Thursday, March 14 $*$ **Solutions** $*$ *Introduction to multiple integrals*

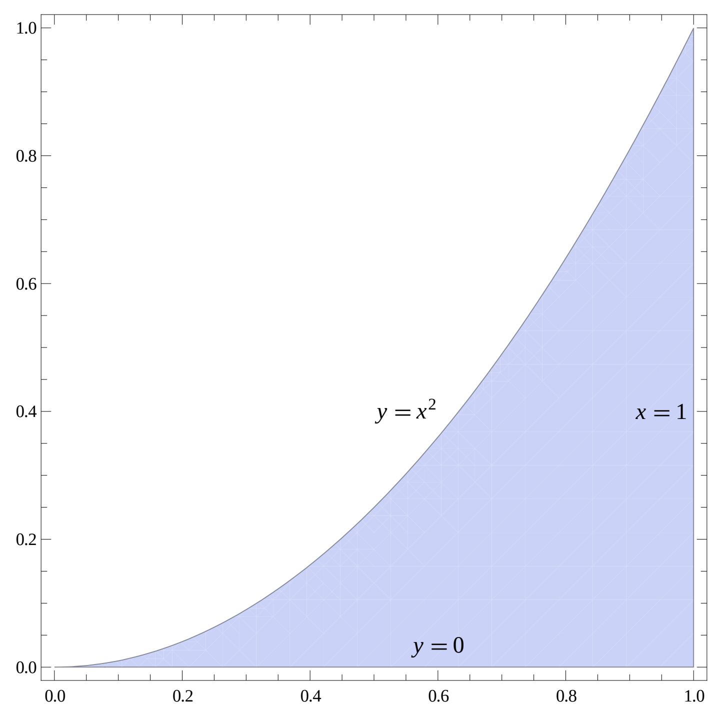
1. Evaluate the following integral by reversing the order of integration:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy.$$

(Hint: When you change to $dx\,dy$, be sure to also change the bounds of integration.)

SOLUTION:

We are integrating over the region below:



Changing the order of integration we get

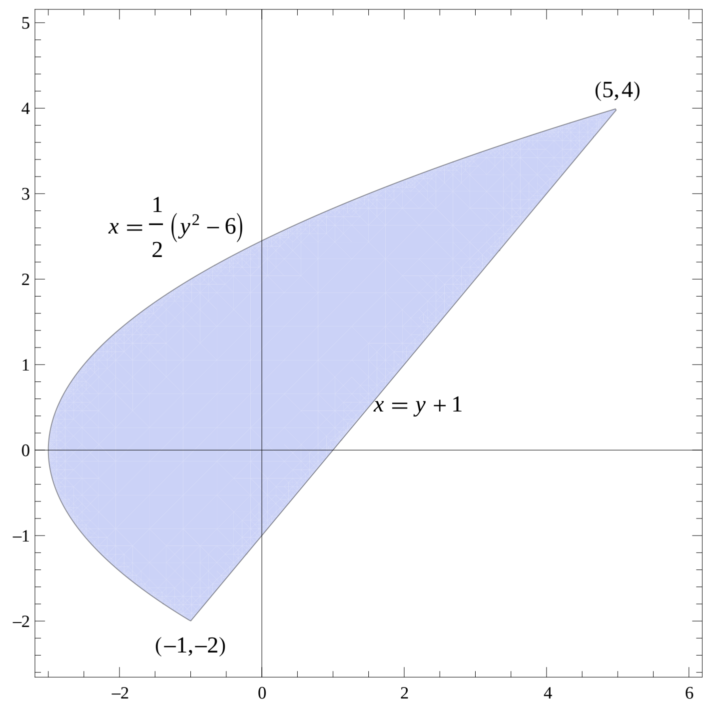
$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx$$

$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^1 x^2\sqrt{x^3+1}\,dx = 2/9[(x^3+1)^{3/2}]_0^1 = 2/9(2^{3/2}-1).$

2. Consider the region bounded by the curves determined by $-2x + y^2 = 6$ and $-x + y = -1$.

   (a) Sketch the region R in the plane.

