

ON TYPE-II SINGULARITIES IN RICCI FLOW ON \mathbb{R}^{n+1}

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ABSTRACT. In each dimension $n+1 \geq 3$ and for each real number $\lambda \geq 1$, we construct complete solutions to Ricci flow on \mathbb{R}^{n+1} which encounter global singularities at a finite time T . The singularities are forming arbitrarily slowly with the curvature blowing up arbitrarily fast at the rate $(T-t)^{-(\lambda+1)}$. Near the origin, blow-ups of such a solution converge uniformly to the Bryant soliton. Near spatial infinity, blow-ups of such a solution converge uniformly to the shrinking cylinder soliton. As an application of this result, we prove that there exist standard solutions of Ricci flow on \mathbb{R}^{n+1} whose blow-ups near the origin converge uniformly to the Bryant soliton.

1. INTRODUCTION

An important phenomenon in Ricci flow is the formation of finite-time singularities which occurs for a large family of initial metrics. Let (M, g) be a complete Riemannian manifold and $g(t)$ be a solution to the Ricci flow

$$\frac{\partial}{\partial t}g = -2 \operatorname{Ric}(g)$$

for time $t \geq 0$. Suppose $g(t)$ becomes singular at time $T < \infty$. Then this finite-time singularity is called *Type-I* if

$$\sup_{M \times [0, T)} |\operatorname{Rm}(\cdot, t)| (T - t) < \infty,$$

and it is called *Type-II* if

$$\sup_{M \times [0, T)} |\operatorname{Rm}(\cdot, t)| (T - t) = \infty.$$

The simplest example of a Type-I singularity in Ricci flow is the shrinking round sphere. In his seminal paper [27], Hamilton proved that Ricci flow on a compact three-manifold with positive Ricci curvature develops a Type-I singularity and shrinks to a round point. The same is true for Ricci flow on a compact four-manifold with positive curvature operator [28]. By the works of Hamilton [29] and Chow [18], Ricci flow on S^2 with an arbitrary initial metric always develops a Type-I singularity and shrinks to a round point. On a compact n -dimensional manifold for $n \geq 3$, Böhm and Wilking [8] proved that Ricci flow starting at a metric with 2-positive curvature operator (the

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sum of the two smallest eigenvalues of Rm is positive) develops a Type-I singularity and shrinks to a round point. We note that the $n = 4$ case in their result was known earlier [16]. Brendle [10] generalized the result of [8] under a much weaker assumption on the curvature operator. All these Type-I singularities are *global* in the sense that the volume of a manifold shrinks to zero at time T .

In [30], Hamilton sketched intuitively the formation of *local* singularities under Ricci flow. By local we mean that a singularity forms on a compact subset of a manifold while the volume of the manifold remains positive at time T . Rigorous results on finite-time local singularities in Ricci flow were obtained later. On a noncompact warped product $\mathbb{R} \times_f S^n$, Simon [41] showed that there are Ricci flow solutions that encounter finite-time local singularities. For local singularities in Kähler-Ricci flow, the first examples were constructed on holomorphic line bundles over $\mathbb{C}\mathbb{P}^{n-1}$ using $U(n)$ -invariant shrinking gradient Kähler-Ricci solitons [23].

Hamilton's examples of local singularities are the so-called *neckpinches* on a sphere. To describe them precisely, we recall the blow-up technique in singularity analysis. We say that a sequence $\{(x_i, t_i)\}_{i=0}^\infty$ of points and times in a Ricci flow is a *blow-up sequence* at time T if $t_i \nearrow T$ and $|\text{Rm}(x_i, t_i)| \nearrow \infty$ as $i \nearrow \infty$. A blow-up sequence has a *pointed singularity model* if the sequence of parabolically dilated metrics

$$g_i(x, t) := |\text{Rm}(x_i, t_i)| g(x, t_i + |\text{Rm}(x_i, t_i)|^{-1}t)$$

has a complete smooth limiting metric. A Ricci flow solution is said to develop a neckpinch singularity at time $T < \infty$ if there is some blow-up sequence at T whose pointed singularity model exists and is given by the self-similarly shrinking Ricci soliton on the cylinder $\mathbb{R} \times S^n$.

A neckpinch singularity is *nondegenerate* if every pointed singularity model of any blow-up sequence at T is a shrinking cylinder soliton. A nondegenerate neckpinch is a Type-I singularity. The first rigorous examples of finite-time neckpinch singularities in Ricci flow on a compact manifold were produced by Angenent and Knopf [3]. They exhibited a class of rotationally symmetric metrics on S^{n+1} ($n \geq 2$) which develop Type-I neckpinch singularities under Ricci flow. In a subsequent paper [7], the same authors proved the precise asymptotics for such neckpinch singularities.

A neckpinch singularity is *degenerate* if there is at least one blow-up sequence at T with a pointed singularity model that is not a shrinking cylinder soliton. A degenerate neckpinch is expected to be a Type-II singularity. In this paper, we construct Ricci flow solutions that encounter finite-time Type-II singularities, which can be regarded as global degenerate neckpinches on \mathbb{R}^{n+1} . Before stating our main theorem, we first recount the existing results on Type-II singularities in Ricci flow.

Daskalopoulos and Hamilton [21] showed that on \mathbb{R}^2 there exist complete noncompact Ricci flow solutions that form Type-II singularities at the rate $(T-t)^{-2}$. Their proof is particular to dimension two, in which case the Ricci

flow is conformal and the conformal factor u evolves by the logarithmic fast diffusion equation $u_t = \Delta \log u$. Assuming rotational symmetry and additional constraints, Daskalopoulos and del Pino [20] gave a precise description of the extinction profile of this maximal solution in \mathbb{R}^2 : up to proper scaling, it must be a cigar soliton in an inner region, and a logarithmic cusp in an outer region. Daskalopoulos and Šešum [22] proved the same result without assuming rotational symmetry. An extension of the results of [20, 22] was obtained by Hui [31]. The formal asymptotics of the extinction profile were derived by King [33].

In dimension three or higher, if one is willing to assume rotational symmetry of the metrics, then the Ricci flow is reduced to a parabolic equation for a scalar function. Gu and Zhu [26] proved the existence of Type-II singularities on S^{n+1} , although their work shed little light on the geometric details of such solutions. Garfinkle and Isenberg [24, 25] have conducted numerical investigations on the formation of Type-II singularities modeled by degenerate neckpinches on S^3 .

In their recent works, Angenent, Isenberg, and Knopf [5, 6] demonstrated the existence of rotationally symmetric Ricci flow solutions on S^{n+1} that develop finite-time Type-II degenerate neckpinches. Their solutions become singular at the rate $(T-t)^{-2+2/k}$ for $k \in \mathbb{N}$ and $k \geq 3$. Moreover, they were able to describe the asymptotic profiles of these solutions. The techniques in [5, 6] have been applied to study singularity formation in other geometric flows. For example, Angenent and Velázquez [1] studied the asymptotic shape of cusp singularities in the curve shortening flow. The same authors [2] constructed solutions with degenerate neckpinches to the mean curvature flow.

In this paper, we consider rotationally symmetric Riemannian metrics on \mathbb{R}^{n+1} ($n \geq 2$). We first note that Ricci flow on \mathbb{R}^{n+1} can encounter finite-time singularity. For example, take a metric on S^{n+1} as constructed in [3] and conformally open up the north pole of the sphere. This produces an initial geometry on \mathbb{R}^{n+1} , which one expects to develop finite-time Type-I neckpinch singularity under Ricci flow. Similarly, one expects that there are Ricci flow solutions that form finite-time Type-II singularities on \mathbb{R}^{n+1} . Indeed, this happens on \mathbb{R}^2 [21].

We now state our main result.

Theorem 1.1. *In each dimension $n+1 \geq 3$ and for each real number $\lambda \geq 1$, there exists an open set of complete rotationally symmetric metrics \mathcal{G}_{n+1} on \mathbb{R}^{n+1} such that the Ricci flow starting at $g_0 \in \mathcal{G}_{n+1}$ has a unique solution $g(t)$ for $t \in [0, T)$, $T < \infty$. The solution $g(t)$ develops a finite-time global singularity at time T with the following properties.*

- (1) *The singularity is Type-II with*

$$\sup_{x \in \mathbb{R}^{n+1}} |\text{Rm}(x, t)| = \frac{C}{(T-t)^{(\lambda+1)}}$$

attained at the origin, where C is a constant depending on n .

- (2) If one rescales a solution so that the distance from the origin dilates at the rate $(T - t)^{-(\lambda+1)/2}$, then the metric converges uniformly on intervals of order $(T - t)^{(\lambda+1)/2}$ to the Bryant soliton.
- (3) If one rescales a solution at the parabolic rate $(T - t)^{-1/2}$, then the metric converges uniformly to the shrinking cylinder soliton near spatial infinity.

Furthermore, the solutions exhibit the asymptotic behavior of the formal solution described in Section 3.

Remark 1.1. The singular time T is determined only by the initial radius of the asymptotic cylinder at spatial infinity. In terms of the rescaled time τ_0 (cf. Proposition 5.1), $T = e^{-\tau_0}$.

Theorem 1.1 is inspired by [5, 6]. To prove it, we begin by constructing a family of formal solutions to Ricci flow on \mathbb{R}^{n+1} with curvature blow-up rate of $(T - t)^{-(\lambda+1)}$ near the origin for each $\lambda \geq 1$, and of $(T - t)^{-1}$ near spatial infinity. Using each formal solution, we construct upper and lower barriers to the Ricci flow PDE and prove a comparison principle. Before the first singular time T , the curvatures are bounded and so the Ricci flow solution exists and is unique [40, 15]. For all initial data between the barriers, we obtain unique complete solutions to the Ricci flow whose asymptotic properties are the same as those of the formal solution.

