Last time: For $\vec{F} = \langle P, Q, R \rangle$ a vector field on $D \subset \mathbb{R}^3$

we defined $\quad \operatorname{curl} \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$\quad \operatorname{div} \vec{F} = P_x + Q_y + R_z.$

Practice: Find $\operatorname{curl} \langle P(x,y), Q(x,y), 0 \rangle$.

§ VECTOR FORMS OF GREEN'S THEOREM (§16.4)

- Identify $\mathbb{R}^2$ with $\{z = 0\} \subset \mathbb{R}^3$

  This lets us view $\vec{F}(x,y) = \langle P, Q \rangle$ as a vector field on $\mathbb{R}^3$,

  setting $\vec{F}(x,y,z) = \langle P(x,y), Q(x,y), 0 \rangle$.

  Suppose $\vec{F}$ is the velocity field of a fluid flow in $\mathbb{R}^3$.

§ CURL OF $\vec{F}$.

Let C be an oriented path such that $C = \partial B$, $C \subset \{z=0\}$.

Let $\vec{T}$ be the unit tangent vector of C

(recall: for a parametrization $\vec{r}$ of C,

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|}, \quad \text{and} \quad \underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{integral of a vector field}} = \int_C \underbrace{\vec{F} \cdot \vec{T}}_{\text{integral of a function}}\, ds\,)$$

Definition: The circulation of $\vec{F}$ along C is $\int_C \vec{F} \cdot \vec{T}\, ds$.

- positive if the fluid flows with C
- negative if the fluid flows against C.

Theorem: $$\iint_B (\operatorname{curl} \vec{F}) \cdot \vec{k}\, dA = \int_C \vec{F} \cdot \vec{T}\, ds.$$

proof. We showed $(\operatorname{curl} \vec{F}) \cdot \vec{k} = Q_x - P_y$.

$$\Rightarrow \iint_B (\operatorname{curl} \vec{F}) \cdot \vec{k}\, dA = \iint_B Q_x - P_y\, dA$$

$$= \int_{\partial B} \langle P, Q \rangle \cdot d\vec{r} \quad \text{(By Green's theorem!)}$$

$$= \int_C \vec{F} \cdot \vec{T}\, ds. \qquad \square$$

Let B be a tiny disk, and imagine a ball placed at the centre.

C

- If the fluid flows in the direction of C,
  - $\int_{\partial B} \vec{F}\cdot\vec{T}\, ds > 0$
  - $\iint_B (\text{curl}\,\vec{F})\cdot k > 0$

    $\Leftrightarrow \text{curl}\,\vec{F}\cdot k = Q_x - P_y > 0$, so $\text{curl}\,\vec{F}$ points upward (for $\underline{\underline{\text{tiny}}}$ B).

  - ball rotates counter clockwise.

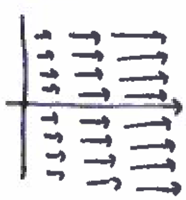
Similarly, the ball rotates clockwise $\Leftrightarrow$ curl $\vec{F}$ points downward.

slide.

For general $\vec{F}$ on $\mathbb{R}^3$:

- $\text{curl}(\vec{F})(P)$ points in the direction of axis of rotation of a small ball at P.
- the direction is determined by the Right Hand Rule.
- the magnitude $|\text{curl}\,\vec{F}\,(P)|$ corresponds to the speed of rotation.

<u>Examples:</u> 1) Let $\vec{F} = \langle x, 0, 0\rangle$.



<u>Geometrically</u>:

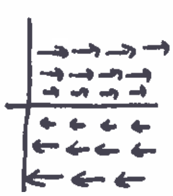
No rotation $\Rightarrow$ $\text{curl}\,\vec{F} = \langle 0,0,0\rangle$.

<u>Check algebraically</u>:

$$\text{curl}\,\vec{F} = \langle 0-0, (x)_z - 0, 0-(x)_y\rangle$$

$$= \langle 0, 0, 0\rangle$$

2) Let $\vec{F} = \langle y, 0, 0\rangle$



Q Does curl $\vec{F}$ point up, point down, or vanish?

