STURM-LIOUVILLE PROBLEMS

As a result of using separation of variables in linear PDEs such as the Heat Equation or Laplace's Eqn, we come across the EIGENVALUE PROBLEM

$$\boxed{\begin{aligned} y'' + \lambda y &= 0 \\ y(0) = y(1) &= 0 \end{aligned}} \qquad (1)$$

If $\lambda = 0$ or $\lambda < 0 \Rightarrow$ TRIVIAL SOLUTIONS

For non-trivial solutions

$$\lambda_n = (n\pi)^2 > 0 \longrightarrow \text{EIGENVALUES}$$

$$y_n(x) = \sin(n\pi x) \longrightarrow \text{EIGENFUNCTIONS}$$

for $n = 1, 2, 3 \ldots$

Defining $L \equiv \dfrac{d^2}{dx^2} \Rightarrow \boxed{Ly = -\lambda y}$

Similar to a linear algebra eigenvalue problem.

MAIN PROPERTIES

1) The eigenfunctions are <u>ORTHOGONAL</u> with respect to the inner product

$$\langle y_n, y_m \rangle = \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx$$

$$= \begin{cases} 0, & n \neq m \\ 1/2, & n = m \end{cases}$$

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2) The eigenfunctions form a <u>COMPLETE SET</u>. i.e: given a function $f(x)$ we can write

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

FOURIER SINE SERIES

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3) Orthogonality allows us to calculate the FOURIER COEFFICIENTS

$$c_n = \langle f, y_n \rangle = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

The eigenvalue problem (1) is a special case of the more general problem

$$\boxed{\begin{aligned} &\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0 \\ &\text{in} \quad x_0 \leq x \leq x_1 \\ &\text{with BCs} \quad \alpha_0 y(x_0) + \beta_0 y'(x_0) = 0 \\ &\qquad\qquad\quad\ \alpha_1 y(x_1) + \beta_1 y'(x_1) = 0 \end{aligned}} \qquad (2)$$

This is called a STURM–LIOUVILLE problem.

It is convenient to define the S–L operator $L(y) = (py')' + qy$

Then, the DE becomes

$$\boxed{L(y) = -\lambda r(x) y} \qquad (3)$$

(A GENERALIZED eigenvalue problem).

NOTE that the eigenvalue problem (1) is a special case of (2) with

$$p(x) \equiv 1 \qquad \beta_0 = \beta_1 = 0$$
$$q(x) \equiv 0 \qquad \alpha_0 = \alpha_1 = 1$$
$$r(x) \equiv 1 \qquad x_0 = 0$$
$$x_1 = 1$$

A S-L Problem is called <u>REGULAR</u> if the functions p, p', q, r are CONTINUOUS on $x_0 \le x \le x_1$ and

$$p(x) > 0 \quad \text{in } [x_0, x_1]$$
$$r(x) > 0$$

Otherwise is called <u>SINGULAR</u>.

EXAMPLES

1) Vibrating String problem leads to

$$y'' + \lambda y = 0 \quad , \quad y(0) = y(1) = 0$$

$\Rightarrow$ TRIGONOMETRIC Functions [REGULAR]

2) Problems in Cylindrical coordinates may lead to BESSEL'S Equation

$$(xy')' + \left(-\frac{n^2}{x} + k^2 x\right) y = 0$$

$$0 \leq x \leq R$$

$\Rightarrow$ BESSEL Functions [SINGULAR]

3) Problems in Spherical coordinates may lead to LEGENDRE'S Equation

$$[(1-x^2)y']' + \lambda y = 0 \quad , \quad -1 \leq x \leq 1$$

$\Rightarrow$ LEGENDRE Polynomials

[SINGULAR]

LAGRANGE'S IDENTITY

Lagrange's identity is fundamental to the study of S-L problems. It will allow us to prove ORTHOGONALITY of the EIGENFUNCTIONS without solving the ODE.

Let y_1 & y_2 be functions having continuous second derivatives on $[x_0, x_1]$. Calculate the expression:

$$\int_{x_0}^{x_1} y_1 L(y_2)\,dx = \int_{x_0}^{x_1} \left[y_1 (p y_2')' + q y_1 y_2 \right] dx$$

$$= \underbrace{\int_{x_0}^{x_1} y_1 (p y_2')'\,dx}_{\text{INTEGRATE BY PARTS TWICE}} + \int_{x_0}^{x_1} q y_1 y_2\,dx.$$

$$= p\left[y_1 y_2' - y_2 y_1'\right]_{x_0}^{x_1}$$

$$+ \int_{x_0}^{x_1} \underbrace{\left[y_2 (p y_1')' + y_2 y_1 q\right]}_{y_2 L(y_1)} dx$$

$$= p\left[y_1 y_2' - y_2 y_1'\right] + \int_{x_0}^{x_1} y_2 L(y_1)\, dx$$

$$\boxed{\int_{x_0}^{x_1} \left[y_1 L(y_2) - y_2 L(y_1)\right] dx = p\left[y_1 y_2' - y_2 y_1'\right]_{x_0}^{x_1}}$$

which is LAGRANGE'S IDENTITY.

ORTHOGONALITY OF EIGENFUNCTIONS

We now use Lagrange's Identity to show that the eigenfunctions of a Regular S-L Problem corresponding to DISTINCT eigenvalues, are ORTHOGONAL.

PROOF

Consider two eigenfunctions y_n, y_m of a Regular S-L problem with corresponding eigenvalues λ_n, λ_m such that $\lambda_n \neq \lambda_m$.

Using the Operator form (3)

$\Rightarrow$

$$L(y_m) = -\lambda_m r(x) y_m \quad \ldots (a)$$

$$L(y_n) = -\lambda_n r(x) y_n \quad \ldots (b)$$

where y_n, y_m satisfy the BCs in equation (2).

Therefore:

$$y_m\, x(b) - y_n\, x(a)$$

$$\Rightarrow \quad y_m L(y_n) - y_n L(y_m)$$

$$= (\lambda_m - \lambda_n)\, r(x)\, y_m\, y_n$$

Integrating on $[x_0, x_1]$ gives

$$\int_{x_0}^{x_1} \left[y_m L(y_n) - y_n L(y_m) \right] dx$$

$$= (\lambda_m - \lambda_n) \int_{x_0}^{x_1} r(x)\, y_m\, y_n\, dx.$$

Using Lagrange's identity together with the BCs in (2), it is not hard to show that $LHS \equiv 0$

$$\Rightarrow (\lambda_m - \lambda_n) \int_{x_0}^{x_1} r(x)\, y_m\, y_n\, dx = 0$$

$$\Rightarrow \boxed{\int_{x_0}^{x_1} r(x)\, y_m\, y_n\, dx = 0}$$

$\lambda_n \neq \lambda_m$ ORTHOGONALITY CONDITION