Our result is interesting in several aspects. First of all, this shows that Type-II singularities in Ricci flow on \mathbb{R}^{n+1} can occur arbitrarily slowly with curvatures blowing up at arbitrarily fast rate. This complements the works of [5, 6]. The $\lambda = 1$ case in Theorem 1.1 can be viewed as a higher dimensional version of the result of Daskalopoulos and Hamilton [21] for rotationally symmetric solutions. The asymptotics in Theorem 1.1 can be compared to those in [20, 22]. Secondly, solutions in [6] become singular at the set of discrete rates $(T - t)^{-2+2/k}$, where $k \in \mathbb{N}$ and $k \geq 3$. In contrast, the curvature blow-up rates of the Ricci flow solutions in Theorem 1.1 form a continuum since $\lambda \in [1, \infty)$. In particular, the $\lambda = 1$ case can be thought of as the limiting case of [6, Main Theorem] as $k \nearrow \infty$. Thirdly, the analysis in [24, 25, 26] suggest that the formation of Type-II singularity on a compact manifold is an unstable property. As a result, the proof in [6] uses the somewhat indirect Wazewski retraction method. In comparison, we use a comparison principle to give a direct proof of Theorem 1.1. So one may regard the formation of Type-II singularity on a noncompact manifold to be a stable property.

Our solutions are modeled by the Bryant soliton near the origin. This is reasonable because blow-ups of Ricci flow singularities are expected, and in many cases proved, to be Ricci solitons. On \mathbb{R}^{n+1} ($n \geq 2$), the Bryant soliton is a rotationally symmetric gradient steady Ricci soliton with positive curvature operator [13, 19]. In dimension three, Bryant [13] showed that there are no other rotationally symmetric steady Ricci solitons. Other

non-rotationally symmetric solitons exist in higher dimensions [32]. Perelman [38] asked if any three-dimensional steady Ricci soliton is necessarily rotationally symmetric. Brendle [12] answered this question affirmatively by proving that on \mathbb{R}^3 , any steady gradient Ricci soliton which is κ -noncollapsed and non-flat must be rotationally symmetric. Brendle [11] also proved a higher dimensional version of this theorem. There are other uniqueness results for Bryant solitons under the additional assumption such as local conformal flatness [14], or suitable asymptotics near spatial infinity [9], or half conformal flatness [17].

In [39], Perelman described a special family of Ricci flow solutions on \mathbb{R}^3 , the so-called *standard solutions*, which are complete rotationally symmetric metrics asymptotic to a round cylinder at spatial infinity. The standard solutions are used to construct long-time solutions and to study Ricci flow with surgery [39, 34]. We will see that a subset of Ricci flow solutions in Theorem 1.1 are in fact standard solutions in the sense defined in [36]. More precisely, we have the following result. We refer the reader to Section 7 for the definition of a standard solution.

Theorem 1.2. *Let $n + 1 \geq 3$. Let \mathcal{G}_{n+1} be given as in Theorem 1.1. There exists an open set $\mathcal{G}_{n+1}^* \subset \mathcal{G}_{n+1}$ such that the Ricci flow starting at $g_0 \in \mathcal{G}_{n+1}^*$ has a unique standard solution $g(t)$ on \mathbb{R}^{n+1} for $t \in [0, T)$, $T < \infty$. Moreover, the solution $g(t)$ satisfies all the properties described in Theorem 1.1.*

Bennett Chow and Gang Tian have conjectured [35] the following.

Conjecture 1.2 (Chow-Tian). *Sequences of appropriately scaled standard solutions (as defined in [36]) with marked origins converge to Bryant solitons in a suitable sense.*

A corollary of Theorem 1.2 gives evidence in favor of the Chow-Tian conjecture.

Corollary 1.3. *In dimension $n + 1 \geq 3$, there exist standard solutions to Ricci flow whose blow-ups near the origin converge uniformly (cf. part (2) of Theorem 1.1) to the Bryant soliton.*

The paper is organized as follows. In Section 2, we describe the basic set-up and the coordinates which we will use. In Section 3, we construct a family of formal solutions with matched asymptotics. We construct sub- and supersolutions to the Ricci flow PDE in Section 4, and use these functions to construct upper and lower barriers in Section 5. In Section 6, we prove a comparison principle for the Ricci flow PDE and use it to prove Theorem 1.1. In Section 7, we relate our solutions to the standard solutions and prove Theorem 1.2.

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2. PRELIMINARIES

Let g_{sph} be the metric of constant sectional curvature one on S^n . We puncture \mathbb{R}^{n+1} at the origin and identify the remaining manifold with $(0, \infty) \times S^n$. For $x \in (0, \infty)$, we define a warped product metric

$$g := g(t, x) = \varphi^2(t, x)dx^2 + \psi^2(t, x)g_{\text{sph}}.$$

The distance s to the origin is

$$s(t, x) := \int_0^x \varphi(t, y)dy.$$

In the s -coordinate, the metric becomes

$$(2.1) \quad g = ds^2 + \psi^2(s, t)g_{\text{sph}}.$$

Extending the metric g to a smooth complete rotationally symmetric metric, which we still denote by g , on \mathbb{R}^{n+1} , then ψ necessarily satisfy the boundary conditions

$$\lim_{x \searrow 0} \psi = 0 \quad \text{and} \quad \lim_{x \searrow 0} \psi_s = 1.$$

We let $\partial_t|_x$ and $\partial_t|_s$ denote taking time derivatives while keeping x and s fixed, respectively. Then

$$[\partial_t|_x, \partial_s] = -n \frac{\psi_{ss}}{\psi} \partial_s.$$

In the s -coordinate, the Ricci flow is reduced to the scalar equation

$$(2.2) \quad \partial_t|_x \psi = \psi_{ss} - (n-1) \frac{1 - \psi_s^2}{\psi^2}.$$

The function φ , which is suppressed in the s -coordinate, evolves by

$$\partial_t|_x(\log \varphi) = n \frac{\psi_{ss}}{\psi}.$$

Let K be the sectional curvature of a two-plane orthogonal to the sphere $\{x\} \times S^n$, and let L be the sectional curvature of a tangential two-plane. Then

$$K = -\frac{\psi_{ss}}{\psi}, \quad L = \frac{1 - \psi_s^2}{\psi^2}.$$

In particular, $|\text{Rm}|^2 = 2nK^2 + n(n-1)L^2$.

Since $\lim_{x \searrow 0} \psi_s = 1$ and the metric is smooth, $\psi_s > 0$ in a neighborhood of the origin. So we can use ψ as a new coordinate there, writing

$$(2.3) \quad g = z(\psi, t)^{-1}d\psi^2 + \psi^2 g_{\text{sph}},$$

where $z(\psi, t) := \psi_s^2$. Then the sectional curvatures are computed by

$$K = -\frac{z_\psi}{2\psi}, \quad L = \frac{1-z}{\psi^2}.$$

Under the Ricci flow, the metric (2.3) evolves by (cf. [6, Section 2.2])

$$(2.4) \quad \partial_t|_\psi z = \mathcal{E}_\psi[z],$$

where \mathcal{E}_ψ is the purely local quasilinear operator

$$\mathcal{E}_\psi[z] := zz_\psi\psi - \frac{1}{2}z_\psi^2 + (n-1-z)\frac{z_\psi}{\psi} + 2(n-1)\frac{(1-z)z}{\psi^2}.$$

We can split \mathcal{E}_ψ into a linear and a quadratic term:

$$\mathcal{E}_\psi[z] = \mathcal{L}_\psi[z] + \mathcal{Q}_\psi[z],$$

where

$$(2.5) \quad \mathcal{L}_\psi[z] := (n-1)\left(\frac{z_\psi}{\psi} + 2\frac{z}{\psi^2}\right),$$

$$(2.6) \quad \mathcal{Q}_\psi[z] := zz_\psi\psi - \frac{1}{2}z_\psi^2 - \frac{zz_\psi}{\psi} - 2(n-1)\frac{z^2}{\psi^2}.$$

The quadratic part defines a symmetric bilinear operator

$$(2.7) \quad \hat{\mathcal{Q}}_\psi[z_1, z_2] := \frac{1}{2}[z_1(z_2)_{\psi\psi} + z_2(z_1)_{\psi\psi} - (z_1)_\psi(z_2)_\psi] \\ - \frac{z_1(z_2)_\psi + z_2(z_1)_\psi}{2\psi} - 2(n-1)\frac{z_1z_2}{\psi^2}.$$

In particular, $\mathcal{Q}_\psi[z] = \hat{\mathcal{Q}}_\psi[z, z]$.

Throughout this paper, we use C_k ($k \in \mathbb{N}$) to denote a constant that may change from line to line. The expression “ $f \lesssim g$ ” means $f \leq C_k g$ for some constant C_k .

3. THE FORMAL SOLUTION

We first briefly review the formal solution in [5, 6]. Introducing the coordinates consistent with a parabolic cylindrical blow-up:

$$(3.1) \quad u := \frac{\psi}{\sqrt{2(n-1)(T-t)}}, \quad \sigma := \frac{s}{\sqrt{T-t}}, \quad \tau := -\log(T-t),$$

then in these coordinates, equation (2.2) becomes

$$(3.2) \quad \partial_\tau|_\sigma u = u_{\sigma\sigma} - \left(\frac{\sigma}{2} + nI[u]\right)u_\sigma + \frac{u-u^{-1}}{2} + (n-1)\frac{u_\sigma^2}{u},$$

where

$$I[u](\sigma, \tau) := \int_0^\sigma \frac{u_{\hat{\sigma}\hat{\sigma}}(\hat{\sigma}, \tau)}{u(\hat{\sigma}, \tau)} d\hat{\sigma}.$$

For bounded σ , the solution to equation (3.2) is approximated by

$$u \approx 1 + \sum_{m=0}^{\infty} a_m e^{(1-m/2)\tau} h_m(\sigma),$$

where h_m is the m -th Hermite polynomial. In [5, 6], the authors assume a nondegenerate neckpinch occurs at the equator of S^{n+1} in such a way that

the Ricci flow solution *does not* approach a cylinder too quickly. So the term with $m = k$ is dominant for some specified $k \geq 3$. They then construct a formal solution with matched asymptotics in four connected regions: the outer, parabolic, intermediate, and tip regions. Their construction starts in the parabolic region which models a nondegenerate neckpinch near the equator of S^{n+1} , and ends in the tip region which models a degenerate neckpinch at one of the poles of S^{n+1} .

