p-Jones-Wenzl projectors

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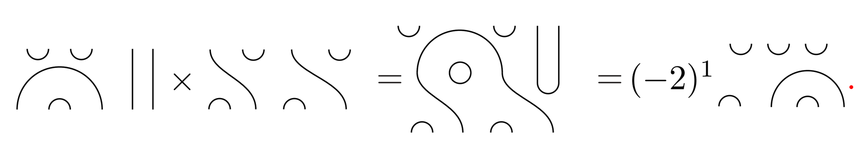
1 The Temperley-Lieb algebra

Let $n \in \mathbb{N}$. The *Temperley-Lieb algebra* TL_n on n strands is the $\mathbb{Q}$-vector space with basis the set of all non-crossing pairings on a rectangle with n marked points on the top and n marked points on the bottom. It is also a $\mathbb{Q}$-algebra. The multiplication of two basis elements is understood in the following example.

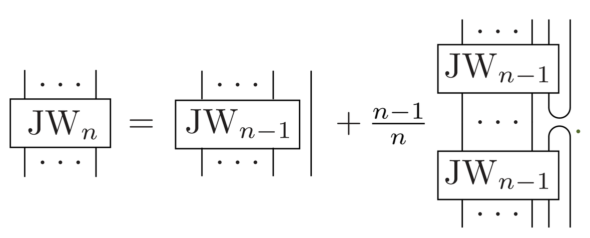
Example 1.1. In TL_8, we have the multiplication of the following two basis elements.



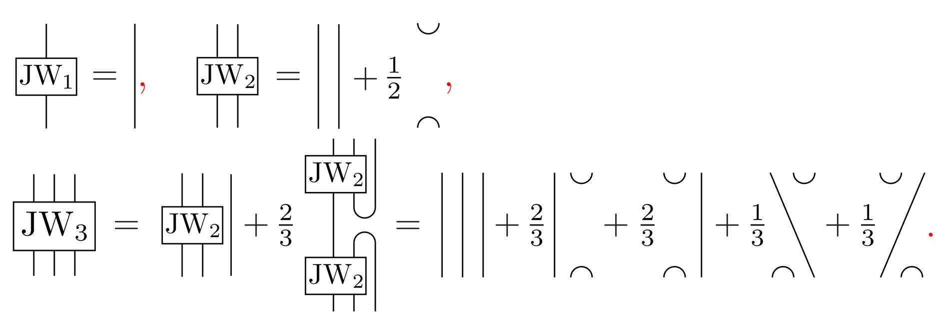
If a number of r loops (pieces isomorphic to $\mathbb{S}^1$) are deleted, we put a $(-2)^r$ to the diagram without loops. Here $r = 1$.

2 Jones-Wenzl projectors

Proposition 2.1 ([Wen87]). *There is a unique non-zero idempotent* $\mathrm{JW}_n \in TL_n$*, called the* Jones-Wenzl projector on n strands*, such that the following recursion follows. Let us define* $\mathrm{JW}_1 \in TL_1$ *as a single vertical line, and for* $n \geq 2$ *we have*



Example 2.2. Let us compute the first three Jones-Wenzl projectors in terms of the $\mathbb{Q}$-basis



Then JW_2 represents the vector $(1, 1/2)$ in $\mathbb{Q}^2$, and JW_3 represents the vector $(1, 2/3, 2/3, 1/3, 1/3)$ in $\mathbb{Q}^5$.

Question 2.3. *Is it possible to imitate this definition of* JW_n *in such a way the respective vectors in* $\mathbb{Q}^N$ *are defined over the field* $\mathbb{F}_p$ *for* p *prime? (Note* JW_3 *above is not defined over* $\mathbb{F}_3$*, since 3 is not invertible in* $\mathbb{F}_3$*.)*

Answer. Yes. We can use the p-adic expansion of n.

