LAPLACE'S EQUATION FOR A CIRCULAR DISK

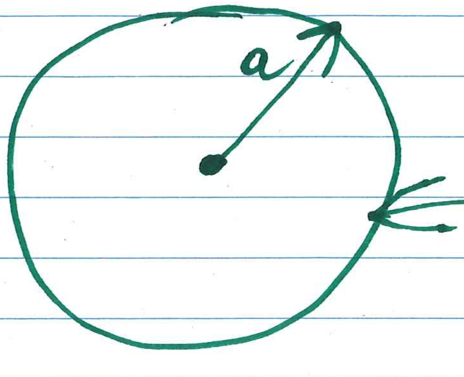
We use Laplace's eqn in POLAR coordinates

(1)

$$\boxed{\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

where $x = r\cos\theta$, $y = r\sin\theta$

Find $u = u(r,\theta)$ which satisfies $\nabla^2 u = 0$ in a DISK of radius $\underline{a}$.



B.C. $\boxed{u(a,\theta) = f(\theta)}$

For $\boxed{\begin{array}{c} 0 \le r \le a \\ -\pi \le \theta \le \pi \end{array}}$

The solution must also satisfy (non-physical) conditions at $r = 0$ and

$$\theta = \pm\pi.$$

BOUNDEDNESS

a) $u(r,\theta)$ must remain BOUNDED as $r \longrightarrow 0$

PERIODICITY

b) $u(r,-\pi) = u(r,\pi)$ for $0 \le r \le a$

c) $\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$ $\quad 0 \le r \le a$

POSSIBLE APPLICATIONS: 1) the STEADY-STATE temperature distribution in a circular disk, with specified temperature distribution $u(a,\theta)$ on the boundary.

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2) FLUID FLOW past a circular cylinder

We will sketch the solution to this problem in the next lecture.

SEPARATION OF VARIABLES

Let $u(r,\theta) = R(r)\,G(\theta)$
and substitute into Laplace's eqn:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(RG)\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}(RG) = 0$$

$$G\,\frac{1}{r}\frac{d}{dr}(rR') + \frac{R}{r^2}G'' = 0$$

$$\times\left(\frac{r^2}{RG}\right) \Longrightarrow$$

$$\frac{r}{R}\frac{d}{dr}(rR') = -\frac{G''}{G} = K$$

K = SEPARATION CONSTANT

$\Longrightarrow$ 2 ODEs

$$\boxed{G'' + KG = 0} \quad (2) \quad G = G(\theta)$$

$$\boxed{r\frac{d}{dr}(rR') - KR = 0} \quad R = R(r)$$

(3)

SOLVE THE Θ EQN

$$G'' + KG = 0$$

Recall that we want periodic BCs in θ $\Rightarrow$ want periodic solutions ($\sin\theta$ & $\cos\theta$)

$$\Rightarrow K = \lambda^2 \qquad \lambda > 0$$

$$\Rightarrow \boxed{G'' + \lambda^2 G = 0}$$

$$\Rightarrow \boxed{G(\theta) = A\cos\lambda\theta + B\sin\lambda\theta}$$

A, B arbitrary constants.

Apply the periodic BCs.

$$u(r,-\pi) = u(r,\pi) \Rightarrow \boxed{G(-\pi) = G(\pi)}$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \Rightarrow \boxed{G'(-\pi) = G'(\pi)}$$

$$G(-\pi) = A\cos\lambda\pi - B\sin\lambda\pi$$

$$G(\pi) = A\cos\lambda\pi + B\sin\lambda\pi$$

$$G(-\pi) = G(\pi) \implies \boxed{2B\sin\lambda\pi = 0}$$

Similarly

$$G'(-\pi) = \lambda A\sin\lambda\pi + B\lambda\cos\lambda\pi$$

$$G'(\pi) = -\lambda A\sin\lambda\pi + B\lambda\cos\lambda\pi$$

$$G'(-\pi) = G'(\pi) \implies \boxed{2\lambda A\sin\lambda\pi = 0}$$

In both cases, for NON-TRIVIAL SOLUTIONS, we must have

$$\sin\lambda\pi = 0$$

$$\implies \boxed{\lambda = n}, \quad n = 1, 2, 3 \ldots$$

NOTE that for $\lambda = 0$
the BCs are also satisfied
and $G(\theta) = A_0$ (arbitrary constant)

Therefore, the solution for $G(\theta)$
is :

$$G_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$$
$n = 1, 2, 3 \ldots.$

$$G_0(\theta) = A_0 \qquad n = 0.$$

Which can be combined as:

$$\boxed{G_n(\theta) = A_n \cos n\theta + B_n \sin n\theta} \tag{4}$$

$n = 0, 1, 2 \ldots\ldots$

SOLVE THE $R(r)$ EQN

Letting $K = \lambda^2 = n^2$

in eqn (3) gives

$$r \frac{d}{dr}(rR') - n^2 R = 0$$

Expanding $\Longrightarrow$

$$\boxed{r^2 R'' + rR' - n^2 R = 0} \qquad (5)$$

This is an EULER–CAUCHY type ODE.

For $n \neq 0$ TRY:

$$\boxed{R = r^p}$$

$$\Rightarrow R' = p r^{p-1} \quad ; \quad R'' = p(p-1) r^{p-2}$$

Substitute into (5)

$$\Rightarrow r^2 p(p-1) r^{p-2} + r p r^{p-1} - n^2 r^p = 0$$

Dividing by r^p gives

$$p(p-1) + p - n^2 = 0$$

$$\Rightarrow \quad p^2 - n^2 = 0$$

$$\Rightarrow \quad p = \pm n$$

$$\Rightarrow \quad \boxed{R(r) = A r^n + \frac{B}{r^n}} \qquad n \neq 0$$

A, B arbitrary constants.

Since we must have

$u(r,\theta)$ BOUNDED as $r \to 0$

$$\Rightarrow \quad B = 0$$

$$\Rightarrow \quad \boxed{R(r) = A r^n} \qquad (6)$$

$n = 1, 2, \ldots$

CASE $n = 0$

Here (5) becomes

$$r^2 R'' + r R' = 0$$

$$\Rightarrow \quad \frac{R''}{R'} = -\frac{1}{r}$$

Integrating both sides

$$\Rightarrow \quad \ln R' = -\ln r + C_1$$

Taking EXPONENTIALS on both sides

$$\Rightarrow \quad R' = \frac{C}{r}$$ $C =$ Arbitrary constant.

$$\Rightarrow \quad \boxed{R(r) = C \ln r + D}$$

C, D arbitrary constants.

For $u(r, \theta)$ to remain BOUNDED as $r \to 0 \Rightarrow C = 0$

$$\Rightarrow \quad \boxed{R(r) = A_0} \quad \text{(constant)} \qquad (7).$$

Therefore

$$u_n(r,\theta) = \begin{cases} A_0 , & n = 0 \\ A_n r^n \cos n\theta + B_n r^n \sin n\theta & n = 1, 2 \ldots \end{cases}$$

Using SUPERPOSITION.

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta}$$

$$0 \leq r \leq a$$

$$-\pi \leq \theta \leq \pi$$

APPLICATION OF BC $u(a,\theta) = f(\theta)$.

$$u(a,\theta) = \sum_{n=0}^{\infty} \left[A_n a^n\right] \cos n\theta + \sum_{n=1}^{\infty} \left[B_n a^n\right] \sin n\theta$$

$$= f(\theta)$$

This is a FOURIER SERIES for the function $f(\theta)$

$\therefore$

$$A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \qquad n = 1, 2 \ldots$$

$$B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \qquad n = 1, 2 \ldots$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta .$$