In this paper, we are interested in solutions that approach a cylinder “quickly”. This leads us to the following construction. We first build a model for degenerate neckpinch near the origin of \mathbb{R}^{n+1} , we then work our way out to the rest of the manifold. We will see that our formal solution is defined in two connected regions: the *interior* and *exterior* regions. It turns out, cf. the proof of Lemma 6.3, these are enough to define complete metrics on \mathbb{R}^{n+1} . One may compare to the compact case and think that in the noncompact case the parabolic and outer regions are pushed to spatial infinity.

3.1. Approximate solution in the interior region. In the interior region, which is to be specified in (4.2), we expect u to be small and introduce the variable

$$r := e^{\gamma\tau}u,$$

where $\gamma > 0$ is a constant to be specified.

In the u -coordinate, by the change of variable formulae:

$$\begin{aligned} \partial_t|_{\psi z} &= \{\partial_\tau|_{uz} + z_u(\partial_\tau|_{\psi u})\} \frac{d\tau}{dt} = \left(\partial_\tau|_{uz} + \frac{1}{2}\psi z_\psi\right) e^\tau, \\ \mathcal{E}_\psi[z] &= \frac{1}{2(n-1)} e^\tau \mathcal{E}_u[z], \end{aligned}$$

equation (2.4) becomes

$$(3.3) \quad \partial_\tau|_{uz} = \frac{1}{2(n-1)} \mathcal{E}_u[z] - \frac{1}{2}uz_u,$$

where we have used $\psi z_\psi = uz_u$.

In the r -coordinate, since

$$\begin{aligned} \partial_\tau|_{rz} &= \partial_\tau|_{uz} + z_u(\partial_\tau|_{ru}) = \partial_\tau|_{uz} - \gamma uz_u, \quad uz_u = rz_r, \\ \mathcal{E}_u[z] &= e^{2\gamma\tau} \mathcal{E}_r[z], \end{aligned}$$

equation (3.3) becomes

$$\mathcal{J}_r[z] = 0,$$

where

$$(3.4) \quad \mathcal{J}_r[z] := e^{-2\gamma\tau} \left\{ \partial_\tau|_{rz} + \left(\frac{1}{2} + \gamma\right) rz_r \right\} - \frac{1}{2(n-1)} \mathcal{E}_r[z].$$

For sufficiently large τ , the term involving $e^{-2\gamma\tau}$ becomes negligible and the equation $\mathcal{J}_r[z] = 0$ is approximated by

$$\mathcal{E}_r[z] = 0,$$

whose solutions, subject to the boundary conditions $z(0) = 1$ and $z(\infty) = 0$, are the Bryant soliton profile functions

$$z(r) = \mathfrak{B}(ar),$$

where $a > 0$ is an arbitrary constant. Each member of the one-parameter family of complete smooth metrics given by

$$g = \mathfrak{B}^{-1}(ar)dr^2 + r^2g_{\text{sph}}$$

is a scaled version of the Bryant soliton.

The function $\mathfrak{B}(r)$ is smooth and strictly monotone decreasing for all $r > 0$. Near $r = 0$, $\mathfrak{B}(r)$ has the asymptotic expansion

$$(3.5) \quad \mathfrak{B}(r) = 1 - b_2r^2 + b_3r^4 + b_3r^6 + \dots \quad \text{as } r \searrow 0,$$

where b_k 's are constants. In particular, $b_2 > 0$. Near $r = \infty$, $\mathfrak{B}(r)$ has the asymptotic expansion

$$(3.6) \quad \mathfrak{B}(r) = r^{-2} + c_2r^{-4} + c_3r^{-6} + \dots \quad \text{as } r \nearrow \infty,$$

where c_k 's are constants. In this paper, we normalize $\mathfrak{B}(r)$ by setting $c_2 = 1$. For more information on $\mathfrak{B}(r)$, we refer the reader to [5, Appendix B].

Refining the approximate solution by considering an expansion of the form

$$(3.7) \quad z = \mathfrak{B}(ar) + e^{-\tilde{\gamma}\tau}\beta_1(r) + e^{-\tilde{\gamma}\tau}\beta_2(r) + \dots,$$

where $\tilde{\gamma} > 0$, then

$$z \sim a^{-2}r^{-2} \quad \text{as } \tau \nearrow \infty, r \nearrow \infty,$$

which, in terms of the u -coordinate, is

$$z \sim a^{-2}e^{-2\gamma\tau}u^{-2} \quad \text{as } \tau \nearrow \infty, u \text{ small.}$$

In Lemma 4.1, we will use $z = \mathfrak{B}(ar) + e^{-\tilde{\gamma}\tau}\beta_1(r)$ with $\tilde{\gamma} = \lambda$, where λ is introduced in equation (3.8), to construct sub- and supersolutions in the interior region.

3.2. Approximate solution in the exterior region. We expect the exterior region, which is to be specified in (4.8), to be a time-dependent subset of the neighborhood of the origin where $1 > z > 0$ and $0 < u < 1$. In this region, z evolves by equation (3.3), i.e.,

$$\partial_\tau|_uz = \frac{1}{2(n-1)}\mathcal{E}_u[z] - \frac{1}{2}uz_u.$$

To construct a formal solution to this equation, we try the series

$$(3.8) \quad z = e^{-\lambda\tau}Z_1(u) + e^{-2\lambda\tau}Z_2(u) + \dots = \sum_{m \geq 1} e^{-m\lambda\tau}Z_m(u),$$

where $\lambda > 0$ is a constant to be chosen. We substitute this expansion into the equation above and split $\mathcal{E}_u[z]$ into linear and quadratic parts given in (2.5) and (2.6) respectively. By comparing the coefficient of $e^{-m\lambda\tau}$ in the resulting equation, we find Z_m must satisfy the ODE

$$(3.9) \quad \frac{1}{2}(u^{-1} - u) \frac{dZ_m}{du} + (u^{-2} + m\lambda) Z_m = -\frac{1}{2(n-1)} \sum_{i=1}^{m-1} \hat{\mathcal{Q}}_u[Z_i, Z_{m-i}].$$

When $m = 1$, equation (3.9) is a linear homogeneous equation

$$(3.10) \quad \frac{1}{2}(u^{-1} - u) \frac{dZ_1}{du} + (u^{-2} + \lambda) Z_1 = 0,$$

whose solutions are

$$(3.11) \quad Z_1(u) = bu^{-2}(1 - u^2)^{1+\lambda},$$

where b is an arbitrary constant that will be determined by matching considerations.

When $m = 2$, equation (3.9) becomes

$$(3.12) \quad \frac{1}{2}(u^{-1} - u) \frac{dZ_2}{du} + (u^{-2} + 2\lambda) Z_2 = \mathcal{Q}_u[Z_1],$$

where

$$(3.13) \quad \mathcal{Q}_u[Z_1] = 2b^2u^{-6}(1 - u^2)^{2\lambda} \{4 - n(1 - u^2)^2 + 2u^2(\lambda - 3) + u^4(\lambda - 1)^2\}.$$

So the solutions of equation (3.12) are

$$Z_2(u) = u^{-2}(1 - u^2)^{1+2\lambda} f(u),$$

where

$$f(u) = C_1 - 2b^2 \left(\frac{4-n}{u^2} - \frac{\lambda^2 - 1}{1 - u^2} \right) - 4(1 + \lambda)b^2 (\log(1 - u^2) - 2 \log u)$$

for an arbitrary constant C_1 .

By direction computation, we have the following.

Lemma 3.1. *If $\lambda \geq 1$, then*

$$\lim_{u \nearrow 1} \left| Z_2(u)/Z_1(u) \right| = 0.$$

So for any $\lambda \geq 1$,

$$z(u, \tau) \approx e^{-\lambda\tau} bu^{-2}(1 - u^2)^{1+\lambda} + O\left(e^{-2\lambda\tau} Z_2(u)\right)$$

is a valid approximation for $u \nearrow 1$ and τ sufficiently large. Going in the other direction, as $u \searrow 0$,

$$Z_1(u) \approx bu^{-2}, \quad Z_2(u) = O(u^{-4}),$$

so then

$$z \approx e^{-\lambda\tau} b u^{-2} + O\left(e^{-2\lambda\tau} u^{-4}\right).$$

This approximation is valid as long as

$$|e^{-\lambda\tau} b u^{-2}| \gg |e^{-2\lambda\tau} u^{-4}|,$$

which is when

$$u \gg e^{-\lambda\tau/2},$$

or equivalently, in the r -coordinate,

$$r \gg e^{(\gamma-\lambda/2)\tau}.$$

From now on, given $\lambda \geq 1$, we choose $\gamma = \lambda/2$.

3.3. Matching condition. We now match the formal solutions in the interior and the exterior regions when τ is sufficiently large. At $r = A \gg 1$, the formal solution in the interior region is approximately

$$\begin{aligned} z(A) &\approx \mathfrak{B}(aA) + e^{-\lambda\tau} \beta(A) \\ &\approx \mathfrak{B}(aA) \\ &\approx (aA)^{-2}. \end{aligned}$$

At $u = e^{-\lambda\tau/2} A$, the formal solution in the exterior region is approximately

$$\begin{aligned} z\left(e^{-\lambda\tau/2} A\right) &\approx e^{-\lambda\tau} Z_1\left(e^{-\lambda\tau/2} A\right) \\ &\approx bA^{-2}(1 - e^{-\lambda\tau} A^2)^{1+\lambda} \\ &\approx bA^{-2}. \end{aligned}$$

Thus, matching the two expressions implies that for a given constant $a > 0$, we ought to have

$$b \approx a^{-2}.$$

This relation is made more precise in Lemma 5.4.