3 p-Fathers, p-Adams, and p-supports

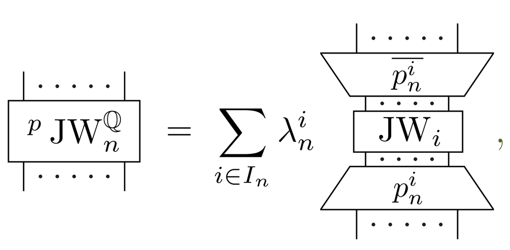
Let n be a fixed natural number and p a prime. If $n+1 = \sum_{i=m}^{r} a_i p^i$ is the p-adic expansion of $n+1$ with $a_m \neq 0$, then the *p-father* $f_p[n]$ (or just $f[n]$) of n is defined as $-1 + \sum_{i=m+1}^{r} a_i p^i$. We denote $f[n] \succ n$. If $n+1 = jp^i$ for some $0 < j < p$ and some $i \in \mathbb{N}$, we say that n is a *p-Adam* (because it has no father).

The relation $\succ$ extends to a partial order $>_p$ on $\mathbb{N}$ with p-Adams as maximal elements. We define the *p-support* of n as the set $I_n = \{a_i p^i \pm a_{i-1} p^{i-1} \pm \cdots \pm a_1 p \pm a_0 - 1\}$.

Example 3.1.
- For $p = 2$. We can compute $f[6]$. Since $6+1 = 4+2+1$, then $f[6] = (4+2)-1 = 5$. In a similar way, $f[5] = 3$, and 3 is a 2-Adam (and also the *grandfather* of 6). Therefore, $3 \succ 5 \succ 6$. The supports are: $I_6 = \{0, 2, 4, 6\}$, $I_5 = \{1, 5\}$, and $I_3 = \{3\}$.
- For $p = 3$. We can compute $f[13]$. Since $13+1 = 9+3+2$, then $f[13] = (9+3)-1 = 11$. In a similar way, $f[11] = 8$, and 8 is a 3-Adam. Therefore, $8 \succ 11 \succ 13$. The supports are: $I_{13} = \{3, 7, 9, 13\}$, $I_{11} = \{5, 11\}$, and $I_8 = \{8\}$.

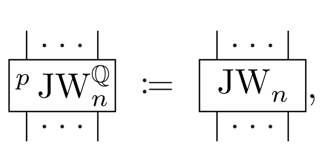
4 Recursive definition of the p-Jones-Wenzl projectors

We fix a prime number p. The *rational p-Jones-Wenzl idempotent on n strands* ${}^p\mathrm{JW}_n^{\mathbb{Q}}$ will be defined using induction on n with respect to $<_p$. Let us write it down in the form



with $\lambda_n^i \in \mathbb{Q}$, p_n a crossingless matching from n points to i points, and $\overline{p_n}$ the symmetric of p_n.

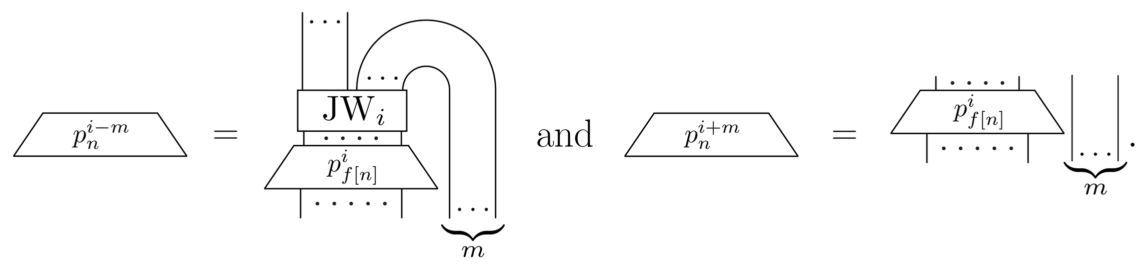
If n is a p-Adam, we define



or to be more precise, as $I_n = \{n\}$, we define $\lambda_n^n = 1$ and $p_n^n = 1_n \in TL_n$.

