Tuesday, February 19 $*$ **Solutions** $*$ *Taylor series, the* 2^{nd} *derivative test, and changing coordinates.*

1. Consider $f(x,y) = 2\cos x - y^2 + e^{xy}$.

   (a) Show that $(0,0)$ is a critical point for f.

   SOLUTION:

   $\frac{\partial f}{\partial x}|_{(0,0)} = (-2\sin x + ye^{xy})|_{(0,0)} = 0$ and $\frac{\partial f}{\partial y}|_{(0,0)} = (-2y + xe^{xy})|_{(0,0)} = 0$

   (b) Calculate each of f_{xx}, f_{xy}, f_{yy} at $(0,0)$ and use this to write out the 2^{nd}-order Taylor approximation for f at $(0,0)$.

   SOLUTION:

   The second order Taylor approximation of a function $f(x,y)$ at $(0,0)$ is given by $T_2(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + (f_{xx}(0,0)/2)x^2 + (f_{yy}(0,0)/2)y^2 + f_{xy}(0,0)xy$. For this problem we have $f_{xx} = -2\cos x + y^2e^{xy}$, $f_{yy} = -2 + x^2e^{xy}$, and $f_{xy} = e^{xy} + xye^{xy}$. So $f_{xx}(0,0) = -2 = f_{yy}(0,0)$ and $f_{xy}(0,0) = 1$. Also $f(0,0) = 3$. So the second order Taylor approximation for f at $(0,0)$ is $g(x,y) = 3 - x^2 - y^2 + xy$.

2. Let $g(x,y)$ be the approximation you obtained for $f(x,y)$ near $(0,0)$ in 1(b). It's not clear from the formula whether g, and hence f, has a min, max, or a saddle at $(0,0)$. Test along several lines until you are convinced you've determined which type it is. In the next problem, you'll confirm your answer in two ways.

   SOLUTION:

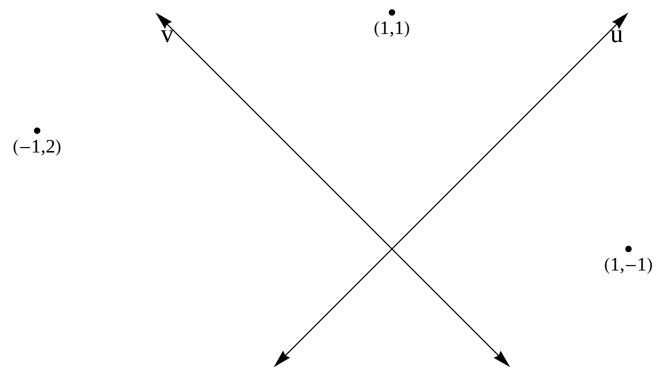
   Let's test a general line $y = mx$ which goes through $(0,0)$ as $x \to 0$. Then $g(x,mx) = 3 - x^2 - m^2x^2 + mx^2 = 3 - (1 - m + m^2)x^2$. The polynomial $1 - m + m^2$ is always positive (it opens upward and has its global minimum at $m = 1/2$ where $1 - m + m^2 > 0$). So $g(x,mx)$ is always a downward opening parabola. This suggests that $(0,0)$ is a relative maximum.

3. Consider alternate coordinates (u,v) on $\mathbb{R}^2$ given by $(x,y) = (u - v, u + v)$.

   (a) Sketch the u- and v-axes relative to the usual x- and y-axes, and draw the points whose (u,v)-coordinates are: $(-1,2)$, $(1,1)$, $(1,-1)$.

   SOLUTION:

   If we express u and v in terms of x and y we get $u = 1/2(x + y)$ and $v = 1/2(y - x)$. So the u-axis is given in x and y coordinates by all multiples of the vector $(1,1)$ and the v-axis is given by all multiples of the vector $(-1,1)$. The two axes and the points are shown below.



(b) Express g as a function of u and v, and expand and simplify the resulting expression.

SOLUTION:

$3-x^2-y^2+xy = 3-(u-v)^2-(u+v)^2+(u-v)(u+v) = 3-(u^2-2uv+v^2)-(u^2+2uv+v^2)+u^2-v^2 = 3-u^2-3v^2$.