3.4. Features of the formal solution. Our formal solution is valid for all dimensions $n+1 \geq 3$, and it is defined in the interior and the exterior regions. Cf. the proof of Lemma 6.3, the metric corresponding to the formal solution is complete on \mathbb{R}^{n+1} , and one approaches spatial infinity as $u \nearrow 1$. Also, as $u \nearrow 1$, $z(u) \searrow 0$, i.e., $\psi_s \searrow 0$, so the metric (2.1) is approaching that of a round cylinder near spatial infinity. As $u \searrow 0$ and $\tau \nearrow \infty$, $z(u) \nearrow 1$ and the formal solution z is asymptotic to a Bryant soliton profile function.

The norm of the curvature tensor achieves its maximum value at the origin O [6], where we have

$$|\text{Rm}(O, t)| = \frac{C}{(T-t)^{(\lambda+1)}}$$

for some constant C depending on n . Thus, the curvature of a Ricci flow solution that asymptotically approaches this formal solution necessarily blows up at the same rate.

4. SUB- AND SUPERSOLUTIONS

A metric of the form (2.3) evolving under the Ricci flow is determined by a profile function z which, in the u -coordinate, satisfies the quasilinear parabolic equation (3.3). In this section, we construct sub- and supersolutions to this equation in the interior and the exterior regions, respectively.

4.1. In the interior region. In the r -coordinate, where $r = e^{\lambda\tau/2}u$, z satisfies the equation $\mathcal{T}_r[z] = 0$, where the operator \mathcal{T}_r is defined in (3.4). We call z a subsolution (supersolution) of $\mathcal{T}_r[z] = 0$ if $\mathcal{T}_r[z] \leq 0$ (≥ 0).

Lemma 4.1. *Let $\lambda \geq 1$. For any $A_1 > 0$, there exist a bounded function $\beta : (0, \infty) \rightarrow \mathbb{R}$, a sufficiently small $B_1 > 0$, and a sufficiently large $\tau_1 < \infty$, all depending only on A_1 such that the functions*

$$(4.1) \quad z_{int}^\pm := \mathfrak{B}(A_1 r) \pm e^{-\lambda\tau} \beta(r)$$

are sub- (z_{int}^-) and super- (z_{int}^+) solutions of $\mathcal{T}_r[z] = 0$ in the interior region

$$(4.2) \quad \Omega_{int} := \{0 \leq r \leq B_1 e^{\lambda\tau/2}\}$$

for all $\tau \geq \tau_1$.

Proof. Let $\mathbf{B}(r) := \mathfrak{B}(A_1 r)$. For $z(r) = \mathbf{B}(r) + e^{-\lambda\tau} \beta(r)$ to be a supersolution, it suffices to show $\mathcal{T}_r[z] \geq 0$. Since $\mathbf{B}(r)$ solves $\mathcal{E}_r[z] = 0$,

$$\begin{aligned} \mathcal{T}_r [z_{int}^+] &= e^{-\lambda\tau} \left\{ -\frac{\mathcal{L}_r[\beta] + 2\hat{\mathcal{Q}}_r[\mathbf{B}, \beta]}{2(n-1)} + \frac{\lambda+1}{2} r \mathbf{B}' \right\} \\ &\quad + e^{-2\lambda\tau} \left\{ -\lambda\beta + \frac{\lambda+1}{2} r \beta' - \frac{\mathcal{Q}_r[\beta]}{2(n-1)} \right\}. \end{aligned}$$

Set $\hat{A} := 1 + \frac{\lambda+1}{2}$, and let β solve the linear inhomogeneous ODE

$$(4.3) \quad \mathcal{L}_r[\beta] + 2\hat{\mathcal{Q}}_r[\mathbf{B}, \beta] = 2(n-1)\hat{A}r\mathbf{B}'.$$

Using the definitions of \mathcal{L}_r and $\hat{\mathcal{Q}}_r$ in (2.5) and (2.6) respectively, equation (4.3) becomes

$$(4.4) \quad \begin{aligned} \mathbf{B}\beta'' + \left\{ \frac{n-1}{r} - \mathbf{B}' - \frac{\mathbf{B}}{r} \right\} \beta' \\ + \left\{ \mathbf{B}'' - \frac{\mathbf{B}'}{r} + 2(n-1)\frac{1-2\mathbf{B}}{r^2} \right\} \beta = 2(n-1)\hat{A}r\mathbf{B}'. \end{aligned}$$

Recall the asymptotic expansions of $\mathbf{B}(r)$ near $r = 0$ and $r = \infty$ given by (3.5) and (3.6), respectively. Then near $r = 0$, equation (4.4) is approximated by

$$\beta'' + \frac{n-2}{r}\beta' - \frac{2(n-1)}{r^2}\beta = -C_1r^2 \quad (C_1 > 0),$$

whose solution is

$$\beta_0 = C_2r^{1-n} + C_3r^2 - C_4r^4,$$

where C_2, C_3 are arbitrary constants and C_4 is a constant depending on C_1 . Discarding the unbounded solution and choosing $C_3 = 1$, then there exists a solution β_p of equation (4.3) with

$$\beta_p(r) = r^2 + o(r^2) \quad \text{as } r \searrow 0.$$

Near $r = \infty$, the ODE (4.4) is a perturbation of the equation

$$\frac{1}{(A_1r)^2}\beta'' + \frac{n-1}{r}\beta' + \frac{2(n-1)}{r^2}\beta = -\frac{4(n-1)\hat{A}}{(A_1r)^2},$$

whose general solution is

$$\beta_\infty(r) = C_5re^{-\alpha r^2} + C_6r \int_1^r \rho^{-2}e^{-\alpha(r^2-\rho^2)}d\rho - \frac{2\hat{A}}{A_1^2},$$

with $\alpha := \frac{n-1}{2}A_1^2$ and arbitrary constants C_5, C_6 . The second term in this expression is $O(r^{-2})$. So every solution of equation (4.3), in particular $\beta_p(r)$ given above, has the following asymptotic expansions:

$$(4.5) \quad \beta(r) = \begin{cases} r^2 + o(r^2) & \text{as } r \searrow 0, \\ -2\hat{A}/A_1^2 + o(1) & \text{as } r \nearrow \infty. \end{cases}$$

Also, the asymptotic expansions

$$-r\mathbf{B}'(r) = \begin{cases} C_7r^2 + o(r^2) & \text{as } r \searrow 0, \\ C_8r^{-2} + o(r^{-2}) & \text{as } r \nearrow \infty, \end{cases}$$

imply that

$$-r\mathbf{B}'(r) \geq C_9 \min\{r^2, r^{-2}\}.$$

Then in view of (4.5), we have for $0 < r \leq 1$,

$$\left| -\lambda\beta + \frac{\lambda+1}{2}r\beta' - \frac{\mathcal{Q}_r[\beta]}{2(n-1)} \right| \leq C_{10}r^2,$$

and hence

$$\begin{aligned} \mathcal{J}_r[z_{\text{int}}^+] &\geq -e^{-\lambda\tau}r\mathbf{B}'(r) - e^{-2\lambda\tau}C_{10}r^2 \\ &\geq e^{-\lambda\tau}r^2 \left(C_9 - e^{-\lambda\tau}C_{10} \right) \\ &> 0, \end{aligned}$$

for all $\tau \geq \tau_1$ with τ_1 sufficiently large. And for $r \geq 1$,

$$\left| -\lambda\beta + \frac{\lambda+1}{2}r\beta' - \frac{\mathcal{Q}_r[\beta]}{2(n-1)} \right| \leq C_{11},$$

so then

$$\begin{aligned} \mathcal{T}_r [z_{\text{int}}^+] &\geq -e^{-\lambda\tau} r \mathbf{B}'(r) - e^{-2\lambda\tau} C \\ &\geq e^{-\lambda\tau} \left(C_9 r^{-2} - e^{-\lambda\tau} C_{11} \right) \\ &> 0, \end{aligned}$$

if $r < B_1 e^{\lambda\tau/2}$ with constant $B_1 := \sqrt{C_9/C_{11}}$.

Therefore, z_{int}^+ is indeed a supersolution. That z_{int}^- is a subsolution is proved similarly. \square

4.2. In the exterior region. In the u -coordinate, z evolves by equation (3.3), which we rewrite as $\mathcal{D}_u[z] = 0$, where

$$(4.6) \quad \mathcal{D}_u[z] := \partial_\tau|_u z - \frac{1}{2}(u^{-1} - u)z_u - u^{-2}z - \frac{\mathcal{Q}_u[z]}{2(n-1)}.$$

We call z a subsolution (supersolution) of $\mathcal{D}_u[z] = 0$ if $\mathcal{D}_r[z] \leq 0$ (≥ 0).

Lemma 4.2. *Let $\lambda \geq 1$. Define $Z_1(u) := u^{-2}(1-u^2)^{1+\lambda}$. Given $A_2 > 0$, there exist a function $\zeta : (0, 1) \rightarrow \mathbb{R}$, a constant $B_2 > 0$, a sufficiently large $\tau_2 < \infty$, and a constant $A_3^* < \infty$ depending only on A_2 such that for any $A_3 \geq A_3^*$, the functions*

$$(4.7) \quad z_{\text{ext}}^\pm(u, \tau) := e^{-\lambda\tau} A_2 Z_1(u) \pm e^{-2\lambda\tau} A_3 \zeta(u)$$

are sub- (z_{ext}^-) and super- (z_{ext}^+) solutions of $\mathcal{D}_u[z] = 0$ in the exterior region

$$(4.8) \quad \Omega_{\text{ext}} := \left\{ B_2 \sqrt{\frac{A_3}{A_2}} e^{-\lambda\tau/2} \leq u < 1 \right\},$$

for $\tau \geq \tau_2$ where τ_2 depends only on A_2 and A_3 .