If n is not a p-Adam, we set $m := n - f[n]$. As $I_n = (I_{f[n]} - m) \sqcup (I_{f[n]} + m)$, for each $i \in I_{f[n]}$ we define

$$\lambda_n^{i-m} = (-1)^m \cdot \frac{i+1-m}{i+1} \lambda_{f[n]}^i, \quad \lambda_n^{i+m} = \lambda_{f[n]}^i, \quad \text{and}$$

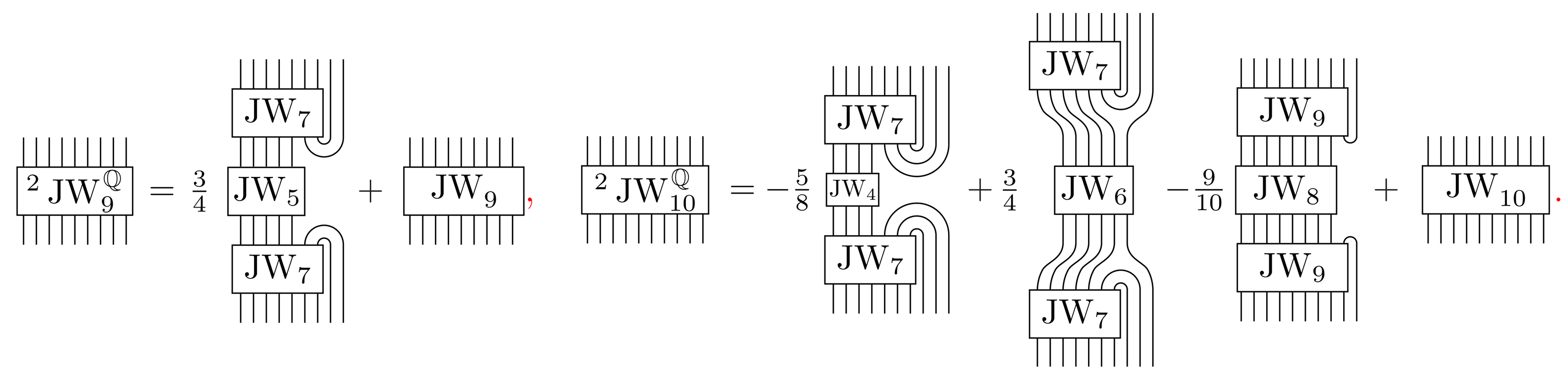


With this we finish the definition of ${}^p\mathrm{JW}_n^{\mathbb{Q}}$.

Theorem 4.1 (Main theorem [BLS19]). *For all* $n \in \mathbb{N}$*, the morphism* ${}^p\mathrm{JW}_n^{\mathbb{Q}} \in TL_n$ *is idempotent. Furthermore, if we express* ${}^p\mathrm{JW}_n^{\mathbb{Q}}$ *in the* $\mathbb{Q}$*-basis of crossingless matchings, and write each of its coefficients as an irreducible fraction* a/b*, then* p *does not divide* b*.*

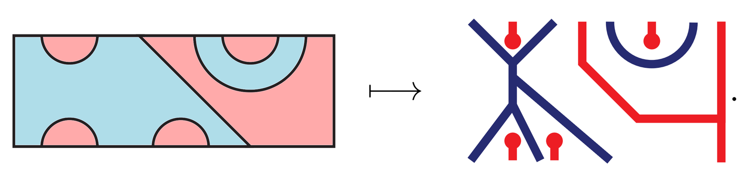
Definition 4.2 (Main definition [BLS19]). We define the *p-Jones-Wenzl projector on n-strands* ${}^p\mathrm{JW}_n \in TL_n(\mathbb{F}_p)$ as the expansion of ${}^p\mathrm{JW}_n^{\mathbb{Q}} \in TL_n$ in the $\mathbb{Q}$-basis of crossingless matchings but replacing each of the coefficients a/b (expressed as irreducible fractions) by $\overline{a} \cdot \left(\overline{b}\right)^{-1} \in \mathbb{F}_p$, where the bar means reduction modulo p.