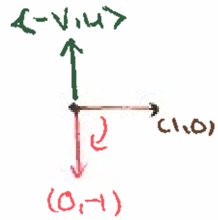
§ What about Div $\vec{F}$??

Let $\langle u,v\rangle$ be any vector in $\mathbb{R}^2$.

Rotating $\langle u,v\rangle$ 90 degrees clockwise or counter clockwise gives a vector of the same length, orthogonal to $\langle u,v\rangle$

clockwise: $\boxed{\langle v,-u\rangle}$

counter clockwise $\boxed{\langle -v,u\rangle}$



E.g. consider B and ∂B, a curve parametrized by

$$\vec{r}(t) = \langle x(t), y(t)\rangle.$$

· if we ~~rotate~~ have the unit tangent vector

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{\langle x', y'\rangle}{\sqrt{(x')^2 + (y')^2}}$$

· if we rotate 90° clockwise, we get

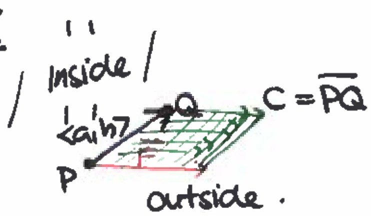
$$\boxed{\vec{n} = \frac{\langle y', -x'\rangle}{\sqrt{(x')^2 + (y')^2}}}$$ ← "unit outward normal vector"

- points away from B.

Definition: the flux of $\vec{F}$ across ∂B is $\displaystyle\int_{\partial B} \vec{F}\cdot\vec{n}\, ds$.

Geometric meaning: the flux is the amount of fluid flowing out of B in unit time.

Why?



Suppose $\vec{F}$ is constant.

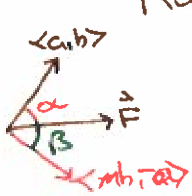
The fluid flowing across $C = \overline{PQ}$ in unit time is the (signed) area of the parallelogram with edges $\langle a,b\rangle = \overline{PQ}$ and $\vec{F}$.

The signed area is

$$|\langle a,b\rangle|\cdot|\vec{F}|\ \sin\alpha = |\langle a,b\rangle|\cdot|\vec{F}|\ \cos(\beta)$$

$$= |\langle b,-a\rangle|\cdot|\vec{F}|\cos(\beta)$$

$$= \vec{F}\cdot\langle b,-a\rangle$$



Theorem $\iint_B \operatorname{div}\vec{F}\, dA = \int_{\partial B} \vec{F}\cdot\vec{n}\, ds.$

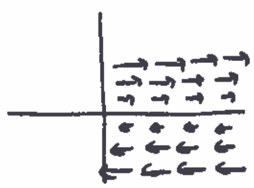
proof. $\int_{\partial B} \vec{F}\cdot\vec{n}\, ds = \int_a^b (\vec{F}\cdot\vec{n})|\vec{r}\,'(t)|\, dt$

$= \int_a^b \frac{Py' - Qx'}{|\vec{r}\,'|}|\vec{r}\,'|\, dt = \int_a^b (Py' - Qx')\, dt$

$= \int_{\partial B} P\,dy - Q\,dx = \iint_B (P_x + Q_y)\, dA = \iint_B \operatorname{div}\vec{F}\cdot dA$ ∎.

$\vec{G} = \langle -Q, P\rangle$ Green's Theorem!

Example: $\vec{F} = \langle y, 0, 0\rangle$

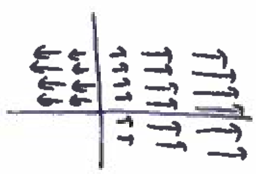


· fluid going in = fluid coming out

$\Rightarrow$ we guess $\operatorname{div}\vec{F} = 0$.

Check: $\operatorname{div}\vec{F} = y_x + 0_y + 0_z = 0$ ☺.

$\boxed{L}$ $\vec{F} = \langle x, 0, 0\rangle$



if $\operatorname{div}\vec{F}$ positive, negative, or zero?