Proof. Since $A_2 Z_1$ is a solution of the ODE (3.10), we have

$$\begin{aligned} e^{2\lambda\tau} \mathcal{D}_u[z_{\text{ext}}^+] &= A_3 \left\{ -\frac{1}{2}(u^{-1} - u)\zeta' - (u^{-2} + 2\lambda)\zeta \right\} - \frac{A_2^2}{2(n-1)} \mathcal{Q}_u[Z_1] \\ &\quad - \frac{A_2 A_3}{n-1} e^{-\lambda\tau} \hat{\mathcal{Q}}_u[Z_1, \zeta] - \frac{A_3^2}{n-1} e^{-2\lambda\tau} \mathcal{Q}_u[\zeta]. \end{aligned}$$

Since $0 < u < 1$, the definition of Z_1 implies that

$$(4.9) \quad |Z_1'| \leq \frac{C_1}{u(1-u^2)} Z_1, \quad |Z_1''| \leq \frac{C_2}{u^2(1-u^2)^2} Z_1,$$

and from (3.13),

$$(4.10) \quad \left| \mathcal{Q}_u[Z_1] \right| \leq C_3 u^{-6} (1-u^2)^{2\lambda}.$$

Let $\zeta : (0, 1) \rightarrow \mathbb{R}$ be a solution of the inhomogeneous ODE

$$(4.11) \quad -\frac{1}{2}(u^{-1} - u)\zeta' - (u^{-2} + 2\lambda)\zeta = u^{-6}(1 - u^2)^{2\lambda}.$$

Then we solve the ODE to obtain

$$\zeta(u) = u^{-4}(1 - u^2)^{2\lambda}h(u),$$

where

$$h(u) = 1 - 2u^2 + C_4u^2(1 - u^2) + 2u^2(1 - u^2) [\log(1 - u^2) - 2\log u]$$

for an arbitrary constant C_4 . This implies that ζ has the asymptotic behavior

$$(4.12) \quad \zeta(u) = \begin{cases} u^{-4} + O(u^{-2} \log u) & \text{as } u \searrow 0, \\ -(1 - u^2)^{2\lambda} + O\left((1 - u^2)^{1+2\lambda} \log(1 - u^2)\right) & \text{as } u \nearrow 1. \end{cases}$$

We then have the following estimates. For $0 < u < 1/2$,

$$(4.13) \quad \left| \hat{\mathcal{Q}}_u[Z_1, \zeta] \right| \leq C_5u^{-8}, \quad \left| \mathcal{Q}_u[\zeta] \right| \leq C_6u^{-10}.$$

For $1/2 \leq u < 1$,

$$(4.14) \quad \left| \hat{\mathcal{Q}}_u[Z_1, \zeta] \right| \leq C_7(1 - u^2)^{3\lambda-1}, \quad \left| \mathcal{Q}_u[\zeta] \right| \leq C_8(1 - u^2)^{4\lambda-2}.$$

Using the definition of ζ and the estimate (4.10), we have

$$e^{2\lambda\tau} \mathcal{D}_u[z_{\text{ext}}^+] \geq (A_3 - C_3A_2^2) u^{-6}(1 - u^2)^{2\lambda} - \frac{A_2A_3}{n-1} e^{-\lambda\tau} \left| \hat{\mathcal{Q}}_u[Z_1, \zeta] \right| - \frac{A_3^2}{n-1} e^{-2\lambda\tau} \left| \mathcal{Q}_u[\zeta] \right|.$$

We choose $A_3^* = 2C_3A_2^2$. Then for $A_3 \geq A_3^*$, we have the following. For $0 < u \leq 1/2$, there exists a constant $B_2 < \infty$ such that (4.13) implies

$$e^{2\lambda\tau} \mathcal{D}_u[z_{\text{ext}}^+] \geq C_9u^{-6} \left(A_2^2 - C_5A_2A_3u^{-2}e^{-\lambda\tau} - C_6A_3^2u^{-4}e^{-2\lambda\tau} \right) \geq 0$$

for $e^{\lambda\tau}u^2 \geq B_2^2A_3/A_2$, that is,

$$B_2 \sqrt{\frac{A_3}{A_2}} e^{-\lambda\tau/2} \leq u \leq \frac{1}{2}.$$

For $1/2 \leq u < 1$, writing $v := 1 - u^2$, then in view of (4.14),

$$e^{2\lambda\tau} \mathcal{D}_u[z_{\text{ext}}^+] \geq C_{10} \left(A_2^2 - C_7A_2A_3e^{-\lambda\tau}v^{\lambda-1} - C_8A_3^2e^{-2\lambda\tau}v^{2\lambda-2} \right) v^{2\lambda} \geq 0$$

if $\tau \geq \tau_2$ with τ_2 sufficiently large.

Therefore, z_{ext}^+ is indeed a supersolution. That z_{ext}^- is a subsolution is proved similarly. \square

5. BARRIERS

We denote by z_{form} the formal solution constructed in Section 3. A lower (upper) barrier is a subsolution (supersolution) that lies below (above) z_{form} in an appropriate space-time region. The main result of this section is the following.

Proposition 5.1. *There exist a sufficiently large $\tau_0 < \infty$ and positive piecewise smooth functions $z^\pm(u, \tau)$, $0 < u < 1$ and $\tau \geq \tau_0$, such that the following are true.*

- (B1) z^\pm are upper (+) and lower (-) barriers to equation (3.3).
- (B2) $z^-(u, \tau_0) < z_{\text{form}}(u, \tau_0) < z^+(u, \tau_0)$ for $u \in (0, 1)$.
- (B3) Near $u = 0$, $z^\pm = z_{\text{int}}^\pm$; near $u = 1$, $z^\pm = z_{\text{ext}}^\pm$.
- (B4) At any $\tau \in (\tau_0, \infty)$, $\lim_{u \searrow 0} z^- = \lim_{u \searrow 0} z^+ = 1$, and $\lim_{u \nearrow 1} z^- = \lim_{u \nearrow 1} z^+ = 0$.
- (B5) At any $\tau \in (\tau_0, \infty)$, there exists a constant K independent of τ such that

$$(5.1) \quad |z_u^\pm / u|, |z_{uu}^\pm| \leq K e^{\lambda\tau},$$

at points where z^\pm are smooth.

The proposition will follow from several lemmata. We first explain the idea behind its proof. We properly order z_{ext}^\pm and z_{int}^\pm so that $z_{\text{int}}^- \leq z_{\text{int}}^+$, $z_{\text{ext}}^- \leq z_{\text{ext}}^+$. We then patch together z_{int}^+ and z_{ext}^+ near the interior-exterior interface to obtain an upper barrier. A similar patching argument yields a lower barrier.

Lemma 5.2. *Let β be defined as in Lemma 4.1. Let A_1^+ and A_1^- denote the constant A_1 in z_{int}^+ and z_{int}^- , respectively. For $A_1^- > A_1^+$, there exists $\tau_3 \geq \tau_1$ such that*

$$z_{\text{int}}^\pm = \mathfrak{B}(A_1^\pm r) \pm e^{-\lambda\tau} \beta$$

are properly ordered in Ω_{int} for all $\tau \geq \tau_3$.

Proof. For $A_1^- > A_1^+$, using the asymptotic expansions of \mathfrak{B} and β (cf. the proof of Lemma 4.1) we have the following. Near $r = 0$, with $b_2 > 0$,

$$\begin{aligned} z_{\text{int}}^+ - z_{\text{int}}^- &= \left\{ b_2 [(A_1^-)^2 - (A_1^+)^2] + 2[1 + o(1)]e^{-\lambda\tau} \right\} r^2 + O(r^4) \\ &> 0 \quad \text{as } r \searrow 0. \end{aligned}$$

Near $r = \infty$, with $\hat{A} = 1 + \frac{\lambda-1}{2}$,

$$\begin{aligned} z_{\text{int}}^+ - z_{\text{int}}^- &= [(A_1^+)^{-2} - (A_1^-)^{-2}] \left\{ r^{-2} - 2[\hat{A} + o(1)]e^{-\tau} \right\} + O(r^{-4}) \\ &> 0 \end{aligned}$$

for sufficiently large τ and r . On any bounded interval $c < r < C$ and for sufficiently large τ , it is straightforward to check that $z_{\text{int}}^+ > z_{\text{int}}^-$. Thus, the lemma follows. \square

Lemma 5.3. *Let Z_1 and ζ be defined as in Lemma 4.2. Let A_2^+ and A_2^- denote the constant A_2 in z_{ext}^+ and z_{ext}^- , respectively. For $A_2^+ > A_2^-$, if we relabel $A_3 := \max\{A_3(A_2^+), A_3(A_2^-)\}$, $B_2 := \max\{B_2(A_2^+), B_2(A_2^-)\}$, $\tau_2 := \max\{\tau_2(A_2^+), \tau_2(A_2^-)\}$, then there exists $\tau_4 \geq \tau_2$ such that*

$$z_{\text{ext}}^\pm(u, \tau) = e^{-\lambda\tau} A_2^\pm Z_1(u) \pm e^{-2\lambda\tau} A_3 \zeta(u)$$

are properly ordered in Ω_{ext}^1 for all $\tau \geq \tau_4$.

Proof. For $A_2^+ > A_2^-$, the asymptotic expansions (4.12) of ζ imply the following. As $u \searrow 0$, $z_{\text{ext}}^+ > z_{\text{ext}}^-$. As $u \nearrow 1$,

$$\begin{aligned} z_{\text{ext}}^+ - z_{\text{ext}}^- &= e^{-\lambda\tau} (1 - u^2)^{1+\lambda} \left\{ (A_2^+ - A_2^-) u^{-2} - 2A_3 e^{-\lambda\tau} (1 - u^2)^{\lambda-1} \right\} \\ &\quad + e^{-2\lambda\tau} O\left((1 - u^2)^\lambda \log(1 - u^2)\right) \\ &> 0 \end{aligned}$$

for all τ sufficiently large. On any interval $0 < a \leq u \leq b < 1$ and for sufficiently large τ , $z_{\text{ext}}^+ > z_{\text{ext}}^-$ by a direct computation. Thus, the lemma is proved. \square

For sufficiently large τ , Ω_{int} and Ω_{ext} intersect. In below, we state and prove a patching lemma for z_{int}^+ and z_{ext}^+ . We omit the patching lemma for z_{int}^- and z_{ext}^- , since its statement and proof are entirely analogous. To shorten notations, we write A_1^+ and A_2^+ as A_1 and A_2 .