Example 4.3. Let us compute ${}^2\mathrm{JW}_{10}^{\mathbb{Q}}$. We first note that $f_2[10] = 9$, $f_2[9] = 7$ and 7 is a 2-Adam. We have,



5 Relation to the 3-canonical basis for $\tilde{A}_1$

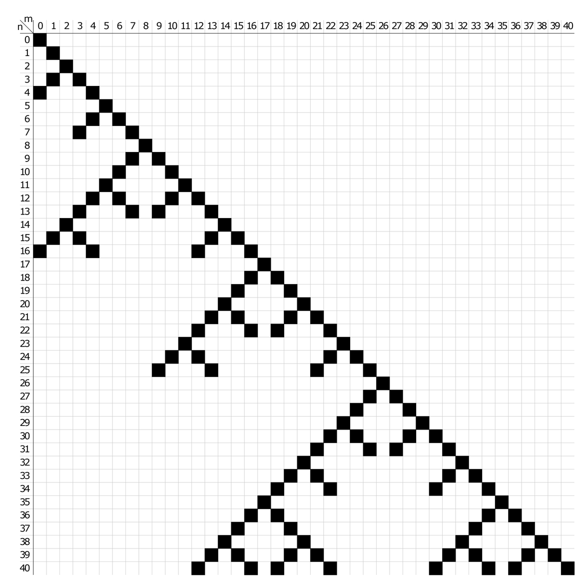
Consider the Coxeter system $\tilde{A}_1$ with $S = \{s, t\}$. Let $\mathbf{H}$ be the Hecke algebra with canonical basis $\{b_w\}$. Let $\mathcal{H}$ be the diagrammatic Hecke category. Let $(s_1, s_2, \ldots, s_k)$ be a reduced expression of w and r a simple reflection. Then $b_w b_r = b_{wr} + b_{ws_k}$, if $k > 1$ and $r = s_{k-1}$, this is the *Dyer's relation*. This equation is lifted by the recursion in section 2, using the functor of Ben Elias [Eli16] which maps the Jones-Wenzl projectors into idempotents of $\mathcal{H}$. A visualisation of the functor is



On the other hand. As is shown in [JW17] the $\tilde{A}_1$ p-canonical basis can be expressed in terms of the canonical basis in the following way:

$${}^p b_{\underline{n+1}} = \sum_{i \in -1 + I_n} b_{\underline{i}},$$

where $\underline{n}$ is the unique element w of length n such that $sw < w$. For $p = 3$, this relation can be visualised as a "fractal" relation.



To read this picture, for example, in the row labeled by 12, it means ${}^3 b_{\underline{13}} = b_{\underline{3}} + b_{\underline{7}} + b_{\underline{9}} + b_{\underline{13}}$. Note that we computed before $I_{13} = \{3, 7, 9, 13\}$.

Our recursion defined in the previous section lifts this relation for the 3-canonical basis in terms of the canonical basis in the same way the recursion in section 2 lifts the Dyer's relation.

References

[BLS19] Gaston Burrull, Nicolas Libedinsky, and Paolo Sentinelli. p-Jones-Wenzl idempotents. *Adv. Math.*, 352:246–264, 2019.

[Eli16] Ben Elias. The two-color Soergel calculus. *Compositio Mathematica*, 152(2):327–398, 2016.

[JW17] Lars Thorge Jensen and Geordie Williamson. The p-canonical basis for Hecke algebras. *Categorification and Higher Representation Theory, Contemp. Math*, 683:333–361, 2017.

[Wen87] Hans Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.