   SOLUTION:



   (b) Set up and evaluate an integral of the form $\iint_R dA$ that calculates the area of R.

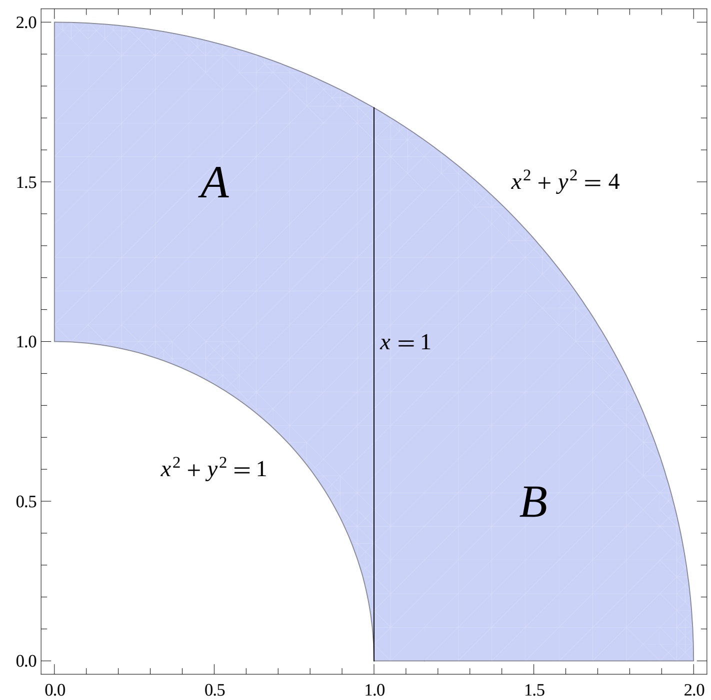
   SOLUTION:

$$\int_{-2}^{4}\int_{\frac{y^2-6}{2}}^{y+1} dx\,dy = \int_{-2}^{4} y+1-\frac{y^2-6}{2}\,dy = \left[-1/6y^3+1/2y^2+4y\right]_{-2}^{4} = 18$$

3. Consider the region R in the first quadrant which lies above the x-axis and between the circles of radius 1 and 2 centered at $(0,0)$. *Without using polar coordinates*, evaluate

$$\iint_R y\,dA.$$

SOLUTION: Notice that both the function y and the region R are symmetric about the y-axis, so we can integrate y over the half of R which lies in the first quadrant (Call this R') and double our answer. R' is shown below.



Break up R' into two parts A and B as above. Integrating, we obtain

$$\iint_R y\,dA = \iint_A y\,dA + \iint_B y\,dA = \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} y\,dy\,dx$$

$$= \int_0^1 [y^2/2]_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}\,dx + \int_1^2 [y^2/2]_0^{\sqrt{4-x^2}}\,dx = \int_0^1 3/2\,dx + \int_1^2 1/2(4-x^2)\,dx$$

$$= 7/3$$

Now double this value to get $14/3$, which is the integral over the entire region R.

