, let s_r be the first generator in the unique reduced expression for y , and write $y = s_r y'$. Since $w_{ij}y$ is reduced we have $r \notin \{j, n\}$. If $r \in J$ then $p = njr y'$, and so again $\deg \mathcal{Q}_J(p) \leq 0$. If $r = 0$ then $y' = 123 \dots$ or $y' = n(n-1)(n-2) \dots$ and so $p = nj0123 \dots$ or $p = nj0n(n-1)(n-2) \dots$. In the first case the remainder of the path consists of positive crossings staying in \mathcal{A}_J , and hence $\deg \mathcal{Q}_J(p) = 0$. In the second case we have $s_0 s_n \notin \mathcal{A}_J$ and hence $p = nj0n(n-1)(n-2) \dots$, and again by Lemma 6.5 it follows that $\deg \mathcal{Q}_J(p) \leq 0$.

Hence the result. \square

Corollary 6.8. *Let $x \in \Gamma_!$ and $y \in \Gamma_! \cup \{e\}$ and $0 \leq i < j \leq n$. Let w_{ij} be the longest element of $W_{ij} = \langle s_i, s_j \rangle$, and suppose that $\ell(xw_{ij}y) = \ell(x) + \ell(w_{ij}) + \ell(y)$. Then $\pi_J(T_w) \in \text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}])$.*

Proof. We have

$$\pi_J(T_{xw_{ij}y}) = \pi_J(T_x)\pi_J(T_{w_{ij}y}).$$

Let s_r be the last generator in the unique reduced expression for x . By Corollary 6.6 we have $\pi_J(T_x) \in \mathfrak{q}^a E_{k,r+1} + \text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}])$ for some $1 \leq k \leq n+1$ and $a \in \mathbb{Z}$. Moreover, by Lemma 6.7 $\pi_J(T_{w_{ij}y})$ satisfies

$$\pi_J(T_{w_{ij}y}) \in \sum_{l=1}^{n+1} (f_l E_{i+1,l} + g_l E_{j+1,l}) + \mathfrak{q}^{-1} \text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}]),$$

for some $f_l, g_l \in \mathbb{Z}[\zeta, \zeta^{-1}]$. Since $\ell(xw_{ij}y) = \ell(x) + \ell(y) + \ell(w_{ij})$ we have $r \neq i, j$, and hence $\pi_J(T_x)\pi_J(T_{w_{ij}y}) \in \text{Mat}_{n+1}(\mathbb{Z}[\mathfrak{q}^{-1}])$ as required. \square

Corollary 6.9. *If $w \in W \setminus \Gamma_!$ then $\deg \pi(T_w) \leq 0$.*

Proof. The result is clear if $w = e$. If $w \neq e$ choose any reduced expression for w , and write this expression in the form $w = y_1 \mathbf{w}_1 y_2 \mathbf{w}_2 y_3 \cdots y_m \mathbf{w}_m y_{m+1}$ where $y_k \in \Gamma_! \cup \{e\}$ and \mathbf{w}_k is either of the form sts with $m_{st} = 3$ or st with $m_{st} = 2$. We have

$$\pi_J(T_w) = \pi_J(T_{y_1 \mathbf{w}_1 y_2}) \pi_J(T_{\mathbf{w}_2}) \pi_J(T_{y_3 \mathbf{w}_3 y_4}) \cdots,$$

and applying Lemma 6.7(1) and Corollary 6.8 to see that each matrix is in $\text{Mat}_{n+1}(\mathbb{Z}[\mathbf{q}^{-1}])$, hence the result. \square

The following theorem shows, in particular, that Conjectures 5.15 and 5.16 hold for the case $\Phi = A_n$ and $J = \{1, 2, \dots, n-1\}$. Recall the definition of leading matrices from Definition 5.3.

Theorem 6.10. *Let $\Phi = A_n$ and $J = \{1, 2, \dots, n-1\}$. Let F be any fundamental domain for the action of \mathbb{T}_J on \mathbb{W}^J , and let $\Pi = (\pi_J, M_J, \mathbf{B}_F)$.*

- (1) *The matrix representation Π is bounded with bound $\mathbf{a}_\Pi = 1$.*
- (2) *We have $\Gamma_\Pi = \{w\sigma \mid w \in \Gamma_!, \sigma \in \Sigma\}$.*
- (3) *The leading matrices $\{\mathbf{c}(w) \mid w \in \Gamma_\Pi \cap W\}$ form a basis of the \mathbb{Z} -module $\text{Mat}_{n+1}(\mathbb{Z}[\zeta, \zeta^{-1}])$.*

Proof. Corollaries 6.6 and 6.9 show for $w \in W$ the degree of the matrix entries of $\pi_J(T_w)$ is bounded by 1, and that this bound is attained in the matrix $\pi_J(T_w)$ if and only if $w \in \Gamma_!$. Since $\pi_J(T_\sigma)$ is a monomial matrix with entries in $\mathbb{Z}[\zeta, \zeta^{-1}]$ both (1) and (2) follow. The statement (3) follows from Corollary 6.6. \square

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