Lemma 5.4. *Let $R_D := D\sqrt{A_3/A_2}$ where $D > 0$ is arbitrary. Suppose A_1 and A_2 satisfy the following inequality*

$$(5.2) \quad \left(1 + \frac{3}{8}D^{-2}\right)A_2 < A_1^{-2} < \left(1 + \frac{1}{2}D^{-2}\right)A_2.$$

Then there exists $\tau_5 := \max\{\tau_3, \tau_4\}$ sufficiently large such that

$$(5.3) \quad z_{\text{int}}^+ \leq z_{\text{ext}}^+ \quad \text{at } r = R_D,$$

$$(5.4) \quad z_{\text{int}}^+ \geq z_{\text{ext}}^+ \quad \text{at } r = 2R_D,$$

for $\tau \geq \tau_5$.

Proof. At the interface of the interior and the exterior regions, we have the following when $\tau \geq \tau_5$. From the interior region, as $r \nearrow \infty$, $\mathfrak{B}(r) = r^{-2} + c_2 r^{-4} + O(r^{-6})$, and so

$$z_{\text{int}}^+ = A_1^{-2} r^{-2} + c_2 A_1^{-4} r^{-4} + O(r^{-6}) + O(e^{-\lambda\tau}) \quad \text{as } r \nearrow \infty.$$

From the exterior region, as $u \searrow 0$, using $u = r e^{-\lambda\tau/2}$ and (4.12), we have on any compact r -interval,

$$\begin{aligned} z_{\text{ext}}^+ &= A_2 e^{-\lambda\tau} u^{-2} (1 - u^2)^{\lambda+1} + A_3 e^{-2\lambda\tau} u^{-4} (1 + O(u^2 \log u)) \\ &= A_2 r^{-2} + A_3 r^{-4} + O(\tau e^{-\lambda\tau}). \end{aligned}$$

¹In the definition (4.8) of Ω_{ext} we replace A_2 with A_2^- since $A_2^+ > A_2^-$.

Then on bounded r -interval, one has

$$r^2 (z_{\text{int}}^+ - z_{\text{ext}}^+) = (A_1^{-2} - A_2) + (c_2 A_1^{-4} + O(r^{-2}) - A_3) r^{-2} + O(\tau e^{-\lambda\tau}).$$

We can choose a constant \hat{C} so large that for

$$A_3 \geq \hat{C} A_1^{-4} \quad \text{and} \quad A_3 \geq \hat{C} \sqrt{A_2},$$

we have

$$\left| \frac{c_2 A_2}{A_3 A_1^4} + O\left(\frac{A_2^2}{A_3^2}\right) \right| \leq \frac{A_2}{2}.$$

Then at $r = R_D$,

$$\begin{aligned} R_D^2 (z_{\text{int}}^+ - z_{\text{ext}}^+) &= (A_1^{-2} - A_2) + \left[\frac{c_2 A_2}{A_3 A_1^4} + O\left(\frac{A_2^2}{A_3^2}\right) - A_2 \right] D^{-2} + O(\tau e^{-\lambda\tau}) \\ &\leq A_1^{-2} - \left(1 + \frac{1}{2} D^{-2}\right) A_2 + O(\tau e^{-\lambda\tau}), \end{aligned}$$

and at $r = 2R_D$,

$$\begin{aligned} 4R_D^2 (z_{\text{int}}^+ - z_{\text{ext}}^+) &= (A_1^{-2} - A_2) + \left[\frac{c_2 A_2}{A_3 A_1^4} + O\left(\frac{A_2^2}{A_3^2}\right) - A_2 \right] \frac{D^{-2}}{4} + O(\tau e^{-\lambda\tau}) \\ &\geq A_1^{-2} - \left(1 + \frac{3}{8} D^{-2}\right) A_2 + O(\tau e^{-\lambda\tau}). \end{aligned}$$

Now choose A_1 and A_2 according to (5.2), then the lemma follows for $\tau \geq \tau_5$. \square

Lemmata 5.2, 5.3, and 5.4 allow us to construct barriers for equation (3.3). From now on, we denote by z^+ a function defined by

$$(5.5) \quad z^+(u, \tau) := \begin{cases} z_{\text{int}}^+, & \text{if } 0 < u \leq e^{-\lambda\tau/2} R_D, \\ \min\{z_{\text{int}}^+, z_{\text{ext}}^+\}, & \text{if } e^{-\lambda\tau/2} R_D \leq u \leq 2e^{-\lambda\tau/2} R_D, \\ z_{\text{ext}}^+, & \text{if } 2e^{-\lambda\tau/2} R_D \leq u < 1, \end{cases}$$

for $\tau \geq \tau_5$. We define z^- analogously using z_{int}^- and z_{ext}^- . In particular, for $e^{-\lambda\tau/2} R_D \leq u \leq 2e^{-\lambda\tau/2} R_D$, $z^- = \max\{z_{\text{int}}^-, z_{\text{ext}}^-\}$.

Lemma 5.5. *Let $\tau \in (\tau_5, \infty)$. There exists a constant K independent of τ such that*

$$|z_u^\pm / u|, |z_{uu}^\pm| \leq K e^{\lambda\tau}$$

at points where z^\pm are smooth.

Proof. At a point where z^+ is smooth, z^+ is either z_{int}^+ or z_{ext}^+ .

Suppose z^+ is smooth at $u \in (0, 2e^{-\lambda\tau/2} R_D)$ and $z^+ = z_{\text{int}}^+$, then

$$\begin{aligned} z^+ &= \mathfrak{B}(A_1 r) + e^{-\lambda\tau} \beta(r) \\ &= 1 + C_1 r^2 + o(r^2) + e^{-\lambda\tau} (r^2 + o(r^2)) \quad \text{as } r \searrow 0, \\ &= 1 + C_1 e^{\lambda\tau} u^2 + e^{\lambda\tau} o(u^2) + u^2 + o(u^2) \quad \text{as } u \searrow 0. \end{aligned}$$

So then

$$\begin{aligned} z_u^+ &= e^{\lambda\tau} (C_2 u + o(u)) + u + o(u) \quad \text{as } u \searrow 0, \\ z_{uu}^+ &= e^{\lambda\tau} (C_3 + o(1)) + 1 + (o(1)) \quad \text{as } u \searrow 0. \end{aligned}$$

Thus, there exists a constant K_1 such that for $0 < u < 2e^{-\lambda\tau/2}R_D$,

$$(5.6) \quad |z_u^+/u|, |z_{uu}^+| \leq K_1 e^{\lambda\tau}.$$

Suppose z^+ is smooth at $u \in (e^{-\lambda\tau/2}R_D, 1)$ and $z^+ = z_{\text{ext}}^+$, then

$$z^+ = e^{-\lambda\tau} A_2 Z_1(u) + e^{-2\lambda\tau} A_3 \zeta(u),$$

where $Z_1(u) = u^{-2}(1-u^2)^{\lambda+1}$ for $\lambda \geq 1$, and $\zeta(u)$ is a smooth solution to the ODE (4.11). So then

$$\begin{aligned} |z_u^+/u| &\lesssim e^{-\lambda\tau} |Z_1'/u| + e^{-2\lambda\tau} |\zeta'/u|, \\ |z_{uu}^+| &\lesssim e^{-\lambda\tau} |Z_1''| + e^{-2\lambda\tau} |\zeta''|. \end{aligned}$$

From the definition of Z_1 , we compute

$$(5.7) \quad Z_1'/u = -2(u^{-4} + \lambda u^{-2})(1-u^2)^\lambda,$$

$$(5.8) \quad Z_1'' = 2(1-u^2)^{\lambda-1} [3u^{-4} + 3(\lambda-1)u^{-2} + (2\lambda-1)\lambda].$$

From equation (4.11), we have

$$-\frac{1}{2}\zeta' = \frac{(1-u^2)^{2\lambda-1}}{u^5} + \frac{u^{-1} + \lambda u}{(1-u^2)}\zeta.$$

Then using (4.12) we obtain, writing $v := 1-u^2$,

$$(5.9) \quad |\zeta'/u| \lesssim \begin{cases} u^{-6} + O(u^{-4} \log u) & \text{as } u \searrow 0, \\ v^{2\lambda-1} + O(v^{2\lambda} \log v) & \text{as } u \nearrow 1. \end{cases}$$

and similarly,

$$(5.10) \quad |\zeta''| \lesssim \begin{cases} u^{-6} + O(u^{-4} \log u) & \text{as } u \searrow 0, \\ v^{2(\lambda-1)} + O(v^{2\lambda-1} \log v) & \text{as } u \nearrow 1. \end{cases}$$

Thus, by (5.7)–(5.10), there exist constants K_2, K_3 such that for $e^{-\lambda\tau/2}R_D < u < 1$,

$$|z_u^+/u| \leq K_2 e^{\lambda\tau}, \quad |z_{uu}^+| \leq K_3 e^{\lambda\tau}.$$

Choose $K = \max\{K_1, K_2, K_3\}$, then the lemma is true for z^+ . The proof for z^- is similar. \square

We can now prove Proposition 5.1.

Proof of Proposition 5.1. Since $\lim_{u \searrow 0} z_{\text{int}}^\pm = 1$, $z_{\text{int}}^\pm > 0$ on $0 < r \leq 2R_D$ for sufficiently small D . Since $Z_1(u) \geq 0$, there exists a sufficiently large $\tau_0 \geq \tau_5$ such that $z_{\text{ext}}^\pm > 0$ on $e^{-\lambda\tau/2}R_D \leq u < 1$. Thus, z^\pm are positive piecewise smooth functions for $0 < u < 1$ and $\tau \geq \tau_0$. The minimum (maximum) of two supersolutions (subsolutions) is still a supersolution (subsolution), so

(B1) is true. One checks (B2)–(B4) directly using the definition of z^\pm and the properties of z_{int}^\pm and z_{ext}^\pm . (B5) follows from Lemma 5.5. \square

6. EXISTENCE AND UNIQUENESS OF COMPLETE SOLUTIONS

We first prove a comparison principle for equation (3.3). Similar results have appeared in [4, 37].