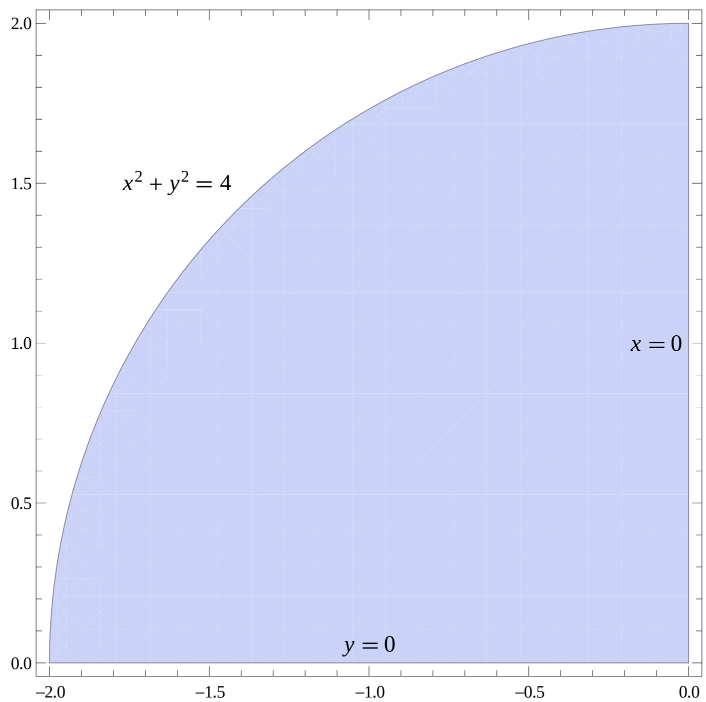
4. Evaluate

$$\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx.$$

Hint: don't do it directly.

SOLUTION:

The region over which we are integrating is:



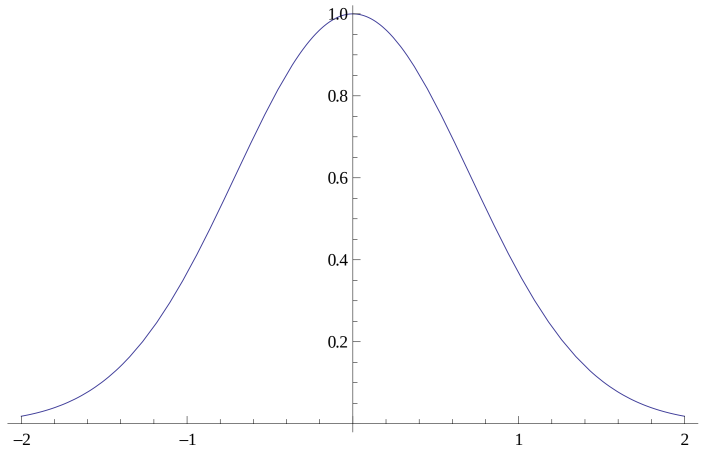
Converting to polar we get

$$\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx = \int_{\pi/2}^{\pi}\int_{0}^{2} (r^2) r\,dr\,d\theta = 2\pi$$

5. The function $P(x) = e^{-x^2}$ is fundamental in probability.

(a) Sketch the graph of $P(x)$. Explain why it is called a "bell curve."

SOLUTION:



(b) Compute $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ using the following brilliant strategy of Gauss.

i. Instead of computing I, compute $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

ii. Rewrite I^2 as an integral of the form $\iint_R f(x,y)\,dA$ where R is the entire Cartesian plane.

SOLUTION:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx$$

iii. Convert that integral to polar coordinates.

SOLUTION:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta$$

iv. Evaluate to find I^2. Deduce the value of I.

SOLUTION:

$$\int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta = 2\pi\int_0^{\infty} re^{-r^2}\,dr = 2\pi\lim_{t\to\infty}\int_0^{t} re^{-r^2}\,dr = 2\pi\lim_{t\to\infty}\left[-1/2e^{-r^2}\right]_0^t$$

$$= \pi\lim_{t\to\infty}(-e^{-t^2}+1) = \pi$$

So $I = \sqrt{\pi}$.

6. Compute $\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\, dx\, dy$.

SOLUTION:

As in the previous problem, let's convert to polar coordinates.

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2}\, dx\, dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2}\, dr\, d\theta = \pi/2 \int_0^\infty \frac{r}{(1+r^2)^2}\, dr$$

This is an improper integral, so

$$\pi/2 \int_0^\infty \frac{r}{(1+r^2)^2}\, dr = \pi/2 \lim_{t\to\infty} \int_0^t \frac{r}{(1+r^2)^2}\, dr = \pi/4 \lim_{t\to\infty} \left[-\frac{1}{1+r^2}\right]_0^t$$

$$= \pi/4 \lim_{t\to\infty} \left(-\frac{1}{1+t^2}+1\right) = \pi/4$$