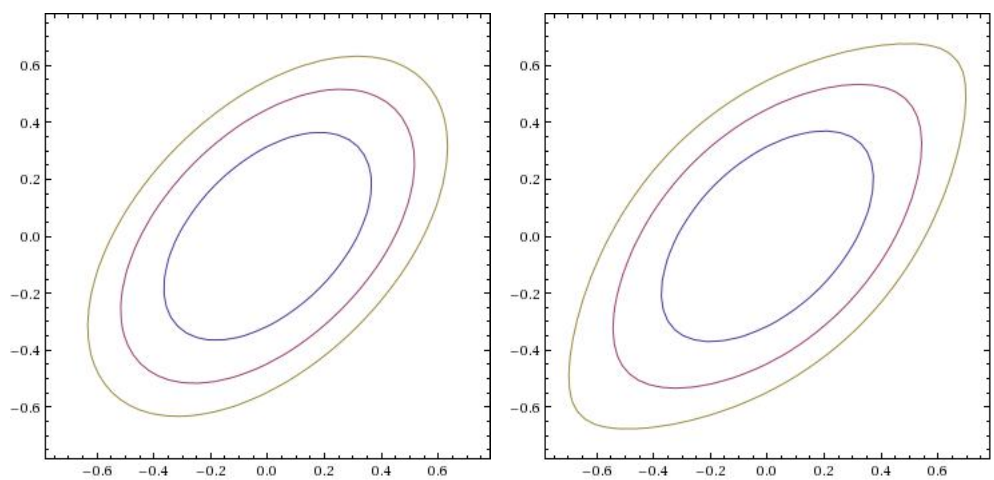
(c) Explain why your answer in 3(b) confirms your answer in 2.

SOLUTION:

This is an elliptic paraboloid (in uv coordinates) opening downward with maximum at $(0,0,3)$, so it confirms that $(0,0)$ is a local maximum ($(0,0)$ goes to $(0,0)$ under the transformation, so this reasoning makes sense).

(d) Sketch a few level sets for g. What do the level sets of f look like near $(0,0)$?

SOLUTION: The level sets are sketched for $g = 2.7, 2.8, 2.9$ on the left and for $f = 2.7, 2.8, 2.9$ on the right. The level sets for g are ellipses that approximate the level sets of f close to $(0,0)$. The ellipses shrink as they get closer to $g(x,y) = 3$, which consists of the single solution $(x,y) = (0,0)$.



(e) It turns out that there is always a similar change of coordinates so that the Taylor series of a function f which has a critical point at $(0,0)$ looks like $f(u,v) \approx f(0,0) + au^2 + bv^2$. In fact this is why the 2^{nd} derivative test works.

Double check your answer in 2 by applying the 2^{nd}-derivative test directly to f.

SOLUTION:

The Hessian $f_{xx}f_{yy}-(f_{xy})^2$ is $(-2)(-2)-1^2=3>0$ at $(0,0)$ and $f_{xx}(0,0)=-2<0$. So f has a relative maximum at $(0,0)$ as suspected.

4. Consider the function $f(x,y)=3xe^y-x^3-e^{3y}$.

   (a) Check that f has only one critical point, which is a local maximum.

   SOLUTION:

   $f_x=3e^y-3x^2$ and $f_y=3xe^y-3e^{3y}$. $f_y=0$ only if $x=e^{2y}$ and $f_x=0$ only if $e^y=x^2$. Solving these simultaneously we see that x must satisfy $(x^2)^2=(e^y)^2=x$, so $x=0,-1$, or 1. But $x=e^{2y}>0$ so the only critical point is $x=1, y=0$. Calculating, we see that $f_{xx}(1,0)=f_{yy}(1,0)=-6$ and $f_{xy}(1,0)=3$. So the Hessian $f_{xx}f_{yy}-(f_{xy})^2=36-9=27>0$ at $(1,0)$. Since $f_{xx}(1,0)<0$, the second derivative test tells us that $f(1,0)=1$ is a local maximum.

   (b) Does f have an absolute maxima? Why or why not?

   SOLUTION:

   f does not have an absolute maximum. For instance if we take the trace curve $y=0$ we get $f(x,0)=3x-x^3-1$, which is unbounded as $x\to\infty$. Absolute maxima and minima are only guaranteed over a closed and bounded set in the domain. The plane $\mathbb{R}^2$ is closed but not bounded, so there is no guarantee that a continuous function will achieve an absolute maximum or minimum over $\mathbb{R}^2$.