Lemma 6.1. *Let $\bar{\tau} \in [\tau_0, \infty)$ be arbitrary. Let z^\pm be two nonnegative sub- (+) and supersolutions (–) of equation (3.3) respectively. Suppose there exists a constant K such that either $|z_u^-/u|, |z_{uu}^-|$ or $|z_u^+/u|, |z_{uu}^+|$ are bounded by $Ke^{\lambda\tau}$. Moreover, assume*

(C1) $z^-(u, \tau_0) < z^+(u, \tau_0)$ for $0 < u < 1$;

(C2) $z^-(0, \tau) \leq z^+(0, \tau)$, and $z^-(1, \tau) \leq z^+(1, \tau)$ for all $\tau \in [\tau_0, \bar{\tau}]$.

Then $z^-(u, \tau) \leq z^+(u, \tau)$ in $[0, 1] \times [\tau_0, \bar{\tau}]$.

Remark 6.2. *In this lemma, we assume z^\pm are smooth. The result holds for piecewise smooth z^\pm . When applying the comparison principle, we will only evaluate z^\pm at “points of first contact with a given smooth function” which are necessarily smooth points of z^\pm for each $\tau \geq \tau_0$.*

Proof of Lemma 6.1. Suppose $|z_u^+/u|, |z_{uu}^+| \leq Ke^{\lambda\tau}$. For $\mu > 0$ to be chosen and arbitrary $\varepsilon > 0$, define a function

$$w := e^{-\mu e^{\lambda\tau}} (z^+ - z^-) + \varepsilon.$$

Then $w > 0$ on the parabolic boundary of the evolution by assumptions (C1) and (C2). We claim that $w > 0$ in $(0, 1) \times [\tau_0, \bar{\tau}]$. Suppose the contrary, there must be an interior point u_* and a first time τ_* such that $w(u_*, \tau_*) = 0$ and $w_\tau(u_*, \tau_*) \leq 0$. Moreover, at (u_*, τ_*) , we have

$$z^+ = z^- - \varepsilon e^{-\mu e^{\lambda\tau_*}}, \quad z_u^+ = z_u^-, \quad z_{uu}^+ \geq z_{uu}^-.$$

Then at (u_*, τ_*) ,

$$\begin{aligned} 0 &\geq e^{\mu e^{\lambda\tau_*}} w_\tau = (z_\tau^+ - z_\tau^-) - \lambda\mu e^{\lambda\tau_*} (z^+ - z^-) \\ &= (z^+ - z^-) \left(u^{-2} - \lambda\mu e^{\lambda\tau_*} \right) + \frac{\mathcal{Q}_u[z^+] - \mathcal{Q}_u[z^-]}{2(n-1)} \\ &= (z^- - z^+) \left\{ \lambda\mu e^{\lambda\tau_*} + \frac{(z_u^+/u) - z_{uu}^+}{2(n-1)} + \frac{z^+ + z^- - 1}{u^2} \right\} \\ &\quad + z^- (z_{uu}^+ - z_{uu}^-) \\ &\geq \varepsilon e^{-\mu e^{\lambda\tau_*}} \left\{ \lambda\mu e^{\lambda\tau_*} - \frac{Ke^{\lambda\tau_*}}{(n-1)} - \frac{1}{u_*^2} \right\}. \end{aligned}$$

Choose μ so large that $\lambda\mu > K/(n-1) + u_*^{-2}e^{-\lambda\tau_*}$, then at (u_*, τ_*) ,

$$0 \geq w_\tau > 0,$$

which is a contradiction. This proves the case $|z_u^+/u|, |z_{uu}^+| \leq Ke^\tau$.

The case when $|z_u^-/u|, |z_{uu}^-| \leq Ke^{\lambda\tau}$ is proved analogously because at the interior first contact point (u_*, τ_*) , we have

$$e^{\mu e^{\lambda\tau_*}} w_\tau = (z^- - z^+) \left\{ \mu\lambda e^{\lambda\tau_*} + \frac{(z_u^-/u) - z_{uu}^-}{2(n-1)} + \frac{z^+ + z^- - 1}{u^2} \right\} + z^+ (z_{uu}^+ - z_{uu}^-).$$

Therefore, the lemma is proved. \square

Now for any solution z of equation (3.3) we have the following.

Lemma 6.3. *Suppose $0 < z \leq z^+$, then z defines a complete rotationally symmetric metric $g := z^{-1}d\psi^2 + \psi^2 g_{sph}$ on \mathbb{R}^{n+1} .*

Proof. By definition g is rotationally symmetric. To see g is a complete metric, it suffices to show that any radial geodesic η starting from the origin has infinite length in the s -coordinate. The length of η in s -coordinate is a function of u and τ given by

$$s(u, \tau) = e^{-\tau/2} \sigma(u) = e^{-\tau/2} \int_0^u \frac{d\sigma}{d\hat{u}} d\hat{u}.$$

Since $z = \psi_s^2 = 2(n-1)u_\sigma^2$, and $0 < z \leq z^+$ by hypothesis, we have

$$\frac{\sigma(u)}{\sqrt{2(n-1)}} \geq \int_{u_0}^u \frac{1}{\sqrt{z}} d\hat{u} \geq \int_{u_0}^u \frac{1}{\sqrt{z^+}} d\hat{u}.$$

As $u \nearrow 1$,

$$\begin{aligned} z_{\text{ext}}^+ &= e^{-\lambda\tau} A_2 u^{-2} (1-u^2)^{1+\lambda} + e^{-2\lambda\tau} A_3 \zeta(u), \\ &= e^{-\lambda\tau} A_2 u^{-2} (1-u^2)^{\lambda+1} \\ &\quad + e^{-2\lambda\tau} A_3 \left\{ -(1-u^2)^{2\lambda} + O\left((1-u^2)^{2\lambda+1} \log(1-u^2)\right) \right\}. \end{aligned}$$

So for u_0 and τ_0 sufficiently large, $z^+ = z_{\text{ext}}^+$ in $[u_0, 1) \times [\tau_0, \infty)$ with

$$z_{\text{ext}}^+ \leq e^{-\lambda\tau} u^{-2} (1-u^2)^{1+\lambda} \left(\frac{3A_2}{2} \right).$$

It follows that

$$\begin{aligned} \frac{s(u, \tau)}{\sqrt{2(n-1)}} &\geq e^{-\tau/2} \int_{u_0}^u \frac{1}{\sqrt{z^+}} d\hat{u} = e^{-\tau/2} \int_{u_0}^u \frac{1}{\sqrt{z_{\text{ext}}^+}} d\hat{u} \\ &\geq \sqrt{\frac{3A_2}{2}} e^{(\lambda-1)\tau/2} \int_{u_0}^u \frac{\hat{u}}{(1-\hat{u}^2)^{(1+\lambda)/2}} d\hat{u}. \end{aligned}$$

Hence,

$$\frac{s(u, \tau)}{\sqrt{3(n-1)A_2}} \geq \begin{cases} e^{(\lambda-1)\tau/2} \{ \log(1-u_0^2) - \log(1-u^2) \}, & \lambda = 1, \\ \frac{e^{(\lambda-1)\tau/2}}{(\lambda-1)} \{ (1-u^2)^{(1-\lambda)/2} - (1-u_0^2)^{(1-\lambda)/2} \}, & \lambda > 1. \end{cases}$$

Therefore, for each $\tau \geq \tau_0$, $\lim_{u \nearrow 1} s(u, \tau) = \infty$, thus proving the lemma. \square

We are now ready to prove our main result.

Proof of Theorem 1.1. Let \hat{z}_0 be the function obtained by patching together $\mathfrak{B}(\tilde{A}_1 r)$ and $\tilde{A}_2 Z_1(u)$, where $A_1^+ < \tilde{A}_1 < A_1^-$ and $A_2^- < \tilde{A}_1 < A_2^+$. Because $z^-(u, \tau_0) < z^+(u, \tau_0)$, we can smooth out \hat{z}_0 to obtain a smooth initial profile z_0 with $0 < z^-(u, \tau_0) < z_0 < z^+(u, \tau_0)$ for $0 < u < 1$. By Lemma 6.3, z_0 determines a complete rotationally symmetric metric g^0 on \mathbb{R}^{n+1} . It is straightforward to check that g^0 has bounded sectional curvatures. Since the sectional curvatures depend smoothly on the metric, there is a neighborhood \mathcal{G}_{n+1} of g^0 in the C^2 -topology that corresponds to an open set of z all of which lie between $z^-(u, \tau_0)$ and $z^+(u, \tau_0)$.

Let $g_0 \in \mathcal{G}_{n+1}$. There exists a unique solution $g(t)$ to Ricci flow for $t \in [0, T_0)$ with $g(0) = g_0$ [40, 15]. By expression (2.1), g_0 has a ψ -profile function $\psi(s, 0) < r_0$ for some constant $r_0 > 0$. Since the metric $\tilde{g}_t = ds^2 + \tilde{\psi}(t)^2 g_{\text{sph}}$ with $\tilde{\psi}(0) \equiv r_0$ is a shrinking cylinder solution to Ricci flow on $\mathbb{R} \times S^n$, $\psi(t) \leq \tilde{\psi}(t)$ where $\tilde{\psi}(t) \searrow 0$ in finite time. So $g(t)$ encounters a global singularity.

The profile $z(u, \tau)$ of $g(t)$ is the unique solution of equation (3.3) for $0 < u < 1$ and $\tau \geq \tau_0$, with boundary conditions $z(0, \tau) = 1$ and $z(1, \tau) = 0$, and initial condition $z(u, \tau_0) = z_0$. The barriers z^\pm satisfy the hypotheses of Lemma 6.1, so $z^- \leq z(u, \tau) \leq z^+$ by the comparison principle for $\tau_0 \leq \tau < \infty$. So for $0 \leq t < T = e^{-\tau_0}$, $g(t)$ corresponding to $z(u, \tau)$ is a complete metric on \mathbb{R}^{n+1} by Lemma 6.3.

The sectional curvatures of $g(t)$ at the origin O are

$$K|_O = L|_O = \lim_{x \searrow 0} \frac{1 - \psi_s^2}{\psi^2} = \lim_{r \searrow 0} \frac{1 - z}{r^2} e^{(\lambda+1)\tau} = \frac{C}{(T-t)^{(\lambda+1)}},$$

where C is a positive constant depending on n . So part (1) of Theorem 1.1 is proved.

Since $z^- \leq z(u, \tau) \leq z^+$ for any $\tau < \infty$, the solution $z(u, \tau)$ exhibits the asymptotic behavior of z^\pm . Near the origin, $z(u, \tau)$ converges uniformly to the Bryant soliton profile function for $0 < u < R_D e^{-\lambda\tau}$ and $\tau \nearrow \infty$. Near spatial infinity, $u \nearrow 1$ while $z(u, \tau) \searrow 0$. Thus, $g(t)$ has asymptotic behavior described in parts (2) and (3) of Theorem 1.1. \square

7. RELATION TO THE STANDARD SOLUTIONS

In [39], Perelman described a special family of Ricci flow solutions, the so-called standard solutions, on \mathbb{R}^3 . These solutions are complete rotationally symmetric with nonnegative sectional curvature, and split at infinity as the metric product of a ray and the round 2-sphere of constant scalar curvature.

Consider a rotationally symmetric metric g_0 on \mathbb{R}^{n+1} with the following properties:

- (P1) $\text{Rm}_{g_0} \geq 0$ everywhere with $\text{Rm}_{g_0} > 0$ at some point.
- (P2) The curvature $|\text{Rm}_{g_0}|$ and its derivatives $|\nabla^i \text{Rm}_{g_0}|$, $i = 1, 2, 3, 4$ are bounded.

- (P3) There is a sequence of points $y_k \rightarrow \infty$ in \mathbb{R}^{n+1} such that $(\mathbb{R}^{n+1}, g_0, y_k)$ converges to $\mathbb{R} \times S^n(r_0)$, where $r_0 > 0$ is some constant, in pointed C^3 Cheeger-Gromov topology.

Following [36], a Ricci flow solution $g(t)$ whose initial condition satisfies (P1)–(P3) is called a standard solution. A standard solution of Ricci flow is unique up to the first singular time [36, 15].

Lemma 7.1. *Let \mathcal{G}_{n+1} be as in Theorem 1.1. There is an open set $\mathcal{G}_{n+1}^* \subset \mathcal{G}_{n+1}$ of metrics that satisfy properties (P1)–(P3).*

Proof. Define

$$\mathcal{G}_{n+1}^* := \{g_0 \in \mathcal{G}_{n+1} : g_0 \text{ satisfies P(1)–P(3)}\}.$$

We first show \mathcal{G}_{n+1}^* is nonempty.

Let $\tau = \tau_0$ correspond to $t = 0$. By the proof of Theorem 1.1, there exists \hat{z}_0 which is obtained by patching scaled copies of \mathfrak{B} and Z_1 . Let \hat{g}_0 be the metric determined by the profile function \hat{z}_0 . For \hat{g}_0 , $K = -(z_u/2u)e^{\tau_0} = -(z_r/2r)e^{(\lambda+1)\tau_0} > 0$ at the origin. Observe that the patching occurs in $R_D \leq r \leq 2R_D$, where $R_D = D\sqrt{A_3/A_2}$ for an arbitrary constant $D > 0$. So by the continuity of K there exists D_0 such that $K > 0$ for $0 < r \leq 2R_0$, where $R_0 := R_{D_0}$. On the other hand, where $\hat{z}_0 = A_2u^{-2}(1-u^2)^{1+\lambda}$, we have

$$K = -\frac{-z_\psi}{2\psi} = \left(-\frac{-z_u}{2u}\right)e^\tau = A_2u^{-4}(1-u^2)^\lambda(1+\lambda u^2)e^\tau > 0.$$

Hence, the piecewise smooth function \hat{z}_0 determines a metric \hat{g}_0 for which $K > 0$ in the interior of \mathbb{R}^{n+1} where \hat{g}_0 is smooth, and $K \searrow 0$ as $u \nearrow 1$, i.e., as one approaches spatial infinity. Since $z^- < z^+$, we can smooth \hat{z}_0 to obtain a smooth metric g_0 for which $K \geq 0$ everywhere with $K > 0$ at the origin, and $g_0 \in \mathcal{G}_{n+1}$. Also for this metric g_0 , because $L = (1-z)/\psi^2$, $L \geq 0$ everywhere with $L > 0$ at the origin, and $L \rightarrow 1/\psi^2$ as we approach spatial infinity. Thus, g_0 satisfies (P1).

To check (P2), we first note that $|\text{Rm}_{g_0}|$ is bounded by the proof of Theorem 1.1. The derivatives $\nabla^i \text{Rm}_{g_0}$, $i \in \mathbb{N}$, are determined by $\partial_s^i K$ and $\partial_s^i L$. Recall that $s(u, \tau) = e^{-\tau/2}\sigma(u)$ and $z = 2(n-1)u_\sigma^2$, then at time τ_0 ,

$$\frac{\partial s}{\partial u} = \frac{\partial \sigma}{\partial u} e^{-\tau_0/2} = e^{-\tau_0/2} \frac{\sqrt{2(n-1)}}{\sqrt{z_0}}.$$

Since $0 < z^- < z_0 < z^+$, arguing as in the proof of Lemma 6.3, there exists $u_0 \in (0, 1)$ such that for $u_0 \leq u < 1$,

$$(7.1) \quad \frac{\partial u}{\partial s} \lesssim \sqrt{z_{\text{ext}}^+} \lesssim (1-u^2)^{(\lambda+1)/2}, \quad \lambda \geq 1.$$

By the chain rule that $\partial_s = (\partial u / \partial s) \partial_u$, one checks that

$$|K_s| \lesssim (1-u^2)^{(3\lambda-1)/2}, \quad |L_s| \lesssim (1-u^2)^{(\lambda+1)/2} + O\left((1-u^2)^{(3\lambda+1)/2}\right).$$

So K_s and L_s are bounded. Similarly, direct computation shows that $|\partial_s^i K|$ and $|\partial_s^i L|$ are bounded for $i = 2, 3, 4$. If $0 < u \leq u_0$, then we are looking at a compact subset of \mathbb{R}^{n+1} where $|\nabla^i \text{Rm}_{g_0}|$ are bounded for any $i \in \mathbb{N}$ because g_0 is smooth. Thus, g_0 satisfies (P2).

To check (P3), we let y_k to be a sequence of points whose s -coordinates $s_k \nearrow \infty$ as $k \nearrow \infty$. Let $U_k := (-k, \infty) \times S^n(r_0)$ be an exhaustion of the cylinder $\mathbb{R} \times S^n(r_0)$. Then the translation map $s \mapsto (s + 2k)$ defines an embedding $\psi_k : U_k \rightarrow \mathbb{R}^{n+1}$, $V_k := \psi_k(U_k) = (k, \infty) \times S^n(r_0)$. We need to show for $g_0 = ds^2 + \psi(s, \tau_0)^2 g_{\text{sph}}$,

$$(7.2) \quad g_0|_{V_k} \xrightarrow{C^3} g_{\text{cyl}} \text{ on compact subsets of } \mathbb{R} \times S^n(r_0),$$

where $g_{\text{cyl}} = ds^2 + r_0^2 g_{\text{sph}}$ is the standard metric on the round cylinder. Without loss of generality, assume $r_0 = 1$. For all sufficiently large k , the u -coordinate of y_k is bounded between u_0 and 1. At initial time, $\psi \lesssim u$, so $\partial_s^i \psi \lesssim \partial_s^i u$, $i \in \mathbb{N}$. Then at $\tau = \tau_0$, as $s_k \nearrow \infty$, $\psi \lesssim u \nearrow 1$, and hence from (7.1), we obtain

$$\begin{aligned} \psi_s &\lesssim u_s \lesssim (1 - u^2)^{\frac{(\lambda+1)}{2}} \searrow 0, \\ \psi_{ss} &\lesssim u_{ss} \lesssim (1 - u^2)^\lambda \searrow 0, \\ \psi_{sss} &\lesssim u_{sss} \lesssim (1 - u^2)^{\frac{(3\lambda-1)}{2}} \searrow 0. \end{aligned}$$

This shows (7.2)², and hence g_0 satisfies (P3).

Therefore, $g_0 \in \mathcal{G}_{n+1}^*$. Since the sectional curvatures depend smoothly on the metric, there is an open set \mathcal{G}_{n+1}^* of g_0 in C^6 -topology such that any $g \in \mathcal{G}_{n+1}^*$ satisfies P(1)–P(3). Hence, the lemma follows. \square

We now prove Theorem 1.3.

Proof of Theorem 1.3. By Lemma 7.1, the Ricci flow solution $g(t)$ on \mathbb{R}^{n+1} starting at $g_0 \in \mathcal{G}_{n+1}^*$ is a standard solution. Since $g_0 \in \mathcal{G}_{n+1}$, Theorem 1.1 applies to $g(t)$, and so Theorem 1.3 follows. \square

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²One checks that $\partial_s^i \psi \lesssim \partial_s^i u \lesssim (1 - u^2)^{1+i\frac{(\lambda-1)}{2}} \searrow 0$ as $u \nearrow 1$, so we in fact have convergence in pointed C^∞ Cheeger-Gromov topology.

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